Vv285 Recitation Class 9
Potential & Vector Calculus

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July 15, 2022

Outline

Gradient and Directional Derivatives

$f : \mathbb{R}^n \to \mathbb{R}$ is called a **scalar function** or **potential**. Its first derivative is a map in the dual space of $\mathbb{R}^n$, i.e, $(\mathbb{R})^*$. We suppose f is differentiable.

1. The transpose of the Jacobian of f is called **gradient**:

$$\nabla f(x) := (J_f(x))^T = (\frac{\partial f}{\partial x_1}|_x, ..., \frac{\partial f}{\partial x_n}|_x)^T$$

   Note that the gradient is a vector in $\mathbb{R}^n$.

2. Suppose $h \in \mathbb{R}^n$ and $\|h\| = 1$. The directional derivative is defined as:

$$D_h f|_x := \frac{d}{dt} f(x + th)|_{t=0} = \langle \nabla f(x), h \rangle$$

3. Normal derivative is a special kind of directional derivative. Suppose $\mathcal{S}$ is a hypersurface in $\mathbb{R}^n$, $p \in \mathcal{S}$ and denote the normal vector at p as $N(p)$. The **normal derivative** is defined as $\frac{\partial f}{\partial n}|_p := D_{N(p)}|_p$

Exercise - Gradient and Directional Derivatives

TASK

1. $f(x, y) = xe^{xy} + y$. *Calculate the gradient at* (x_0, y_0) *and the directional derivative at* $(2, 0)$, *which is in direction of* $\theta = \frac{2\pi}{3}$.
2. *Calculate the normal derivative at* $p = (2, 2)$, *which is on the circle* $x^2 + y^2 = 4$.

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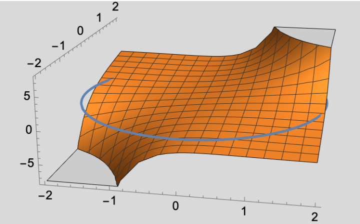

Figure: 1

Line Integral for Vector Fields

1. Given $F\colon \Omega \to \mathbb{R}^n,\ \Omega \in \mathbb{R}^n$ a continuous vector fields and $\mathcal{C}^*$ an **oriented** open, smooth curve in $\mathbb{R}^n$, the line integral of F along $\mathcal{C}^*$ is given by:

$$\int_{\mathcal{C}^*} F d\vec{l} = \int_{\mathcal{C}^*} \langle F, T \rangle \, dl = \int_I \big\langle F \circ \gamma(t), \gamma'(t) \big\rangle \, dt$$

where γ is a parameterization of curve $\mathcal{C}^*$.

2. F is **conservative** if $\oint_{\mathcal{C}^*} F d\vec{l} = 0$ for any closed curve.
3. F is a **potential field** if there exists a differentiable potential function U s.t. $F(x) = \nabla U(x)$.
4. Potential fields are automatically conservative fields, but not vise versa.
5. If F is a potential field defined on a **connected open set**, then $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$. The converse is true when F is defined on a **simply connected open set**.

Practice - Line Integral

TASK

Given $F(x,y) = (x+y, 1-x)$, $\mathcal{C}^*$ *is the portion of* $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ *that is in the forth quadrant with counter clockwise orientation. Evaluate* $\displaystyle\int_{\mathcal{C}^*} F \cdot d\vec{l}$.

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TASK

Evaluate $\int_{\mathcal{C}^*} F \cdot d\vec{l}$ *where* $F(x,y) = (2x^3y^4 + x, 2x^4y^3 + y)$ *and* $\mathcal{C}^*$ *is given by a parameterization* $\gamma(t) = (t\cos(\pi t) - 1, \sin(\frac{\pi t}{2}))$, $t \in [0,1]$

Circulation and Divergence

$\Omega \in \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ is a continuously differentiable vector field.

1. Divergence of F: $\Omega \to \mathbb{R}$, $\operatorname{div} F := \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \langle \nabla, F \rangle$, which represents the flux density.
2. Circulation of F (in 3 dimension):

$$\operatorname{rot} F = \nabla \times F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

which represents the circulation density. For 2-D, $\operatorname{rot} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Green's Theorem

Only in $\mathbb{R}^2$, $R \subseteq \mathbb{R}^2$ be a bounded and simple region, then

$$\int_{\partial R^*} F d\vec{l} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx$$

where ∂R^* denotes the boundary of R in **counter-clockwise** orientation.

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TASK

Evaluate $\oint_{\mathcal{C}^*} xy\ dx + x^2 y^3\ dy$ where $\mathcal{C}$ is the triangle with vertices $(0,0),\ (1,0),\ (1,2)$ with positive orientation.

Reminder

Similarly, we can calculate flux using the same way:

$$\int_{\partial R^*} \langle F, N \rangle\, dl = \int_R \operatorname{div} F dx$$

References I

- VV285 slides from Horst Hohberger
- Paul's online note
  `https://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx`