University of Pittsburgh

Algorithms and Data Structures 2
CS 1501

Pitt SCI
5 YEARS

Spring 2023

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(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

Announcements

- Upcoming Deadlines
  - Lab 10: Tuesday 4/11 @ 11:59 pm
  - Homework 11: this Friday @ 11:59 pm
  - Assignment 4: this Friday @ 11:59 pm
    - Support video and slides on Canvas + Solutions for Labs 8 and 9
  - Midterm Question Reattempts: Monday 4/17 @ 11:59 pm
    - up to **7 points** back
    - Please use GradeScope's Regrade Requests for each question **individually**

Previous Lecture …

Weighted Shortest Paths problem

- Dijkstra’s single-source shortest paths algorithm
  - Real-world optimizations
- Bellman-Ford’s shortest paths algorithm
  - correct with negative edge weights
  - negative cycles

This Lecture …

Dynamic Programming

- Unbounded Knapsack
- 0/1 Knapsack
- Subset Sum
- Edit Distance
- Longest Common Subsequence

Let's change focus into a different method of problem solving

We will get back to graphs in the last week!

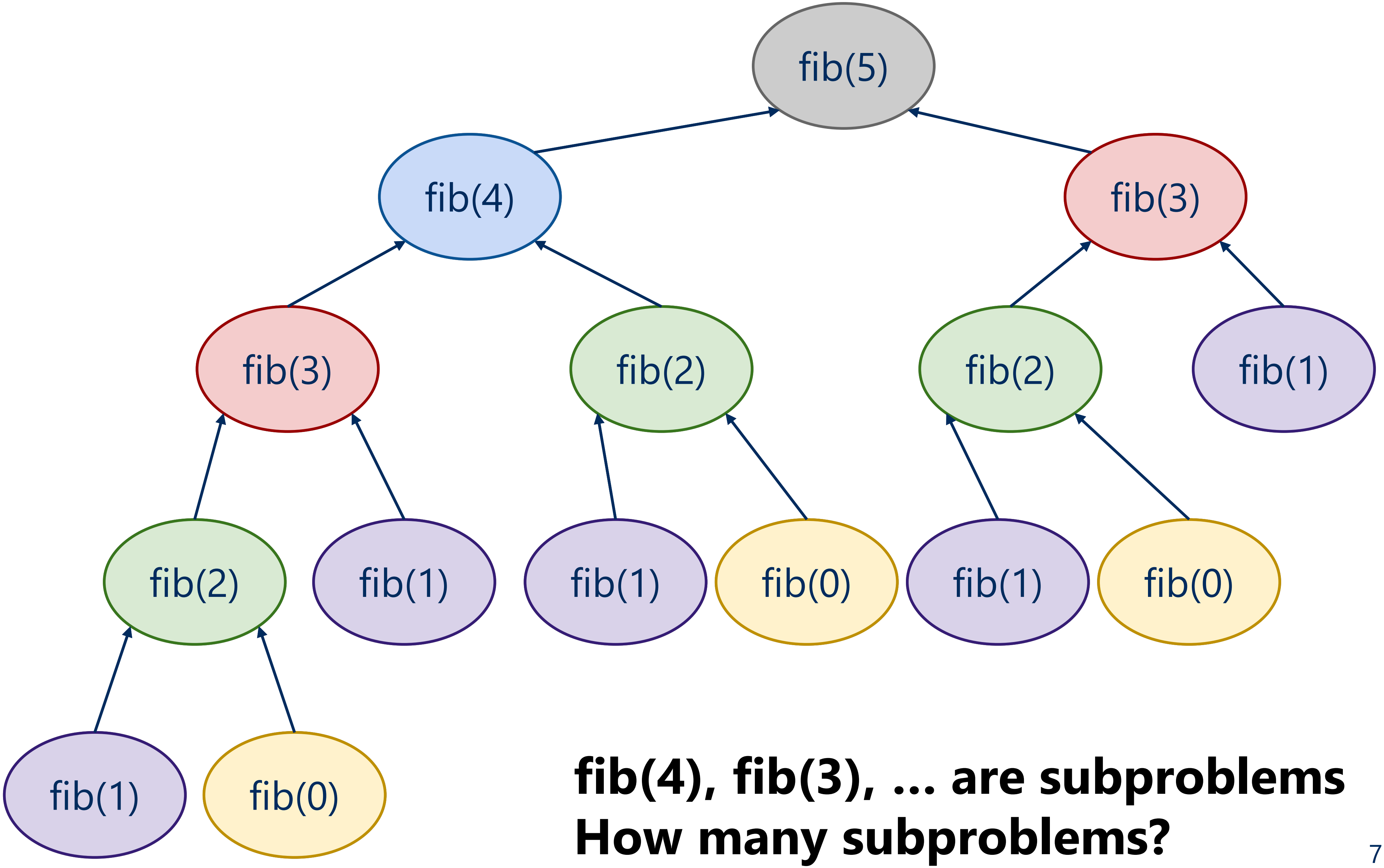
Consider computing the n^{th} Fibonacci number

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, - -

```
int fib(n) {
    if (n == 0) { return 0 };
    else if (n == 1) { return 1 };
    else {
        return fib(n - 1) + fib(n - 2);
    }
}
```

