# Algorithms and Data Structures 2
# CS 1501

Fall 2022

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(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

# Announcements

- Upcoming Deadlines
  - Homework 7: this Friday @ 11:59 pm
  - Lab 6: next Monday 10/31 @ 11:59 pm
  - **Assignment 2: Friday 11/4 @ 11:59 pm**
  - Lab 7: Monday 11/7 @ 11:59 pm
- Live Support Session for Assignment 2
  - This Friday 7-8 pm (https://pitt.zoom.us/my/khattab)
- Weekly Live QA Session on Piazza
  - Friday 4:30-5:30 pm

# Previous lecture

- ADT Graph
  - definitions
  - representations
    - two-arrays
    - adjacency matrix
    - adjacency lists
  - traversals
    - BFS
      - shortest paths based on number of edges
      - connected components

# This Lecture

- ADT Graph
  - traversals
    - DFS
      - finding articulation points of a graph
  - representation
    - Graph compression

# Problem of previous lecture

- **Input**: A file containing LinkedIn (LI) accounts and their connections
  - Account1: Connection1, Connection2, …
  - Account2: Connection1, Connection2, …
  - …

- **Output**: Answer the following questions:
  - Given two LI accounts, how "far" are they from each other?
    - e.g., 1st connection?, 2nd connection?, etc.
  - Are the accounts in the file all ***connected***?
    - If not, how many ***connected components*** are there?
  - For each connected component, are there certain accounts that if removed, the remaining accounts become ***partitioned***?

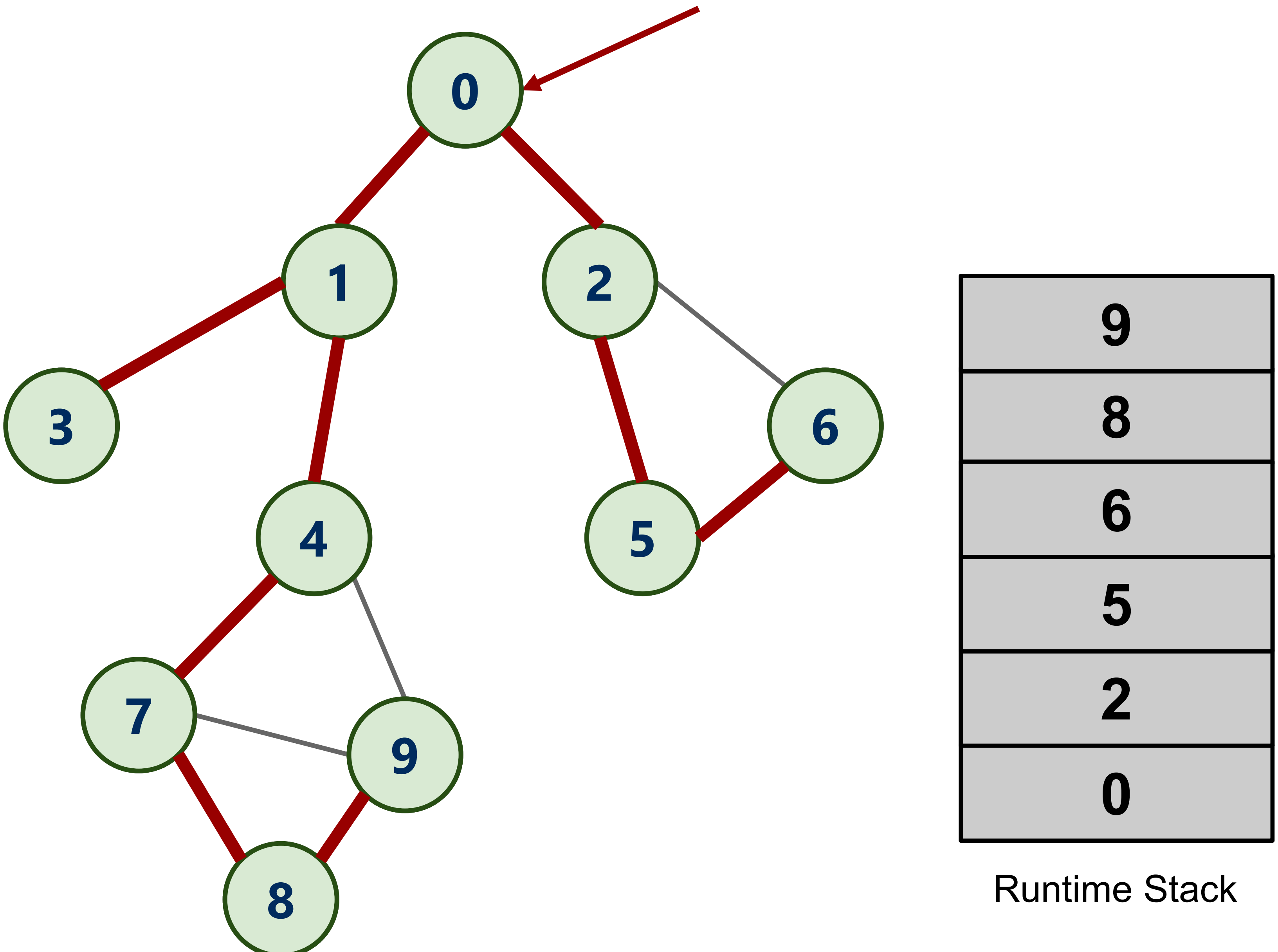
# DFS – Depth First Search

- Already seen and used this throughout the term
  - For Huffman encoding…
    - as we build the codebook out of the Huffman Trie
- Can be easily implemented recursively
  - For each vertex, visit *first* unseen neighbor
  - Backtrack at deadends (i.e., vertices with no unseen neighbors)
    - Try *next* unseen neighbor after backtracking
  - An arbitrary order of neighbors is assumed

# DFS Pseudo-code

```
DFS(vertex v) {
  seen[v] = true //mark v as seen
  for each unseen neighbor w
    parent[w] = v
    DFS(w)
}
```

# DFS example



# When to visit a vertex

```
DFS(vertex v) {
   seen[v] = true //mark v as seen
   visit v //before visiting its children in the spanning tree
   for each unseen neighbor w
      parent[w] = v
      DFS(w)
}
```

# When to visit a vertex

```
DFS(vertex v) {
   seen[v] = true //mark v as seen
for each unseen neighbor w
      parent[w] = v
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 visit v //after visiting its children in the spanning tree
}
```

# When to visit a vertex

```
DFS(vertex v) {
   seen[v] = true //mark v as seen
for each unseen neighbor w
      parent[w] = v
      DFS(w)
      visit v //after processing each child
}
```

# Runtime Analysis of BFS

- Each vertex is added to the queue exactly once and removed exactly once
  - *v* add/remove operations
    - *O(v)* time for vertex processing
- Edges are checked when adding the list of neighbors to the queue
- Each edge is checked at most twice, one per edge endpoint
  - *O(e)* time for edge processing
- Total time: vertex processing time + edge processing time
  - *O(v + e)*

# Runtime Analysis for DFS

- For Adjacency Matrix representation, BFS checks each *possible* edge!
    - $O(v^2)$ time for edge processing with Adjacency Matrix
- Total time: $O(v^2 + v) = O(v^2)$

# Runtime Analysis of DFS

- Each vertex is seen then visited exactly once
  - *O(v)* time for vertex processing
- Edges are checked when finding the list of neighbors
- Each edge is checked at most twice, one per edge endpoint
  - *O(e)* time for edge processing
- Total time: vertex processing time + edge processing time
  - *O(v + e)*

