Vv285 Recitation Class 6
Curve Length, Tangent and Curvature

Yuxiang Chen

June 24, 2022

Outline

Tangent lines and vectors

Suppose $\mathcal{C}$ is a curve in V and γ the parameterization.

1. Tangent line at t_0: $T_p\mathcal{C} = \{x \in V : x = \gamma(t_0) + \gamma'(t_0)t,\ t \in \mathbb{R}\}$
2. Unit tanget vector: $T \circ \gamma(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$

$\gamma = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \gamma' = \begin{pmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{pmatrix}$

TASK

$t \in [0, 2\pi]$

Suppose $\mathcal{C} \in \mathbb{R}^2$ *and* $\gamma(t) = (\sin(t), 2\cos(t))$. *Calculate:*

$\|\gamma'\| = \sqrt{f_1'^2 + f_2'^2}$

1. *The tangent line at* $p_1 = (0, 2)$ *and* $p_2 = (\sqrt{2}/2, \sqrt{2})$
2. *The unit tangent vector field of* $\mathcal{C}$.

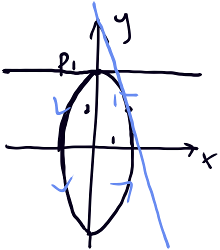
$\gamma(t) = \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$, $\gamma'(t) = \begin{pmatrix} \cos t \\ -2\sin t \end{pmatrix}$, $t \in [0, 2\pi]$

1. tangent line at $P(\sin t_0, 2\cos t_0)$

$$T_P C = \begin{pmatrix} \sin t_0 \\ 2\cos t_0 \end{pmatrix} + \underbrace{\begin{pmatrix} \cos(t_0)\cdot t \\ -2\sin(t_0)\cdot t \end{pmatrix}}_{\gamma'(t)} = \begin{pmatrix} \cos(t_0)\cdot t + \sin(t_0) \\ -2\sin(t_0)\cdot t + 2\cos(t_0) \end{pmatrix}, \quad t \in \mathbb{R}$$

(θ) $t_1 = 0$
$P_1(0, 2)$. $\quad T_{P_1} C = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1\cdot t \\ 0\cdot t \end{pmatrix} = \begin{pmatrix} t \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$

$P_2\left(\frac{\sqrt{2}}{2}, \sqrt{2}\right)$. $\quad T_{P_2} C = \begin{pmatrix} \frac{\sqrt{2}}{2}(t+1) \\ \sqrt{2}(-t+1) \end{pmatrix}, \quad t \in \mathbb{R}$



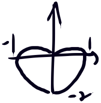
2. unit tangent vector field

$$t = \underbrace{\frac{\gamma'(t)}{\|\gamma'(t)\|} = \begin{pmatrix} \cos t \\ -2\sin t \end{pmatrix} \cdot \frac{1}{\sqrt{1 + 3\sin^2 t}}}_{t \in [0, 2\pi]}$$

Curve length and line integral

Suppose $\mathcal{C}$ is a curve in V and $\gamma\colon [b,s] \to \mathcal{C}$ the parameterization. The curve length formula:

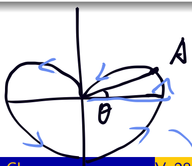
$$l = \int_b^s \|\gamma'(t)\|\, dt$$



TASK

1. Calculate the curve length of the cardioid, i.e, $r = 1 - \sin(\theta)$, using a suitable parameterization.

2. Find the length of graph of $y = x^2$ when $x \in [-2, 2]$



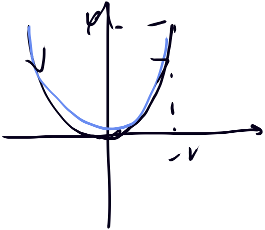
positively or

1. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \gamma: [0, 2\pi], \quad \gamma(\theta) = \begin{pmatrix} (1-\sin\theta)\cos\theta \\ (1-\sin\theta)\sin\theta \end{pmatrix}$

$$\gamma'(\theta) = \begin{pmatrix} \sin\theta + \sin^2\theta - \cos^2\theta \\ \cos\theta - 2\sin\theta\cos\theta \end{pmatrix}$$

$$\int_0^{2\pi} \sqrt{(\sin\theta + \sin^2\theta - \cos^2\theta)^2 + (\cos - 2\sin\theta\cos\theta)^2}\, d\theta \ \ldots$$

2.



