Vv285 Recitation Class 2

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Outline

Review for this week - Linear maps

1. Homogeneity and additivity.
   - Is $L(x) = c,\ x \in \mathbb{R}^n,\ c \neq 0$ a linear map?
   - Why the map $z \mapsto \bar{z}$ is not linear when $\mathbb{C}$ is a complex vector space?
2. Important linear maps
   - The orthogonal projection map P_U: U is a subspace of V, $P_U v = u + w$ for every $v \in V$, in which $v = u + w$ s.t. $u \in U$ and $w \in U^\perp$
   - The inclusion map i: U is a subspace of V, $i(u) = u$ for all $u \in U$ (what's its range and kernel?)

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   - The inclusion map i: U is a subspace of V, $i(u) = u$ for all $u \in U$ (what's its range and kernel?)
3. Theorem that uniquely defines a homomorphism (linear map): define what the basis in the original vector space is mapped into.
4. The space of linear map $\mathcal{L}(U, V)$ is a vector space. (what is the dimension of this space?)
   - The space of *linear functional*: $\mathcal{L}(V, \mathbb{F})$ is called the *dual space* of V, denoted by V^* or V' (notation in Axler)
5. Important: **range and kernel** of $L \in \mathcal{L}(U, V)$
   - In which space lies $\operatorname{ran} L$ ($\ker L$) ?

Exercise - Homogeneity or additivity ?

Q1 Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

Q2 Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ s.t.

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but φ is not linear ($\mathbb{C}$ is viewed as a complex vector space).

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Hint

For 1, consider $f(x, y) = \sqrt{xy}$; *for 2, consider* $g(z) = \operatorname{Re} z$

Review for this week - Isomorphisms

1. Definition: $L \in \mathcal{L}(U, V)$ is a bijective linear map
    - Property: L maps a basis to another basis (is the reverse still valid ?)
    - $\dim U = \dim V$, i.e, U and V are isomorphic .
2. Important: **Dimension formula**

T or **F** $L \in \mathcal{L}(U, V)$ is injective iff it's surjective.

T or **F** $L \in \mathcal{L}(U, V)$, then $U = (\ker L) \oplus (\operatorname{ran} L)$.

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3. Quick proof: Suppose $T \in \mathcal{L}(\mathbb{F}^4, \mathbb{F}^2)$ such that $\ker T = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2,\ x_3 = 7x_4\}$. Show that T is surjective

Exercise - range and kernel

Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{ran} S \subseteq \ker T$$

Prove that $T \circ S = 0$, where 0 denotes the 0 map.

Exercise - Injectiveness and surjectiveness

Suppose $v_1, ..., v_m$ are distinct vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, ..., z_m) = z_1 v_1 + \cdots z_m v_m$$

(a) What property of T makes $v_1, ..., v_m$ span V?

(b) What property of T ensures the linear independence of $v_1, ..., v_m$?

Exercise - Injectiveness and surjectiveness

(a) T is surjective. This is because for any $w \in V$, we must find some $\{\lambda_1, ..., \lambda_m\} \subseteq \mathbb{F}$ such that

$$w = T(\lambda_1, ..., \lambda_m) = \lambda_1 v_1 + \cdots + \lambda_m v_m$$

Therefore, the list $\{v_1, ..., v_m\}$ spans V.
(b) T is injective. In this case, $T(z_1, ..., z_n) = 0$ implies that $z_1 = ... = z_n = 0$. Equivalently $\sum_{i=1}^{n} z_i v_i = 0$ implies that $z_1 = ... = z_n = 0$. Therefore, $v_1, .., v_n$ are linearly independent.

Exercise - Riesz representation

Prove that suppose V is finite-dimensional inner product space and φ is a linear functional on V. Then there is a unique vector $u \in V$ such that

$$\varphi(v) = \langle u, v \rangle$$

for every v.

Exercise - Riesz representation

- Existence:
  let $\{e_1, .., e_n\}$ be an orthonormal basis for V, then we have

$$\begin{aligned}\varphi(v) &= \varphi(\sum_i \langle e_i, v\rangle\, e_i) = \sum_i \langle e_i, v\rangle\, \varphi(e_i) \\ &= \left\langle \overline{\varphi(e_i)} e_i, v \right\rangle \Rightarrow u := \sum_i \overline{\varphi(e_i)} e_i\end{aligned}$$

- Uniqueness:
  Suppose $w \neq u$, and $\langle v, w\rangle = \varphi(v) = \langle v, u\rangle$, then $\langle v, u - w\rangle = 0$ for all v, which implies that $u - w = 0$, i.e, $u = w$.

