Vv285 Recitation Class 6
Curve Length, Tangent and Curvature

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Outline

Tangent lines and vectors

Suppose $\mathcal{C}$ is a curve in V and γ the parameterization.

1. Tangent line at t_0: $T_p\mathcal{C} = \{x \in V : x = \gamma(t_0) + \gamma'(t_0)t, \ t \in \mathbb{R}\}$
2. Unit tanget vector: $T \circ \gamma(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$

TASK

Suppose $\mathcal{C} \in \mathbb{R}^2$ *and* $\gamma(t) = (\sin(t), 2\cos(t))$. *Calculate:*

1. *The tangent line at* $p_1 = (0, 2)$ *and* $p_2 = (\sqrt{2}/2, \sqrt{2})$
2. *The unit tangent vector field of* $\mathcal{C}$.

Curve length and line integral

Suppose $\mathcal{C}$ is a curve in V and $\gamma\colon [b,s] \to \mathcal{C}$ the parameterization. The curve length formula:

$$l = \int_b^s \|\gamma'(t)\| \, dt$$

TASK

1. Calculate the curve length of the cardioid, i.e, $r = 1 - \sin(\theta)$, using a suitable parameterization.

2. Find the length of graph of $y = x^2$ when $x \in [-2, 2]$

Line integral

$\mathcal{C}^* \in V$ is an **oriented** curve and $f\colon \mathcal{C} \to \mathbb{R}$ is a continuous function and $\gamma\colon I \to \mathcal{C}$ is a parameterization. Then, the line integral of f along $\mathcal{C}^*$ is given by:

$$\int_{\mathcal{C}^*} f\,\vec{dl} := \int_I (f \circ \gamma)(t) \cdot \|\gamma'(t)\|\,dt$$

Line Integral Practices

TASK

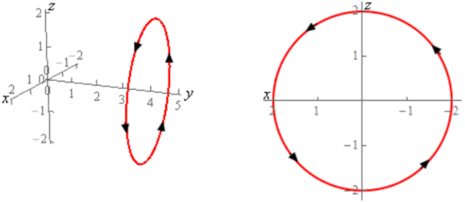
Calculate the following line integrals:

1. $\displaystyle\int_{\mathcal{C}^*} 3x^2 - 2y \; d\vec{l}$ *where* $\mathcal{C}^*$ *is the line segment from* $(3,6)$ *to* $(1,-1)$.
2. *Evaluate* $\displaystyle\int_{\mathcal{C}^*} 1 + x^3 \; dx$ *where* $\mathcal{C}^*$ *is a composition of two curves: one is a half circle centered at the origin with radius 2 and the other a line segment joining* $(0,2)$ *to* $(-3,-4)$*. Suppose it's positively oriented.*

Line Integral Practices

TASK

Evaluate $\int_{\mathcal{C}^*} x^2y^2 \; d\vec{l}$ *where* $\mathcal{C}$ *is the circle centered at* $(0,4,0)$ *with radius 2. The orientation is shown in following figure.*



Curvature

$$\kappa \circ \gamma(t) = \kappa \circ l^{-1}(s)|_{s=\circ\gamma(t)} = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}$$

. So it's enough to calculate curvature just using a parameterization and tangent vectors.

TASK

[1] *Calculate the curvature of the curve with parameterization* $f\colon [0,1] \to \mathbb{R},\ f(x) = (\cos(2\pi x), \sin(2\pi x)).$

[1] This practice is from Leyang Zhang. Vielen Dank!

References I

- Practice questions from Leyang Zhang
- VV285 slides from Horst Hohberger
- Paul's online note
  https://tutorial.math.lamar.edu/Problems/CalcIII/LineIntegralsIntro.aspx