University of Pittsburgh

Algorithms and Data Structures 2
CS 1501

Pitt SCI

Spring 2023

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(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

Announcements

- Upcoming Deadlines
  - **Lab 10**: ~~Tuesday 4/11~~ May 1 @ 11:59 pm
  - **Lab 11**: ~~Tuesday 4/18~~ May 1 @ 11:59 pm
  - **Lab 12**: May 1 @ 11:59 pm
  - **Homework 11**: ~~Friday 4/14~~ May 1 @ 11:59 pm
  - **Homework 12**: May 1 @ 11:59 pm
  - **Assignment 4**: ~~Friday 4/14~~ May 1 @ 11:59 pm
    - Support video and slides on Canvas + Solutions for Labs 8 and 9
  - **Assignment 5**: May 1 @ 11:59 pm
    - to be posted tonight

Final Exam

- **Friday 4/28 12:00-13:50**
  - 169 Crawford Hall
- Same format as midterm
- **Non-cumulative**
- Study guide and practice test on Canvas
- **Review Session** during Finals' Week
  - Date and time TBD
  - recorded

Bonus Opportunities

- **Bonus Lab**
  - worth up to 1%
  - lowest two labs still dropped
- **Bonus Homework**
  - worth up to 1%
  - lowest two homework assignments still dropped
- bonus point for class when

  OMETs response rate >= 80%
  - Currently at 12%
  - Deadline is Sunday 4/23

Previous Lecture

Dynamic Programming examples:

- ○ Longest Common Subsequence
- ○ Reinforcement Learning

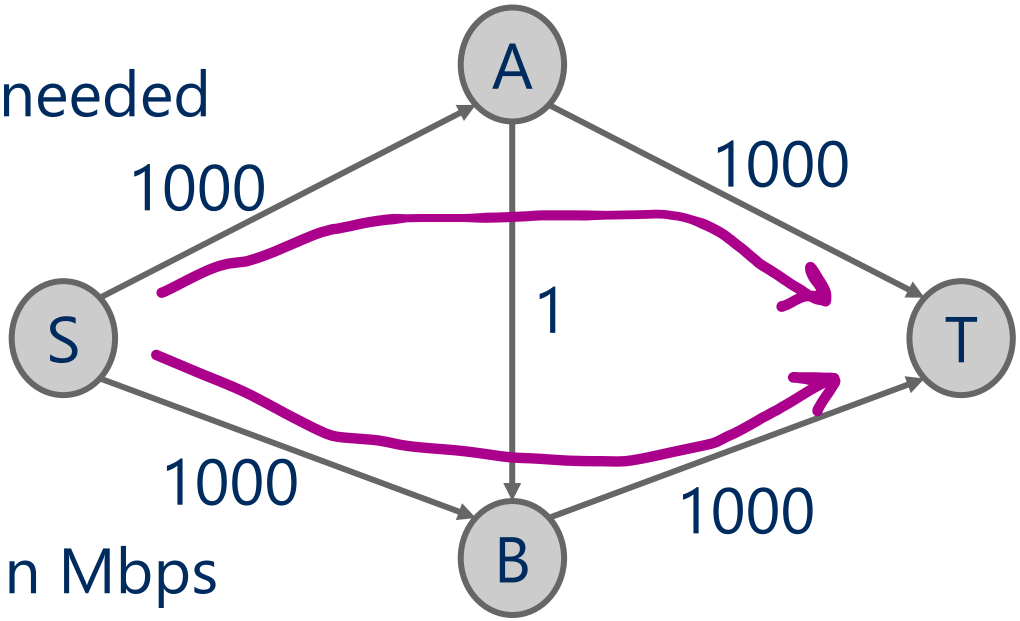
- **Maximum Flow** Problem
  - ○ Ford Fulkerson Framework

This Lecture

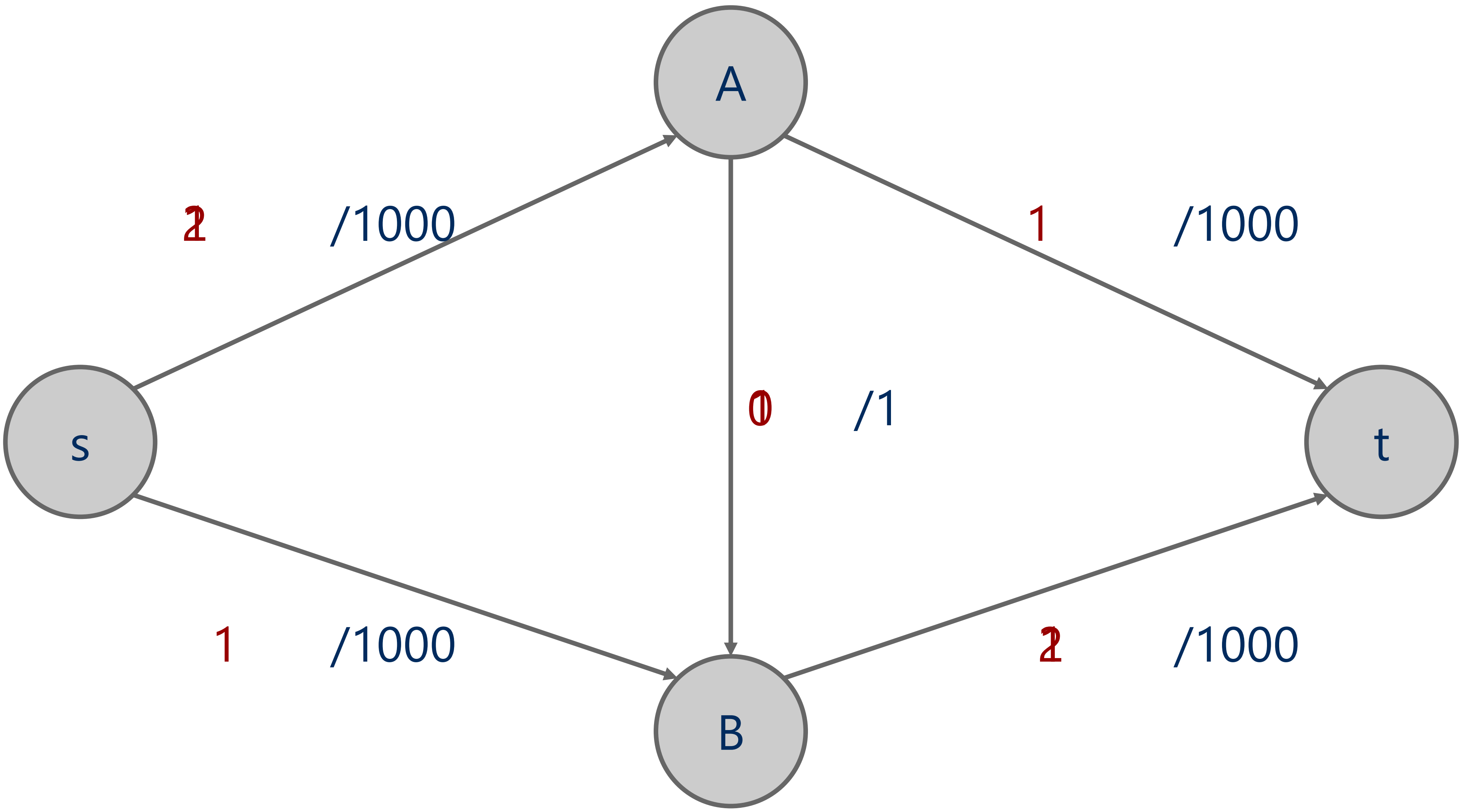
- **Maximum Flow** Problem: useful for general problem solving
  - Edmonds Karp
  - Push Relabel
  - An application of Maximum Flow
- Graph compression
- Local Search

Problem of the Day: Finding Bottlenecks

- send a large file from S to T over a computer network
  - as fast as possible
  - over multiple network paths if needed
- Input:
  - computer network
    - nodes and links
    - links labeled by link speed in Mbps
  - nodes A and B
- Output:
  - The **maximum network speed** possible



Worst-case Runtime for Ford-Fulkerson



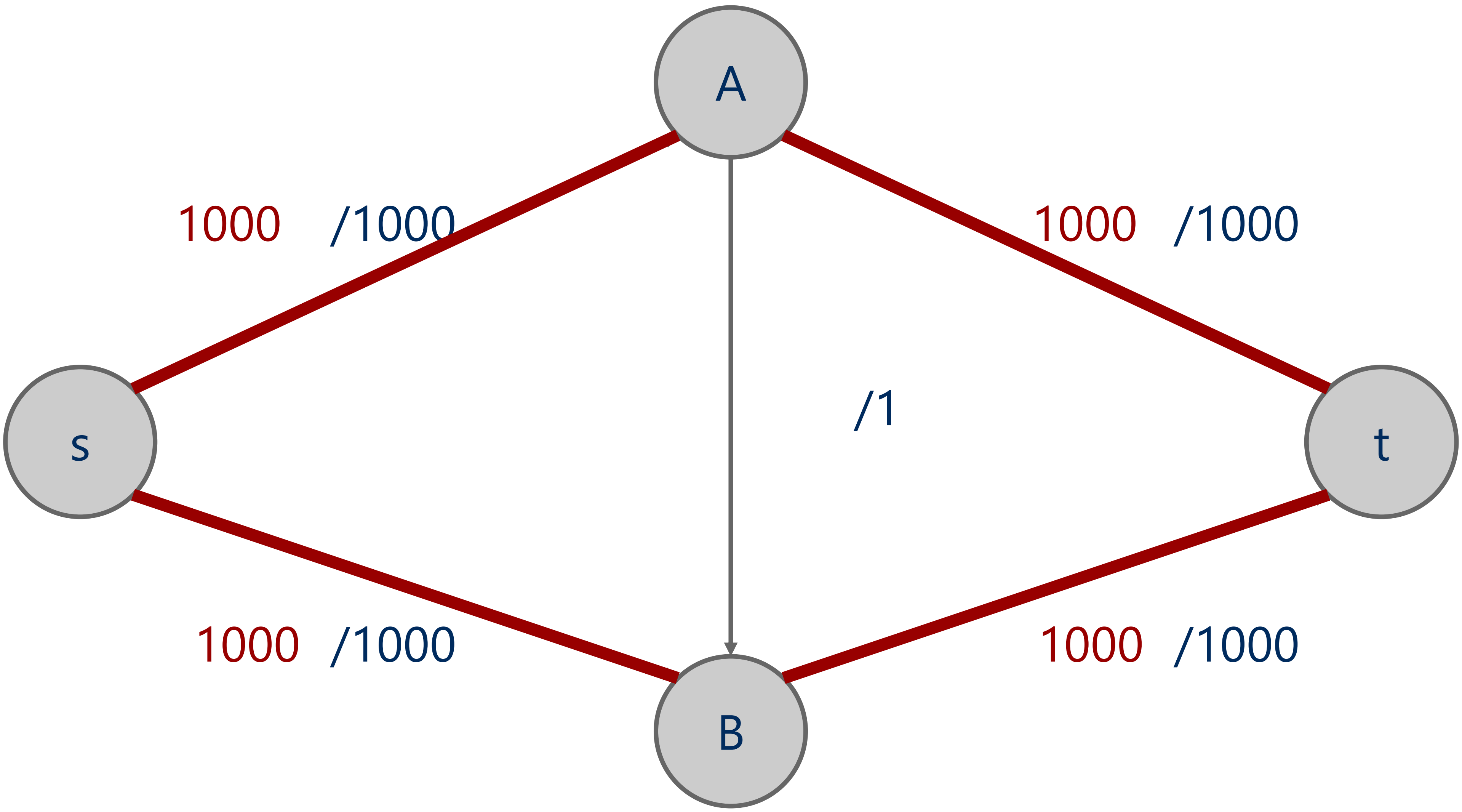
Worst-case Runtime of Ford-Fulkerson

- O(f * (e+v))
  - f: value of maximum flow
  - e+v: time to find an augmenting path
- is that **polynomial** in the ***input size***?
- **No!** f is exponential in **bit length** of f
- Runtime is $O(2^{|f|} * (e+v))$

Edmonds Karp

- How the augmenting path is chosen affects the **performance** of the search for max flow
- Edmonds and Karp proposed a **shortest path heuristic** for Ford Fulkerson
    - Use **BFS** to find augmenting paths

Edmonds Karp using BFS to find augmenting paths



Edmonds Karp

- Running time is $O(e^2 v)$
    - **Proof:** Proposition G in Chapter 6

But our flow graph is weighted...

Edmonds-Karp only uses BFS

- BFS finds spanning trees and shortest paths for **unweighted** graphs
- some **weight-based** measure of priority to find augmenting paths?

Maximum Capacity Path

- Proposed by Edmonds and Karp
- implemented by modifying Dijkstra's shortest paths algorithm
- Define **flow[v]** as the maximum amount of flow from s → v along a **single path**
- Each iteration, set curr as an unmarked vertex with the largest flow
- For each neighbor w of curr:
  - if **min(flow[curr], residual capacity of edge (curr, w))** > flow[w]
    - update flow[w] and parent[w] to be curr

Flow edge implementation (Bonus Lab)

- For each **edge**, we need to store:
  - **from** vertex
  - **to** vertex
  - **edge capacity**
  - **edge flow**
  - residual capacities
    - For **forwards** and **backwards** edges

FlowEdge.java

```
public class FlowEdge {
   private final int v;              // from
   private final int w;              // to
   private final double capacity;    // capacity
   private double flow;              // flow
     …
     public double residualCapacityTo(int vertex) {
      if      (vertex == v) return flow;
      else if (vertex == w) return capacity - flow;
      else throw new
       IllegalArgumentException("Illegal endpoint");
   }
   …
}
```

BFS search for an augmenting path (pseudocode)

```
edgeTo = [v]
marked = [v]
Queue q
q.enqueue(s)
marked[s] = true
while !q.isEmpty():
    v = q.dequeue()
    for each edge in AdjList[v]:
        if residualCapacity(other end-point) > 0:
            if !marked[w]:
                edgeTo[w] = v;
                marked[w] = true;
                q.enqueue(w);
```

Each FlowEdge object is stored in the adjacency list twice:

Once for its forward edge
Once for its backward edge

Push-Relabel Algorithm for Max Flow

- More **efficient** than Edmonds-Karp that uses BFS
  - Running time: **Theta(v^3)** vs. Theta(e^2v)
- **Local** per vertex operations instead of global updates
- Each vertex has a **height** and **excess flow** value

Push-Relabel Algorithm for Max Flow

- **push operation:**
  - Flow **pushed** from higher vertex to lower neighbor
    - height difference of 1 or more
    - over an edge with **residual capacity > 0**
- **relabel operation:**
  - If a vertex's excess flow > 0 and has no lower neighbor
    - **relabel** vertex's height to
      - **1 + min height of neighbors able to receive flow**

Push-Relabel Algorithm for Max Flow

- **push operation:**

- **relabel operation:**

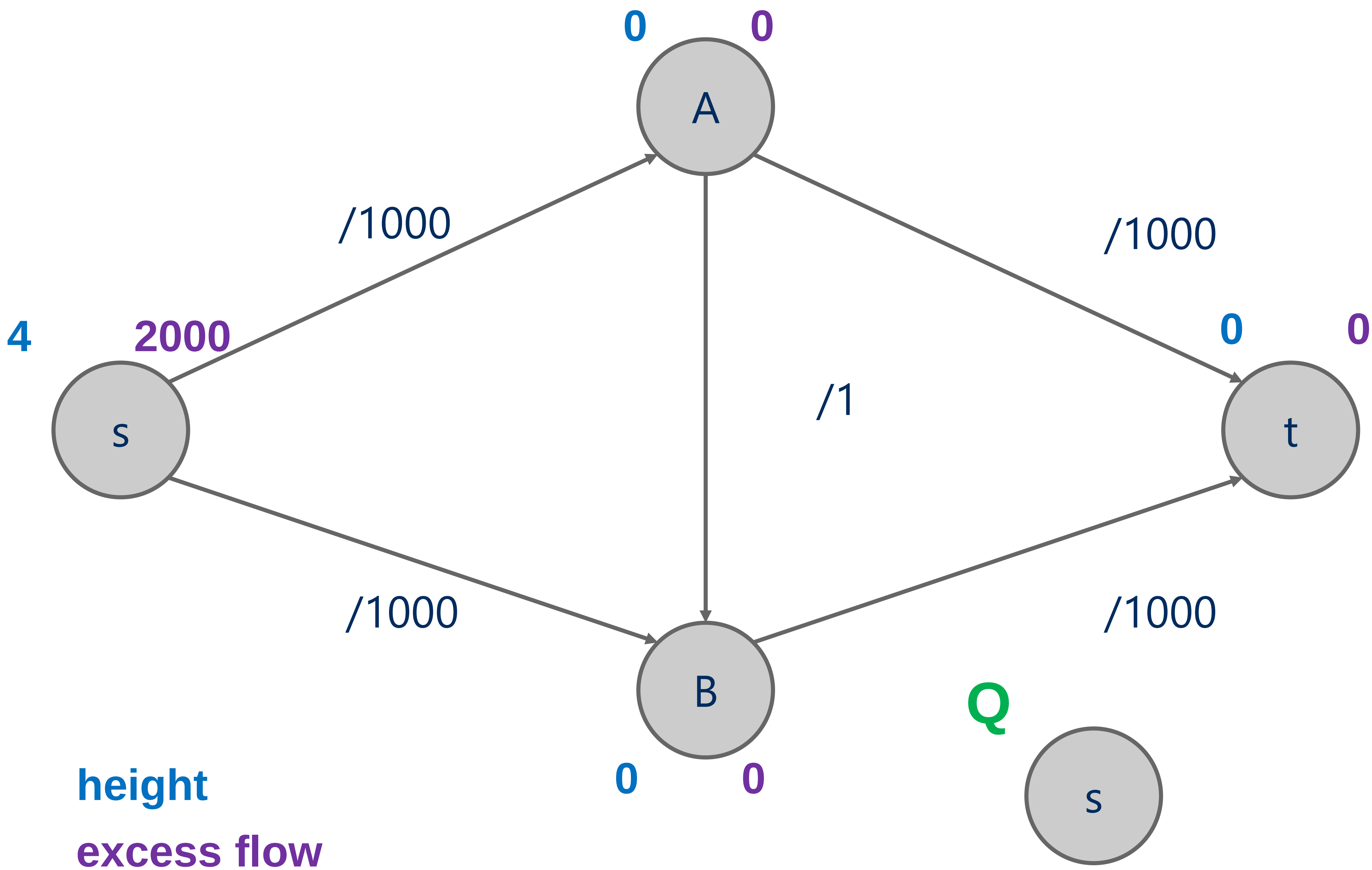
- **Repeat** relabel and push operations until

