Vv285 Recitation Class 3

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May 27, 2022

Outline

Calculus of Linear Maps - Matrices

$\mathbb{F}^6$ $\mathbb{R}^6$

$L \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n) \leftrightarrow j(L) = A$, $h \in \mathcal{L}(\mathbb{P}(5); \mathbb{F})$

1. The column i of a matrix $A \in \mathrm{Mat}(n \times m; \mathbb{F})$ is the **image** of the standard basis b_i in $\mathbb{R}^n$. You may turn every linear maps to a equivalent map from $\mathbb{R}^m$ to $\mathbb{R}^n$.
   - ? Suppose you have a linear map $L \in \mathcal{L}(U, V)$, you want to solve some properties of this map.
   - ➢ Map domain and codomain of a map to $\mathbb{F}^n$. This will be a linear process, you can define $\varphi_1 \in \mathcal{L}(U, X)$ and $\varphi_2 \in \mathcal{L}(V, Y)$, where X and Y are subspace of $\mathbb{R}^n$ and $\mathbb{R}^m$.
   - ➢ turn everything into basis and determine the image of basis in U. Write down the matrix A.
   - ➢ Analyse A instead of L. Then translate the properties of A back to L.

$L \in \mathcal{L}(U, V)$

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_1"] & V \arrow[d, "\varphi_2"] \\ \mathbb{R}^m \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

$m = \dim U$ $\qquad$ $n = \dim V$

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 12 & 13 & 14 \end{pmatrix} \in Mat(4\times 3; \mathbb{R}) \cong L \in \mathcal{L}(\mathbb{R}^3; \mathbb{R}^4)$$

$$L(e_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 12 \end{pmatrix}, \quad L(e_2) = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 13 \end{pmatrix}, \quad L(e_3) = \begin{pmatrix} 7 \\ 8 \\ 9 \\ 14 \end{pmatrix} \qquad \boxed{A = (b_1, b_2, b_3)}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad = b_1 \qquad\qquad = b_2 \qquad\qquad = b_3$$

eg. $V = \mathbb{P}(3)$ $\quad \dim V = 4$, $\quad U \subset V$ (subspace), $U = \operatorname{span}\{1+x, x^2\}$

$L \in \mathcal{L}(U; \mathbb{P}(3))$, $\quad \boxed{L(bx^2 + cx + d) = bx + 2d}$

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_1"] & V \arrow[d, "\varphi_2"] \\ X \subset \mathbb{R}^3 \arrow[r, "A"] & Y \subset \mathbb{R}^2 \end{tikzcd}$$

(rnd')

$\varphi_1 \in \mathcal{L}(U; X)$, $\quad \varphi_1(1+x) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{matrix} \to x^0 \\ \to x^1 \\ \to x^2 \end{matrix}$

$\varphi_1(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$X = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

φ_2 ...

$A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b

$\operatorname{ran} A = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \operatorname{ran} L = \mathbb{P}(1)$

Calculus of Linear Maps - Matrices

1. The column i of a matrix $A \in \mathrm{Mat}(n \times m; \mathbb{F})$ is the **image** of the standard basis b_i in $\mathbb{R}^n$. You may turn every linear maps to a equivalent map from $\mathbb{R}^m$ to $\mathbb{R}^n$.
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   - ➢ turn everything into basis and determine the image of basis in U. Write down the matrix A.
   - ➢ Analyse A instead of L. Then translate the properties of A back to L.
2. Matrix multiplication: row by column (practice this!)

① matrix @ vector

• $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, Av

$L \in \mathcal{L}(\mathbb{R}^2; \mathbb{R}^3)$

$Av = b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\begin{matrix} \\ -1 \\ \downarrow \\ -1 \end{matrix}$

$\overset{(\cdot)^T}{\curvearrowright}$

$(1, 2) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = -1$

$(3, 4) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1$

$b = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

② matrix @ matrix

$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$

$\mathbb{R}^2 \leftarrow AB = C = (c_1, c_2)$

$L_1 \in \mathcal{L}(\mathbb{R}^3; \mathbb{R}^2)$ $\quad L_2 \in \mathcal{L}(\mathbb{R}^2; \mathbb{R}^3)$

$(AB)e_1 = A(\underbrace{Be_1}_{b_1}) = Ce_1 = c_1$

$B = (b_1, b_2)$. $b_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$(\vdots)\, c_1 = A b_1 (\vdots) = \begin{pmatrix} 9 \\ 10 \end{pmatrix}$