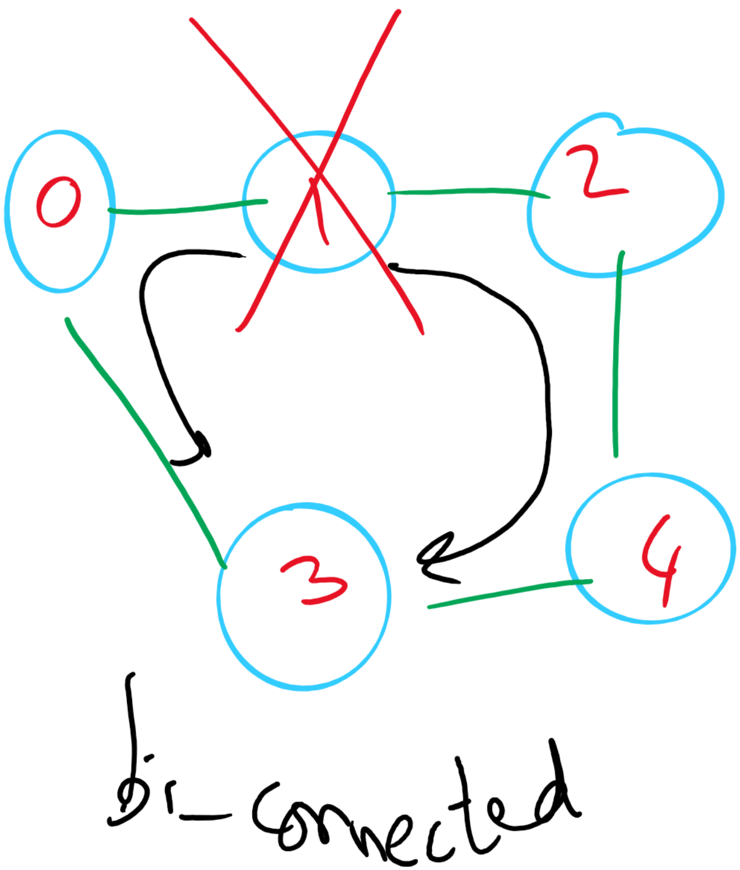
# Runtime Analysis of BFS and DFS

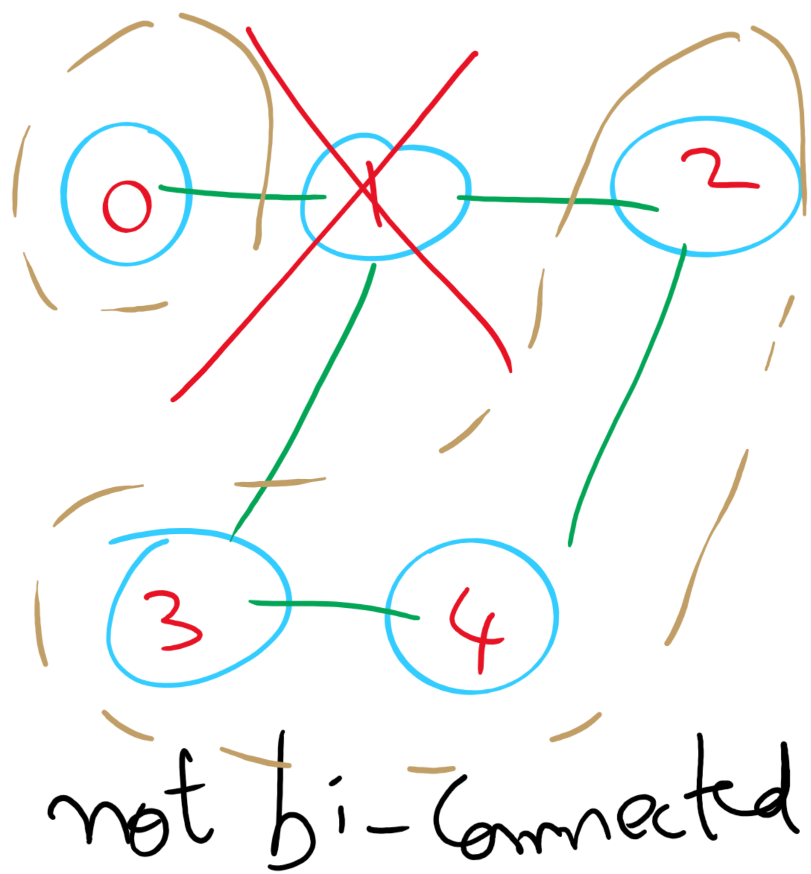
- At a high level, DFS and BFS have the same runtime
  - Each vertex must be seen and then visited, but the order will differ between these two approaches
- The representation of the graph affect the runtimes of of these traversal algorithms?
  - $O(v + e)$ with Adjacency Lists
  - $O(v^2)$ with Adjacency Matrix
  - Note that for a dense graph, $v + e = O(v^2)$

# Biconnected graphs

- A *biconnected graph* has at least 2 distinct paths between all vertex pairs
  - a distinct path shares no common edges or vertices with another path except for the start and end vertices
- A graph is biconnected graph iff it has zero *articulation points*
  - Vertices, that, if removed, will separate the graph

# Biconnected Graph



# Finding articulation points of a graph

- The spanning tree built by a DFS traversal contains one path between each pair of vertices
- If another path exists it must use edges not in the spanning tree
  - we call these back edges
- Consider a vertex v that cannot reach any previous vertices (in the DFS traversal order) **except through its parent**
  - The parent of *v* is an articulation point
  - if *parent[v]* is removed, the graph becomes disconnected because *v* cannot reach at least one vertex
  - Such vertex will exhibit this behavior in *any* DFS traversal of the graph

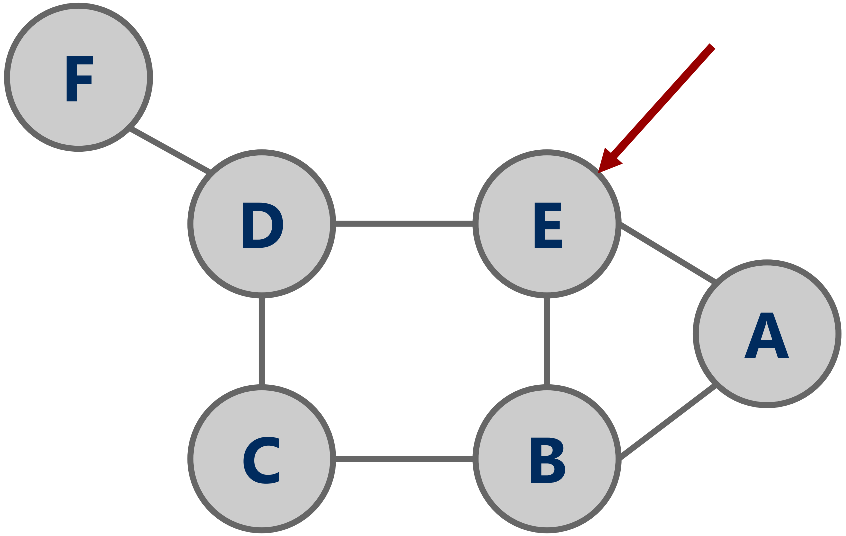
# Finding articulation points of a graph