    - all vertices except source and sink have **0 excess flow**

Push-Relabel Algorithm

- height = 0 and excess = 0 for all vertices
- excess[s] = sum of edge capacities out of s
- height[s] = v
- insert s into Q
- **while Q not empty**
    - v = pop head of queue
    - relabel v if needed
    - **for each neighbor w of v:**
        - push as much of v's excess flow to w
        - increase w's excess flow by the pushed amount
        - add w to Q if not already there

Push Relabel Example



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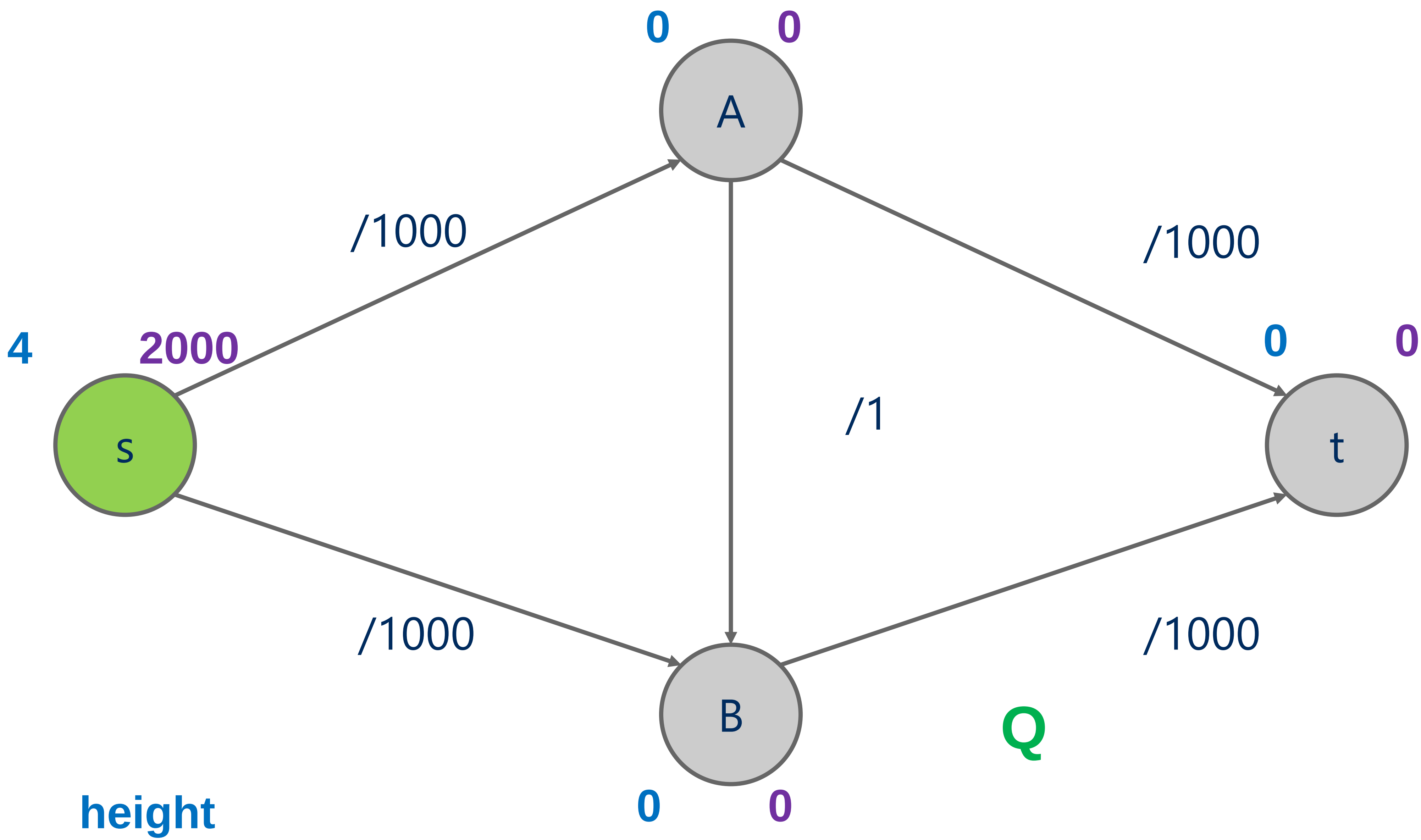

height

excess flow

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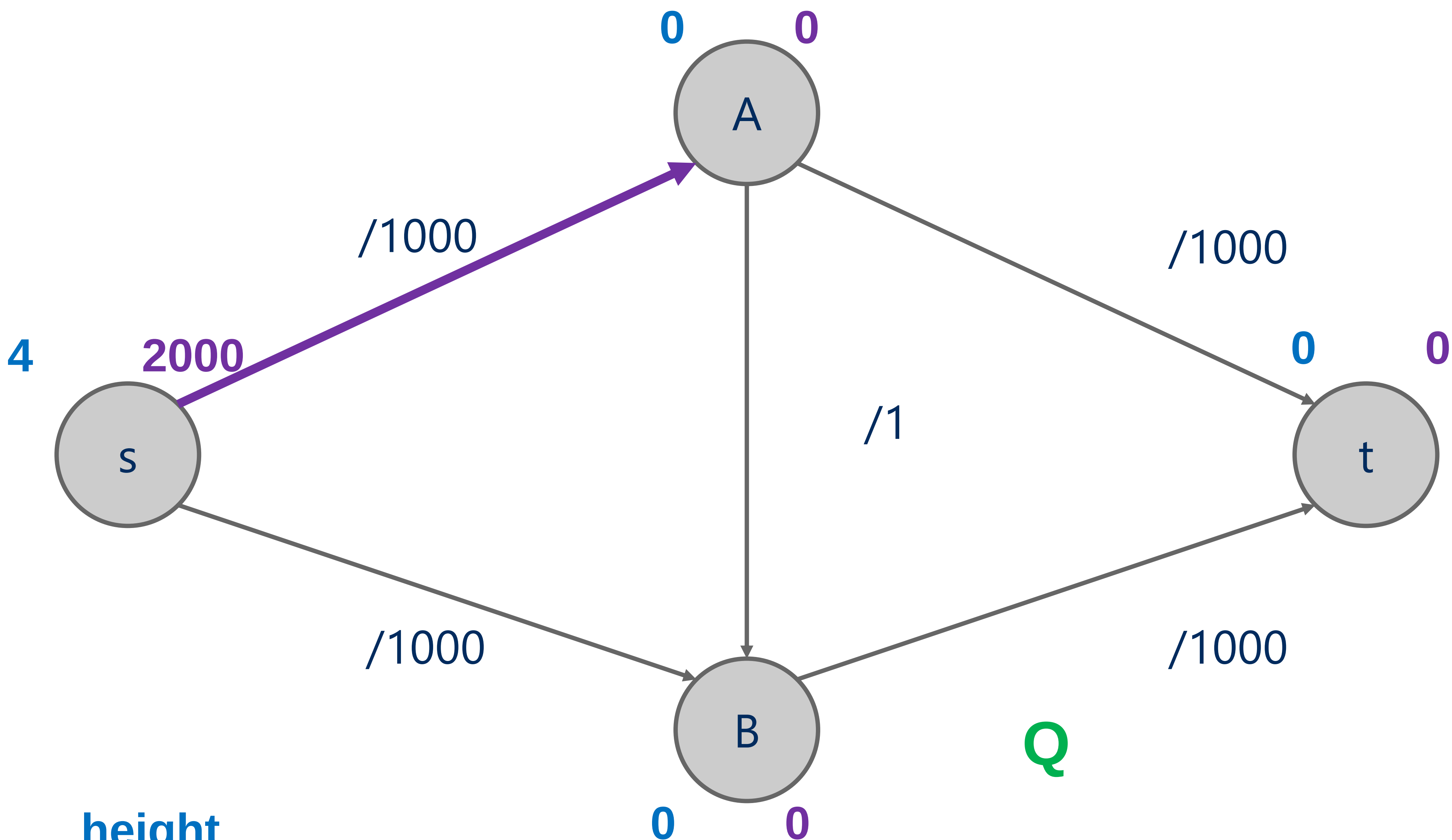

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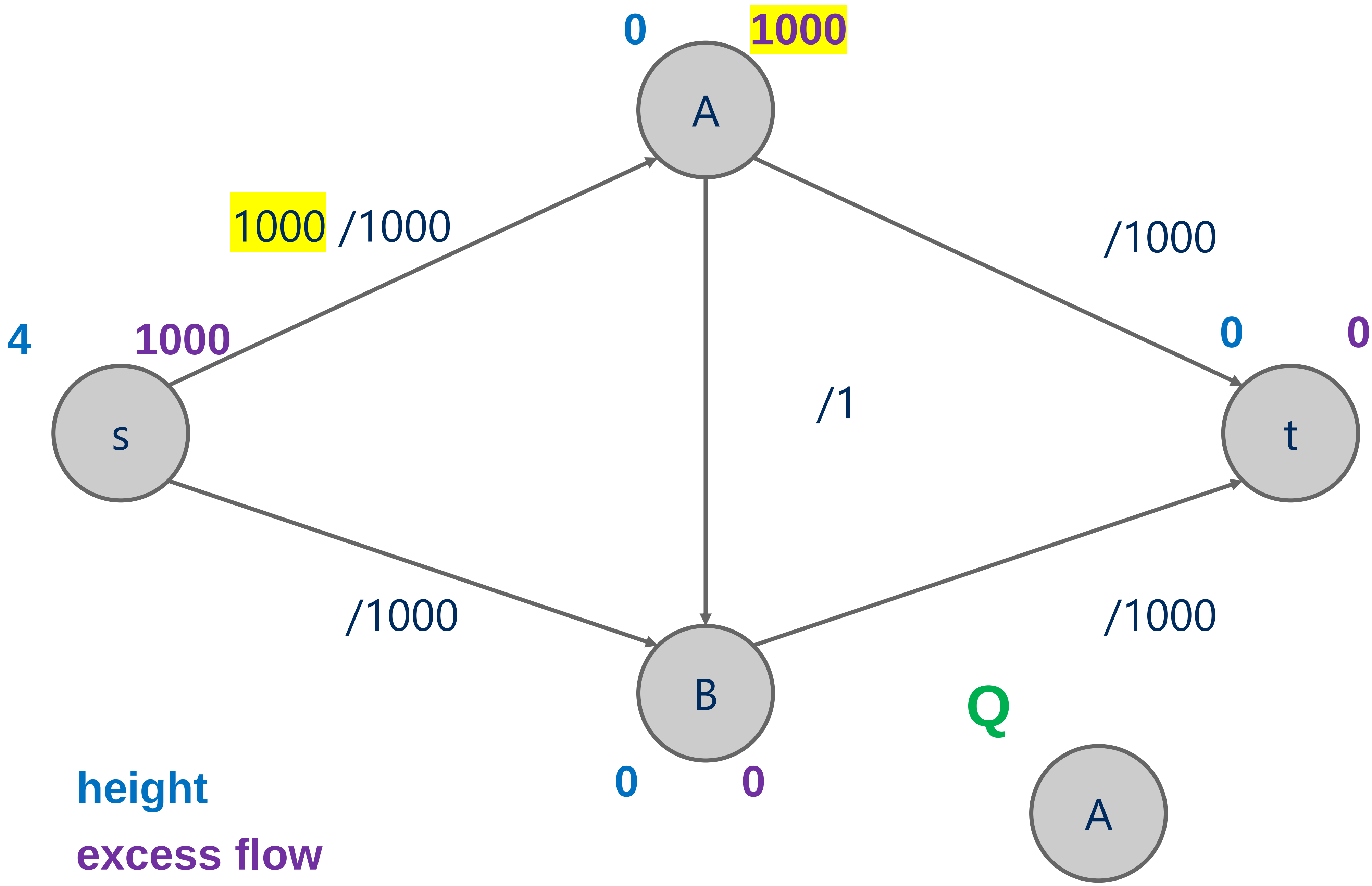

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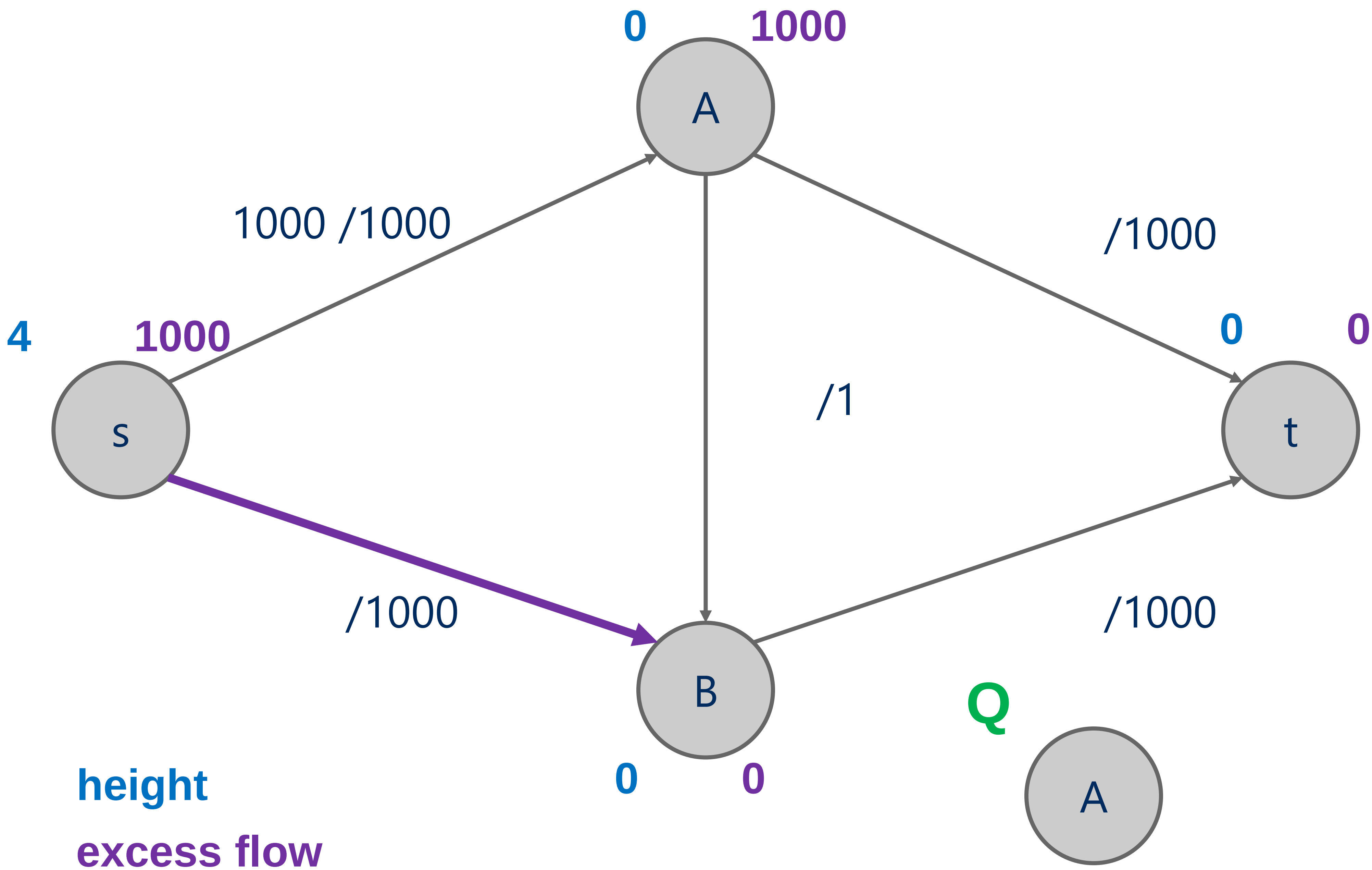