$c_2 = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

$\Rightarrow A \cdot B = \begin{pmatrix} 9 & 8 \\ 10 & 10 \end{pmatrix}$

Recap for this week

$$Ax=b \begin{cases} \text{unique} \to L \text{ injective} \\ \text{infinite-sol} \to L \text{ not injective} \end{cases}$$

1. Fredholm Alternative: equivalent to
   - Either $\ker L = \{0\}$, i.e, L is injective.
   - Or $\ker L$ is not trivial, i.e, L is not injective.
2. Matrix Rank: $\mathrm{rank}\ A =$ row $\mathrm{rank}\ A =$ col $\mathrm{rank}\ A$ (use the Adjoint of a matrix to prove it)

Short Comment - Matrix Ajoint (HW2)

Definition

V is finite dimensional with inner product and $A \in \mathcal{L}(V, V)$. The adjoint $A^* \in \mathcal{L}(V, V)$ is defined by:

$$\text{IF } \langle u, Av \rangle = \langle A^* u, v \rangle \quad \text{for all } u, v \in V$$

Reminder

We don't need L to be endomorphism, we can also define the adjoint of T if $T \in \mathcal{L}(V, W)$. Then $T^* \in \mathcal{L}(W, V)$ and $\langle \underbrace{w}_{\in W}, \underbrace{Tv}_{\in W} \rangle = \langle T^* w, v \rangle$ for all $v \in V$ and $w \in W$

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Reminder

We don't need L to be endomorphism, we can also define the adjoint of T if $T \in \mathcal{L}(V, W)$. Then $T^* \in \mathcal{L}(W, V)$ and $\langle w, Tv \rangle = \langle T^* w, v \rangle$ for all $v \in V$ and $w \in W$

TASK

- *Show A^* is well-defined.*
- *If $V = \mathbb{C}^n$, prove $A^* = \overline{A^T}$.*
- *Show* $(\operatorname{ran} A)^\perp = \ker A^*$ *and* $(\ker A)^\perp = \operatorname{ran} A^*$

1. ① A^* is a function:

suppose $x=y \in V$, $\langle A^*x, v\rangle = \langle x, Av\rangle$

$= \langle y, Av\rangle$

$= \langle A^*y, v\rangle$ for every v

$\langle A^*x - A^*y, v\rangle = 0$ ⊛

$A^*x - A^*y = 0 \Rightarrow A^*x = A^*y$

② A^* is linear map

$\langle u, A^*(\lambda v + \mu w)\rangle = \langle Au, \lambda v + \mu w\rangle$

$= \langle Au, \lambda v\rangle + \langle Au, \mu w\rangle$

$= \langle u, \lambda(A^*v)\rangle + \langle u, \mu(A^*w)\rangle$

Proof $A^* = \overline{A^T}$, $A \in \mathcal{L}(V,V)$

Suppose $\{e_1, e_2, \cdots, e_n\}$ O.N.B standard Basis

$\underline{A^* e_k} \in V$ $= \langle e_1, A^* e_k \rangle e_1 + \cdots + \langle e_n, A^* e_k \rangle e_n$

K's col

$$= \begin{pmatrix} \langle e_1, A^* e_k \rangle \\ \vdots \\ \langle e_n, A^* e_k \rangle \end{pmatrix} = \begin{pmatrix} \langle Ae_1, e_k \rangle \\ \vdots \\ \langle Ae_n, e_k \rangle \end{pmatrix}$$

$$A^* = \begin{array}{c} \begin{array}{ccccc} \text{1 col.} & & \text{k col} & & \text{K col} \end{array} \\ \begin{pmatrix} \langle Ae_1, e_1 \rangle & \cdots & \boxed{\langle Ae_1, e_k \rangle} & \cdots & \langle Ae_1, e_n \rangle \\ \langle Ae_2, e_1 \rangle & & \langle Ae_2, e_k \rangle & \cdots & \langle Ae_2, e_n \rangle \\ \vdots & & \vdots & & \vdots \\ \langle Ae_n, e_1 \rangle & \cdots & \langle Ae_n, e_k \rangle & \cdots & \langle Ae_n, e_n \rangle \end{pmatrix} \end{array}$$