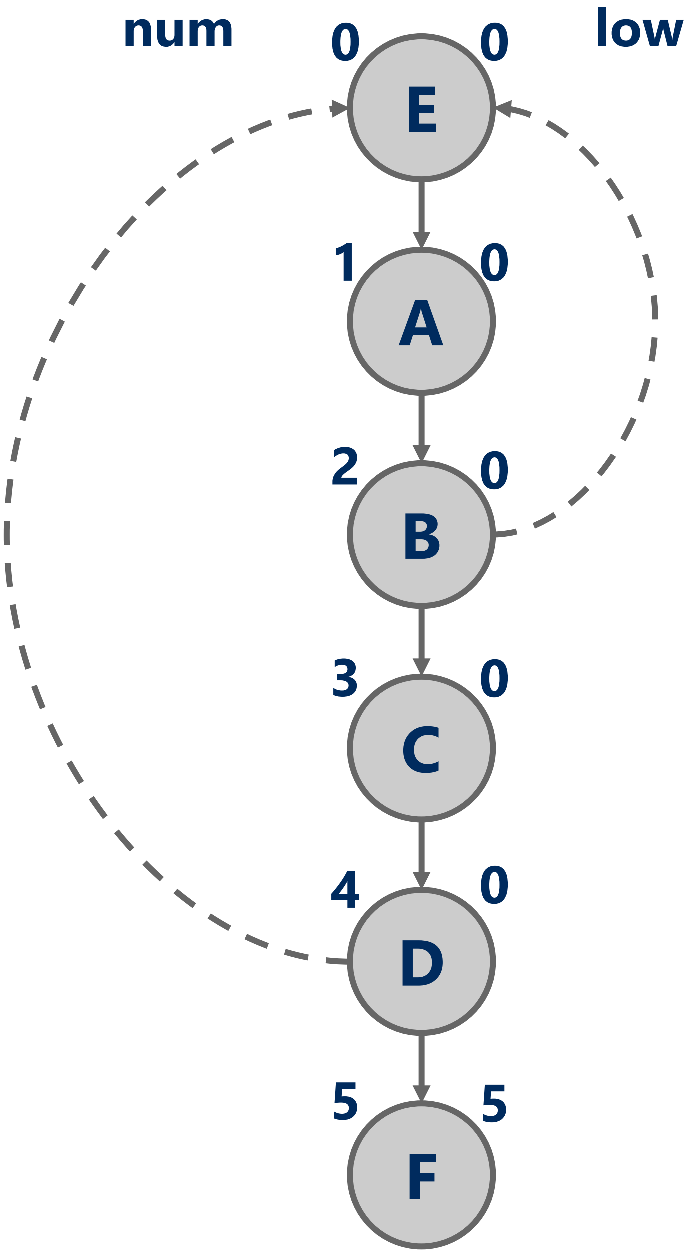
- Consider building up the DFS spanning tree
  - Have it be directed
  - Create "back edges" when considering a vertex that has already been visited in constructing the spanning tree
  - Label each vertex v with with two numbers:
    - num(v) = pre-order traversal order
    - low(v) = lowest-numbered vertex reachable from v using 0 or more spanning tree edges and then at most one back edge

# low(v)

- low(v) = lowest-numbered vertex reachable from v using 0 or more spanning tree edges and then **at most one** back edge
  - Min of:
    - num(v) (the vertex is reachable from itself)
    - Lowest num(w) of all back edges (v, w)
    - Lowest low(w) of all children of *v* (the lowest-numbered vertex reachable from through a child)
- The "at most one back edge" is to ensure distinct paths
- num(v) is computed as we move down the tree
- low(v) is computed as we climb back up the tree
- Recursive DFS is convenient to compute both
  - why?

# Finding articulation points example



# Using DFS to find articulation points

```
int num = 0;
DFS(vertex v) {
    num[v] = num++
    low[v] = num[v] //initially
    seen[v] = true //mark v as seen
    for each neighbor w
        if(w unseen){
            DFS(w)
            low[v] = min(low[v], low[w])
        } else { //back edge
            low[v] = min(low[v], num[w])
        }
}
```

# So where are the articulation points?

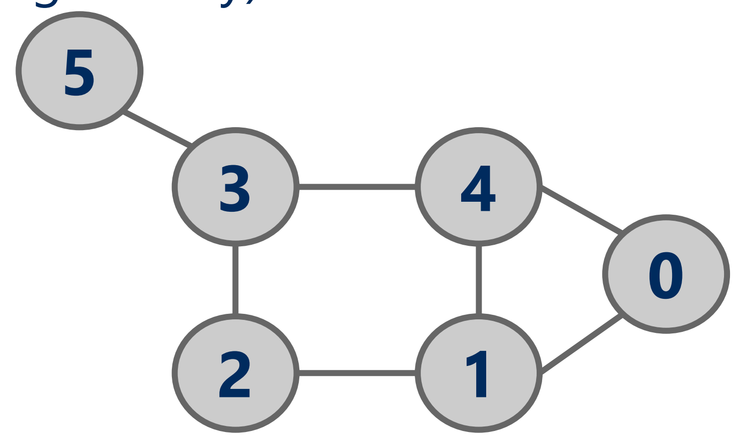
- If any (non-root) vertex v has some child w such that $low(w) \geq num(v)$, v is an articulation point
- What if we start at an articulation point?
  - The starting vertex becomes the root of the spanning tree
  - If the root of the spanning tree has more than one child, the root is an articulation point

# Graph Compression

- Real-life graphs are huge
  - 100’s if not 1000’s of GBs
  - Facebook graph, Google graph, maps, …
- Let’s see one (partial) idea for reducing the size of large graphs

# Graph Compression

- **Step 1:** Construct a Compressed Sparse Row (CSR) representation of the graph
- CSR
  - Edges array concatenates *sorted* neighbor lists of all vertices
  - Offsets array:
    - offsets[v] is the starting index (in the Edges array) for the neighbors of vertex v




**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |

**Offsets**

| | | | | | |
|---|---|---|---|---|---|
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# Graph Compression

- Let's start with one more graph representation
- Compressed Sparse Row (CSR)
- Edges array concatenates *sorted* neighbor lists of all vertices
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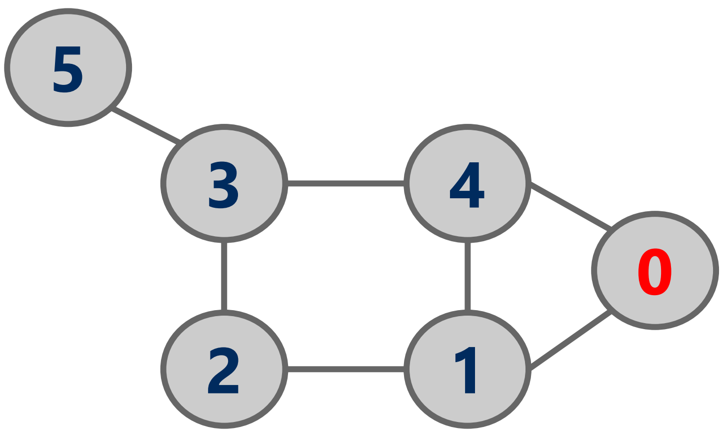