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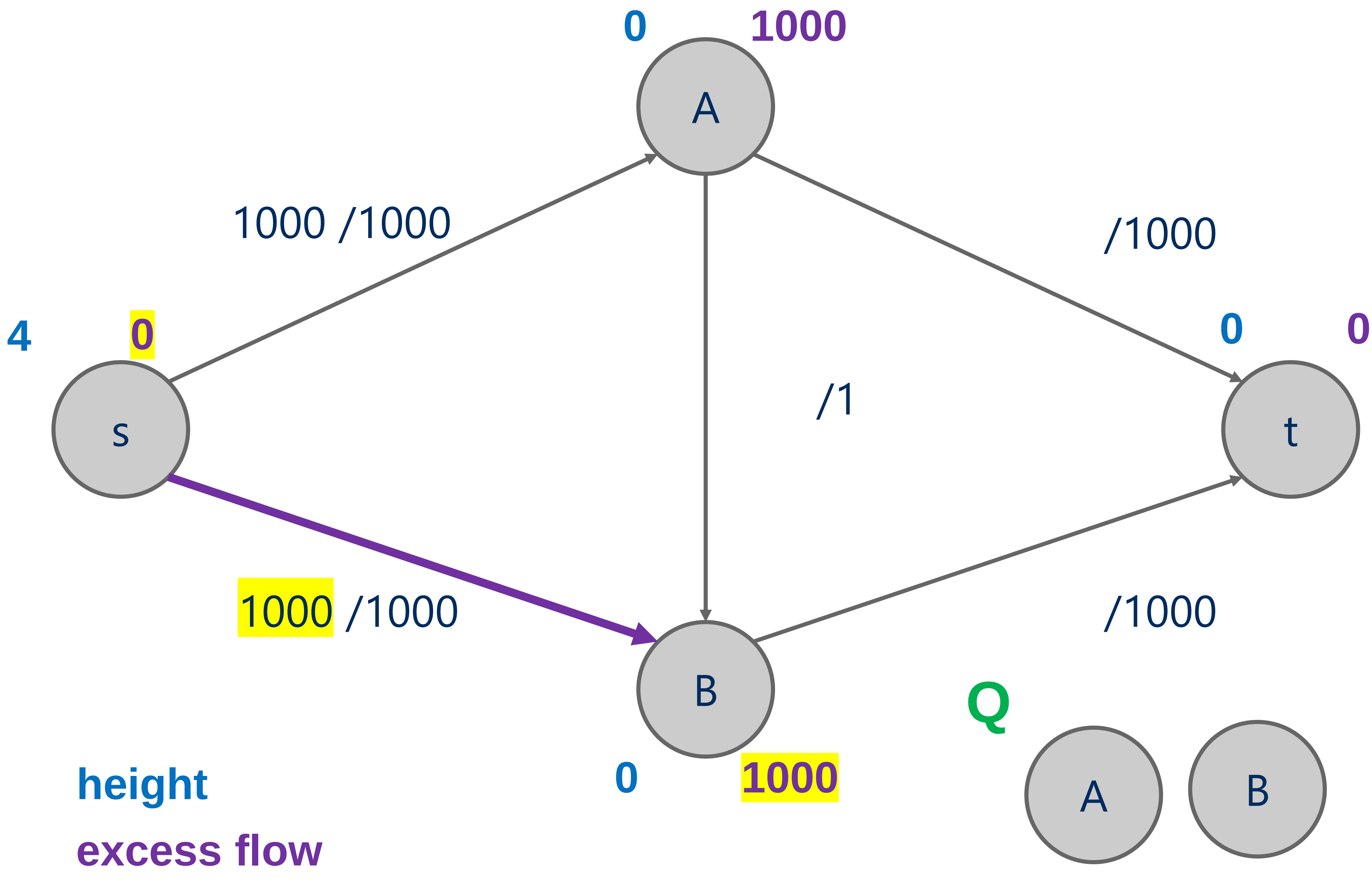

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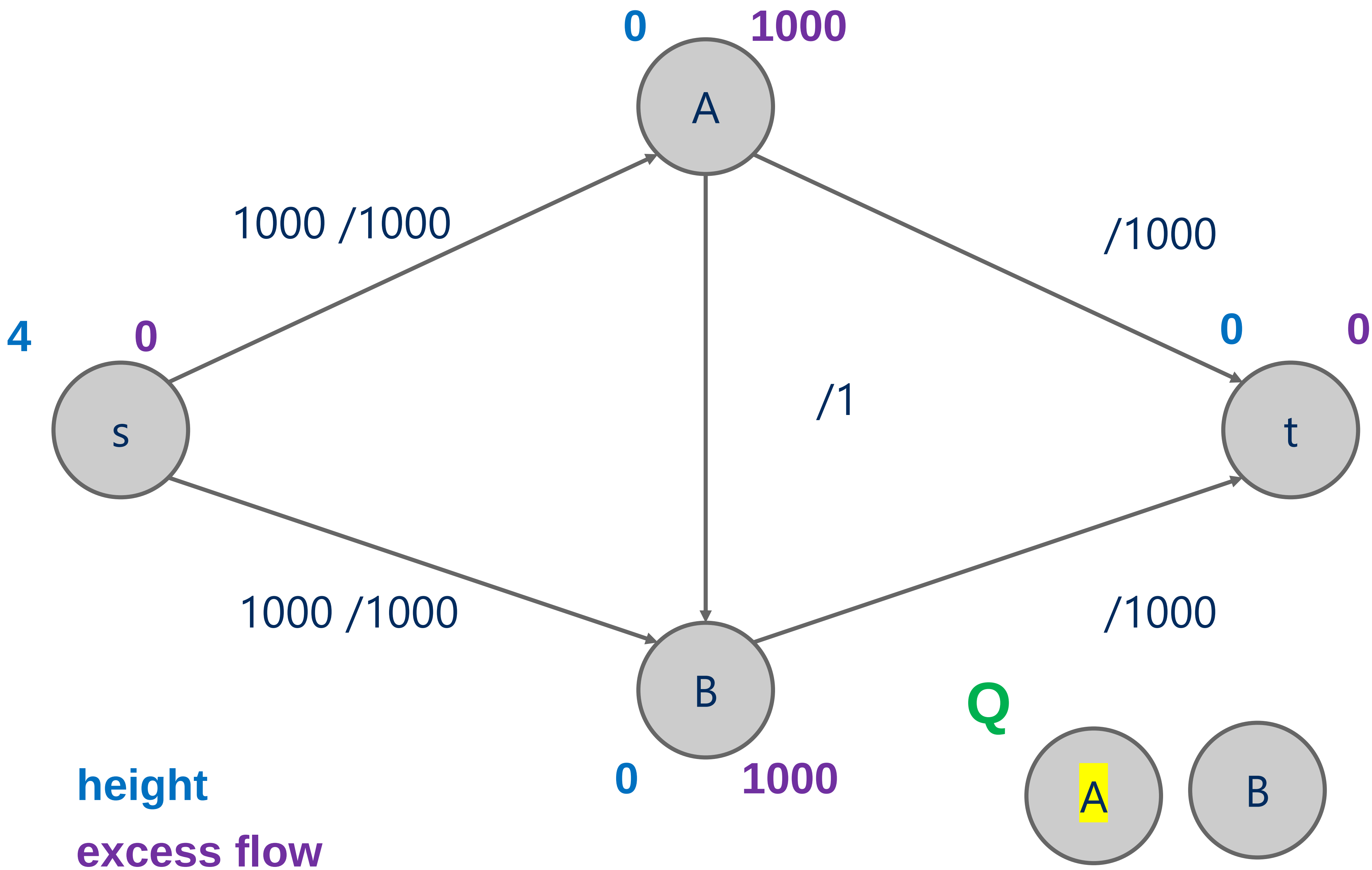

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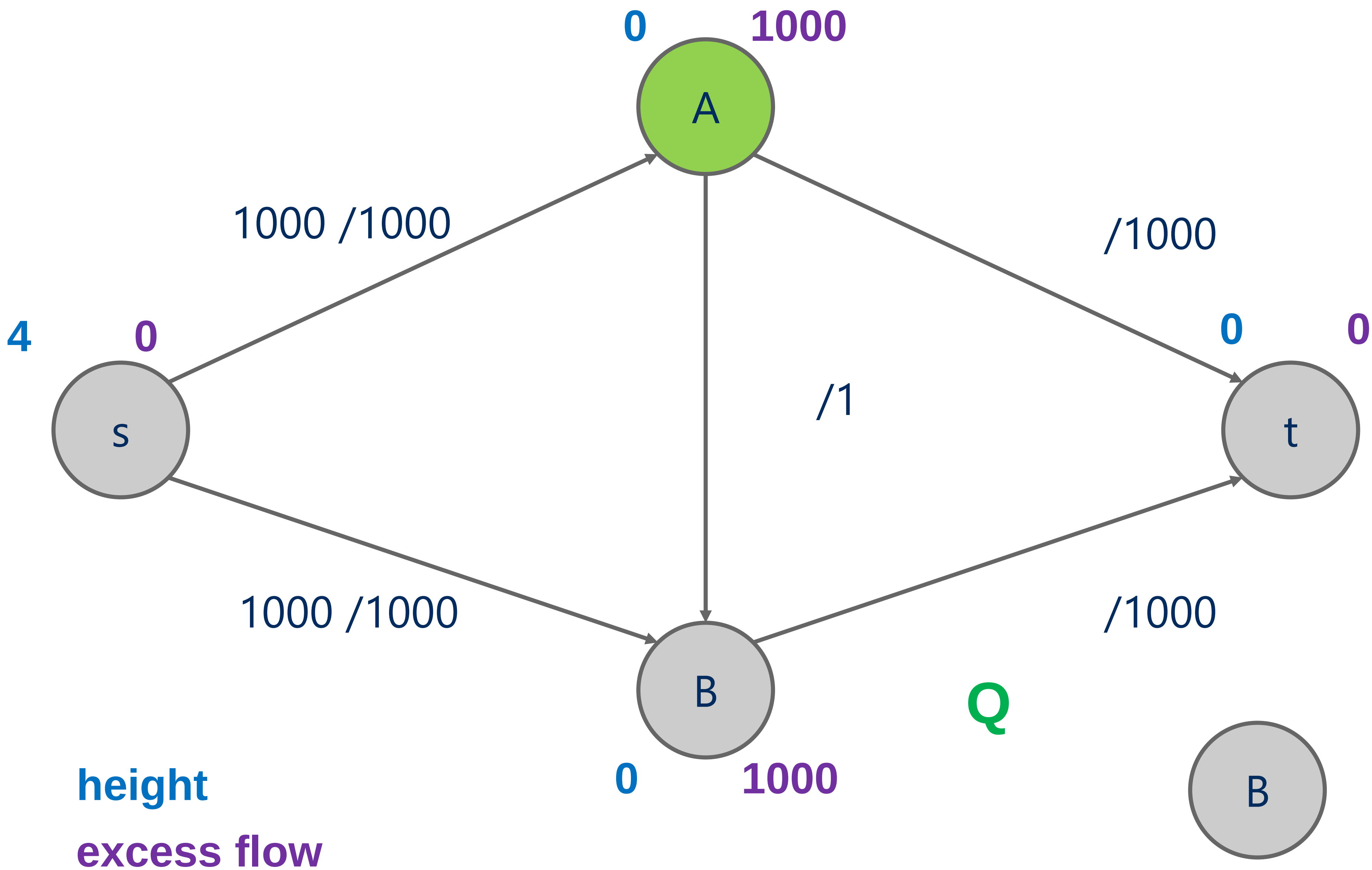

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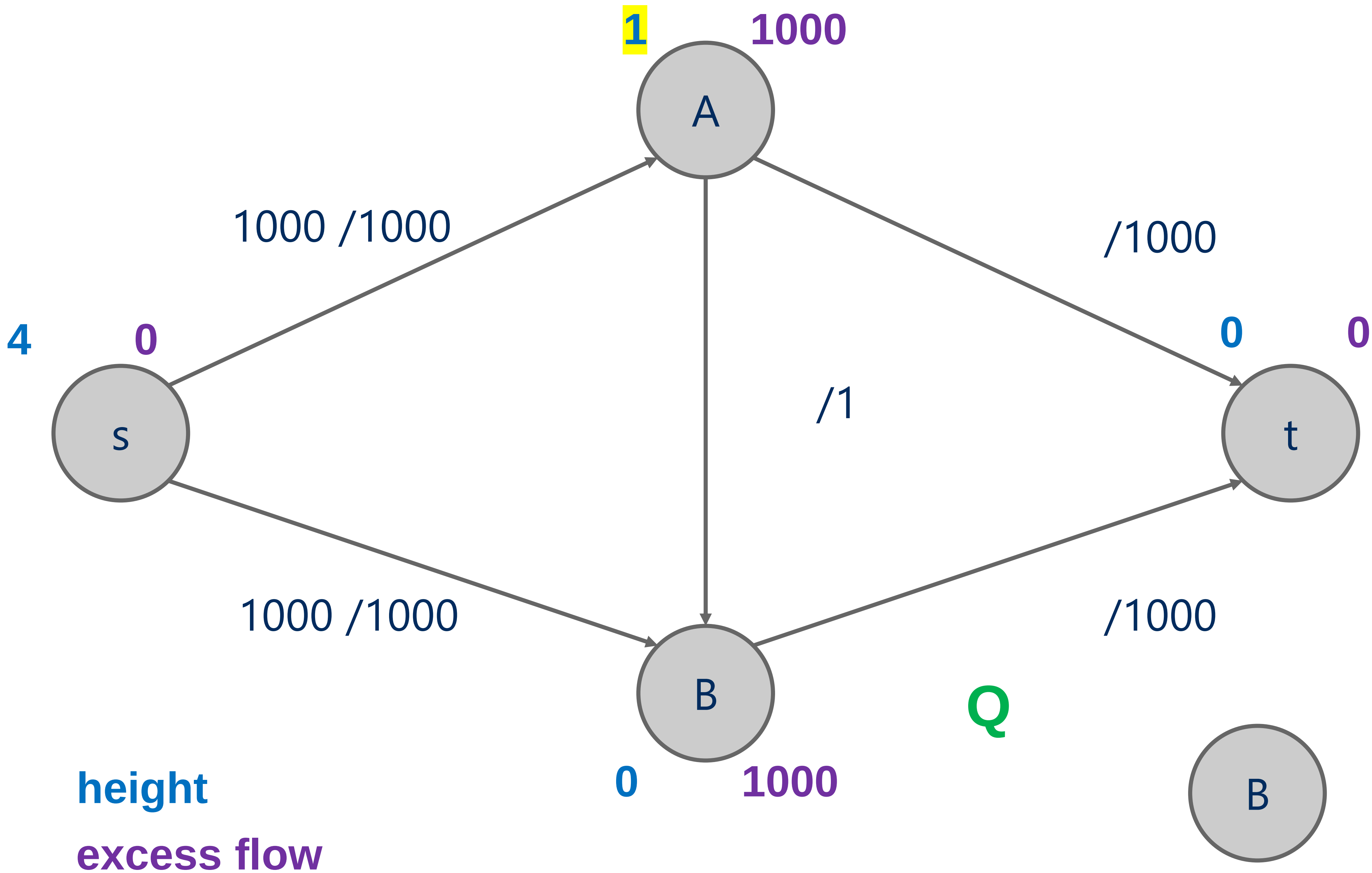

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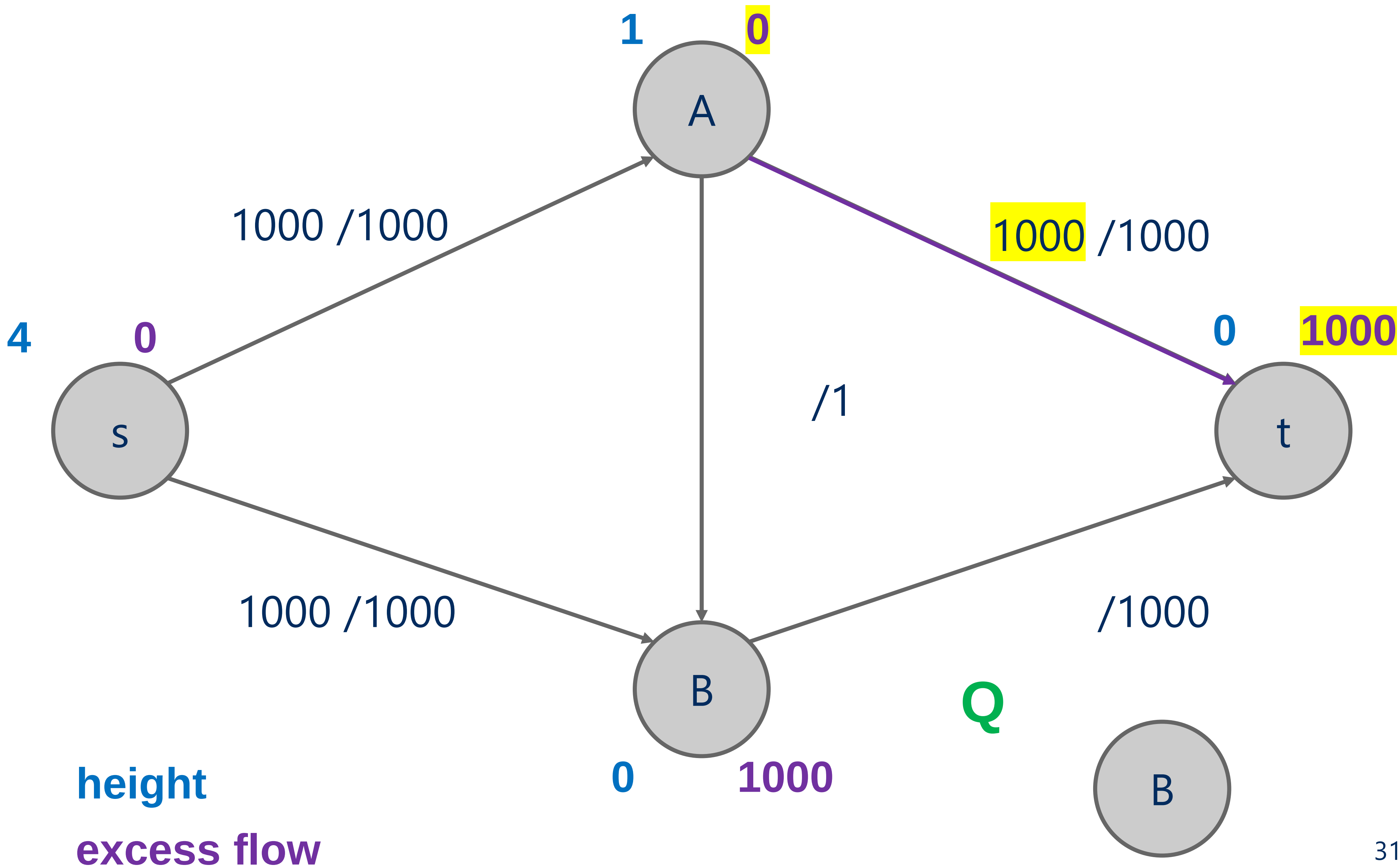