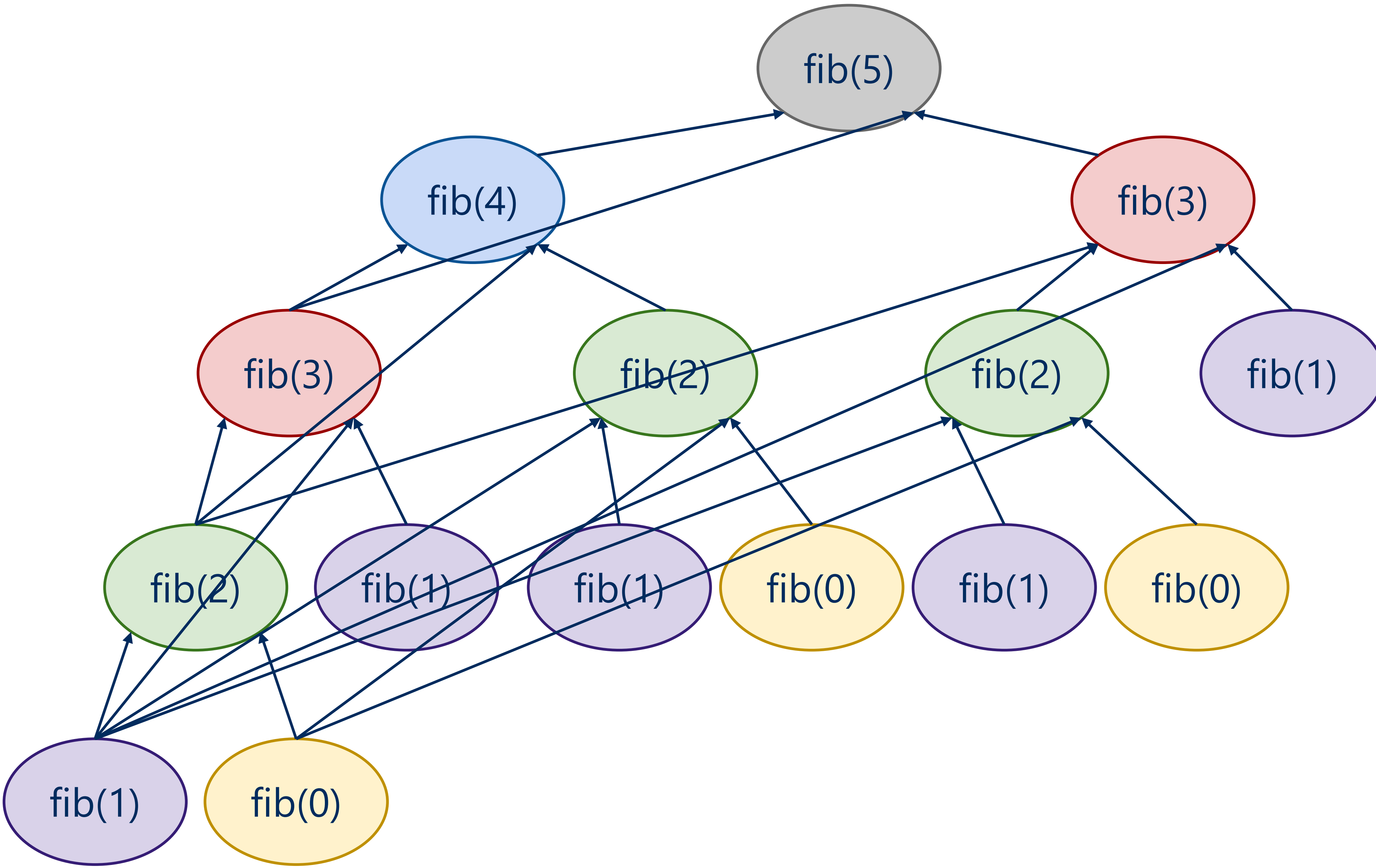
- What is the running time?
- What does the call tree for n = 5 look like?

fib(5)



fib(4), fib(3), … are subproblems
How many subproblems?

How do we improve?



Memoization: save solutions for solved subproblems

```
int[] F = new int[n+1];
F[0] = 0;
F[1] = 1;
for(int i = 2; i <= n; i++) { F[i] = -1 };

int fib_mem(n) {
   if (F[n] == -1) {
         F[n] = fib_mem(n-1) + fib_mem(n-2);
   }
   return F[n];
}
```

- Each subproblem solved once!
- What is the running time?

Note that we can also do this bottom-up!

