Lattice Boltzmann method

Description of the algotithm

Let us consider the Lattice Boltzmann algorithm, which is implemented using the D2Q9 model. This model is two dimensional and involves 9 velocity vectors defined by (see Fig. 1)

$$\vec{e}_0 = (0,0) \tag{1}$$
$$\vec{e}_i = (1,0),(0,1),(-1,0),(0,-1) \quad i = 1,2,3,4 \tag{2}$$
$$\vec{e}_i = (1,1),(-1,1),(-1,-1),(1,-1) \quad i = 5,6,7,8 \tag{3}$$

For each particle on the lattice the probability distribution function $f_i(x, t)$ is associated, which describes the probability of streaming in one particular direction. The macroscopic fluid

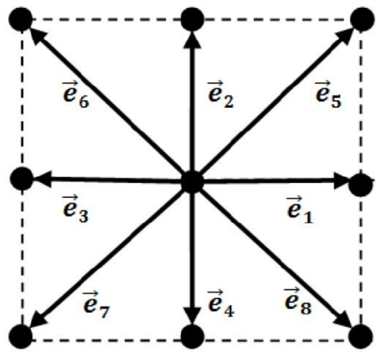

Figure 1: Illustration of a lattice node of the D2Q9 model.

density can be defined as a summation of microscopic particle distribution function

$$\rho(\vec{x}, t) = \sum_{i=0}^{8} f_i(\vec{x}, t) \tag{4}$$

The macroscopic velocity is defined in the following way:

$$\vec{u}(\vec{x}, t) = \frac{1}{\rho} \sum_{i=0}^{8} f_i(\vec{x}, t)\vec{e}_i \tag{5}$$

The key steps in LBM are the streaming and collision processes which are given by

$$f_i(\vec{x} + \vec{e}_i\Delta t, t + \Delta t) - f_i(\vec{x}, t) = -\frac{[f_i(\vec{x}, t) - f_i^{eq}(\vec{x}, t)]}{\tau} \tag{6}$$

Here $f_i^{eq}(x, t)$ is the equilibrium distribution and τ is considered as relaxation time towards equilibrium. We use the Bhatnagar-Gross-Krook collision model where

$$f_i^{eq}(\vec{x}, t) = \omega_i\rho + \rho s_i(\vec{u}(\vec{x}, t)) \tag{7}$$

$$s_i(\vec{u}) = \omega_i\left(3(\vec{e}_i \cdot \vec{u}) + \frac{9}{2}(\vec{e}_i \cdot \vec{u})^2 - \frac{3}{2}(\vec{u} \cdot \vec{u})\right) \tag{8}$$

and the weights $\omega_i = 4/9$ for $i = 0$, $\omega_i = 1/9$ for $i = 1, 2, 3, 4$, $\omega_i = 1/36$ for $i = 5, 6, 7, 8$.

Simulation of the Poiseuille flow by the Lattice Bolzmann approach

We use the following initial and boundary conditions for the velocity components u_x and u_y and density ρ:

$$u_x(x, 0, t) = u_y(x, 0, t) = 0 \tag{9}$$

$$u_x(x, H, t) = u_y(x, H, t) = 0 \tag{10}$$

$$\rho(0, y, t) = \rho_0 \tag{11}$$

$$\rho(L, y, t) = \rho_1 \tag{12}$$

The numerical algorithm can be implemented in the following way:

1. We initialize $f_i = 1/9$ for all nodes.

2. The streaming step: we move f_i in the direction of e_i according to:

$$
\begin{aligned}
f_1[i][j] &= f_1[i-1][j]; & (13)\\
f_2[i][j] &= f_2[i][j-1]; & (14)\\
f_3[i][j] &= f_3[i+1][j]; & (15)\\
f_4[i][j] &= f_4[i][j+1]; & (16)\\
f_5[i][j] &= f_5[i-1][j-1]; & (17)\\
f_6[i][j] &= f_6[i+1][j-1]; & (18)\\
f_7[i][j] &= f_7[i+1][j+1]; & (19)\\
f_8[i][j] &= f_8[i-1][j+1]; & (20)
\end{aligned}
$$

Here $[i]$ corresponds to the position along the x axis, $[j]$ - along the y axis.