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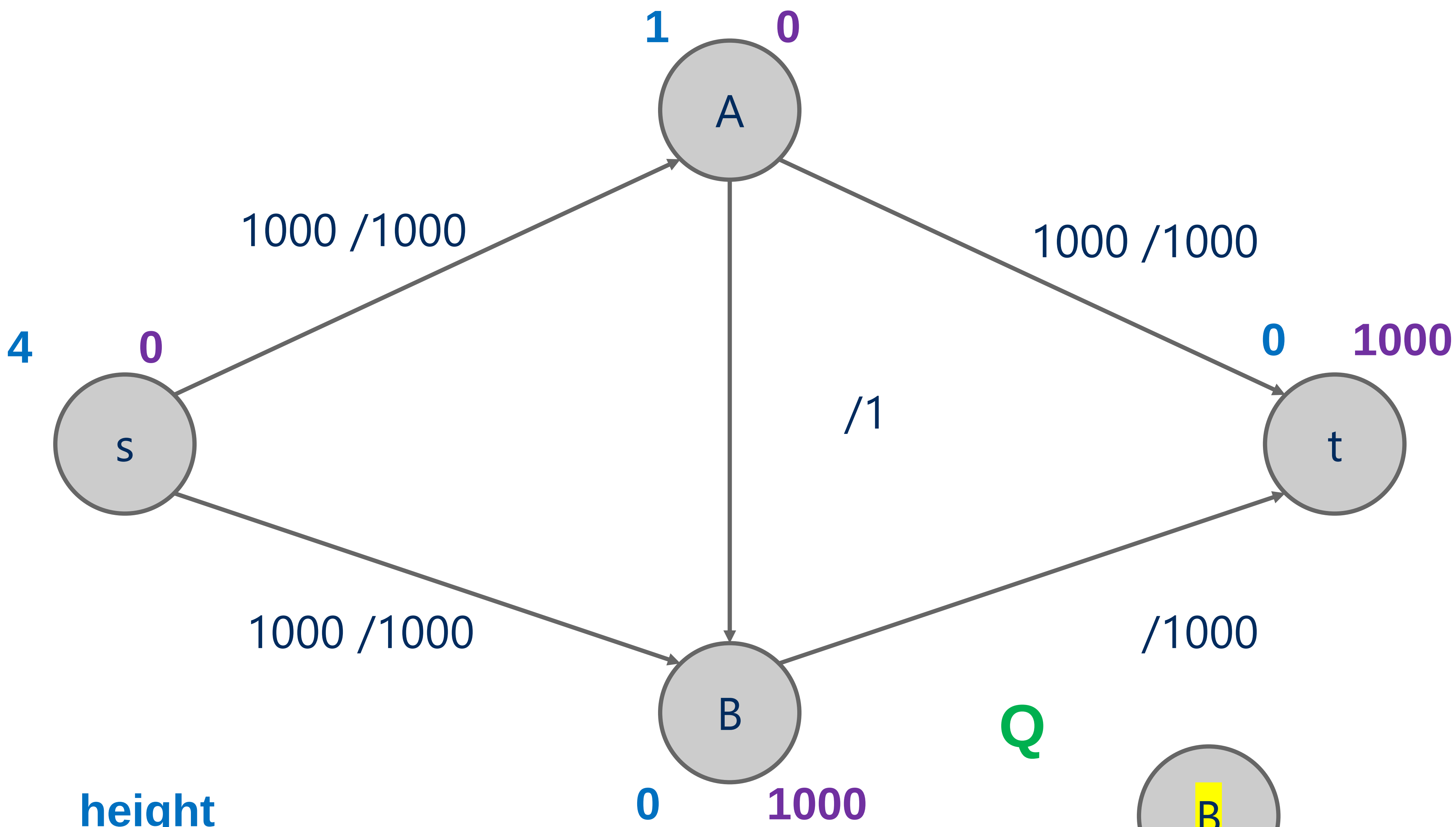

height

excess flow

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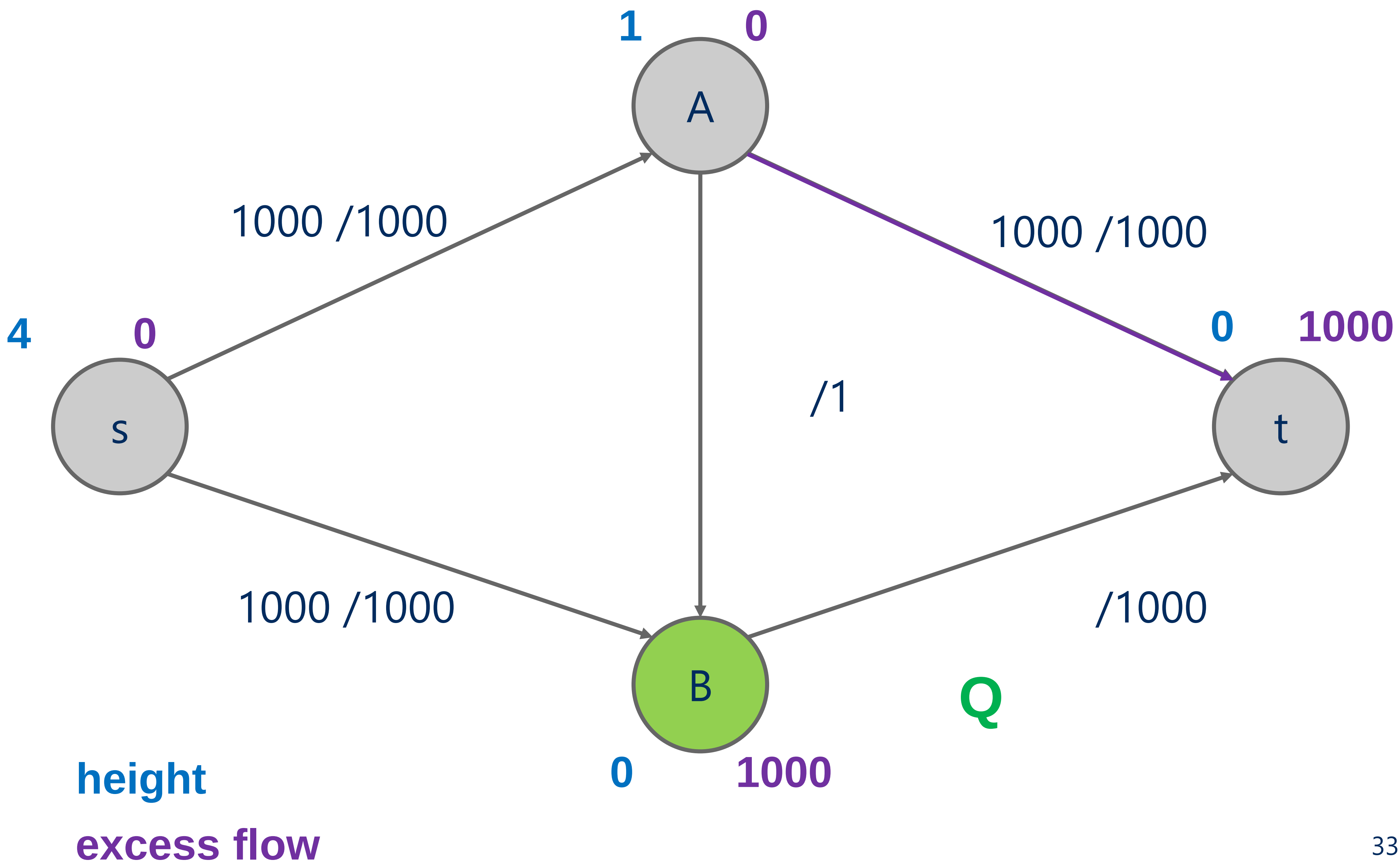

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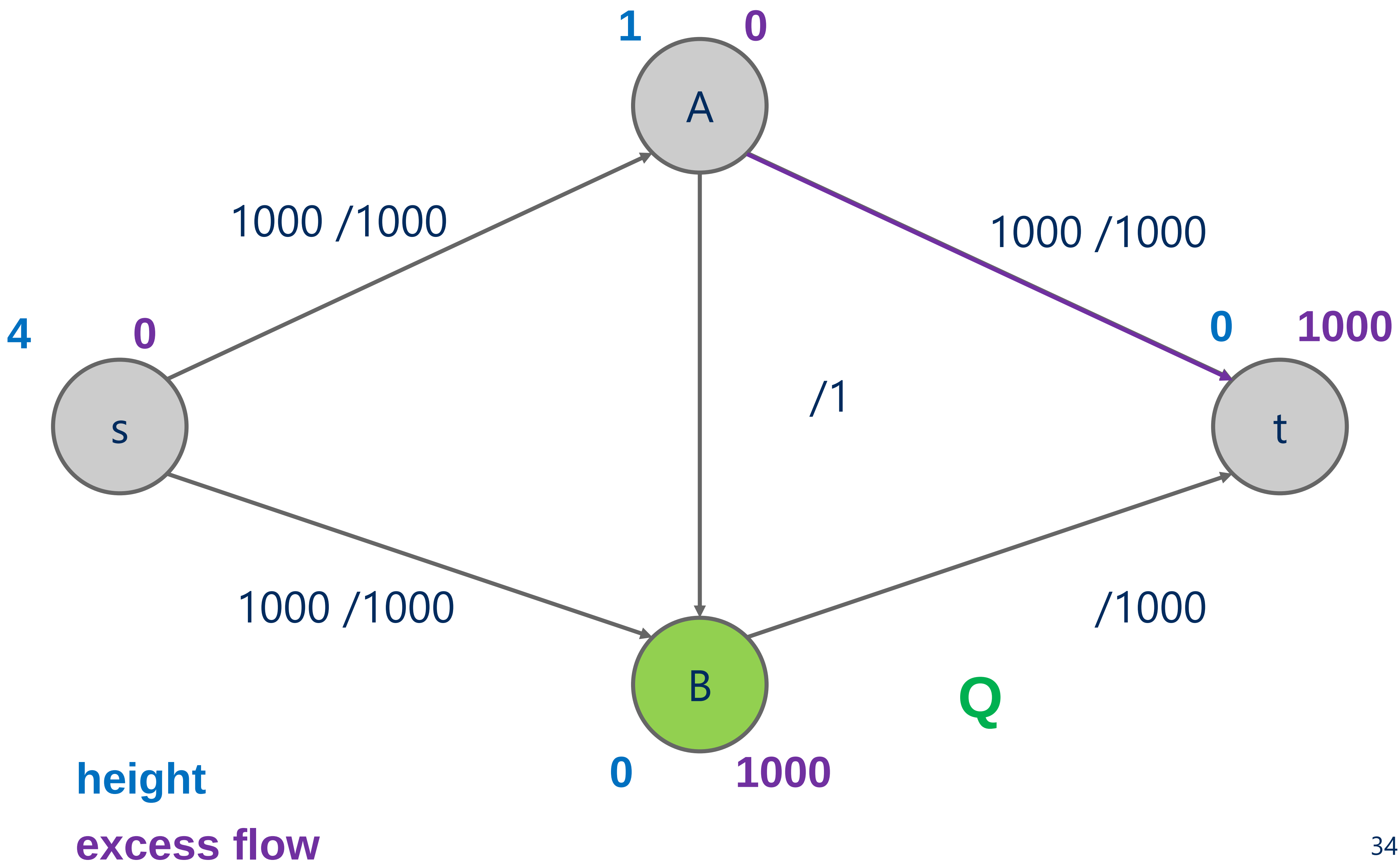

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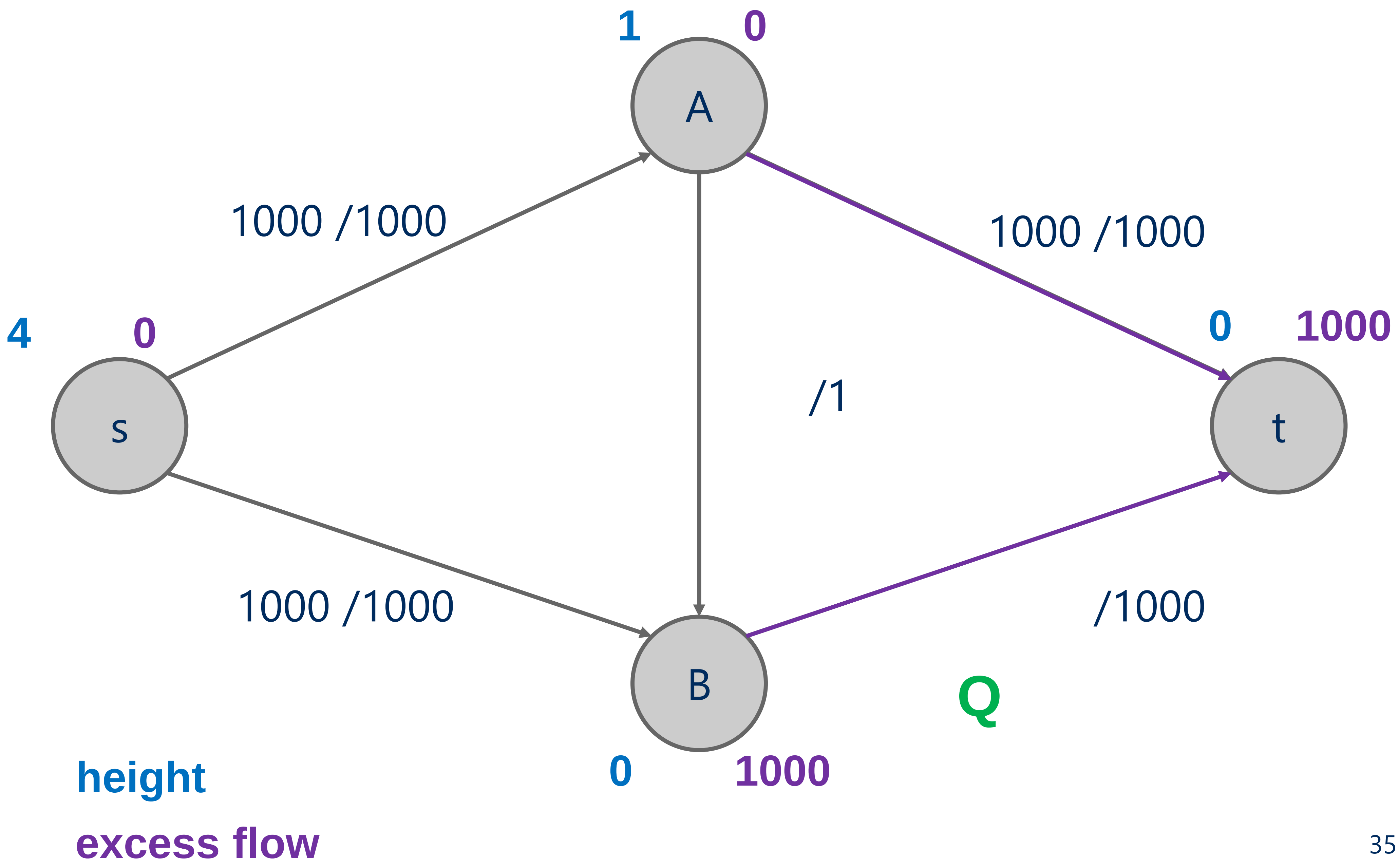