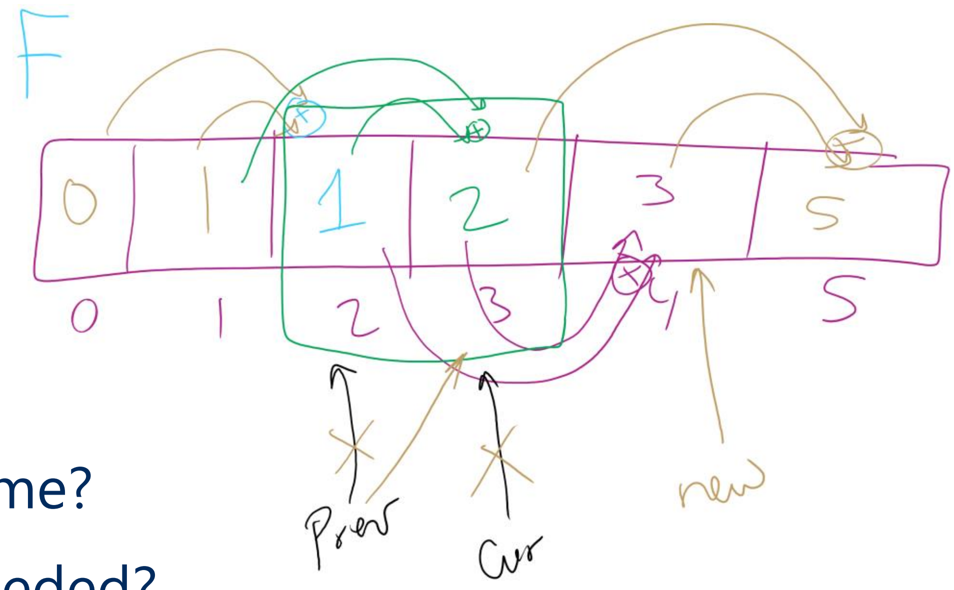
```
int[] F = new int[n+1];
F[0] = 0;
F[1] = 1;

int bottomup_fib(n) {
     for(int i = 2; i <= n; i++) {
          F[i] = F[i-1] + F[i-2];
     }
     return F[n];
}
```

- Each subproblem solved once!
- What is the running time?
- How much space is needed?

Can we improve this bottom-up approach?

```
int improve_bottomup_fib(n) {
    if (n == 0) return 0;
    if (n == 1) return 1;
    int prev = 0; int cur = 1;
    for (int i = 0; i < n; i++){
        int new = prev + cur;
        prev = cur;
        cur = new;
    }
    return cur;
}
```

- What is the running time?
- How much space is needed?



Recap …

- Dynamic Programming
  - avoid solving the same subproblem twice
  - iterative:
    - start with smaller subproblems then larger subproblems, …
  - sometimes possible to optimize space needed

Recap …

- Fibonacci
  - started with inefficient recursive solution
    - solves same subproblems multiple times
  - memoization solution:
    - efficient: solves each subproblem once
    - still recursive
  - dynamic programming:
    - efficient: solves each subproblem once
    - iterative
    - allows for space optimization

Dynamic Programming: a recipe

- What is the **first decision** to make to solve the problem?
  - add fib(n-1) + fib(n-2)
- What **subproblem**(s) emerge out of the that first decision?
  - fib(n-1) and fib(n-2)
- Must **wait** for subproblem solutions to make the first decision?
  - Yes
- start with a recursive solution
- if inefficient, do you have **overlapping** subproblems?
- identify the **unique** subproblems
- Allocate an **array** to hold their solutions
- solve them from **bottom-up** smaller to larger
- Optimize space if possible

Example 2: The unbounded knapsack problem

- a **knapsack** that can hold a **weight limit** L
- a set of n **item types**
  - each has a weight (w_i) and value (v_i)
  - **unbounded** supply of all types
- what is the **maximum value** we can fit in the knapsack?

10 lb. capacity

weight:	6	3	4	2
value:	30	14	16	9

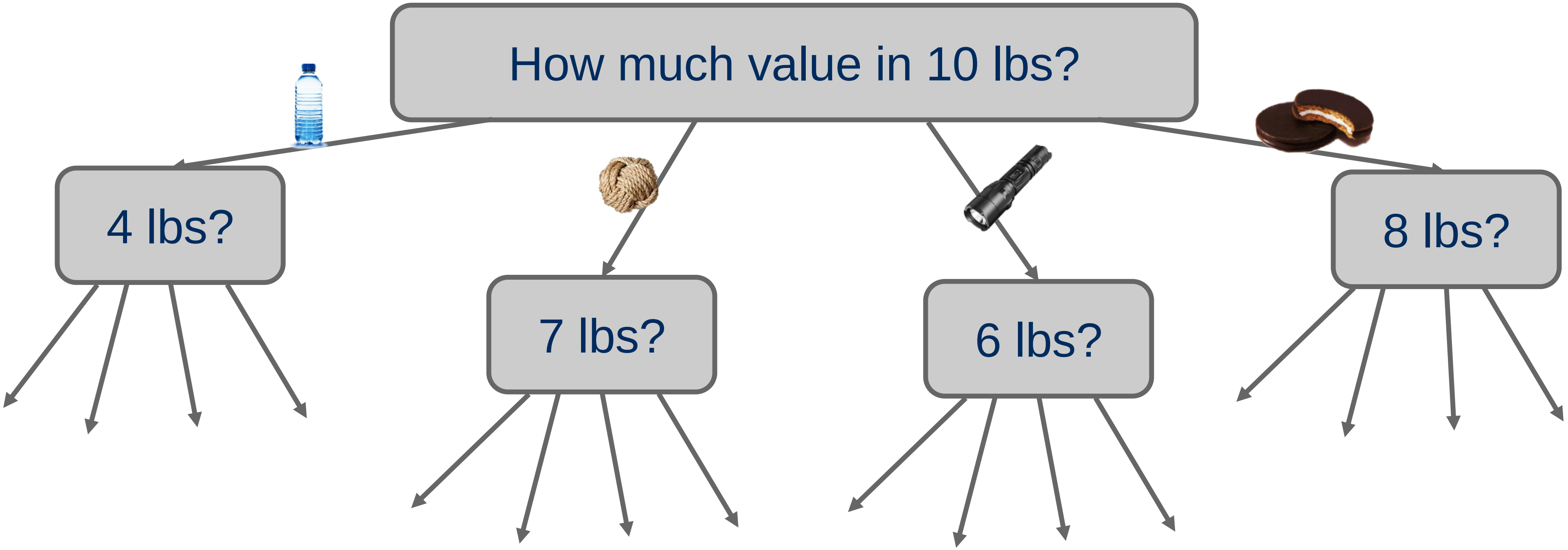
Dynamic Programming: a recipe

- What is the **first decision** to make to solve the problem?
- What **subproblem**(s) emerge out of the that first decision?

Decisions

weight:	6	3	4	2
value:	30	14	16	9

10 lb.
capacity



Dynamic Programming: a recipe

- What is the **first decision** to make to solve the problem?
  - first item to put in the knapsack
- What **subproblem**(s) emerge out of the that first decision?
  - a knapsack with remaining capacity and all items available
- Must **wait** for subproblem solutions to make the first decision?
  - Yes?

Greedy algorithms

- At each step, the algorithm makes a choice that seems to be best **at the moment**
- **Doesn’t wait for subproblem solutions**
- Have we seen greedy algorithms already this term?
  - Yes!
  - Building Huffman tries
  - Prim’s MST algorithm

A greedy algorithm for Unbounded Knapsack

- Add as many copies of **highest value per pound** item as possible:
  - Water: 30/6 = 5
  - Rope: 14/3 = 4.66
  - Flashlight: 16/4 = 4
  - Moon pie: 9/2 = 4.5
- Highest value per pound item? Water
  - Can fit 1 with 4 space left over
- Next highest value per pound item? Rope
  - Can fit 1 with 1 space left over
- No room for anything else
- Total value in the 10 lb. knapsack?
  - 44
  - Is that optimal?

10 lb.
capacity

weight:	6	3	4	2
value:	30	14	16	9

Greedy algorithm doesn’t work for this problem

No optimal solution includes the locally-optimal choices made by the greedy algorithm

Dynamic Programming: a recipe

- Must **wait** for subproblem solutions to make the first decision?
  - Yes!
- start with a recursive solution

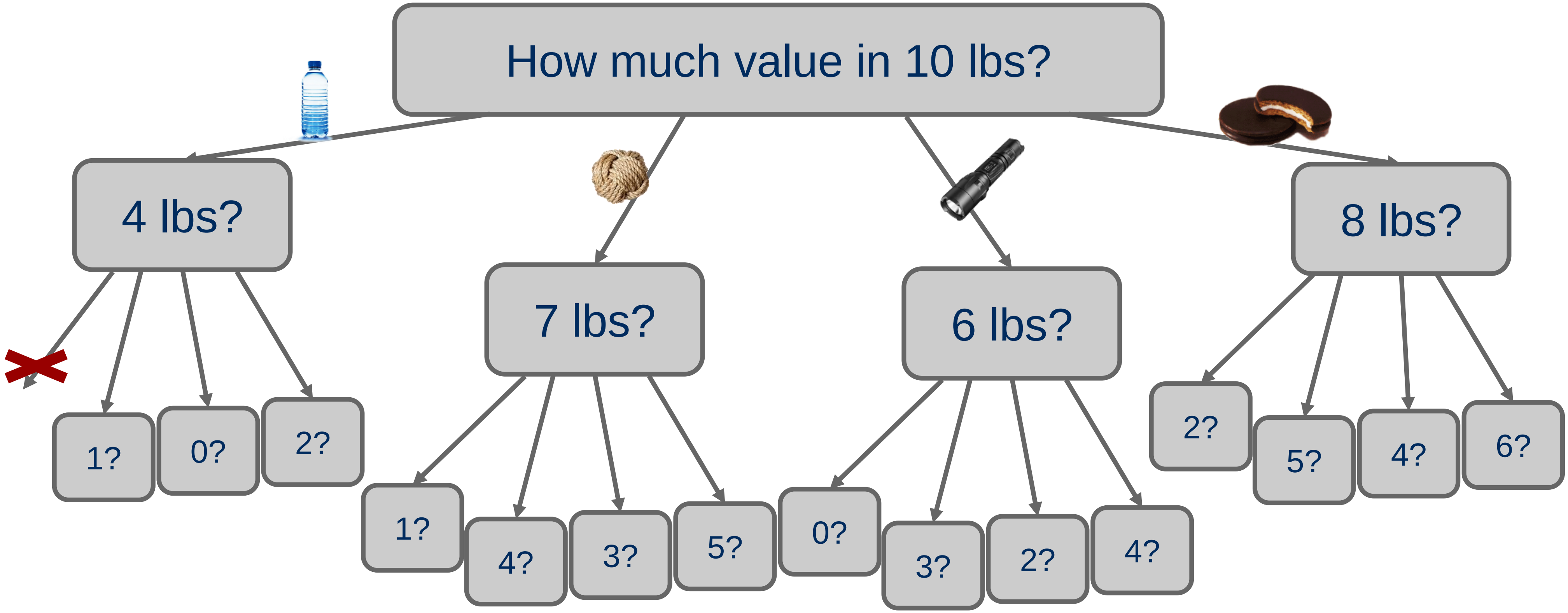
Recursive solution

```
int knapSack(int[] wt, int[] val, int L) {
    if (L == 0) { return 0 };
    int maxValue = 0;
    for(int i=0; i < n; i++){
        if (wt[i] <= L) {
              value = val[i] +
                          knapSack(wt, val, L-wt[i]);
              if (value > maxValue) maxValue = value;
         }
    }
    return maxValue;
}
```