Now, write down A

$$Ae_k = \langle e_1, Ae_k \rangle e_1 + \cdots + \langle e_n, Ae_k \rangle e_n = \begin{pmatrix} \langle e_1, Ae_k \rangle \\ \vdots \\ \langle e_n, Ae_k \rangle \end{pmatrix}$$

$$A = \begin{pmatrix} \langle e_1, Ae_1\rangle & \langle e_1, Ae_2\rangle & \cdots & \langle e_1, Ae_n\rangle \\ \vdots & \vdots & & \vdots \\ \cancel{\langle Ae_1, e_k\rangle} & \cancel{\langle Ae_2, e_k\rangle} & & \cancel{\langle Ae_k, e_k\rangle} \\ \langle e_k, Ae_1\rangle & \langle e_k, Ae_2\rangle & \cdots & \langle e_k, Ae_k\rangle \\ \vdots & \vdots & & \vdots \\ \langle e_n, Ae_1\rangle & \cdots \langle e_n, Ae_2\rangle & \cdots & \langle e_n, Ae_k\rangle \end{pmatrix}$$

k'col

k's row

$\Rightarrow A^* = \overline{A^T}$

③ $(\operatorname{ran} A)^\perp = \ker A^*$

i) prove $\boxed{\ker A^*} \subset (\operatorname{ran} A)^\perp$, pick up w, s.t. $A^* w = 0$
$w \in \ker A^*$

$\langle A^* w, v\rangle = 0$ for $\forall v \in V, v \neq 0$

$\langle w, Av\rangle = 0$ for $\forall v \in V$

$w \perp Av \in \operatorname{ran} A \Rightarrow \ker A^* \subset (\operatorname{ran} A)^\perp$

ii) ..

Recap for this week

1. Fredholm Alternative: equivalent to
   - Either $\ker L = \{0\}$, i.e, L is injective.
   - Or $\ker L$ is not trivial, i.e, L is not injective.
2. Matrix Rank: $\mathrm{rank}\ A = \mathrm{row\ rank}\ A = \mathrm{col\ rank}\ A$
   - $\mathrm{Mat}(m \times n; \mathbb{F})$ need to be a real vector space (why?)
   1. Recap of this proof

A need be a REAL Vector Space.

row rank A = col rank $A^T \overset{?}{=}$ col rank $\overline{A^T}$

= col rank A^*

= dim ran A^*

$\dim (\mathrm{ran}\, A^*)^\perp = \dim \ker A$

$m - \dim \mathrm{ran}\, A^* = \dim \ker A$

$= m - \dim \ker A$

$\dim \ker A + \dim \mathrm{ran}\, A = m$

$= m - (m - \dim \mathrm{ran}\, A)$

= ran A

= col rank A

Practice on calculating linear maps

$A = A^* = \overline{A^T}$

TASK

$Me_1 = e_1$

Let $A \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ which satisfies $A = A^*$ and $\{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$ is in the range of A, such that $\boxed{A(1, 0, 1) = (1, 0, 1)}$

span

(a) *Suppose $\dim \ker A = 2$. Determine the matrix of A w.r.t standard basis on $\mathbb{R}^3$*

(b) *Suppose that $V \subseteq \mathbb{R}^3$ is a subspace such that $\boxed{Ah = 2h}$ for all $h \in V$. Show that $V \perp \{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$*

(c) *Let $\boxed{\{(x, 0, -x) \in \mathbb{R}^3 : x \in \mathbb{R}\} \subseteq V}$ and $Ah = 2h$ for all $h \in V$. Give all possible representations of A (w.r.t standard basis on $\mathbb{R}^3$).*

$\dim \operatorname{ran} A \geq 2$

P1 $\quad A^* = A, \quad (\operatorname{ran} A)^\perp = \ker A^* = \ker A$

$(\operatorname{ran} A) \oplus (\ker A) = \mathbb{R}^3 \quad A \in \mathcal{L}(\mathbb{R}^3; \mathbb{R}^3)$

Basis of $\operatorname{ran} A = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Basis of $\ker A = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\ker A = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\left\{ e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right.$

① change basis (T^{-1})

$e_3 = \dfrac{b_3 - \langle e_1, b_3 \rangle e_1 - \langle e_1, b_2 \rangle e_2}{\| \cdots \|}$