**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
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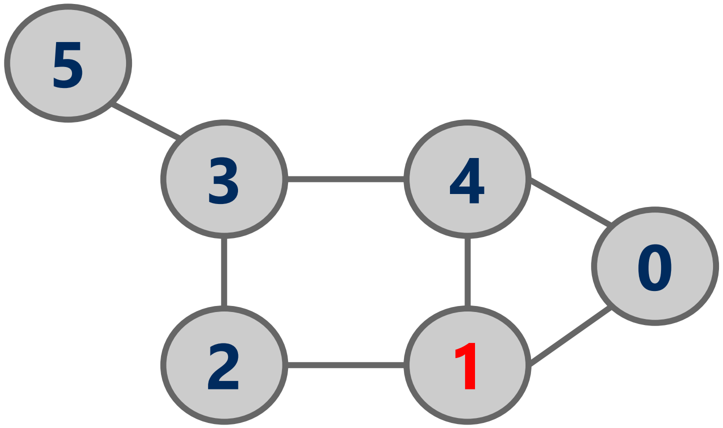


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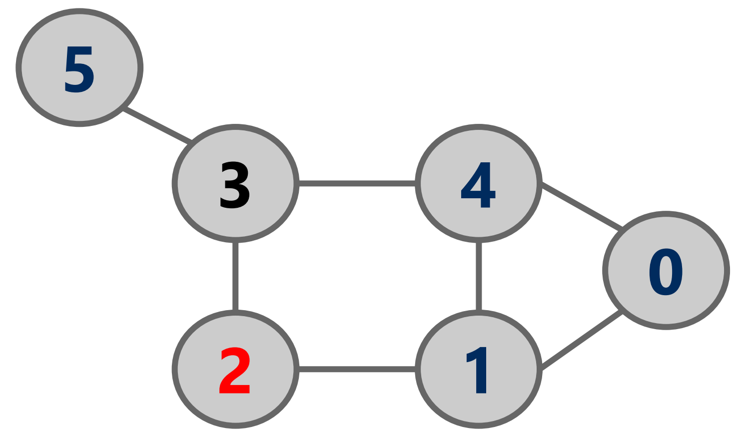


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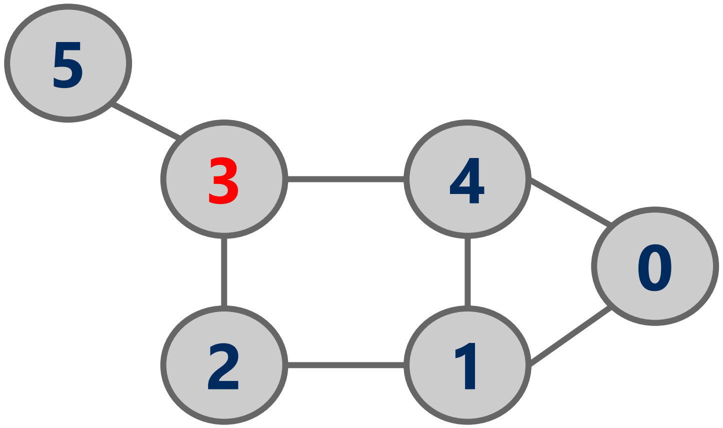


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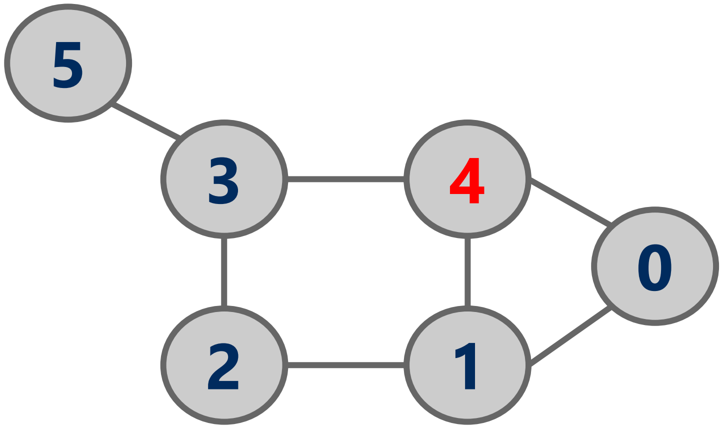


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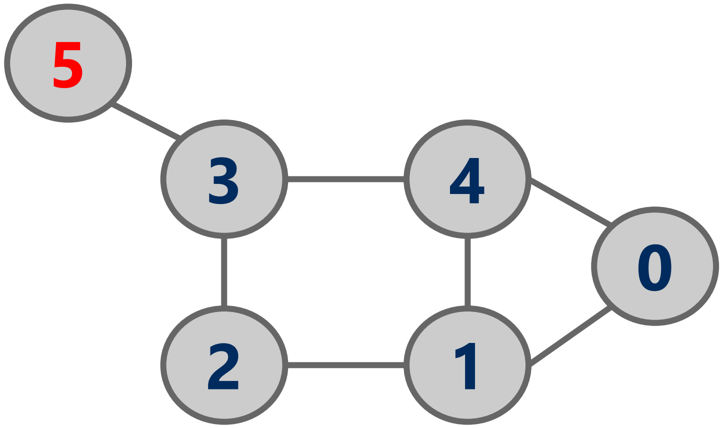


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|---|---|---|---|---|---|
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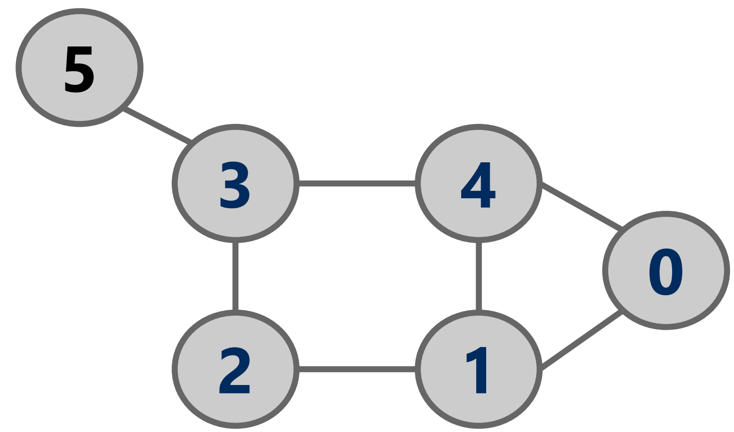
# Graph Compression

- Can we compute the degree of a vertex using the offsets array?
  - Running time?
- What is the required space of this representation?
  - Theta(m + n)
  - Assume 4 bytes per vertex and per edge
  - Total size: *4*v* + *8*e* bytes




**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |

**Offsets**

| | | | | | |
|---|---|---|---|---|---|
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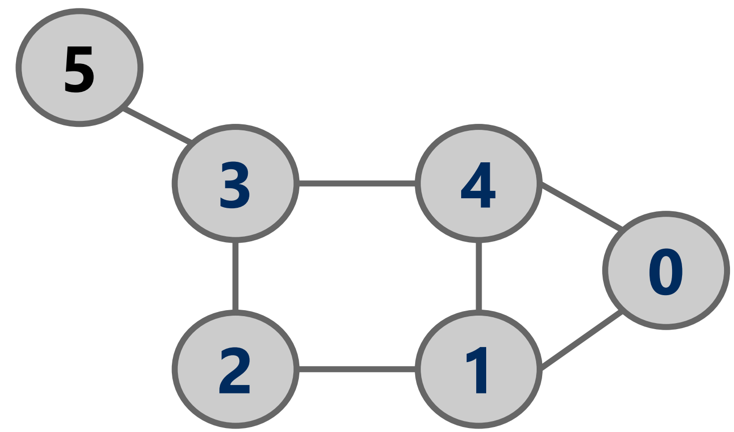
# Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list $v_1$, $v_2$, $v_3$, …
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$




