Vv285 Recitation Class 1

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Outline

Logistics and Style

- Mind-Map track through Horst's slides
  1. More intuitive explanation.
  2. No repetition. Ideal: my RC $=$ (Horst' slides)$^{\perp}$.
  3. Marked importance.
- Exercises with emphasis
  1. 80% of them is close to exam's difficulty
  2. 20% of them is a bit harder, through which I hope to strengthen your understanding for some important concepts
- Interactions and timely-feedback
  1. Hope to see your faces online :)
  2. Welcome to any questions during or after my RC
  3. Don't be shy posting your solution on the chat board or just speaking up.

Vector Space

Definition: A tripe $(V, +, \cdot)$ is called a **real** vector space if

1. V is any set
2. $+ : V \times V \to V$ is a map (addition) with the following properties:
   - Communicativity:
     $u + v = v + u$ for all $u, v \in V$
   - Associativity:
     $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$
   - Additive Identity:
     there exist an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$
   - Additive Inverse:
     for every $v \in V$, there exists $w \in V$ such that $v + w = 0$

Vector Space

3. $\cdot : \mathbb{R} \times V \to V$ is a map (scalar multiplication) with the following properties:
    - Multiplication Identity:
      $1 \cdot u = u$ for all $u \in V$
    - Associativity:
      $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$
    - Distributive Property:
      $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$

Subspace

Definition: Let $(V, +, \cdot)$ be a real or complex vector space. If $U \subseteq V$ and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$

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There is no need to prove the above properties of addition and scalar multiplication of $(U, +, \cdot)$, actually, we only need to verify that

- $0 \in V$
- $u_1 + u_2 \in U$ for all $u_1, u_2 \in U$
- $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ for complex vector space)

In a word, we need to verify if this subset is closed under addition and multiplication, with confirmation of unit element.

you can review the proof on Vv186 *Slide 339*

Exercise - Is this a vector space

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

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Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

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No! Uniqueness of unit element, i.e, $\mathbf{0}$

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$$\begin{aligned} x + y =& (x_1, ..., x_n) + (y_1, ..., y_n) \\ & := (x_1 + y_1, ..., x_n + y_n), \quad x, y \in \mathbb{R} \end{aligned}$$

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- The above vector space is complex if the multiplication factor λ is defined over $\mathbb{C}$.

Exercises

V, U are real (complex) vector spaces, then which of the followings are true

A $V \bigcap U$ is a vector space

B $V \bigcup U$ is a vector space

C $V + U := \{u + v : \forall u \in U,\ \forall v \in V\}$ is a vector space

D Suppose U is a subspace of V, then $V \setminus U$ is a vector space

E Suppose U is a subspace of V, then $(V \setminus U) \bigcup \{0\}$ is a vector space

F Suppose U is a subspace of V and $v \in V$, then $v + U := \{v + u : u \in U\}$ is *always* a vector space

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Answer: A and C

Normed Vector Space

What is ”norm”?

Normed Vector Space

What is "norm"?

Norm is a map (function) defined over a vector space V, $\|\cdot\| : V \to \mathbb{R}$, such that for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ ($\mathbb{C}$ if V is a complex vector space):

1. $\|v\| \leq 0$ and $\|v\| = 0$ if and only if $v = 0$
2. $\|\lambda \cdot v\| = |v| \cdot \|v\|$
3. $\|u + v\| \leq \|u\| + \|v\|$

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If we can define a norm over a vector space V, then we say V is a **normed vector space**.

Normed Vector Spaces - Examples

- p-Norm: $\|x\|_p = (\sum_{j=1}^{n} |x_i|^p)^{1/p}$ for any $p \in \mathbb{N}\backslash\{0\}$ in $\mathbb{R}^n$
  when $p = 2$ we obtain the usual Euclidean norm
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  when $p = 2$ we obtain the usual Euclidean norm
  when $p = \infty$ obtain the max norm $\|x\|_\infty = \max_{1 \le k \le n} |x_k|$
- $C([a, b]),\ [a, b] \subseteq \mathbb{R}$, with $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$.
- Later we'll talk about inner product space and induced norm.

Review for this week - Linear equations & Basis

1. Systems of linear equations - know the proof of fundamental lemma for homogeneous equations
2. Finite dimensional vector space
   - Linear combination and span
   - **Important**: definition of basis (unique presentation), property (1. span the whole space 2. linear independence)
   - Dimension of vector space
   - Basis extension theorem
   - Sums of vector spaces. **Important**: direct sum

Something Horst asks us to do

3.28. Theorem. Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W.$$

The proof will discussed in recitation class.

