Vv285 Recitation Class 7
Integration over $\mathbb{R}^n$

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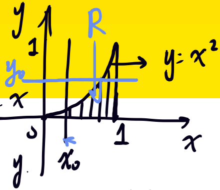
Outline

Determine the Pivot for the Region



$\iint_R f(x,y)\,dx\,dy$ $\quad 0 \le y \le x^2 \to$ w.r.t. x

$\sqrt{y} \le x \le 1 \to$ w.r.t. y

Property

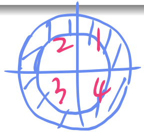
1. $$\iint_R \lambda f(x,y) + \mu g(x,y)\,dA = \lambda \iint_R f(x,y)\,dA + \mu \iint_R g(x,y)\,dA.$$
   Since the integration operator is linear.

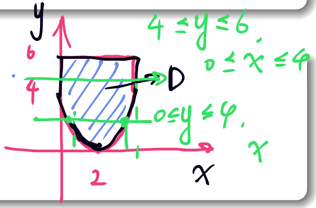
$\iiint_R$

2. *If region R can be split into two disjoint regions D_1, D_2 with $\boxed{D_1 \cap D_2 = \varnothing}$, then*

$$\iint_R f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

TASK





Evaluate $\iint_D \overbrace{42y^2 - 12x}^{P(x,y)}\,dA$ *where*

$D = \{(x,y)\colon 0 \le x \le 4,\ (x-2)^2 \le y \le 6\}$

$$\int_0^4 \int_{(x-2)^2}^{6} (42y^2 - 12x)\, dy\, dx$$

$$= \int_0^4 \left(14y^3 - 12xy\right)\Big|_{(x-2)^2}^{6} dx$$

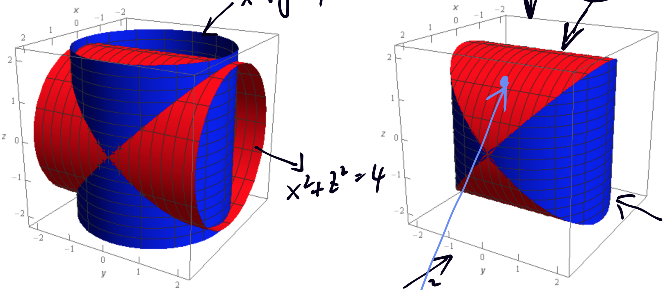
$$= 11136$$

Visualize Geometry

TASK

Find the volume of the region which is formed by the intersection of two cylinders $x^2 + y^2 = 4$ *and* $\boxed{x^2 + z^2 = 4}$ *in* $\mathbb{R}^3$

$x^2+y^2=4$



$x^2+z^2=4$

$\iiint_\Omega 1\, dV$

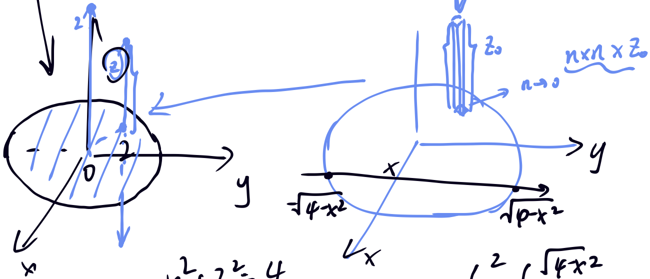
(a) a

2

2

(b) b

Figure: Help your imagination



$x^2+z^2=4$

(upper part) $z=\sqrt{4-x^2}$

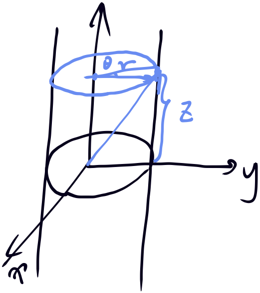
x-o-y plane

$$V=2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx$$

$$=2\int_{-2}^{2}\left(y\cdot\sqrt{4-x^2}\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right)dx$$

$$=\frac{128}{3}$$

Cylindrical Coordinate



Property

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ \boxed{\det(\varphi) = r} \\ r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$

TASK

$x^2+y^2-4x=14 \iff (x-2)^2+y^2=(3\sqrt{2})^2$

1. *Identify the surface defined by the equation:* $\boxed{r^2 - 4r\cos(\theta) = 14}$ $\mathbb{R}^3$
2. *Visualize the surface defined by the equation:* $z = 7 - 4r^2$

$z = 7 - 4x^2 - 4y^2$

$\boxed{4x^2+4y^2+z=7}$

Visualization

$z = 7 - 4x^2$

```
ContourPlot3D[z + 4 x^2 + 4 y^2 == 7, {x, -4, 4}, {y, -4, 4},
  {z, -4, 8}, PlotTheme → "Scientific"]
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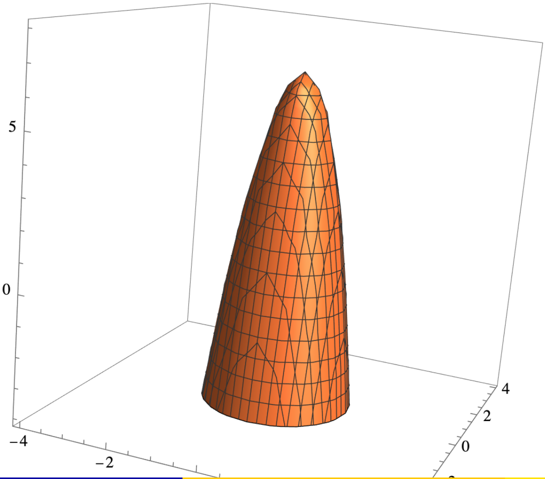
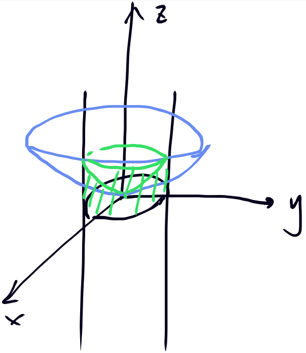


Integration Practice

TASK

Determine the volume of the solid which is inside the cylinder $x^2 + y^2 = 16$, *below* $z = 2x^2 + 2y^2$ *and above the* xy *plane.*

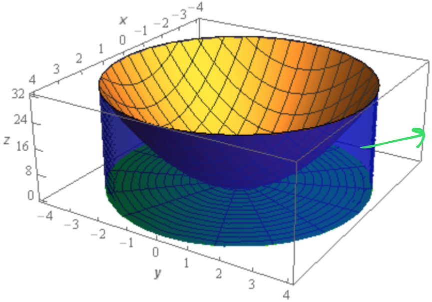
≥ 0



Integration Practice

TASK

Determine the volume of the solid which is inside the cylinder $x^2 + y^2 = 16$*, below* $z = 2x^2 + 2y^2$ *and above the* xy *plane.*



blue

$z = 2x^2 + 2y^2 = 2r^2$

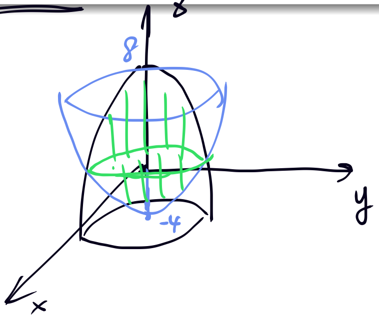
upper bound: $z \le 2r^2$

$$\int_0^{2\pi} \int_0^{4} \int_0^{2r^2} 1 \cdot \underbrace{r}_{|\det J|} \cdot dz\, dr\, d\theta = 256\pi$$

Integration Practice

TASK

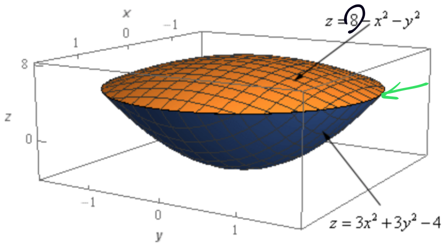
Determine the volume of the solid which is **bounded** *by* $z = 8 - x^2 - y^2$ *and* $z = 3x^2 + 3y^2 - 4$



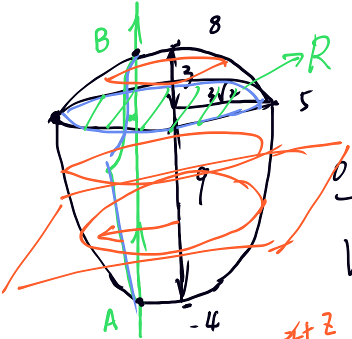
Integration Practice

TASK

Determine the volume of the solid which is **bounded** *by* $z = 8 - x^2 - y^2$ *and* $z = 3x^2 + 3y^2 - 4$



$$\begin{cases} z = 8 - x^2 - y^2 = 8 - r^2 \\ z = 3x^2 + 3y^2 - 4 = 3r^2 - 4 \end{cases} \Rightarrow \begin{cases} z = 5 \\ r = \sqrt{3} \end{cases}$$



$$AB = (8 - r^2) - (3r^2 - 4)$$

$$= \underline{12 - 4r^2 \geq 0}$$

$$\underbrace{0 \leq r \leq \sqrt{3}}$$

$$V = \iint_R (12 - 4r^2)\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} (12 - 4r^2) \cdot \underline{\underline{r}}\, dr\, d\theta$$

$$= \int_0^{2\pi} \underbrace{(6r^2 - r^4)}\Big|_0^{\sqrt{3}}\, d\theta$$

$$= 2\pi \cdot 9 = 18\pi$$

$[-4, 5]$ $\quad r^2 = \dfrac{\cancel{4}+z}{3}$

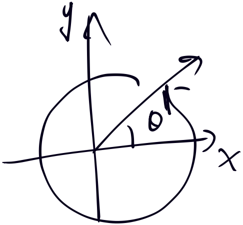
$[5, 8]$ $\quad r^2 = 8 - z$

Spherical Coordinate

Property

$$x = r\cos\theta\sin\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

Suppose f is the transformation from standard coordinate to spherical coordinate, i.e, $f : (x, y, z) \mapsto (r, \theta, \varphi)$, *then* $\boxed{\det Df = r^2 \sin\varphi}$



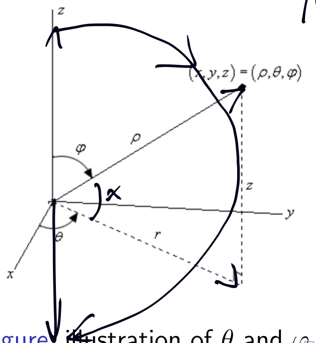