Operator Norm

Properties (you need to remember this!), suppose $T \in \mathcal{L}(U, V)$

1. $\|Tu\| \leq \|T\| \, \|u\|$ for $u \in U$
2. $\|L_2 \circ L_1\| \leq \|L_2\| \cdot \|L_1\|$

$L \in \mathcal{L}(U, V)$ is bounded when $\dim U < \infty$

Theorem (1.1)

Suppose U and V are normed vector space, $\dim U < \infty$ *and* $L \in \mathcal{L}(U, V)$. *Define* $\|L\| = \sup\{\|Le\| : \|e\| = 1\}$ *for* $e \in U$, *then* L *is* **continuous** *and* $\|L\|$ *is* **bounded**.

Theorem (1.2)

Suppose U and V are normed vector space and $L \in \mathcal{L}(U, V)$, if L is continuous, then $\|L\|$ is bounded. (we do not require either U or V is finite dimensional)

Reminder

You're encouraged to learn the proof of these two, but it's not **necessary**. *As long as you remember this conclusion, it'll be helpful to your exams and understandings of behavior of linear maps.*

Proof for Theorem (1.1)

Proof.

First, we select a norm for U. Suppose $\{e_1, .., e_n\}$ is a basis for U and $u = \sum_i \lambda_i e_i$, then $\|u\|_1 := \sum_i |\lambda_i|$. Using the same u, we have:

$$\|Lu\|_V = \left\| L(\sum_i \lambda_i e_i) \right\|_V = \sum_i |\lambda_i| \, \|Le_i\|_V \leq \sum_i |\lambda_i| \cdot C = C \, \|u\|_1$$

$$C := \max\{\|Le_1\|_V, ..., \|Le_n\|_V\}$$

which proves L is bounded when U is endowed with $\|\cdot\|_1$. However, since $\dim U < \infty$, by the equivalence of norms (will be taught in chapter 2), there exist $D > 0$ such that $\|u\|_1 \leq D \|u\|_U$. Therefore, $\|Lu\|_V \leq C \cdot D \|u\|_U$, showing L is bounded. □

Some Preliminaries

Lemma (2.1)

Suppose $T \in \mathcal{L}(X, Y)$, then T is continuous at 0 implies continuity everywhere on X.

2.1.

T continuous at 0 $\Leftrightarrow \forall \epsilon, \exists \delta$ such that $\|(x - y) - 0\| = \|x - y\| \le \delta$ implies $\|T(x - y) - T(0)\| = \|Tx - Ty\| \le \epsilon$, which reads off the continuity at any $x \in X$. $\square$

Proof for Theorem (1.2)

Theorem (1.2)

Suppose $T \in \mathcal{L}(X, Y)$, then T is continuous if and only if T is bounded.

1.2.

($\Rightarrow$) First, suppose T is continuous, then at the origin

$$\|Tx\| = \|T(x) - T(0)\| \leq \epsilon \quad \text{for} \quad \|x\| = \|x - 0\| \leq \delta$$

We simply re-scale x so that $\|Tx\| \leq \dfrac{\epsilon}{\delta}$ for $\|x\| \leq 1$, then T is bounded.
($\Leftarrow$) Then, suppose T is bounded, then $\|Tx\| \leq \|T\| \|x\| \to 0$ as $\|x\| \to 0$, this shows T's continuity at origin. By lemma (2.1), we know T is continuous. $\square$

Don't let your hands get cold !

$$\int \sqrt{x^2 + a^2} dx \quad \text{where } a \in \mathbb{R}$$

References I

- VV186 slides from Horst Hohberger
- VV285 slides from Horst Hohberger
- Linear Algebra Done Right from Axler
- StackExchange `https://math.stackexchange.com/questions/365900/if-u-is-finite-dimensional-then-operator-norm-is-finite`
- Linear Functional Analysis for Scientists and Engineers by Balmohan V.Limaye. Access link from Springer: `https://link.springer.com/chapter/10.1007/978-981-10-0972-3_3`