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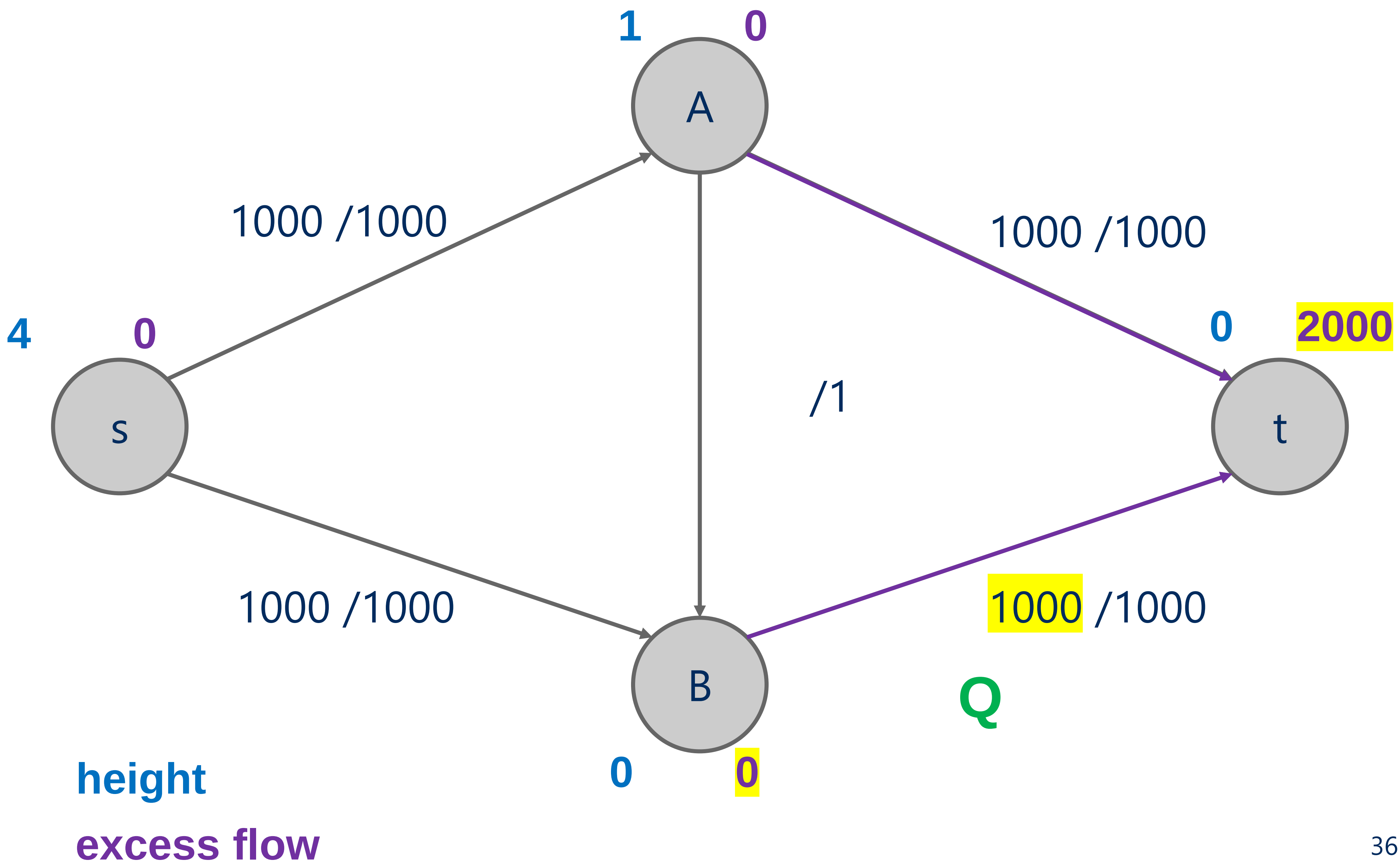

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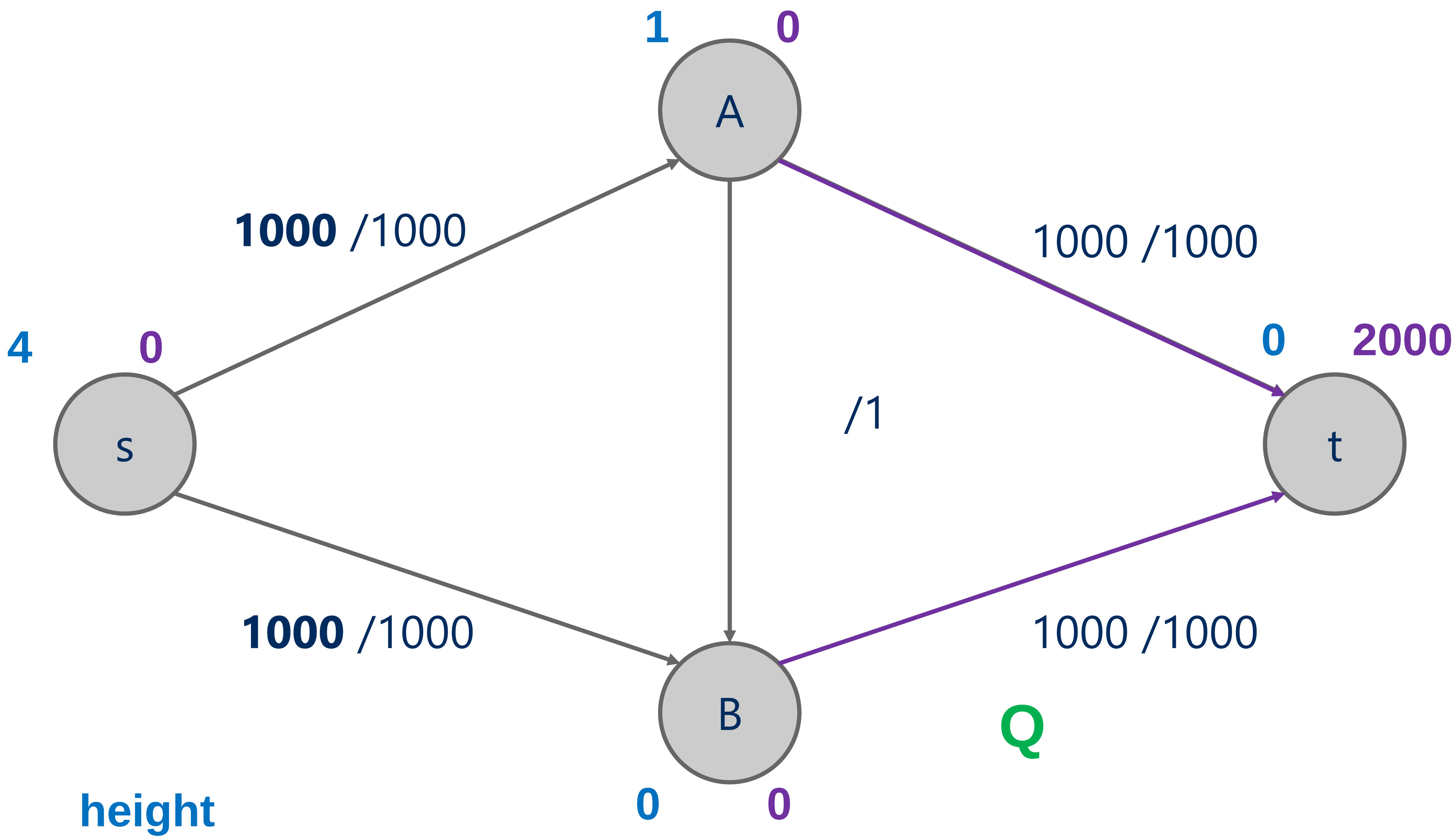

height

excess flow

Problem of the Day: Finding Bottlenecks

- send a large file from S to T over a computer network
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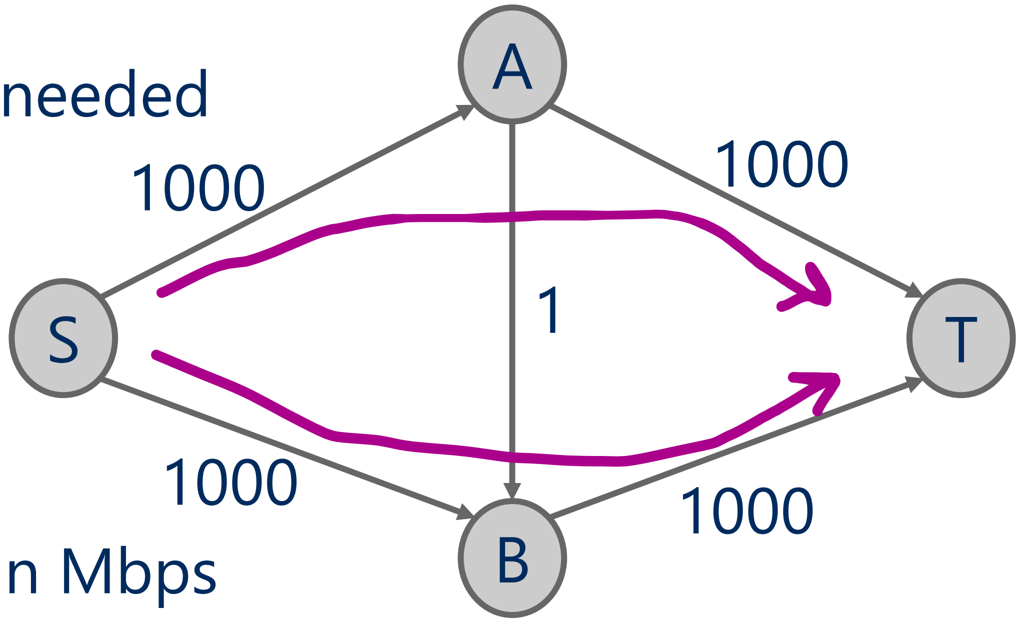
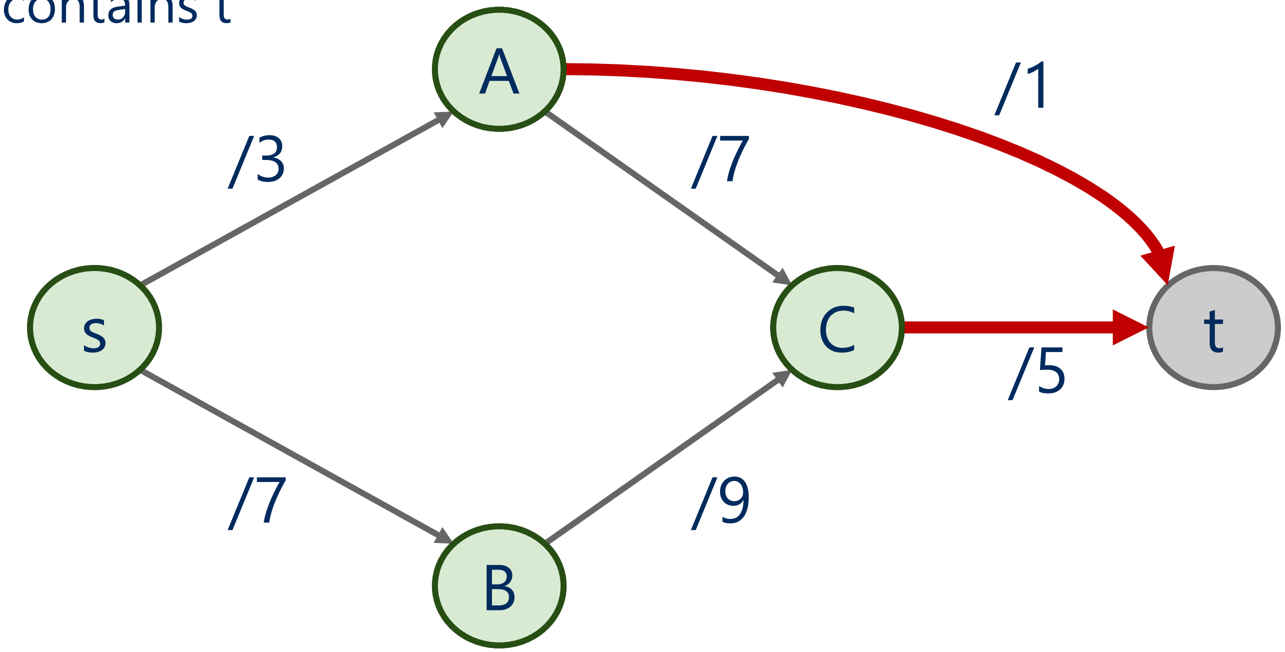


Follow-up Problem

- So, now we found the bottleneck **value**, but which **edges** define the found bottleneck?
    - *Why would you want to know **bottleneck edges**?*

Finding bottleneck edges

- An **st-cut** on G is a set of edges in G that, if removed, will partition the vertices of G into two disjoint sets
  - One contains s
  - One contains t



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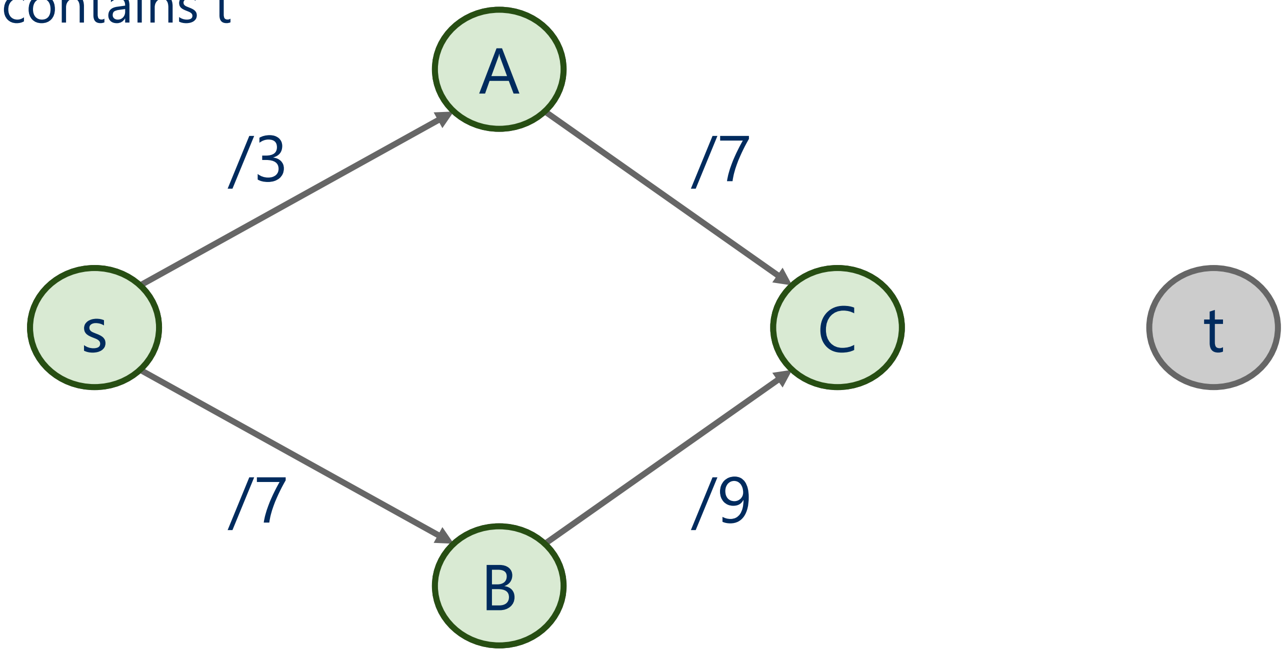

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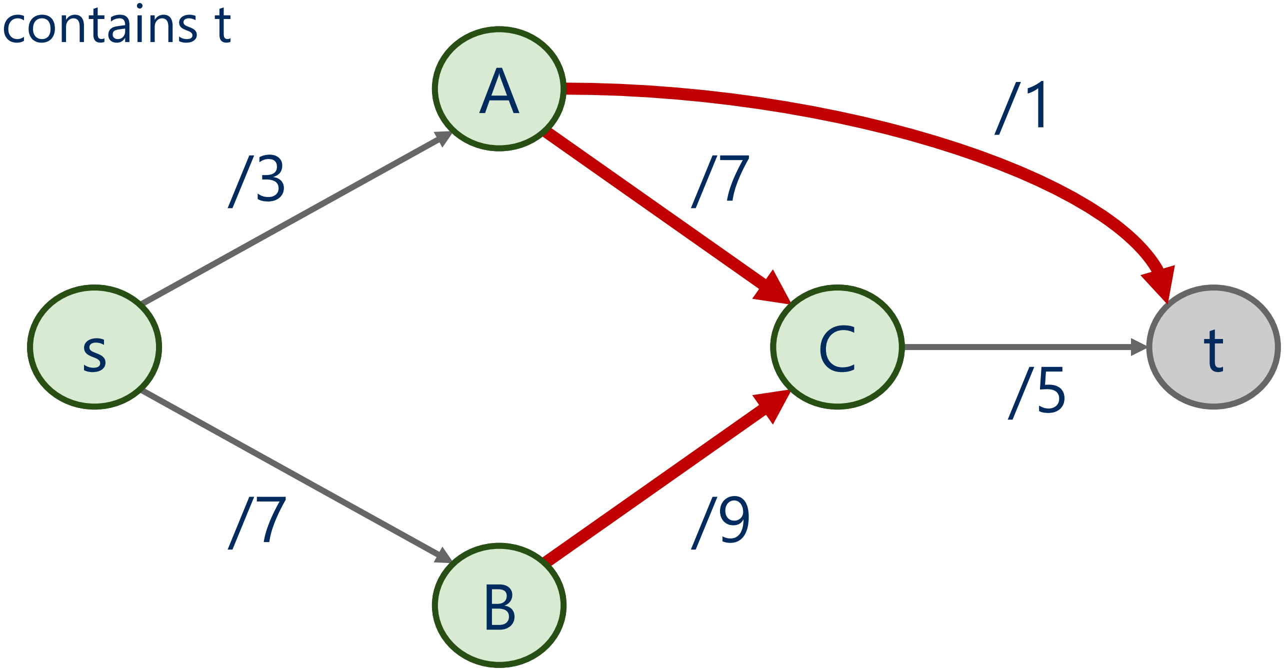