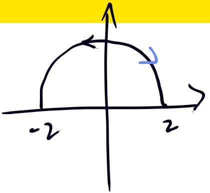
$\gamma: [-2, 2], \quad \gamma(x) = \begin{pmatrix} x^2 \\ x \end{pmatrix}$

$\gamma'(x) = \begin{pmatrix} 2x \\ 1 \end{pmatrix}, \quad \|\gamma'(x)\| = \sqrt{4x^2+1}$

$$\ell = \int_{-2}^{2} \sqrt{4x^2+1}\, dx$$

$$= \frac{\operatorname{arcsinh}(2x)}{4} + \frac{x\sqrt{4x^2+1}}{2} \Bigg|_{-2}^{2} \approx 9.2936$$

Line integral

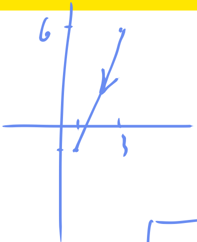


$\boxed{\mathcal{C}^*} \in V$ is an **oriented** curve and $f : \mathcal{C} \to \mathbb{R}$ is a continuous function and $\gamma : I \to \mathcal{C}$ is a parameterization. Then, the line integral of f along $\mathcal{C}^*$ is given by:

$$\int_{\mathcal{C}^*} f\,d\vec{l} := \int_I (f \circ \gamma)(t) \cdot \left\|\gamma'(t)\right\| dt$$

along

Line Integral Practices



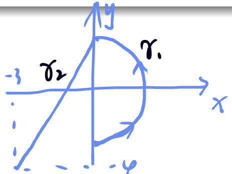
TASK

Calculate the following line integrals:

1. $\int_{\mathcal{C}^*} 3x^2 - 2y \, d\vec{l}$ *where* $\mathcal{C}^*$ *is the line segment from* $(3,6)$ *to* $(1,-1)$.

$\neq \int_{C^*} (1+x^3)\, d\vec{l}$

2. *Evaluate* $\int_{\mathcal{C}^*} 1 + x^3 \boxed{dx}$ *where* $\mathcal{C}^*$ *is a composition of two curves: one is a half circle centered at the origin with radius 2 and the other a line segment joining* $(0,2)$ *to* $(-3,-4)$. *Suppose it's positively oriented.*



Line from $(3,6) \to (1,-1)$

$$r(t) = (1-t)\cdot\begin{pmatrix}3\\6\end{pmatrix} + t\cdot\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3-2t\\6-7t\end{pmatrix}\begin{matrix}\to x\\ \to y\end{matrix}$$

$\boxed{r: [0,1]}$

$$\int_{C^*} \underbrace{3x^2-2y}_{f}\, d\vec{l} = \sqrt{53}\int_0^1 3(3-2t)^2 - 2(6-7t)\, dt$$

$$\|r'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2^2+7^2} = \sqrt{53} \qquad = 8\sqrt{53}$$

step 1: Parameterization ! (note: orientation)

step 2: Calculate $\|r'(t)\|$

step 3. plug in the parameterization,

$2\cos\theta$

$\gamma_1 : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\gamma_1(\theta) = \begin{pmatrix} \\ 2\sin\theta \end{pmatrix}$, $\|\gamma_1'(\theta)\| = 2$

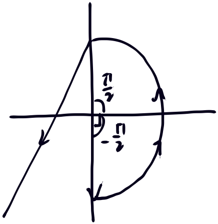
$$\int_{C_1^*} 1+x^3\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1+(2\cos\theta)^3\right]\cdot\left[-2\sin\theta\right] d\theta = 0$$