Review for this week - Inner product

Inner Product Space - Basic properties

- Positivity
- Definiteness
- Additivity in the **second** slot (note the difference with other references!)
- Homogeneity in the **second** slot
- Conjugate symmetry

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Is inner product linear in the first slot ?

Review for this week - Inner product

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- Cauchy-Schwarz Inequality: $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$
  More intuitive proof – using orthogonal projection
- Induced norm. Question: *Is every norm an induced norm?*
- Angel between vectors.
- Important: **Orthogonality**, orthogonal complement, Pythagoras' Theorem (the generality of linear algebra)
- Orthonormal bases (orthogonal + normal (unit norm)), Parseval's Theorem
- Projection theorem: decompose a vector in to U and $U^{\perp}$, which is *unique*
- Bessel's inequality

Review for this week - Inner product

- Simpler proof for best approximation (slides 104)

  Suppose P_v is the projection from $v \in V$ to U, i.e, $P_v = \sum_{i=1}^{r} \lambda_i e_i$, and $u \in U$ is any vector.
  We want to show $\|v - P_v\| \leq \|v - u\|$

Review for this week - Inner product

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  Suppose P_v is the projection from $v \in V$ to U, i.e, $P_v = \sum_{i=1}^{r} \lambda_i e_i$, and $u \in U$ is any vector.
  We want to show $\|v - P_v\| \leq \|v - u\|$
  Proof:

$$\begin{aligned}\|v-u\|^2 &= \Bigg\| \underbrace{v - P_v}_{\in U^\perp} + \underbrace{P_v - u}_{\in U} \Bigg\|^2 = \|v - P_v\|^2 + \|P_v - u\|^2 \\ &\geq \|v - P_v\|^2\end{aligned}$$

- Important: **Gram-Schmidt Orthonormalization** (remember to reduce the norm to unit!)

Exercises - Orthogonal Complement

Suppose U is a subspace for a finite dimension vector space V, prove that $U = (U^{\perp})^{\perp}$

Exercise* - deterministic property of inner product

Let V be a real vector space and $\tau : V \times V \to \mathbb{R}$. Show that if

(a) $\tau(v, v) > 0$ for **some** $v \in V$;

(b) $\tau(v, v) \neq 0$ for any $v \neq 0$;

(c) $\tau(v, \lambda u) = \lambda\tau(v, u)$ for $v, u \in V$ and $\lambda \in \mathbb{R}$;

(d) $\tau(v, u) = \tau(u, v)$;

(e) $\tau(v, u + w) = \tau(v, u) + \tau(v, w)$

then τ is an inner product on V.

Exercise* - deterministic property of inner product

Proof:[1]
Suppose that there is some $u \in V$ with $\tau(u,u) < 0$. Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(t) = \tau(tv + (1-t)u, tv + (1-t)u).$$

g is continuous because of (c) and (d). By the Mean Value Theorem for continuous functions, $g(t_0) = 0$ for some $t_0 \in (0,1)$. Since $v \neq t^{-1}(t-1)u$ (why?), this shows that $\tau(h,h) = 0$ for some $h \neq 0$, contradicting (b). Therefore, $\tau(h,h) > 0$ for all $h \in V \backslash \{0\}$. Then (c) implies that $\tau(0,0) = 0$. It follows that τ is an inner product on V.

[1] This proof is from Leyang Zhang

Exercise - Cauchy-Schwarz Inequality

Prove that

1. $16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})$
2. $(x_1+\cdots+x_n)^2 \leq n(x_1^2+\cdots+x_n^2)$ for all positive integers n and all real numbers $x_1, \ldots, x_n$

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Hint

1. *for 2, try* $u=(1,...,1)$ *and* $v=(x_1,...,x_n)$

Exercise - Gram-Schmidt Orthonormalization

On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx.$$

Apply the Gram–Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

Exercise - Gram-Schmidt Orthonormalization

Solution:
Memorize the algorithm, given $\{v_1, v_2, v_3\}$ an independent set:

$$
\begin{aligned}
e_1 &= \frac{v_1}{\|v_1\|} = 1 \\
e_2 &= \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} = 2\sqrt{3}(x - \frac{1}{2}) \\
e_3 &= \frac{v_3 - \langle v_3, e_2 \rangle e_2 - \langle v_3, e_1 \rangle e_1}{\|v_3 - \langle v_3, e_2 \rangle e_2 - \langle v_3, e_1 \rangle e_1\|} = 6\sqrt{5}(x^2 - x + \frac{1}{6})
\end{aligned}
$$

References I

- VV186 slides from Horst Hohberger
- VV285 slides from Horst Hohberger
- Answers from Leyang Zhang
- Linear Algebra Done Right from Axler