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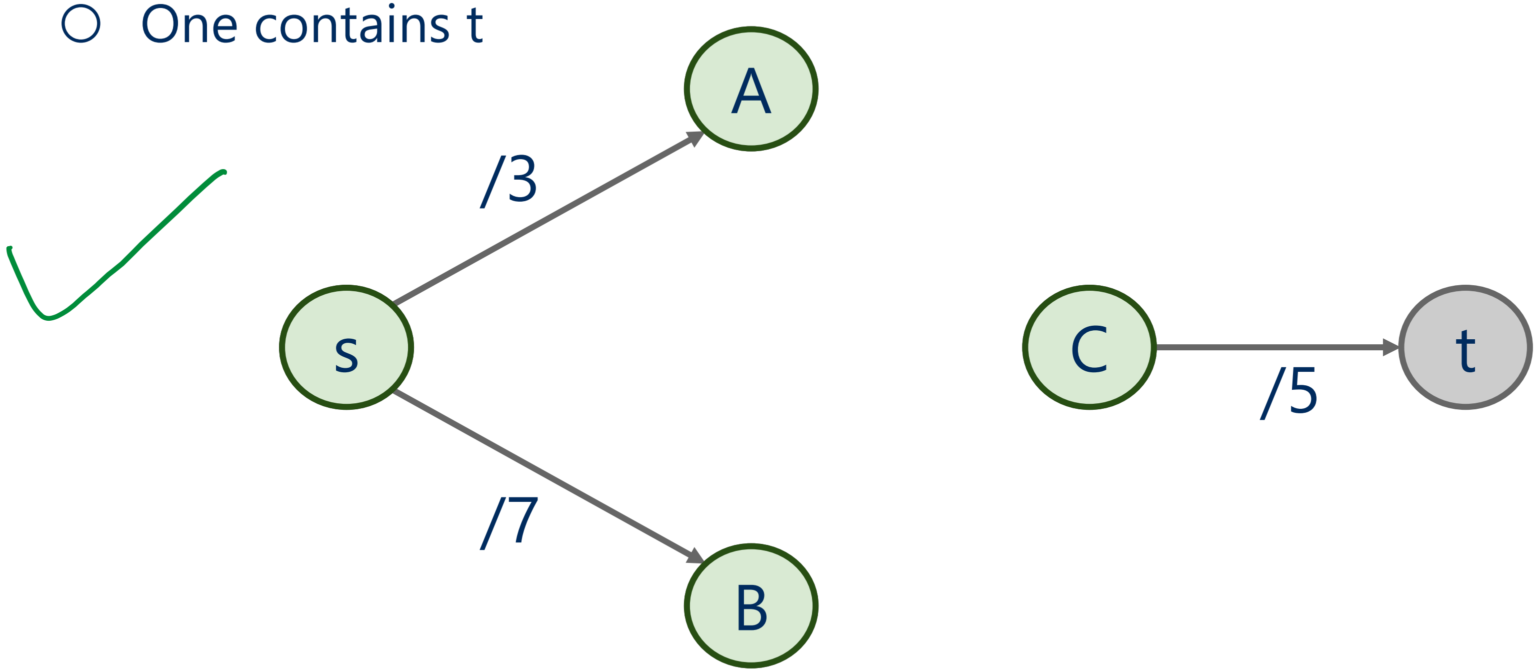

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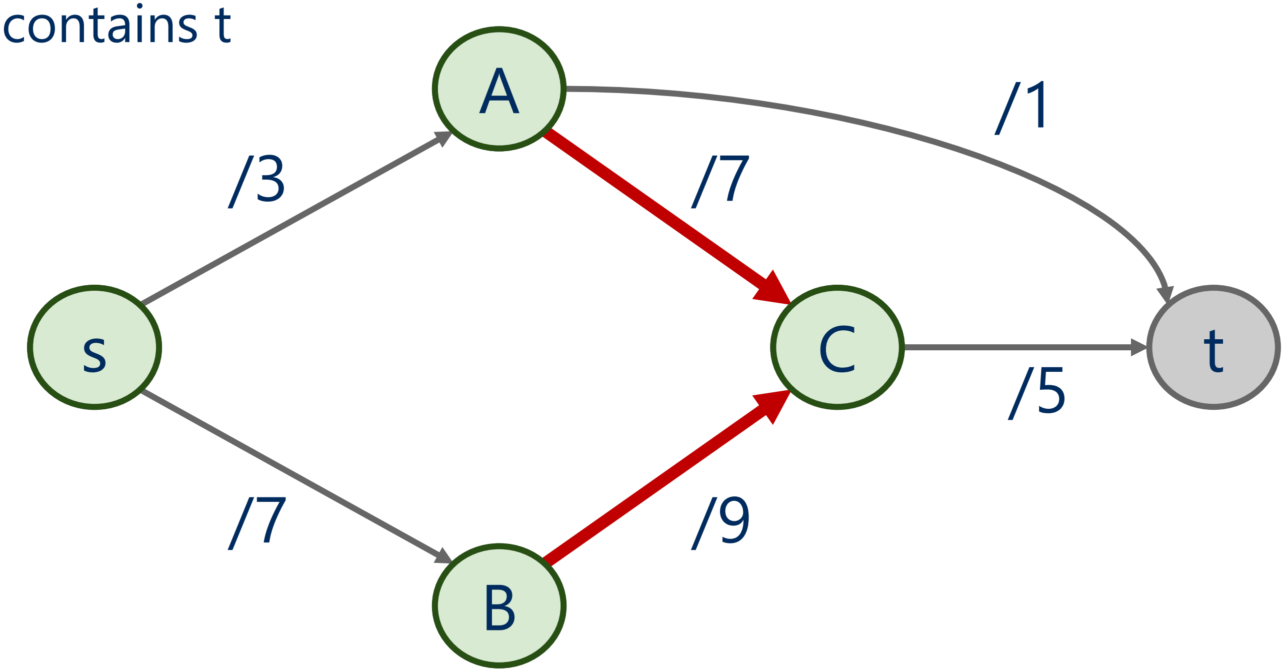

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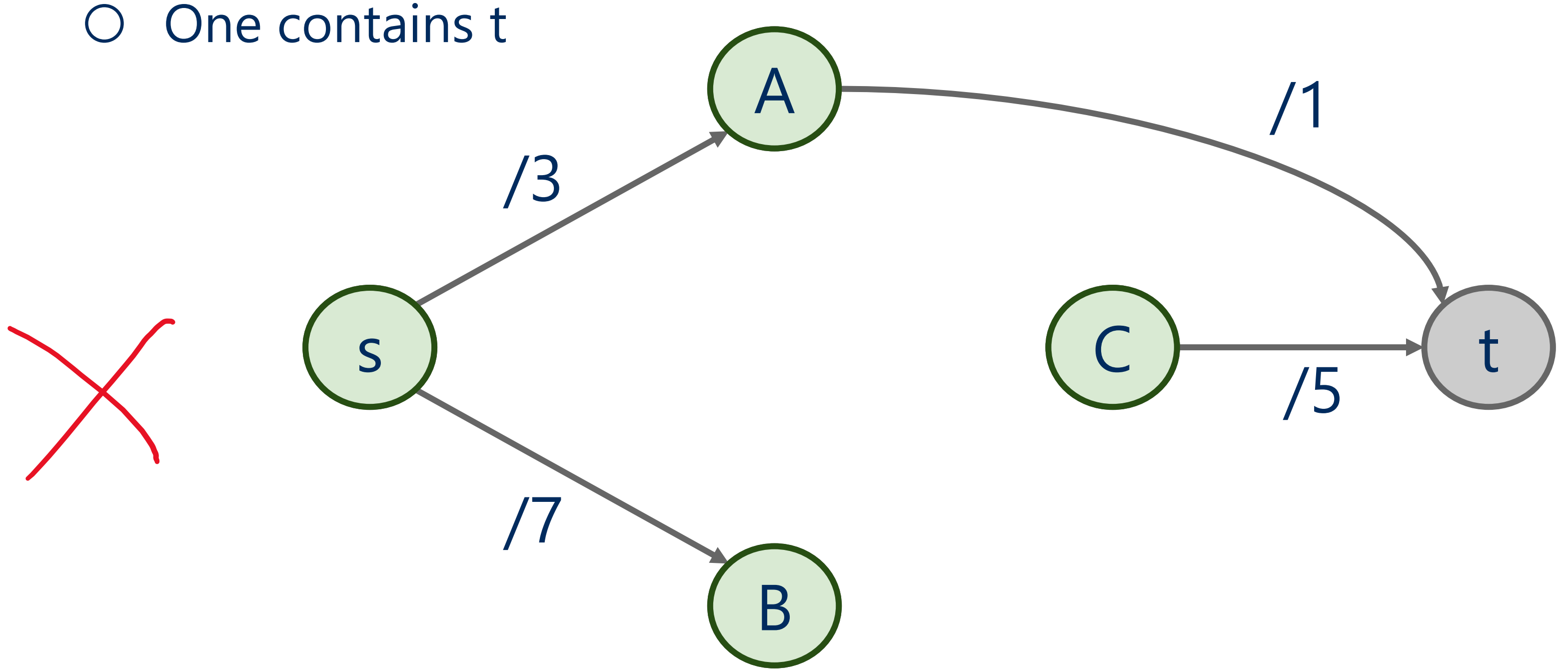


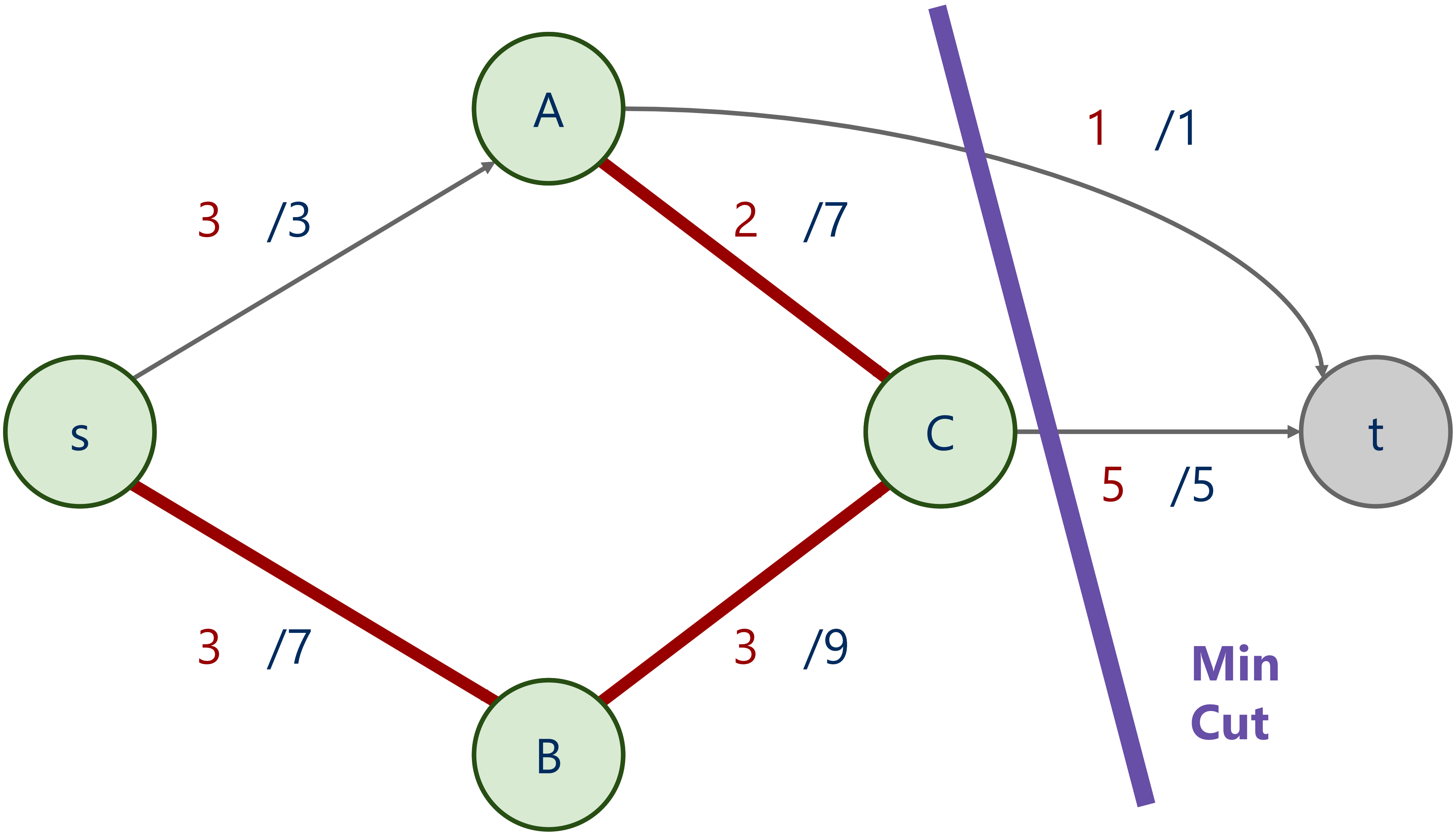
Finding bottleneck edges

- May be many st-cuts for a given graph
- Let’s focus on finding a **minimum st-cut**
  - an st-cut with the **smallest edge capacities**
  - May not be unique

How do we find a min st-cut?

- We could examine residual graphs
  - Specifically, try and allocate flow in the graph until we get to a residual graph with **no augmenting paths**
- The last iteration of Ford-Fulkerson visits every vertex reachable from s
  - Edges with only one endpoint reachable from S comprise a minimum st-cut

Determining the min cut



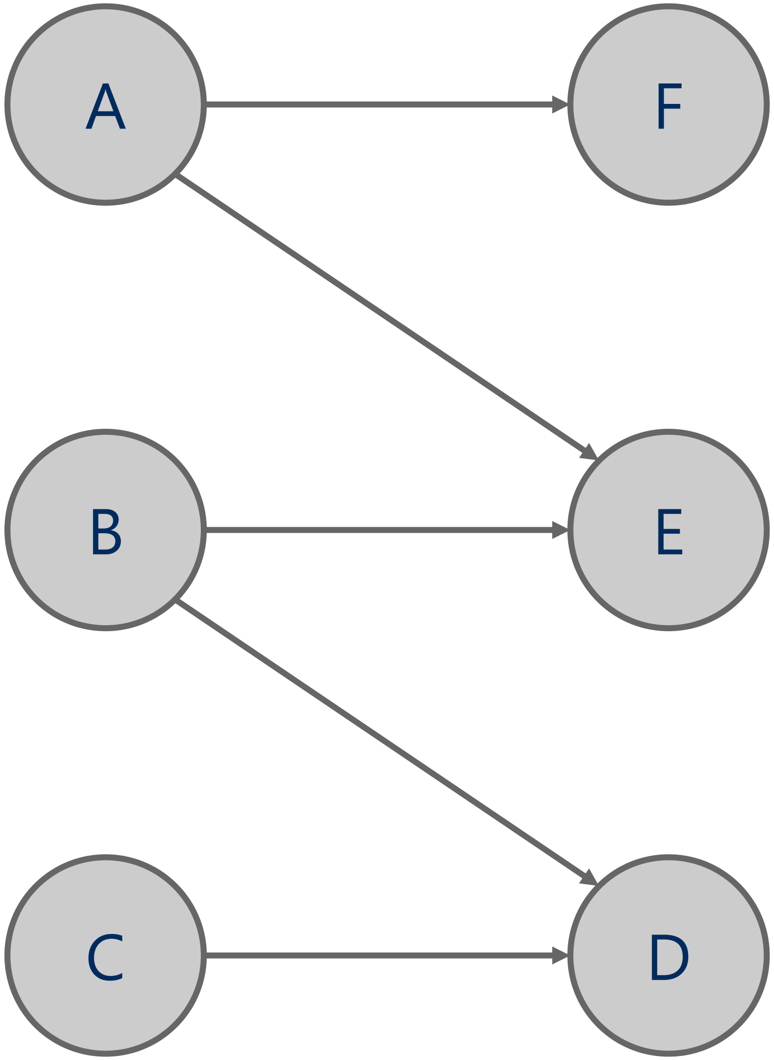
Max flow == Min cut

This is a special case of **duality**

- I.e., look at optimization problem from **two angles**
  - e.g., find the maximum flow or minimum cut
  - The difference between solution values referred to as **duality gap**
    - If the duality gap = 0, **strong duality** holds
      - Max flow/min cut uphold strong duality
    - If the duality gap > 0, **weak duality** holds

Maximum Bipartite Matching

Bipartite Graph: vertices decomposed into **two disjoint sets** such that no two vertices within the same set have an edge between them