$x = 2\cos\theta$

$dx = -2\sin\theta\, d\theta$

$\gamma_2 : [0,1]$, $\gamma_2(t) = (1-t)\begin{pmatrix} 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} -3t \\ 2-6t \end{pmatrix} \rightarrow x$



$$\int_{C_2^*} 1+x^3\,dx = \int_0^1 \left[1+(-3t)^3\right]\cdot(-3)\,dt$$

$dx = -3dt$

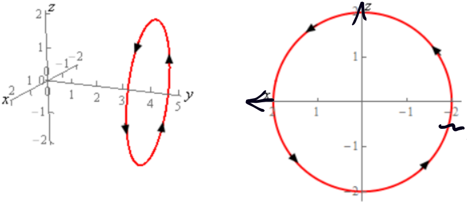
$$= \frac{69}{4}$$

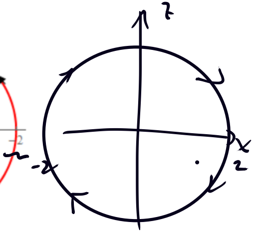
$$\int_{\underset{C_1+C_2}{\underline{C^*}}} (1+x^3)\,dx = \frac{69}{4}$$

Line Integral Practices

TASK

Evaluate $\int_{\mathcal{C}^*} x^2y^2 \; d\vec{l}$ *where* $\mathcal{C}$ *is the circle centered at* $(0,4,0)$ *with radius 2. The orientation is shown in following figure.*





$\gamma : [0, 2\pi]$, $\quad \gamma(\theta) = \begin{pmatrix} 2\cos\theta \\ 4 \\ -2\sin\theta \end{pmatrix}$, $\quad \|\gamma'(\theta)\| = \left\| \begin{pmatrix} -2\sin\theta \\ 0 \\ -2\cos\theta \end{pmatrix} \right\| = \underline{2}$

$$\int_{C^*} x^2 y^2 \, d\ell = \int_0^{2\pi} 64\cos^2(t) \cdot 2 \cdot dt = 128\pi$$

Curvature

$$\kappa \circ \gamma(t) = \kappa \circ l^{-1}(s)|_{s=\circ\gamma(t)} = \boxed{\frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}}$$

. So it's enough to calculate curvature just using a parameterization and tangent vectors.

TASK

$\gamma(t) = (\cos(t), \sin(t)) \quad t \in [0, 2\pi]$

[1] *Calculate the curvature of the curve with parameterization* $f : [0,1] \to \mathbb{R},\ f(x) = (\cos(2\pi x), \sin(2\pi x))$.

[1] This practice is from Leyang Zhang. Vielen Dank!

$$f(x) = \begin{pmatrix} \cos(2\pi x) \\ \sin(2\pi x) \end{pmatrix}. \qquad f'(x) = \begin{pmatrix} -2\pi \sin(2\pi x) \\ 2\pi \cos(2\pi x) \end{pmatrix}$$

$$\underbrace{\|f'(x)\|} = 2\pi \cdot 1 = 2\pi$$

$$(T \circ f)(x) = \frac{f'(x)}{\|f'(x)\|} = \begin{pmatrix} -\sin(2\pi x) \\ \cos(2\pi x) \end{pmatrix}$$

$$\underbrace{(T \circ f)'(x)} = \begin{pmatrix} -2\pi \cos(2\pi x) \\ -2\pi \sin(2\pi x) \end{pmatrix}$$

$$K \circ f(x) = \frac{\|(T \circ f)'(x)\|}{\|f'(x)\|} = \frac{2\pi}{2\pi} = 1$$

curvature of a circle with radius R: $K = \frac{1}{R}$

References I

- Practice questions from Leyang Zhang
- VV285 slides from Horst Hohberger
- Paul's online note
  https://tutorial.math.lamar.edu/Problems/CalcIII/LineIntegralsIntro.aspx