$= \boxed{\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

② Find M in new basis

$Me_1 = e_1$, $Me_2 = Me_3 = 0$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\boxed{\text{ONB}} = \left\{ \begin{array}{ccc} e_1 & e_2 & e_3 \\ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \end{array} \right\}$

$\in \operatorname{ran} A \qquad \in \ker A$

③ change back (T)

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1$

$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2$

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3$

$\Rightarrow T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

$((1,0,0), (0,1,0), (0,0,1))$

$\to (e_1, e_2, e_3)$

$A = T M T^{-1}$

$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^T$$

$$(Mt_1 \; Mt_2 \; Mt_3) \qquad (t_1, t_2, t_3)$$

P2 let $x, y \in \mathbb{R}^3$, $Ax = x$, $Ay = 2y$

$$\langle x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, 2y\rangle = 2\langle x, y\rangle \Rightarrow \langle x, y\rangle = 0$$

P3

$$T = \begin{array}{c} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ \begin{matrix} e_1 & \quad e_2 & \quad e_3 \end{matrix} \end{array} \qquad \underline{T^{-1} = T^T}$$

$Me_1 = e_1$

$Me_2 = ?$

$Me_3 = 2e_3$

$$M = \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 2 \end{pmatrix}$$

$$A = TMT^{-1} = T \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 2 \end{pmatrix} T^{-1}$$

$\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}} \qquad 0 \quad 0 \quad 0$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T$

Invertible Matrices & Linear Maps

1. To verify whether a map (matrix) $\boxed{A}$ is invertible or not, we need to verify that there exists a map B that $AB = \mathbb{I}$ and $BA = \mathbb{I}$ (in infinite dimensional situation, we must show the left inverse and right inverse both exists!)

   $A \in \mathcal{L}(\mathbb{R}^3; C[0,1])$ $\quad$ $\operatorname{ran} A \leq \dim \mathbb{R}^3 = 3$

   - However, if A is on finite dimensional spaces, we only need to show $BA = \mathbb{I}$ or $AB = \mathbb{I}$
   - In finite dimensional spaces, $L \in \mathcal{L}(U, V)$ and $\boxed{\dim U = \dim V}$, L's injectiveness $\Leftrightarrow$ L's surjectiveness $\Leftrightarrow$ L's bijective
   - If A is a square matrix, i.e, $A \in \mathrm{Mat}(n; \mathbb{F})$. Then A is invertible iff $\boxed{\operatorname{rank} A = n}$

     $n \times n$

Practice on Invertibility

TASK

Suppose V *is finite dimensional,* U *is a subspace of* V*, i.e,* $U \subseteq V$*, and* $S \in \mathcal{L}(U, V)$*. Prove that there exists an invertible operator* $T \in \mathcal{L}(V, V)$ *such that* $Tu = Su$ *for every* $u \in U$ *if and only if* S *is injective.*

Practice on Invertibility

Solution

[1] *If S is not injective, any $T \in \mathcal{L}(V, V)$ such that $Tu = Su$ would have non-trivial kernel, whence it cannot be invertible. Conversely, suppose that S is injective. Then the dimension formula says that* $\boxed{\dim \operatorname{ran} S = \dim U}$. *Suppose that* $\dim V = n$ *and* $\dim U = m$. *Let* $\mathcal{U} = \{u_1, \dots, u_m\}$ *be a basis for u and* $\mathcal{W} = \{w_1, \dots, w_m\}$ *be a basis for* $\boxed{\operatorname{ran} S}$. *Extend* $\mathcal{U}$ *to a basis* $\{u_1, \dots, u_n\}$ *of* V *and* $\mathcal{W}$ *to a basis* $\{w_1, \dots, w_n\}$ *of* V. *Define*

$U \subset V$

$i \le m$

$$T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = S\left(\sum_{i=1}^{m} \alpha_i u_i\right) + \sum_{i=m+1}^{n} \alpha_i w_i.$$

$T(\alpha_i u_i) = S(\alpha_i$

$i > m$

Then $\ker T = \{0\}$ *(why?), so* T *is injective and thus is bijective. This completes our proof.*

[1] This proof is from Leyang Zhang

Practice on Invertibility

TASK

Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\ker T_1 = \ker T_2$ *if and only if there exists an invertible operator* $S \in \mathcal{L}(W, W)$ *s.t.* $T_1 = S \circ T_2$.