**Edges**

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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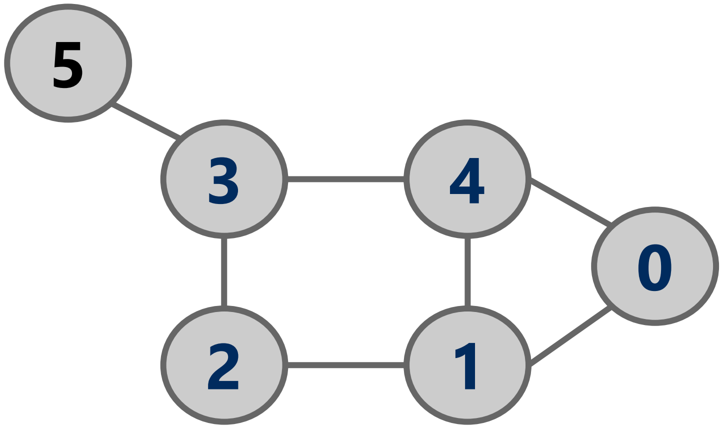


**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
| 1-0 | 4-1 | | | | | | | | | | | | |

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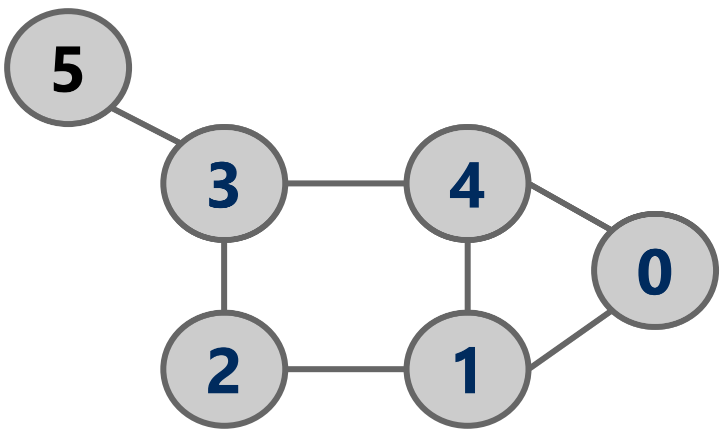


**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
| 1 | 3 | | | | | | | | | | | | |

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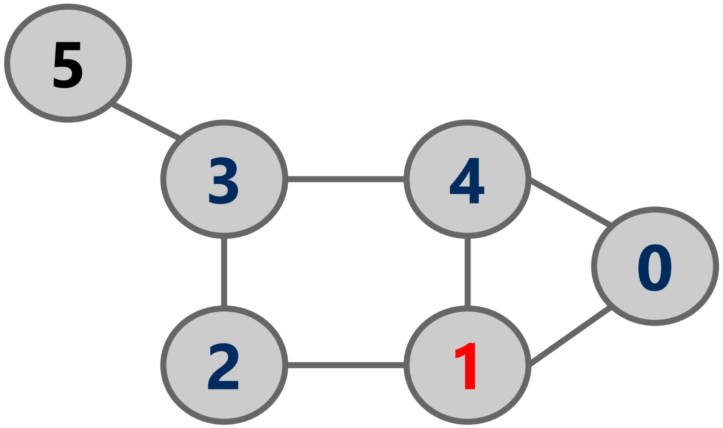


**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
| 1 | 3 | -1 | 2 | 2 | | | | | | | | | |

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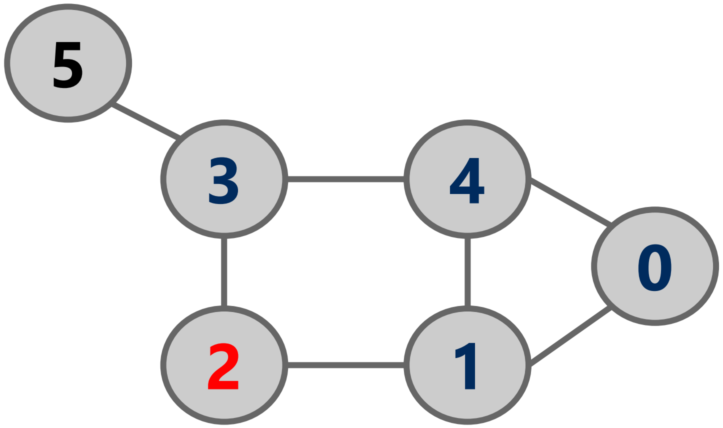


**Edges**

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
| | 1 | 3 | -1 | 2 | 2 | -1 | 2 | | | | | | | |

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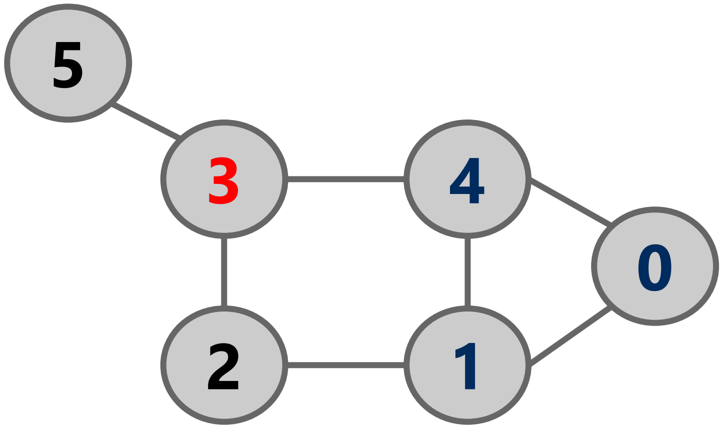


**Edges**

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
| 1 | 3 | -1 | 2 | 2 | -1 | 2 | -1 | 2 | 1 | | | | |

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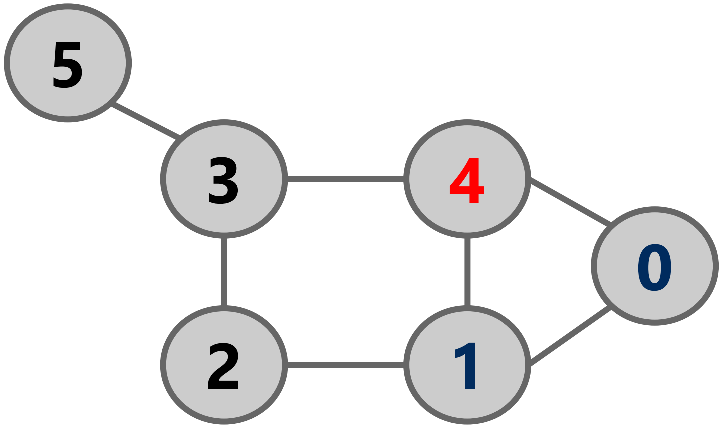