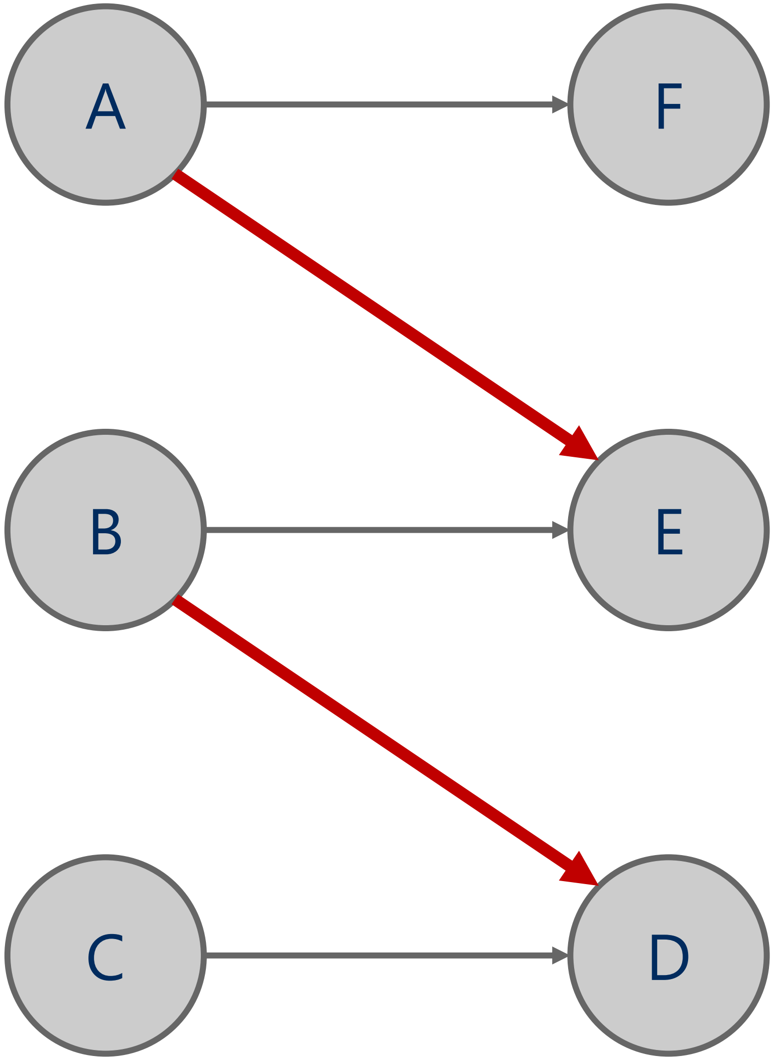


Example: computing tasks and machines

- certain tasks can only run on certain machines
- Run as many tasks as possible

Maximum Bipartite Matching

Bipartite matching: a set of edges such that no two edges share an endpoint



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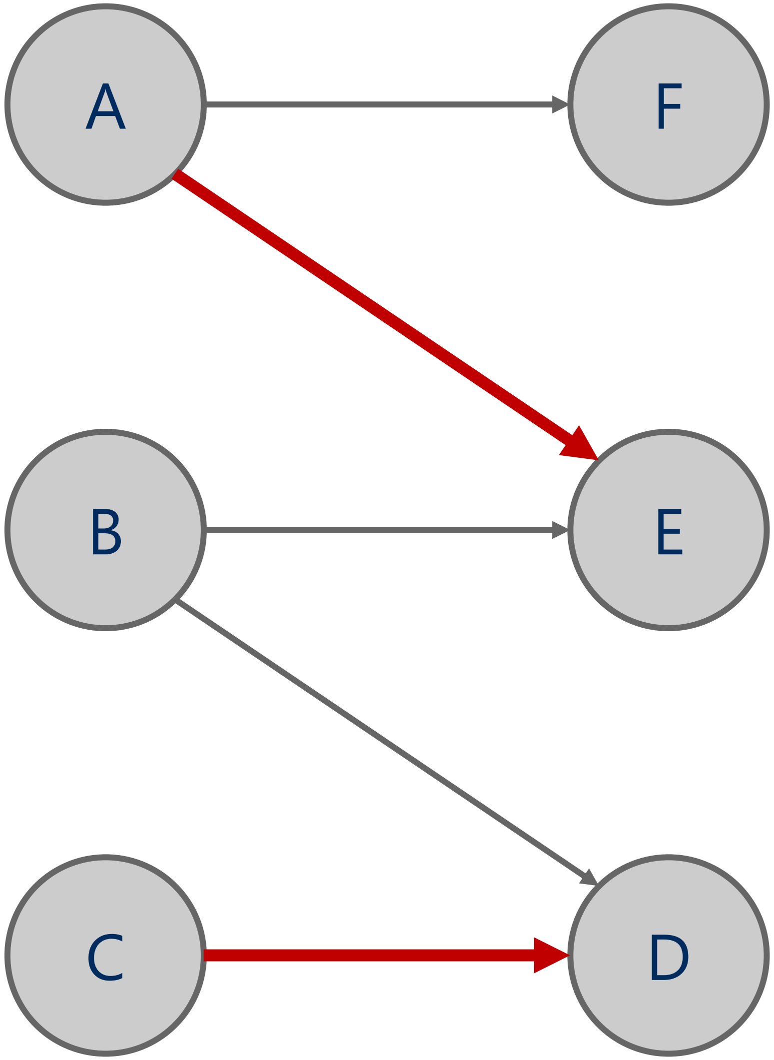
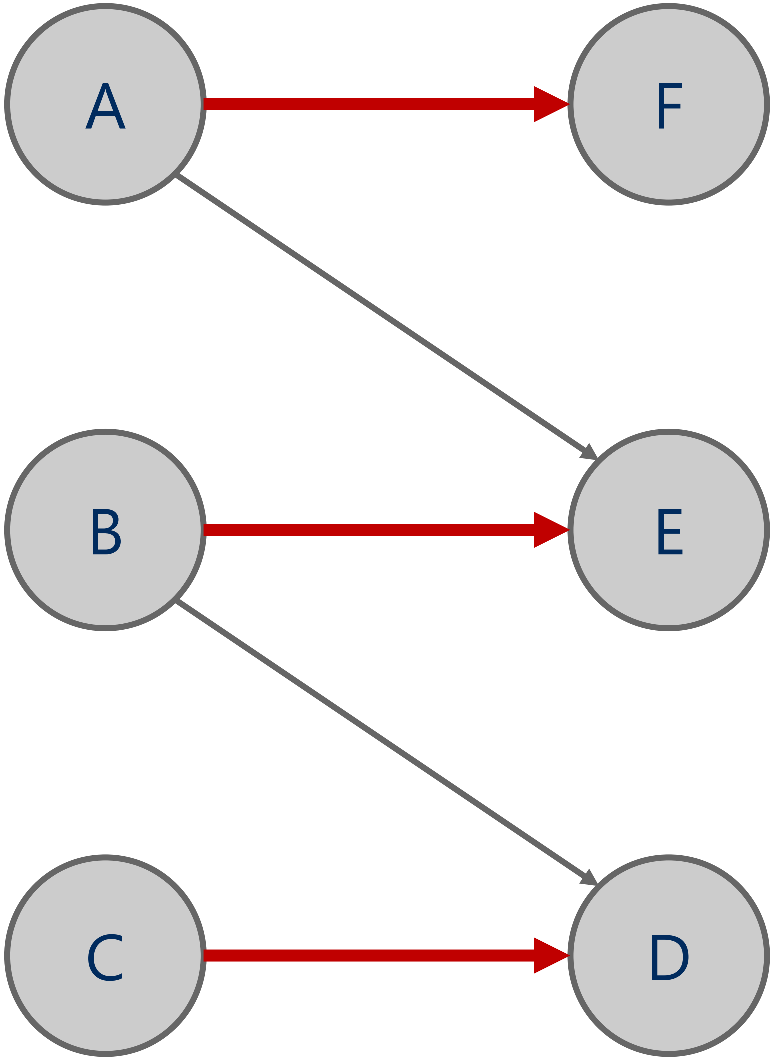


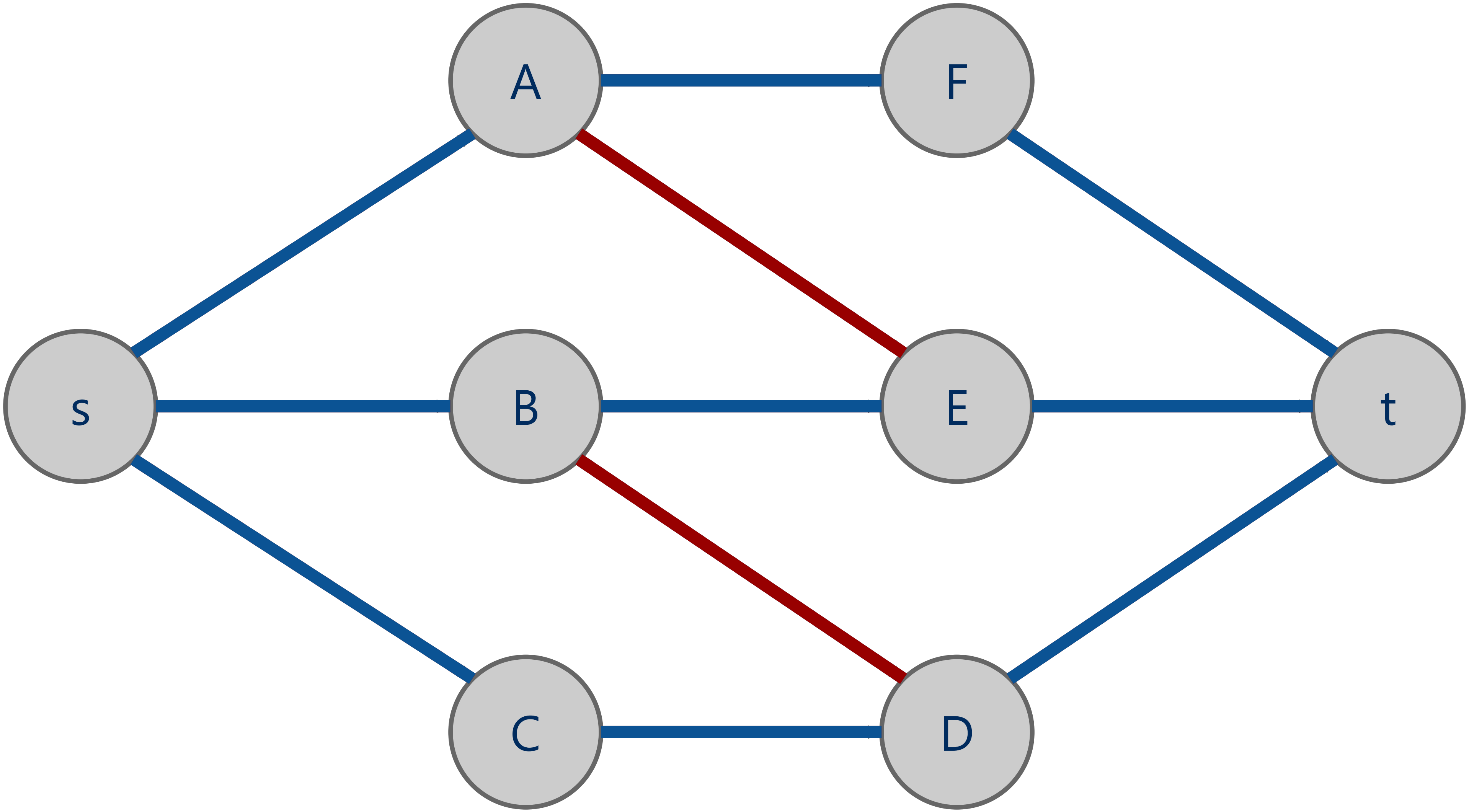
Maximum Bipartite Matching

Maximum matching: the largest possible matching

Let's **reduce** the problem to Maximum Flow!



Solving Maximum Bipartite Matching using Maximum Flow

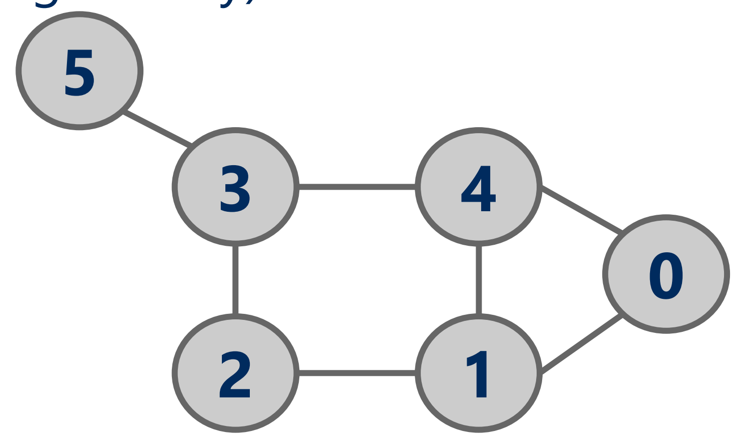


Graph Compression

- Real-life graphs are huge
  - 100’s if not 1000’s of GBs
  - Facebook graph, Google graph, maps, …
- Let’s see one (partial) idea for reducing the size of large graphs

Graph Compression

- **Step 1:** Construct a Compressed Sparse Row (CSR) representation of the graph
- CSR
  - Edges array concatenates ***sorted*** neighbor lists of all vertices
  - Offsets array:
    - offsets[v] is the starting index (in the Edges array) for the neighbors of vertex v



Edges

0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3

Offsets

0	2	5	7	10	13

Graph Compression

- Let's start with one more graph representation
- Compressed Sparse Row (CSR)
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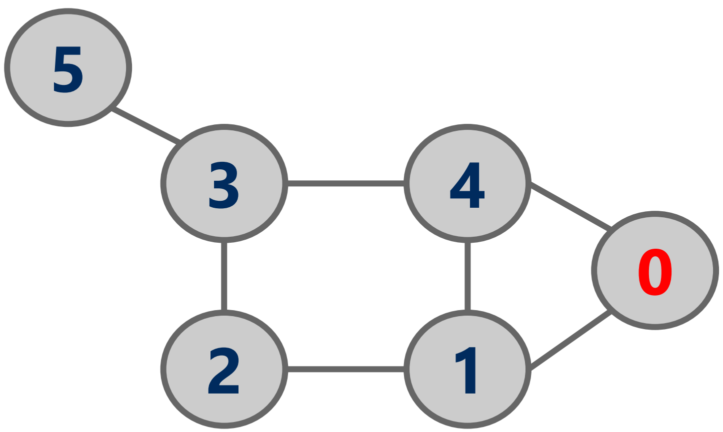

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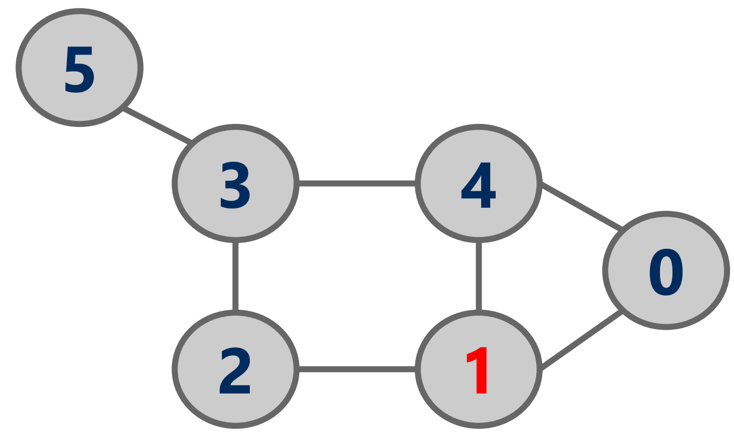

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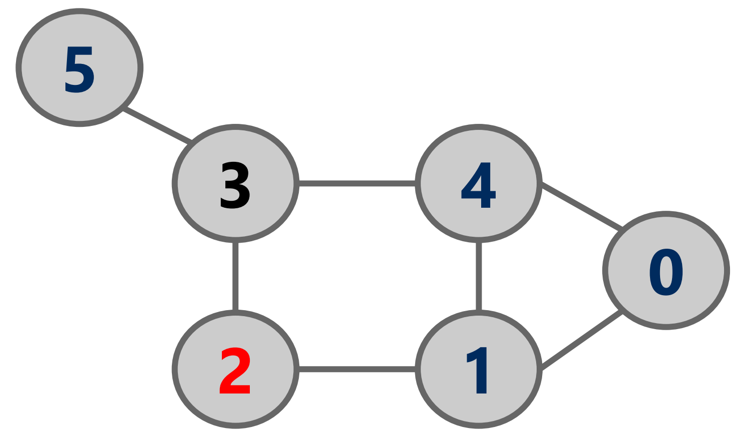

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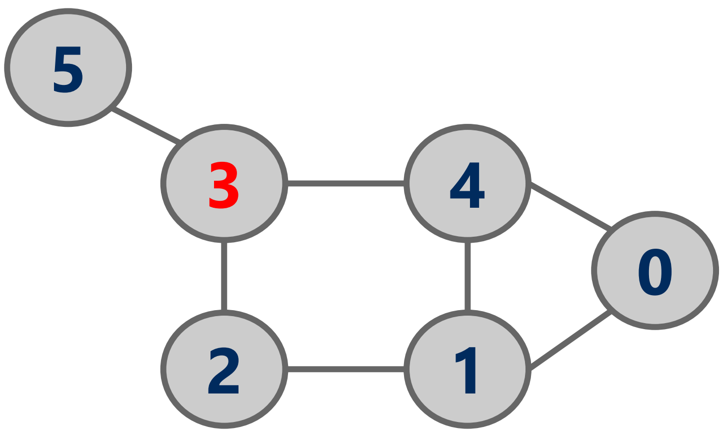