Decisions

weight:	6	3	4	2
value:	30	14	16	9

10 lb.
capacity



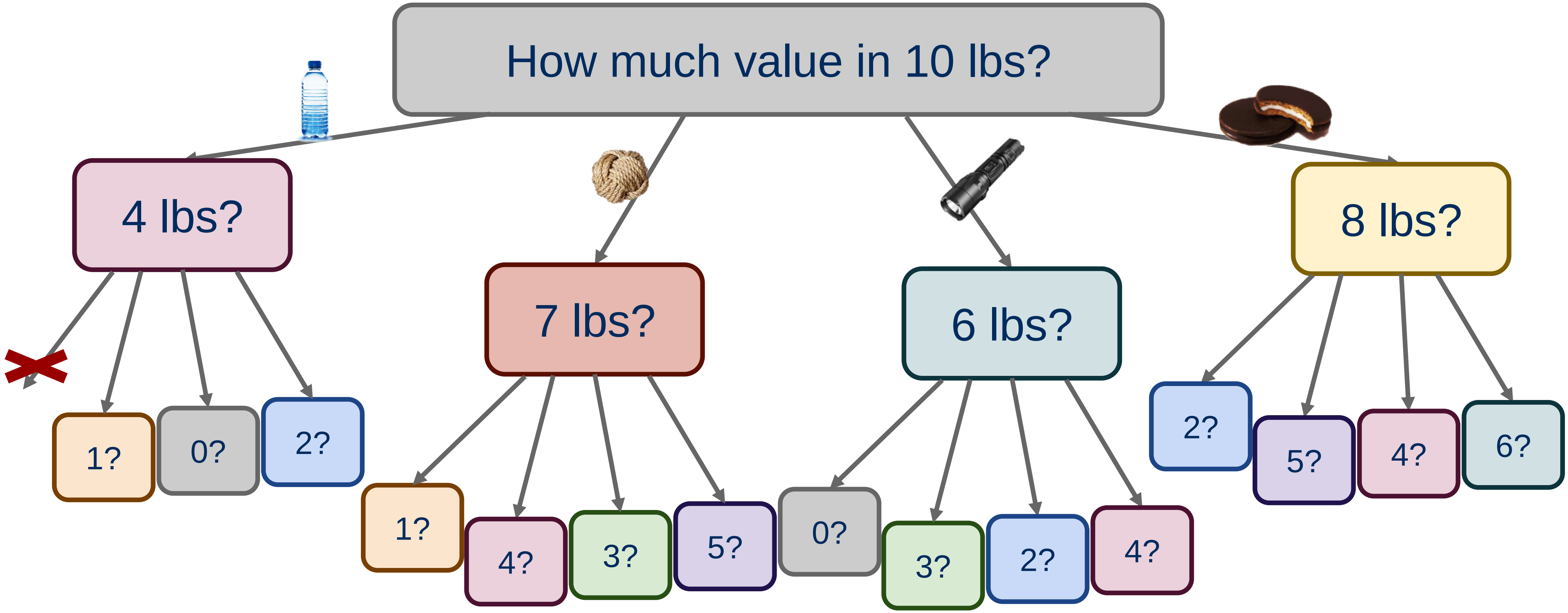
Dynamic Programming: a recipe

- Must **wait** for subproblem solutions to make the first decision?
  - Yes!
- start with a recursive solution
- if inefficient, do you have **overlapping** subproblems?