3. The boundary conditions are implemented in the following way. On the left boundary the density is defined according to $\rho_{in} = 1.5$. On the right boundary $\rho_{out} = 0.5$. On the upper and bottom lines we set the sticking boundary conditions $u_x = 0,\ u_y = 0$.

Lower boundary.

On the lower boundary we define $u_x = 0,\ u_y = 0$. The probability distribution functions f_i are known for $i = 1, 3, 4, 7, 8$. We need to calculate f_i for $i = 2, 5, 6$. Eqs. (4-5) give:

$$
\begin{aligned}
f_2 + f_5 + f_6 &= \rho - (f_0 + f_1 + f_3 + f_4 + f_7 + f_8) & (21)\\
f_5 - f_6 &= f_3 + f_7 - f_1 - f_8 & (22)\\
f_2 + f_5 + f_6 &= f_7 + f_4 + f_8 & (23)
\end{aligned}
$$

In such a way the density ρ can be determined according to:

$$\rho = f_0 + f_1 + f_3 + 2\,(f_4 + f_7 + f_8) \tag{24}$$

f_2 can be defined according to the bounce-back rule for the non-equilibrium part of the particle distribution:

$$f_2 - f_2^{eq} = f_4 - f_4^{eq} \tag{25}$$

According to Eq. (7) $f_2^{eq} = f_4^{eq} = \rho/9$ and

$$f_2 = f_4 \tag{26}$$

Solving Eq. (21) and using Eq. (26) on can get:

$$f_2 = f_4 \tag{27}$$

$$f_5 = f_7 - \frac{1}{2}\,(f_1 - f_3) \tag{28}$$

$$f_6 = f_8 + \frac{1}{2}\,(f_1 - f_3) \tag{29}$$

Left boundary.

On the boundary we define $\rho_{in} = 1.5,\ u_y = 0.$ f_2, f_3, f_4, f_6, f_7 are known, f_1, f_5, f_8 are unknown. They are defined according to

$$\begin{aligned} f_1 + f_5 + f_8 &= \rho_{in} - (f_0 + f_2 + f_3 + f_4 + f_6 + f_7) \quad (30) \\ f_1 + f_5 + f_8 &= \rho_{in} u_x + f_3 + f_6 + f_7 \quad (31) \\ f_5 - f_8 &= -f_2 + f_4 - f_6 + f_7 \quad (32) \end{aligned}$$

Using these rules, one can calculate the velocity

$$u_x = 1 - \frac{f_0 + f_2 + f_4 + 2\,(f_3 + f_6 + f_7)}{\rho_{in}} \quad (33)$$

For the non-equilibrium part the bounceback rule can be used:

$$f_1 - f_1^{eq} = f_3 - f_3^{eq} \quad (34)$$

In such a way, the remaining probability distribution functions can be calculated according to

$$\begin{aligned} f_1 &= f_3 + \frac{2}{3}\rho_{in} u_x \quad (35) \\ f_5 &= f_7 - \frac{1}{2}(f_2 - f_4) + \frac{1}{6}\rho_{in} u_x \quad (36) \\ f_8 &= f_6 + \frac{1}{2}(f_2 - f_4) + \frac{1}{6}\rho_{in} u_x \quad (37) \end{aligned}$$

The boundary conditions in the left upper corner.

Now let us discuss the boundary condition for the corner nodes. Let us consider the left upper corner as an example. Here the probability distributions f_2, f_3 and f_6 are known. The unknown probaility distribution function are determined according to the bounceback condition:

$$\begin{aligned} f_8 &= f_6 \quad (38) \\ f_1 &= f_3 \quad (39) \\ f_4 &= f_2 \quad (40) \\ f_5 &= f_7 = \frac{1}{2}(\rho_{in} - (f_0 + f_1 + f_2 + f_3 + f_4 + f_6 + f_8)) \quad (41) \end{aligned}$$

Literature

1. Qisu Zou, Xiaoyi He. *On pressure and velocity boundary conditions for the lattice Boltzmann BGK model.* Phys. Fluids **9**, 1591 (1997).

2. Yuanxun Bill Bao, Justin Meskas. *Lattice Boltzmann method for fluid simulations.* (2011)