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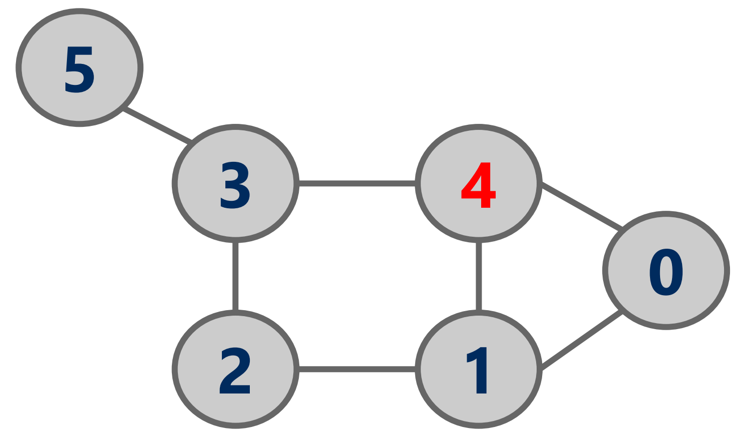

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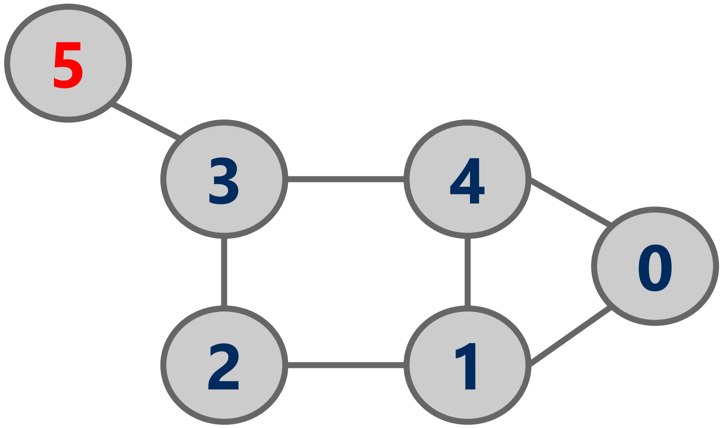

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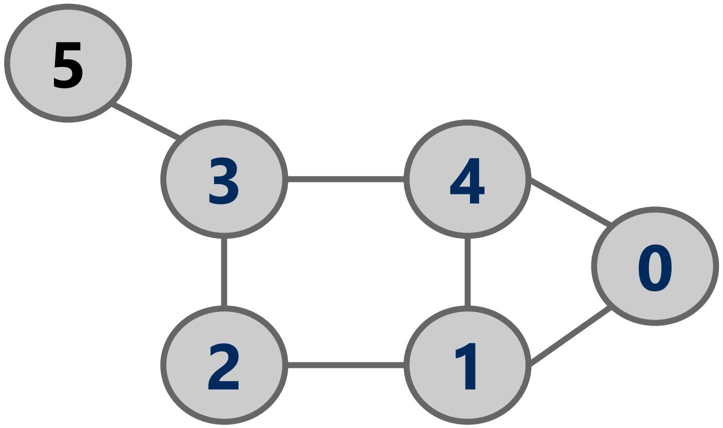
Graph Compression

- Can we compute the degree of a vertex using the offsets array?
  - Running time?
- What is the required space of this representation?
  - Theta(v + e)
  - Assume 4 bytes per vertex and per edge
  - Total size: *4*v* + *8*e* bytes



Edges

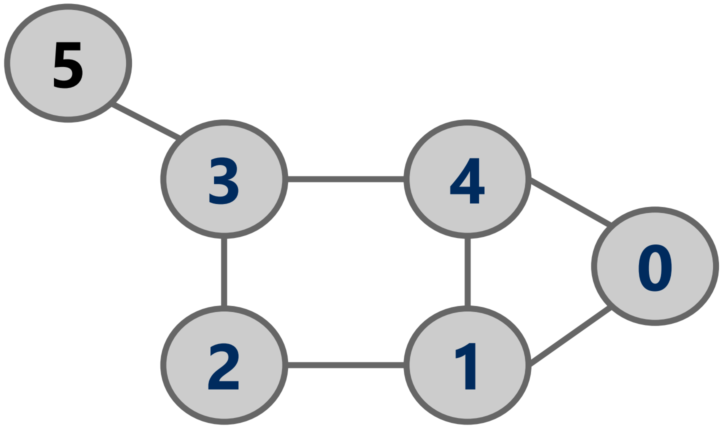
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list $v_1, v_2, v_3, \ldots$
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, …
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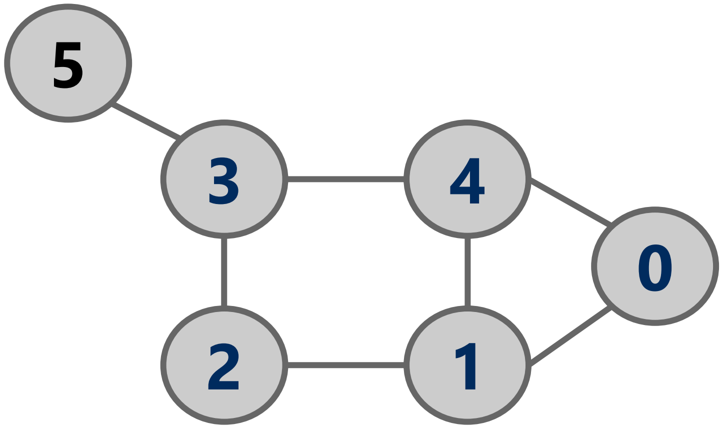
    - $(v_1 - v)$, $(v_2 - v_1)$, $(v_3 - v_2)$, …



Edges

0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1-0	4-1												

Offsets

0	2	5	7	10	13

Graph Compression

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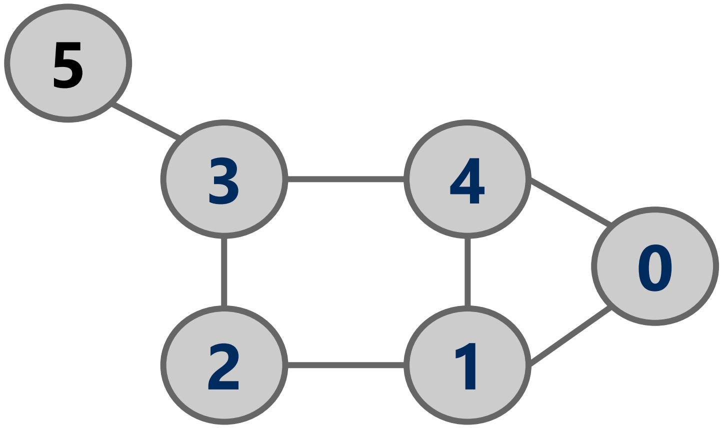
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

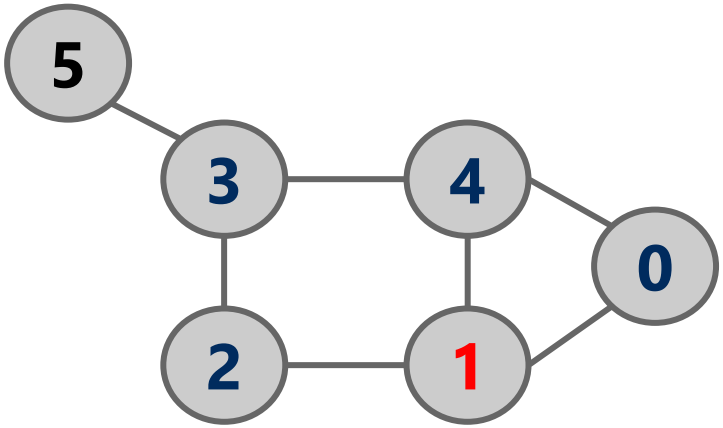
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1	3												

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, …
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

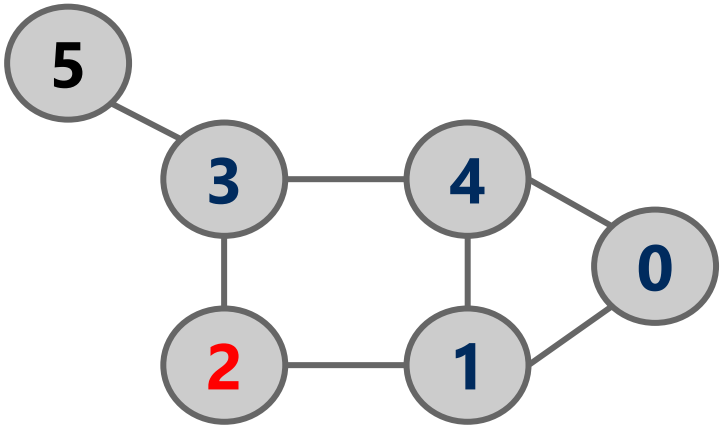
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1	3	-1	2	2									

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, …
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

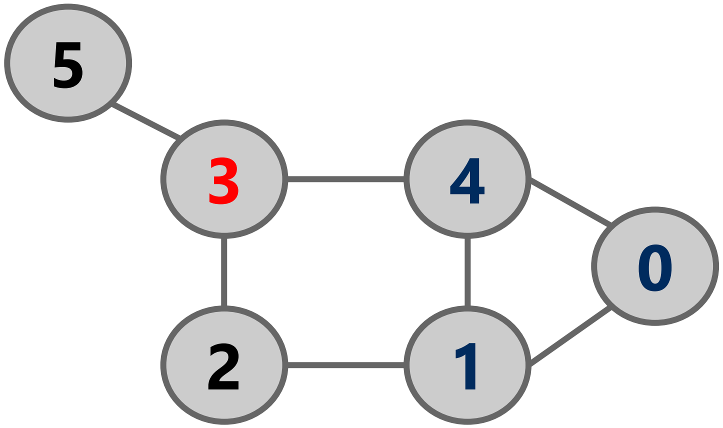
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1	3	-1	2	2	-1	2							

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, ...
  - Store the differences between each two consecutive numbers
    - $(v_1 - v)$, $(v_2 - v_1)$, $(v_3 - v_2)$, ...



Edges

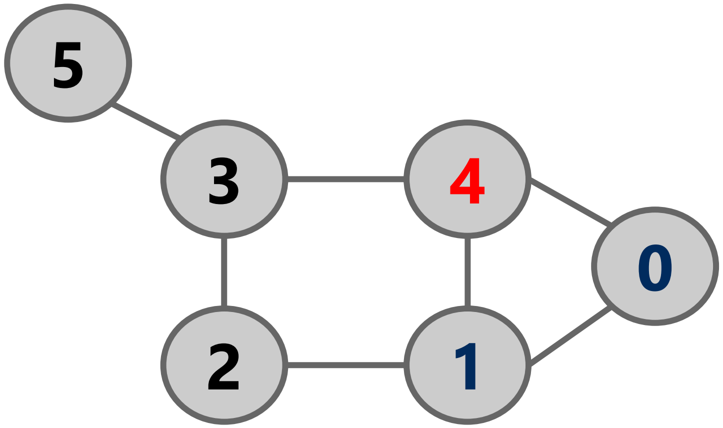
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1	3	-1	2	2	-1	2	-1	2	1				

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, ...
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

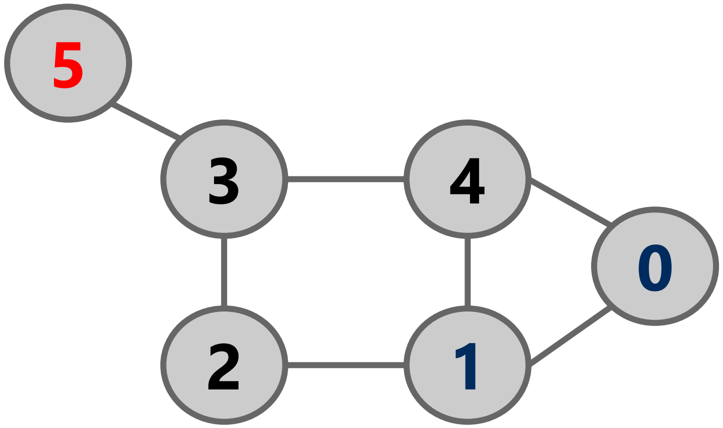
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1	3	-1	2	2	-1	2	-1	2	1	-4	1	2	

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, …
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

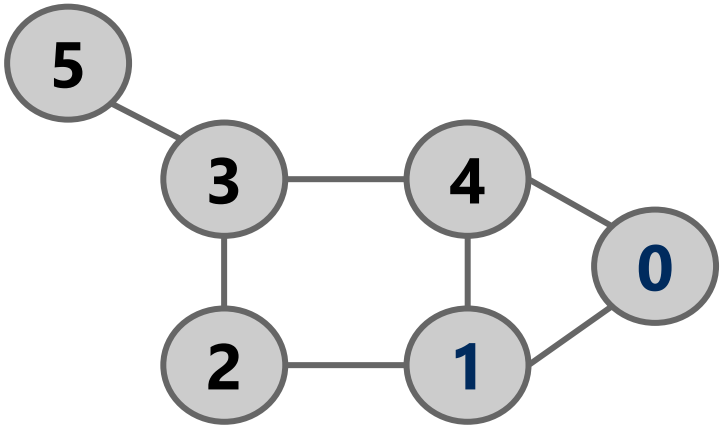
0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	0	2	4	1	3	2	4	5	0	1	3	3
1	3	-1	2	2	-1	2	-1	2	1	-4	1	2	-2

Offsets

0	2	5	7	10	13

Graph Compression

- **Step 2: Difference coding**
  - For each vertex *v*, with a neighbor list v_1, v_2, v_3, …
  - Store the differences between each two consecutive numbers
    - $(v_1 - v), (v_2 - v_1), (v_3 - v_2), \ldots$



Edges

0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	3	-1	2	2	-1	2	-1	2	1	-4	1	2	-2

Offsets

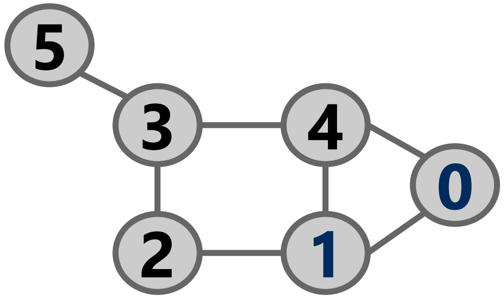
0	2	5	7	10	13

Graph Compression

- **Goal**: make the differences small
    - between vertex labels in each neighbor list small
- For Web Graphs
    - Each vertex is a web page
    - Sort the pages based on **reverse URL** (e.g., edu.pitt.cs.www)
    - Use sorted array indexes as vertex labels
    - Most links are local (within the same domain)
        - neighbors will be close to each other in the sorted list
        - Goal achieved!
- Other graphs can be relabeled to achieve that goal
    - https://www.cs.cmu.edu/~guyb/papers/BBK03.pdf

Graph Compression

- **Step 3: Use Gamma code to compress the differences**



Edges

0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	3	-1	2	2	-1	2	-1	2	1	-4	1	2	-2

Offsets

0	2	5	7	10	13

Gamma Code

- Gamma Code: compress data when
  - small values much more **frequent** than large values
- To encode an integer *x*,
  - T = **largest power of 2 < *x***
  - Encode T as **(log T) zeros** followed by **1**
  - Append the remaining **(log T)** bits of **x – T**
- need **2*floor(log x) + 1** bits << 32 for small x

Gamma Code

- Example: Gamma encode 17: **10001**
  - $T = 16 = 2^4$
  - 1st part of Gamma code: **0000 1**
  - x – T = 17 – 16 = 0001
  - Gamma code: **00001 0001**

Clustering Problem

- **Input**: a set of **n data points** and a **target number of clusters, K**
- **Output**: K clusters
  - K cluster **centroids** (central points)
  - A **label** from the set {1, .., K} for each of the n data points
  - Sum of squared distances from each point to centroid is **minimum**

Clustering Problem

- minimize **distance** defined as:
- for(int i=0; i<n; i++)

  distance += (data[i] – centroid[cluster[i]]) ^{2}

Useful but hard problem!

- unsupervised machine learning algorithms
- dimensionality reduction problems
- **NP-hard**!
  - no efficient solution has been known yet
  - we don’t know yet if an efficient algorithm exists

Lloyd’s Local Search Algorithm

Initialization:

- Start with an **initial** cluster **assignment**
- Compute **initial cluster centroids**

Repeat until no change in centroids and cluster assignments

- Assign each data point to the closest cluster centroid
- Recompute cluster centroids based on new assignment

Limitation of Lloyd’s Algorithm

- Sensitive to initial clustering
- Fix: select initial centroids **as far from each other** as possible

K-Means ++ Algorithm for initial centroids

- Select first centroid with **uniform probability** over all data
- **Repeat** for each of the remaining K-1 initial centroids
  - For each data point
    - Compute distance to nearest centroid
  - Select next centroid with probability that favors data points with **larger distances**