Recursive Solution

weight:	6	3	4	2
value:	30	14	16	9

10 lb.
capacity



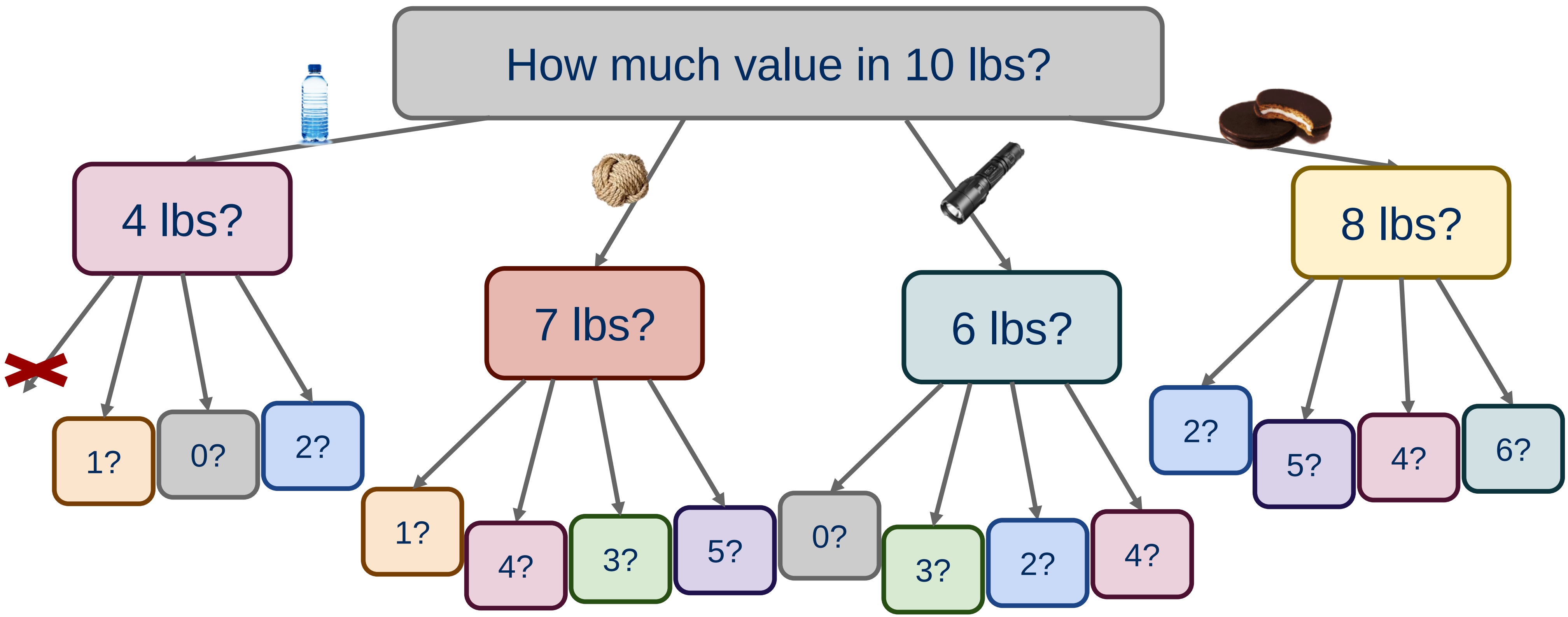
Dynamic Programming: a recipe

- Must **wait** for subproblem solutions to make the first decision?
  - Yes!
- start with a recursive solution
- if inefficient, do you have **overlapping** subproblems?
- identify the **unique** subproblems
- Allocate an **array** to hold their solutions

Ubnique subproblems?

weight:	6	3	4	2
value:	30	14	16	9

10 lb.
capacity



Dynamic Programming: a recipe

- Must **wait** for subproblem solutions to make the first decision?
  - Yes!
- start with a recursive solution
- if inefficient, do you have **overlapping** subproblems?
- identify the **unique** subproblems
- Allocate an **array** to hold their solutions
  - K[] with size L, the knapsack capacity
- solve them from **bottom-up** smaller to larger
  - K[i] holds the maximum value possible with a knapsack of capacity *i*

Bottom-up solution

```
K[0] = 0
for (l = 1; l <= L; l++) {
     int max = 0;
     for (i = 0; i < n; i++) {
          if (wi <= l && vi + K[l - wi]) > max) {
               max = vi + K[l - wi];
          }
     }
     K[l] = max;
}
```

- **Runtime?**
  - **n * L**
    - **L's input size is in bits, hence:**
  - **n * $2^{|L|}$**

