Vv285 Recitation Class 4
Practice Questions on Determinant

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June 3, 2022

Outline

Judge the Followings

[1]Suppose $A \in \mathrm{Mat}(n \times n; \mathbb{C})$ is a matrix with $\det A \neq 0$. Judge T or F:

- F ➢ $\det A \in \mathbb{R}$.
- T ➢ A is invertible.
- T ➢ The row rank of A is n
- T ➢ $\ker A = \{0\}$
- T ➢ For any $y \in \mathbb{C}^n$, $x_0 = A^{-1}y$ is the only solution of $Ax = y$
- F ➢ $\det (A^{-1})^T = \det A$
- F ➢ $\det c \cdot A = c \cdot \det A$, where $c \in \mathbb{R}$

$A = (v_1, \cdots v_n)$, $v_i \in \mathbb{R}^n$

$\det A \neq 0 \iff v_1, \cdots v_n$ independent

$\iff \operatorname{rank} A = n$ (col rank A = row rank A $= n$)

$\iff A$ bijective

$\iff \ker A = \{0\}$, $\operatorname{ran} A = \mathbb{R}^n$

$(\det (A^{-1})) \cdot \det A = 1$

$$B = \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_1+b_2 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{pmatrix} \end{array}, \quad \det B = \underline{0}\ ?$$

$A = (\overset{\in \mathbb{C}^n}{a_1}, a_2, \cdots a_n)$. $cA = (ca_1, ca_2, \cdots ca_n)$ $\det(cA) = c^n \cdot \det A$

[1]This question is provided by Leyang Zhang. Vielen Dank!

Determinant of Vandermonde Matrix

$P \in \mathbb{P}(n-1)$, $P = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ n points

Prove that the determinant of Vandermonde matrix $V_n \in \mathrm{Mat}(n \times n; \mathbb{R})$, which is defined by $V_1 = 1$ and when $n \leq 2$:

$x_i \neq x_j \ (i \neq j)$

$$A = \begin{array}{c} \begin{matrix} & 2 & \cdots & n \end{matrix} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 1 & & B & \\ \vdots & & & \\ 1 & & & \end{pmatrix} \end{array}$$

$\det = \det B$

$$V_n = \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-2} & x_2^{n-1} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-2} & x_n^{n-1} \end{pmatrix} \neq 0$$

$-x_1$

$$V_{n+1} = \begin{pmatrix} 1 & x_1 & \cdots & x_1^n \\ \vdots & & \ddots & \\ 1 & x_n & & x_n^n \end{pmatrix}$$

is $\det V_n = \Pi_{1 \leq i < j \leq n}(x_j - x_i)$

$k = 2$ $\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$ $\det = x_2 - x_1$

when $k = n$ $\det(V_n)$ true

$$V_{nn} = \begin{pmatrix} 1 & x_1 - x_1 & x_1^2 - x_1^2 & \cdots & x_1^n - x_1^{n-1} \cdot x_1 \\ \vdots & x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & \vdots \\ & \vdots & \vdots & & \\ 1 & x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^n - x_n^{n-1} x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & B & \\ 1 & & & \end{pmatrix}$$

$x_2 - x_1$

$$\det(B) = \det \begin{pmatrix} x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_n^2 - x_1^{n-1} x_2 \\ \vdots & \vdots & & \vdots \\ x_n - x_1 & x_n^2 - x_n x_1 & \cdots & x_n^n - x_n^{n-1} x_1 \end{pmatrix}$$

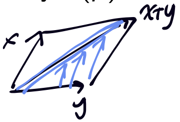
$$= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \det \begin{pmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_n & & \cdots & x_n^{n-1} \end{pmatrix}$$

Linear Transform on Parallelogram

Suppose $P \subseteq \mathbb{R}^2$ is a paralellogram spanned by two vectors $p, q \in \mathbb{R}^2$. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and the denoted the set $L(P)$ as

$$L(P) = \{y \in \mathbb{R}^2 \colon \exists x \in P,\ y = Lx\}$$

1. Prove that $L(P)$ is a parallelogram spanned by $L(p)$ and $L(q)$.
2. Prove that $S_{L(P)} = (\det L) \cdot S_P$
3. Prove that (2)'s result also works in $\mathbb{R}^3$



① $P_{(x,y)} = \{ v : \underbrace{v = \lambda x + \mu y}. \quad \lambda, \mu \in [0,1] \}$

$L(p) = \{ v' : v' = L(v) = L(\lambda x + \mu y) = \lambda(Lx) + \mu(Ly),\ \lambda, \mu \in [0,1] \}$ □.

$\langle a, b \rangle$, $a \times b$

② area $\begin{cases} S_P = \underbrace{\det(x,y)} \\ S_{L(P)} = \det(\underbrace{Lx, Ly}) = \det(\underbrace{L\cdot(x,y)}) = \det(L) \underbrace{\det(x,y)}_{S_P} \end{cases}$

$(x,y) \in Mat(2\times 2 ; \mathbb{R})$

$x, y \in \mathbb{R}^2$

③ $\underset{(x,y,z)}{P^3} = \{ v : \ v = \lambda x + \mu y + \psi z, \ \lambda, \mu, \psi \in [0,1] \}$

volume $\begin{cases} V_{P^3} = \det(x, y, z) \\ V_{L(P^3)} = \det(Lx, Ly, Lz) = \det(L)\cdot \det(x,y,z) = \det L \cdot S_P \end{cases}$

Determinant of block matrices

$\pi(2, 2n) \to (2, 2n)$

$[2, n]$ $[\pi_c, (2, n)]$ $(n+1, 2n)$

$D_1 = \begin{pmatrix} a_{11} & \cdots & a_{1m} & b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots & \vdots & & \\ a_{m1} & \cdots & a_{mm} & b_{m1} & \cdots & b_{mn} \\ 0 & \cdots & 0 & c_1 & \cdots & \\ 0 & \cdots & 0 & & & \end{pmatrix}$

[2] Calculate the determinant of D_1 and D_2: $\pi : [1, n] \to [1, n]$

$[n+1, 2n] \to [n+1, 2n]$

$\begin{pmatrix} A & B \\ \bigcirc_{n \times n} & C \end{pmatrix}$ $D_1 = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$

π

$$|D_1| = \begin{vmatrix} A & B \\ 0 & C \end{vmatrix} \qquad D_2 = \begin{pmatrix} G_1 & * & \cdots & * & * \\ 0 & G_2 & \cdots & * & * \\ 0 & 0 & G_3 & \cdots & * \\ 0 & 0 & 0 & \cdots & G_n \end{pmatrix} \to D_{n-1}$$

$D_n = \begin{pmatrix} D_{n-1} & * \\ 0 & G_n \end{pmatrix}$

where A, B, C are $m \times m$ matrices and $A_1, \ldots, A_n$ are $n \times n$ matrices

$\det(D_1) = \det(A) \cdot \det(C)$

$\det(D_2) = \prod_{i=1}^{n} \det(G_i)$

$D_1 = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ $n+1 - 2n$

$\pi(2) = B_2 \ (n+2)$ $\pi \quad 2 \to n+2$

$\pi \quad n+2 \to 2$

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[2] This exercise is from Leyang Zhang. Vielen Dank!

References I

- VV285 sample exam from Horst Hohberger
- Practice questions from Leyang Zhang