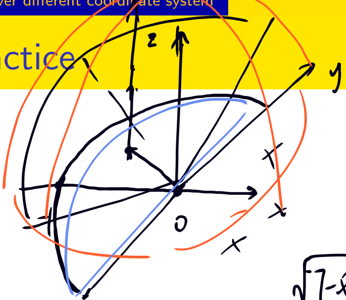


Figure: illustration of θ and φ

Integration Practice



TASK

Evaluate the following triple integral:

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}\,\geq 0}^{\sqrt{7-x^2-y^2}\,\geq 0} \boxed{18y}\, dzdydx \rightarrow \rho(x,y,z)$$

$\sqrt{7-x^2-y^2}=z$

$7=z^2+x^2+y^2$

$6x^2+6y^2=z^2$

$y=0 \quad 6x^2=z^2$

$z=\sqrt{6}\,x$

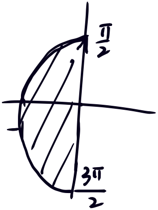
$r\in[0,\sqrt{7}]$

$\varphi:$

$\sqrt{6x^2+6y^2}\le z\le\sqrt{7-x^2-y^2}$

$$\begin{pmatrix} y=r\sin\theta\sin\varphi \\ x=r\cos\theta\sin\varphi \end{pmatrix}$$

$$\underbrace{\boxed{\sqrt{6r^2\sin^2\varphi} \le r\cdot\cos\varphi} \le \sqrt{7-r^2\sin^2\varphi}}$$



$$\sqrt{6}\, r\sin\varphi \le r\cos\varphi$$

$$\tan\varphi \le \frac{1}{\sqrt{6}}$$

$$\underbrace{0 \le \varphi \le \arctan\left(\frac{1}{\sqrt{6}}\right)}$$

$$\underline{\frac{\pi}{2} \le \theta \le \frac{3}{2}\pi}$$

$$\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)}\int_0^{\sqrt{7}} 18\cdot r\sin\theta\sin\varphi\cdot r^2\sin\varphi \; dr\,d\varphi\,d\theta$$

$$= \int_0^{\sqrt{7}}\int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)}\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \left(18r^3\sin^2\varphi\right) \boxed{\sin\theta}\, d\theta\; d\varphi\; dr$$

$: C$

$= 0$

Change of Variables

Property

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v))|\det J_{g,h}|dA'$$

where $x = g(u,v)$ *and* $y = h(u,v)$.

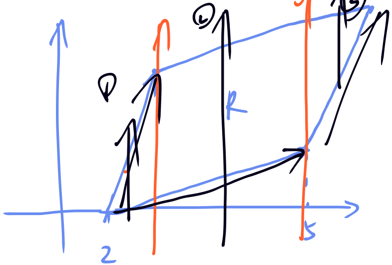
$$\left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\det\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{4}{3} \end{pmatrix}\right| = \frac{1}{3}$$

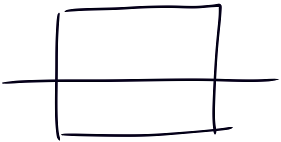
TASK

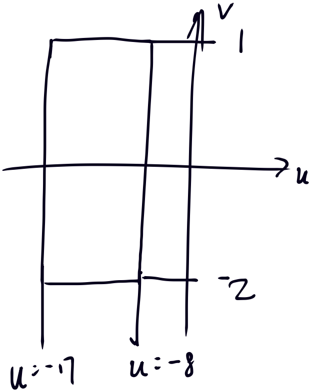
Evaluate $\iint_R 6x - 3ydA$ *where* R *is the parallelogram with vertices* $(2,0),(5,3),(6,7),(3,4)$ *using transformation* $x = \frac{1}{3}(v-u)$ *and* $y = \frac{1}{3}(4v-u)$

$$\int_{-17}^{8}\int_{-v}^{1}(2v+u)\cdot\frac{1}{3}\,dv\,du$$

$$= 243$$







$x-2=y \Rightarrow v=1$

$4x-17=y \Rightarrow u=-17$

$x-2=y \Rightarrow v=-2$

$4x-8=y \Rightarrow u=-8$

References I

- VV285 slides from Horst Hohberger
- Paul's online note
  `https://tutorial.math.lamar.edu/Classes/CalcIII/ChangeOfVariables.aspx`