Bottom-up Solution

weight:	6	3	4	2
value:	30	14	16	9

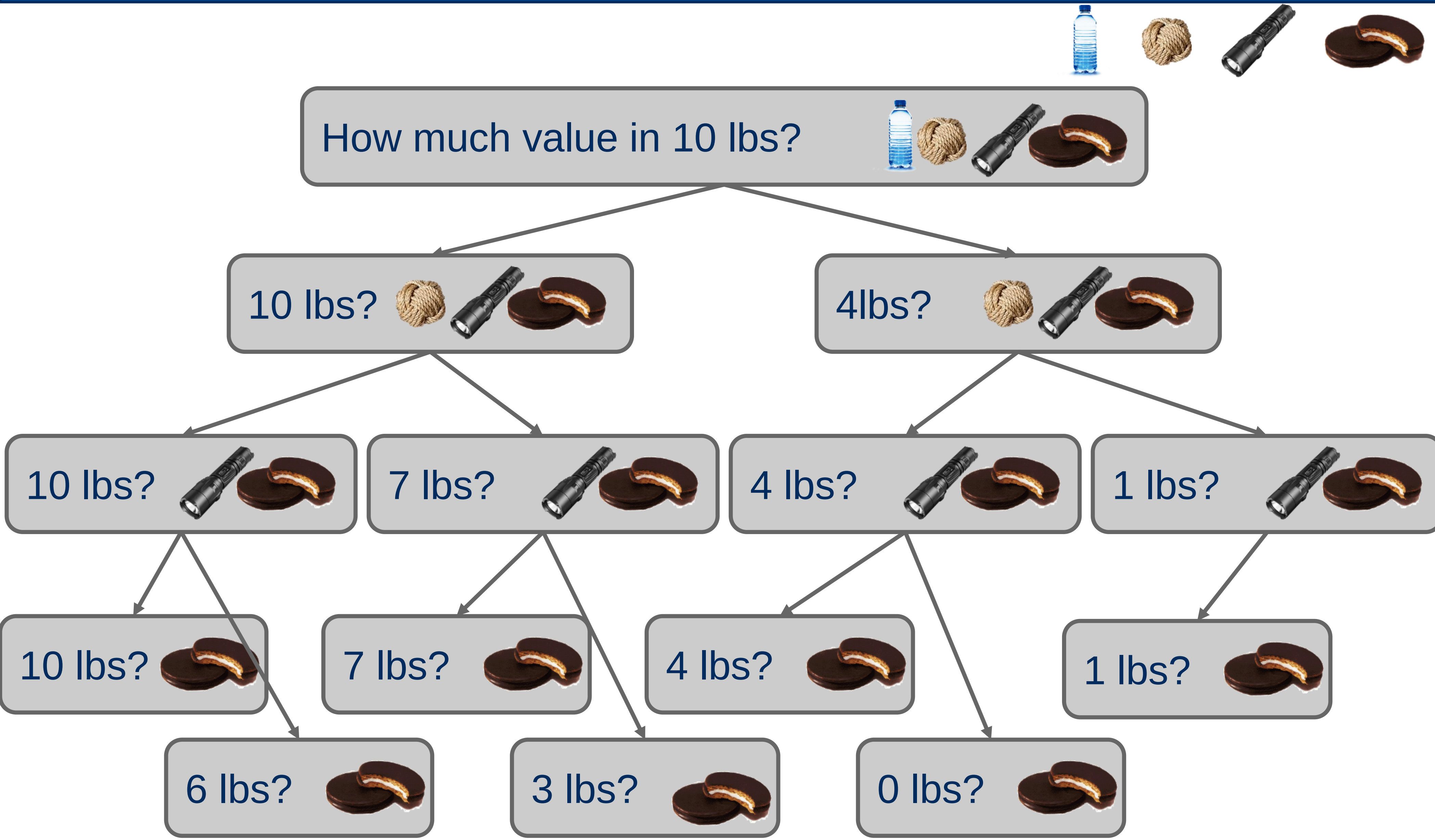
Size:	0	1	2	3	4	5	6	7	8	9	10
Max val:	0	0	9	14	18	23	30	32	39	44	48

Example 3: The 0/1 knapsack problem

- What if we have a finite set of items with a weight and value each?
  - Two choices for each item:
    - Goes in the knapsack or
    - left out
- What would be our first decision?
  - to place or not the first item (or last item)
- What suproblems emerge?
  - if placed, one less item and capacity less by item's weight
  - if placed, one less item and same capacity
  - which choice to take?

Recursive solution

weight:	6	3	4	2
value:	30	14	16	9



Recursive solution

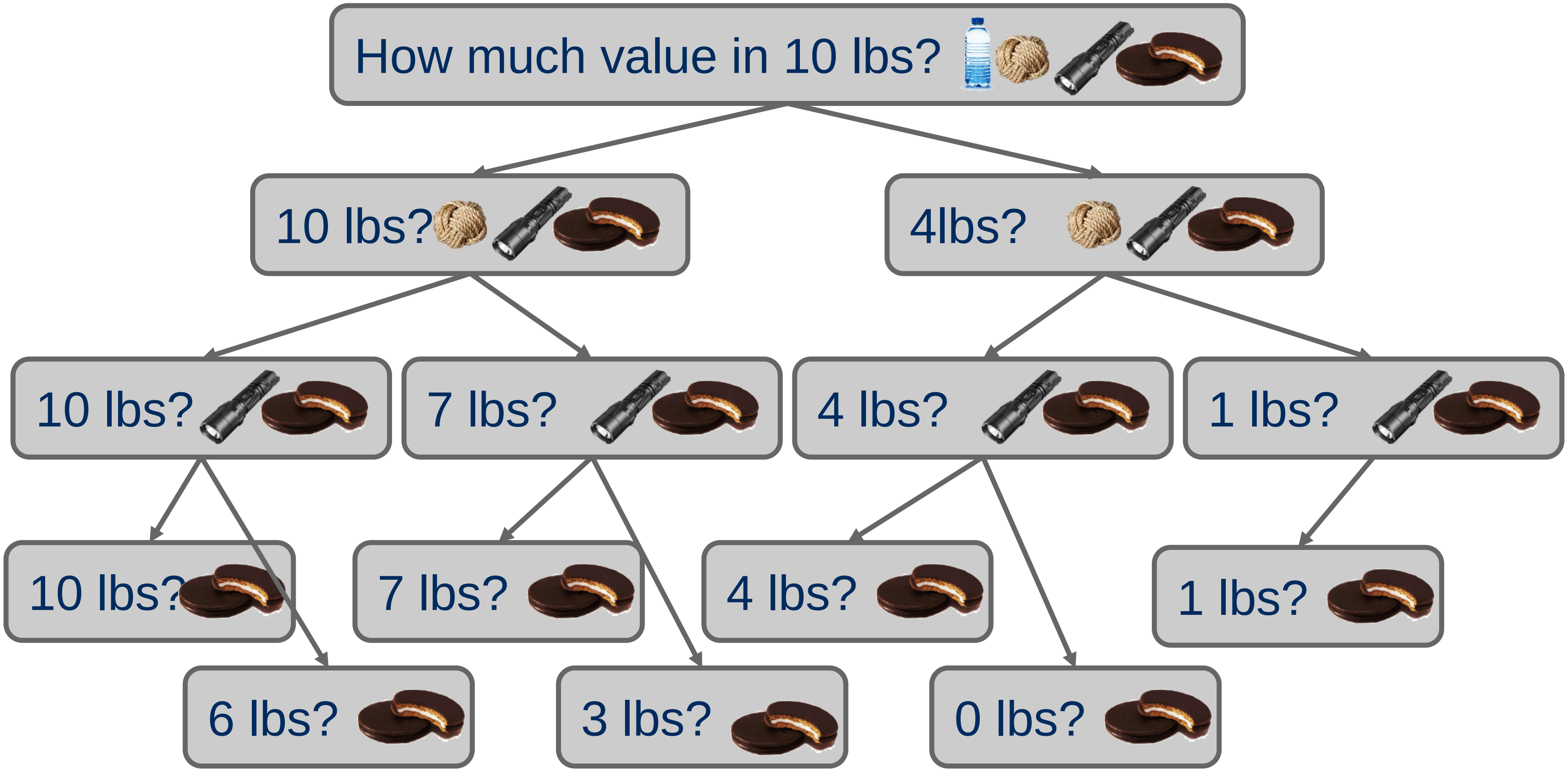
```
int knapSack(int[] wt, int[] val, int L, int n) {
    if (n == 0 || L == 0) { return 0 };
    //try placing the (n-1)st item
    if (wt[n-1] > L) { //cannot place
        return knapSack(wt, val, L, n-1)
    } else {
        return max( val[n-1] + knapSack(wt, val, L-wt[n-1], n-1),
                    knapSack(wt, val, L, n-1)
                  );
    }
}
```