**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
| 1 | 3 | -1 | 2 | 2 | -1 | 2 | -1 | 2 | 1 | -4 | 1 | 2 | |

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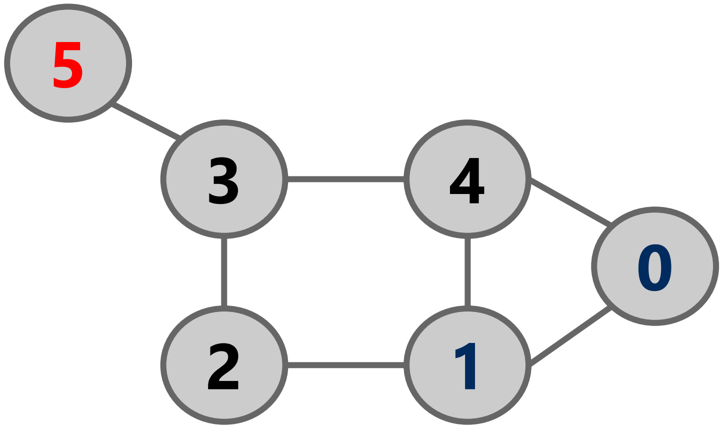


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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Edges | 1 | 4 | 0 | 2 | 4 | 1 | 3 | 2 | 4 | 5 | 0 | 1 | 3 | 3 |
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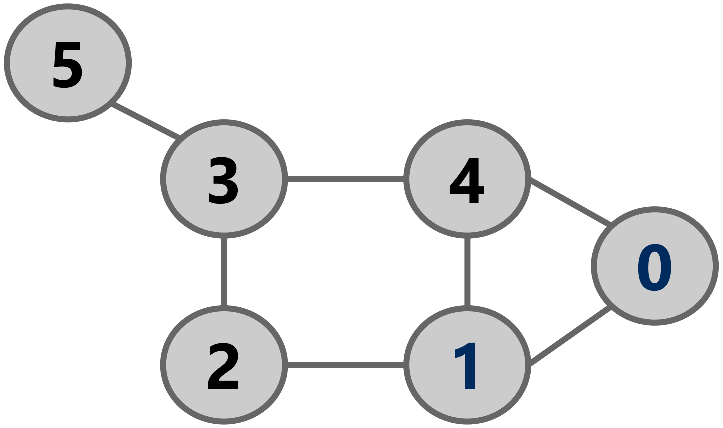


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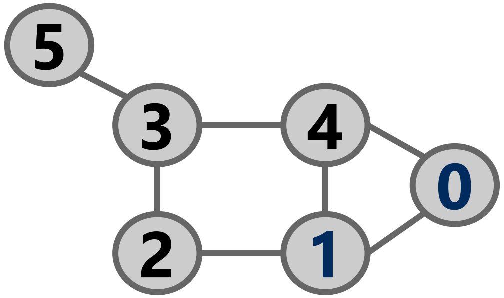
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | -1 | 2 | 2 | -1 | 2 | -1 | 2 | 1 | -4 | 1 | 2 | -2 |

**Offsets**

| | | | | | |
|---|---|---|---|---|---|
| 0 | 2 | 5 | 7 | 10 | 13 |

# Graph Compression

- **Step 3: Use Gamma code to compress the differences**




**Edges**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | -1 | 2 | 2 | -1 | 2 | -1 | 2 | 1 | -4 | 1 | 2 | -2 |

**Offsets**

| | | | | | |
|---|---|---|---|---|---|
| 0 | 2 | 5 | 7 | 10 | 13 |

# Gamma Code

- Gamma Code is used to compress data in which small values are much more frequent than large values
- To encode an integer *x*,
  - find T, the largest power of 2 < *x*
  - Encode T as (log T) zeros followed by 1
  - Append the remaining (log T) binary digits of x
- Example: To encode 17: 10001
  - $T = 16 = 2^4$
  - Gamma code: 0000 1 0001
- 2 floor(log x) + 1
  - much smaller than 32 bits if *x* is small

# Graph Compression

- **Goal**: make the differences between vertex labels in each neighbor list small
  - So that their Gamma codes are much less than 32 bits
- For Web Graphs
  - Each vertex is a web page
  - Sort the pages based on their reverse URL (e.g., www.cs.pitt.edu)
  - Most links are local (within the same domain)
    - neighbors will be close to each other in the sorted list
    - Goal achieved
- Other graphs can be relabeled to achieve that goal
  - https://www.cs.cmu.edu/~guyb/papers/BBK03.pdf

# Neighborhood connectivity Problem

- We want to keep a set of neighborhoods connected with the minimum cost possible
- **Input:** A set of neighborhoods and a file with the following format:
  - neighborhood i, neighborhood j, cost of connecting the two neighborhoods
  - …
- **Output:** A set of neighborhood pairs to be connected and a total cost such that
  - We can go from any neighborhood to any other **(connected)**
  - The total cost should be minimum (i.e., as small as it can be) **(minimal cost)**

# Think Data Structures First!

- How can we structure the input in computer memory?
- Can we use Graphs?
- What about the costs? How can we model that?

# We said spatial layouts of graphs were irrelevant

- We define graphs as sets of vertices and edges
- However, we'll certainly want to be able to reason about bandwidth, distance, capacity, etc. of the real world things our graph represents
  - Whether a link is 1 gigabit or 10 megabit will drastically affect our analysis of traffic flowing through a network
  - Having a road between two cities that is a 1 lane country road is very different from having a 4 lane highway
  - If two airports are 2000 miles apart, the number of flights going in and out between them will be drastically different from airports 200 miles apart

# We can represent such information with edge weights

- How do we store edge weights?
  - Adjacency matrix?
  - Adjacency list?
  - Do we need a whole new graph representation?
- How do weights affect finding spanning trees/shortest paths?
  - The weighted variants of these problems are called finding the *minimum spanning tree* and the *weighted shortest path*

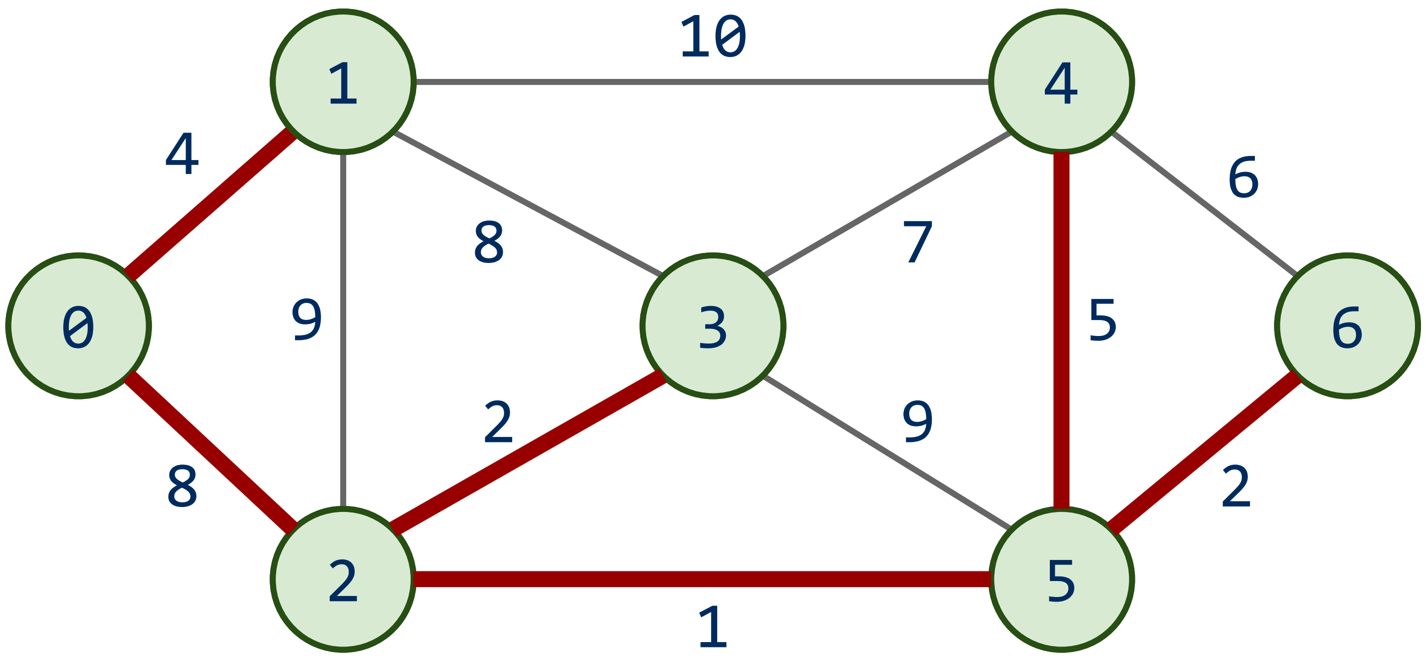
# Minimum spanning trees (MST)