Practice on Invertibility

Solution

We first prove ($\Rightarrow$)*:*
Then we want to find a linear map s.t. $T_1 e = S(T_2 e)$ *for all* $e \in (V \setminus \ker T_1)$. *Clearly,* $S \in \mathcal{L}(\operatorname{ran} T_2, \operatorname{ran} T_1)$. *Since we have:*

$$
\begin{aligned}
\ker T_1 &= \ker T_2 \\
(\Leftrightarrow) \dim \ker T_1 &= \dim \ker T_2 \\
(\Leftrightarrow) \dim V - \dim \ker T_1 &= \dim V - \dim \ker T_2 \\
(\Leftrightarrow) \dim \operatorname{ran} T_1 &= \dim \operatorname{ran} T_2
\end{aligned}
$$

Also, $\ker S = \{0\}$ *(If there is some* $v \in V \setminus \ker T_1$ *s.t.* $0 = S(T_2 v) = T_1 v \neq 0$*, yielding a contradiction. Therefore,* $\ker S = \{0\}$*). Hence,* S *is injective thus invertible on* $\mathcal{L}(\operatorname{ran} T_2, \operatorname{ran} T_1)$. *By the conclusion from the last proof, we can extend* S *to an invertible linear map* $\tilde{S}$ *on* $\mathcal{L}(W, W)$ *s.t.* $\tilde{S}$ *equals to* S *on the latter one's domain.*

Practice on Invertibility

The proof for ($\Leftarrow$) is relatively easy. Since $T_1 e = S(T_2 e)$ for all $e \in V$, thus is also true for $e \in \ker T_1$. Then $0 = T_1 e = S(T_2 e)$. Since $\ker S = \{0\}$, this requires $T_2 e = 0 \Rightarrow e \in \ker T_2$. Therefore, $\ker T_1 = \ker T_2$.

If there is some $S \in \mathcal{L}(W, W)$ with $T_1 = S \circ T_2$, then $T_1 h = 0$ if and only if $T_2 h = 0$, if and only if $h \in \ker T_2$. Thus, $\ker T_1 = \ker T_2$. Conversely, suppose that $\ker T_1 = \ker T_2$. Let U be any subspace of V with $U \oplus \ker T_1 = V$. Let $\{u_1, ..., u_n\}$ be a basis of U. Note that T_1, T_2 restricted to U are invertible. Define

$$\tilde{S} : span\{T_2(u_1), ..., T_2(u_n)\} \to span\{T_1(u_1), ..., T_1(u_n)\}, \tilde{S}\left(\sum_{i=1}^{n} \alpha_i T_2(u_i)\right)$$

It is clear that $\tilde{S}$ is injective. Extend $\tilde{S}$ to an invertible S on W as we do in the last question and we are done.

Practice on Invertibility

$(\Rightarrow)$ $\mathbb{1} - AB$ inv.

$(\mathbb{1} - AB)A = A - ABA = A(\mathbb{1} - BA)$

$(\mathbb{1} - AB)^{-1} A (\mathbb{1} - BA) = A$

$\Rightarrow \mathbb{1} + BA = \mathbb{1} + B(\mathbb{1} - AB)^{-1} \cdot A \cdot (\mathbb{1} - BA)$

$\Rightarrow \mathbb{1} = (\mathbb{1} - BA) + B(\mathbb{1} - AB)^{-1} A \cdot (\mathbb{1} - BA)$

$= (\mathbb{1} - BA)[\mathbb{1} + B(\mathbb{1} - AB)^{-1} \cdot A]$

TASK

[2] *Suppose A, B are linear maps on finite dimensional vector spaces, prove that:*

$$\mathbb{I} - AB \text{ invertible} \quad \Leftrightarrow \quad \mathbb{I} - BA \text{ invertible}$$

$\mathbb{1} - AB$ not inv

$(\mathbb{1} - AB)x = 0, \; x \neq 0$

$(AB)x = x \quad Bx \neq 0$

$A(Bx) = x$

$\Rightarrow BA(Bx) = (Bx)$

$(BA - \mathbb{1})(Bx) = 0$

$(Bx) \in \ker(BA - \mathbb{1})$

$\neq 0$

[2] The solution is provided by Yiwen Tu

References I

- VV285 slides from Horst Hohberger
- Linear Algebra Done Right from Axler
- Answer from Leyang Zhang
- Answer from Yiwen Tu