place the item

don't place the item

Subproblems

- What are the unique subproblems?
- What array should we use to store their solutions?
  - 2-D array!



The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
K[n+1][L+1]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											

K[i][l] is the best (max) value when only the first ***i*** items are available and only ***l*** lbs remain in the knapsack

The 0/1 knapsack dynamic programming solution

```
int knapSack(int wt[], int val[], int L, int n) {
    int[][] K = new int[n+1][L+1];
    for (int i = 0; i <= n; i++) {
        for (int l = 0; l <= L; l++) {
            if (i==0 || l==0){ K[i][l] = 0 };
            //try to add item i-1
            else if (wt[i-1] > l){ K[i][l] = K[i-1][l] };
            else {
                K[i][l] = max(val[i-1] + K[i-1][l-wt[i-1]],
                                K[i-1][l]);
            }
        }
    }
    return K[n][L];
}
```

place the item

don't place
the item

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0										
2	0										
3	0										
4	0										

K[i][l] is the best (max) value when only the first ***i*** items are available and only ***l*** lbs remain in the knapsack

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0					
2	0										
3	0										
4	0										

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0										
3	0										
4	0										

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0								
3	0										
4	0										

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0										
4	0										

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16						
4	0										

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0										

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0									

The 0/1 knapsack dynamic programming solution

```
wt  = [ 6,  3,  4,  2 ]
val = [ 30, 14, 16, 9 ]
```

i\l	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	16	30	30	39	44	46

Example 4: the change making problem

- What is the **minimum** number of coins needed to make up a given change value **k >= 0**?
- If you were working as a cashier, what would your algorithm be to solve this problem?

This is a *greedy algorithm*

- At each step, the algorithm makes the choice that seems to be best **at the moment**

... But wait ...

- Does our greedy change making algorithm solve the change making problem?
  - For US currency ...
    - yes!
  - But what about a currency composed of
    - pennies (1 cent), thrickels (3 cents), and fourters (4 cents)?
    - What denominations would it pick for k=6?

So what changed about the problem?

- For greedy algorithms to produce optimal results, problems must have two properties:
  - **Optimal substructure**: optimal solution to a subproblem leads to an optimal solution to the overall problem
    - best way to make change for 3 cents → best way to make 6 cents
  - The **greedy choice** property
    - Globally optimal solutions assembled from locally optimal choices
    - K = 6: for US currency, the best overall choice will be to use the biggest coin (nickel)
    - With thrickels/fourters, we can't know until we've looked at all possible breakdowns
- Why is optimal substructure not enough?

Let's summarize

- Greedy algorithms
  - elegant but hardly correct
  - need both optimal substructure and greedy choice
- Without the greedy choice property
  - have to solve all **unique** subproblems
  - can be done recursively using Memoization
  - or iteratively using dynamic programming

Where can we apply dynamic programming?

- Problems with two properties:
  - Optimal substructure
    - optimal solution contains optimal solutions of subproblems
  - Overlapping subproblems

Dynamic Programming: a recipe

- What is the **first decision** to make to solve the problem?
- What **subproblem**(s) emerge out of the that first decision?
- Must **wait** for subproblem solutions to make first decision
- start with a recursive solution
- if inefficient, do you have **overlapping** subproblems?
- identify the **unique** subproblems
- Allocate an **array** to hold their solutions
- solve them from **bottom-up** smaller to larger
- Optimize space if possible