- Graphs can potentially have multiple spanning trees
- MST is the spanning tree that has the minimum sum of the weights of its edges

# Prim's algorithm

- Initialize T to contain the starting vertex
  - T will eventually become the MST
- While there are vertices not in T:
  - Find minimum edge-weight edge that connects a vertex in T to a vertex not yet in T
  - Add the edge with its vertex to T

# Prim's algorithm



# Runtime of Prim's

- At each step, check all possible edges
- For a complete graph:
  - First iteration:
    - v - 1 possible edges
  - Next iteration:
    - 2(v - 2) possibilities
      - Each vertex in T shared v-1 edges with other vertices, but the edges they shared with each other already in T
  - Next:
    - 3(v - 3) possibilities
  - ...
- Runtime:
  - $\Sigma_{i\ =\ 1\ \text{to}\ v}$ (i * (v - i)) = Θ(largest term * number of terms)
  - number of terms = *v*
  - largest term is $v^2/4$ (when *i=v/2)*
  - Evaluates to $\Theta(v^3)$

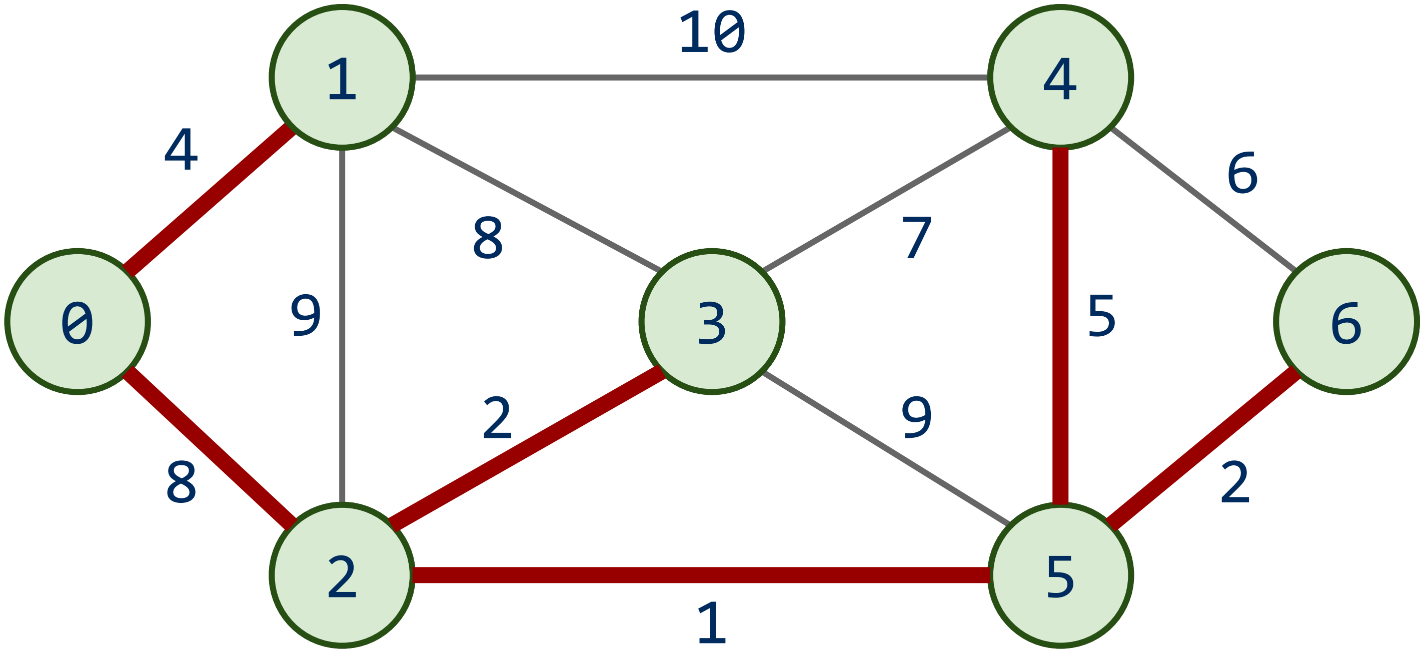
# Do we need to look through all remaining edges?

- No!  We only need to consider the *best* edge possible for each vertex!
  - The best edge of each vertex can be updated as we add each vertex to T

# An enhanced implementation of Prim's Algorithm

- Add start vertex to T
- Search through the neighbors of the added vertex to adjust the parent and best edge arrays as needed
- Search through the best edge array to find the next addition to T
- Repeat until all vertices added to T

# Prim's algorithm




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Parent: | -- | 0 | 0 | 2 | 5 | 2 | 5 |
| Best Edge: | 0 | 4 | 8 | 2 | 5 | 1 | 2 |

# OK, so what's our runtime?

- For every vertex we add to T, we'll need to check all of its neighbors to update their best edges as needed
  - Let's assume we use an **adjacency matrix**:
    - Takes Θ(v) to check the neighbors of a given vertex
    - Time to update parent/best edge arrays?
      - Θ(1)
    - Time to pick next vertex?
      - Θ(v)
    - Total: v*2 Θ(v) = $\Theta(v^2)$

# OK, so what's our runtime?

- For every vertex we add to T, we'll need to check all of its neighbors to update their best edges as needed
  - Let's assume we use **adjacency lists**
    - Takes Θ(d) to check the neighbors of a given vertex
    - Time to update parent/best edge arrays?
      - Θ(1)
    - Time to pick next vertex?
      - Θ(v)
    - Total: $v*\Theta(v + d) = \Theta(v^2)$

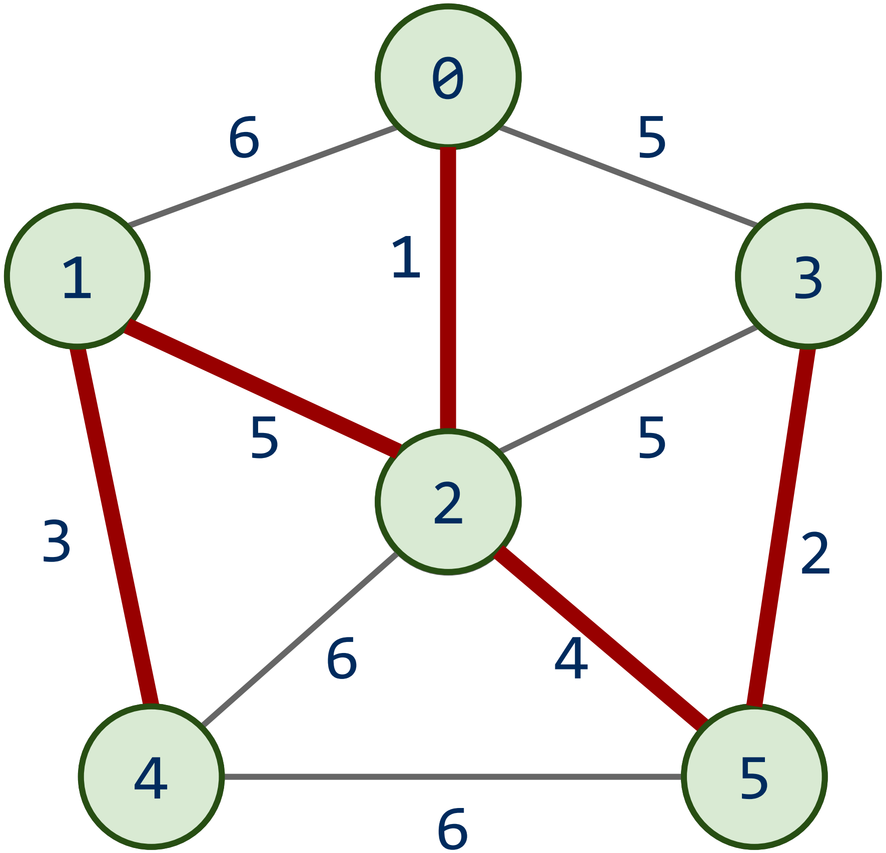
# Prim’s MST Algorithm

- seen, parent, and BestEdge are arrays of size v
- Initialize seen to false, parent to -1, and BestEdge to infinity
- BestEdge[start] = 0
- for i = 0 to v-1
    - Find a vertex w with seen[w] = false and BestEdge[w] is the minimum over all unseen vertices
    - seen[w] = 1
    - for each neighbor x of w
        - if(BestEdge[x] > edge weight of edge (w, x)
            - BestEdge[x] = edge weight of (w, x)
            - parent[x] = w
- The parent array represents the found MST

# What about a faster way to pick the best edge?

- Sounds like a job for a priority queue!
  - Priority queues can remove the min value stored in them in $\Theta(\lg n)$
    - Also $\Theta(\lg n)$ to add to the priority queue
- What does our algorithm look like now?
  - Visit a vertex
  - Add edges coming out of it to a PQ
  - While there are unvisited vertices, pop from the PQ for the next vertex to visit and repeat

# Prim's with a priority queue




PQ:

1: (0, 2)

2: (5, 3)

3: (1, 4)

4: (2, 5)

5: (2, 3)

5: (0, 3)

5: (2, 1)

6: (0, 1)

6: (2, 4)

6: (5, 4)

# Runtime using a priority queue

- Have to insert all e edges into the priority queue
  - In the worst case, we'll also have to remove all e edges
- So we have:
  - $e * \Theta(\lg e) + e * \Theta(\lg e)$
  - $= \Theta(2 * e \lg e)$
  - $= \Theta(e \lg e)$
- This algorithm is known as *lazy Prim's*

# Do we really need to maintain e items in the PQ?

- I suppose we could not be so lazy
- Just like with the best edge array implementation, we only need the best edge for each vertex
    - PQ will need to be indexable to update the best edge
- This is the idea of *eager Prim's*
    - Runtime is $\Theta(e \lg v)$

# Eager Prim's Runtime

$v$ insertions : $v \log v$
$e$ updates : $e \log v$
$v$ removals : $v \log v$

$(e+v) \log v = \Theta(e \log v)$

$e \geq (v-1)$

# Comparison of Prim's implementations

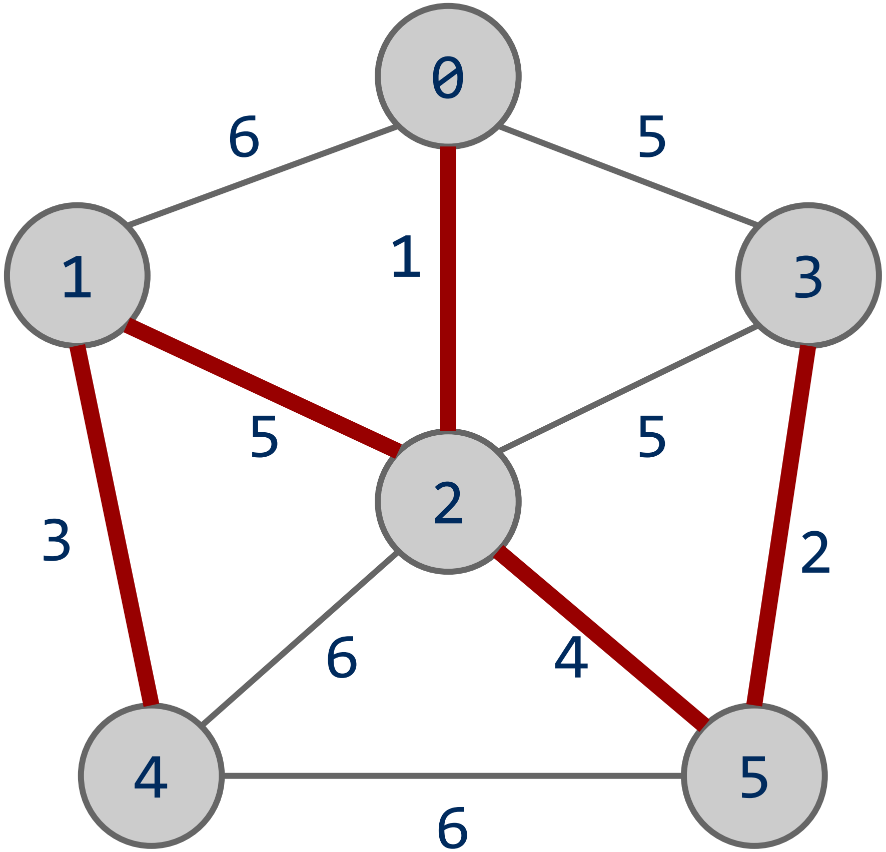
- Parent/Best Edge array Prim's
  - Runtime: $\Theta(v^2)$
  - Space: $\Theta(v)$
- Lazy Prim's
  - Runtime: $\Theta(e \lg e)$
  - Space: $\Theta(e)$
  - Requires a PQ
- Eager Prim's
  - Runtime: $\Theta(e \lg v)$
  - Space: $\Theta(v)$
  - Requires an indexable PQ

How do these compare?

# Another MST algorithm

- Kruskal's MST:
  - Insert all edges into a PQ
  - Grab the min edge from the PQ that does not create a cycle in the MST
  - Remove it from the PQ and add it to the MST

# Kruskal's example




PQ:

1: (0, 2)

2: (3, 5)

3: (1, 4)

4: (2, 5)

5: (2, 3)

5: (0, 3)

5: (1, 2)

6: (0, 1)

6: (2, 4)

6: (4, 5)

# Kruskal's runtime

- Instead of building up the MST starting from a single vertex, we build it up using edges all over the graph
- How do we efficiently implement cycle detection?

# Kruskal's Runtime

$e$ iterations

remove $\log e$

cycle detection $\Theta(v+e)$

DFS/BFS

$e(v+e) = \Theta(e^2)$