Vv285 Mid 2 Recitation Class
Surface Integral

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Outline

Parameterization Using Different Coordinate System

In $\mathbb{R}^3$, $\boxed{f(x,y) = x^2 + e^y \Rightarrow \phi(x,y)} = \begin{pmatrix} x \\ y \\ x^2+e^y \end{pmatrix}$

$\mathbb{R}^n$. $f(x_1, \ldots x_m) = (\cdots)$

$\dim J_f = m$

TASK

Parameterize the following surfaces:

- In $\mathbb{R}^3$, the cylinder $y^2 + z^2 = 25$ $r^2 = 25$ φ_1: $\underset{x}{[-\infty, \infty]} \times \underset{\theta}{[0, 2\pi]}$ $\varphi(x,\theta) = \begin{pmatrix} x \\ 5\sin\theta \\ 5\cos\theta \end{pmatrix}$
- In $\mathbb{R}^3$, the sphere $x^2 + y^2 + z^2 = 30$ ϕ_2: $\underset{\theta}{[0, 2\pi]} \times \underset{\varphi}{[0, \pi]}$ $\phi(\theta, \varphi) = \begin{pmatrix} \sqrt{30}\cos\theta \sin\varphi \\ \sqrt{30}\sin\theta \sin\varphi \\ \sqrt{30}\cos\varphi \end{pmatrix}$
- The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$ $= f(y,z)$ which is in front of the yz-plane

$x \geq 0 \Rightarrow \boxed{5y^2 + 2z^2 \geq 10} \Rightarrow \varphi(y,z) = \begin{pmatrix} 5y^2+2z^2-10 \\ y \\ z \end{pmatrix}$

$5(r\cos\theta)^2 + 2(r\sin\theta)^2 \geq 10$

$5\cos^2\theta + 2\sin^2\theta \geq \frac{10}{r^2}$

$\int_r \int_\theta$

The Area of Surfaces

Property

1. *In $\mathbb{R}^3$, the area of given surface $\mathcal{S}$:* $\varphi(x_1, x_2) = \begin{pmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \end{pmatrix}$, $t_{x_1} = \begin{pmatrix} \frac{\partial}{\partial x_1}(\varphi_1) \\ \frac{\partial}{\partial x_1} \varphi_2 \end{pmatrix}$

$$A = \iint_\Omega \underbrace{\|t_1 \times t_2\|}_{dA} \circ \varphi(x) dx_1 dx_2 \tag{1}$$

$t_{x_2}: \ldots$

Where $t_1,\ t_2$ are the tangent vector at point (x_1, x_2).

Note that you don't need to normalize the tangent vector!

2. *At higher dimension, the infinitesimal area $dA = \sqrt{g(x)}dx_1 dx_2 \ldots dx_m$ (x is high dimensional vector, i.e, $x = (x_1, .., x_n)$.) $g(x)$ is calculated as:*

$t_1, \ldots t_m$

$\downarrow \qquad \downarrow$

$x_1 \qquad x_m$

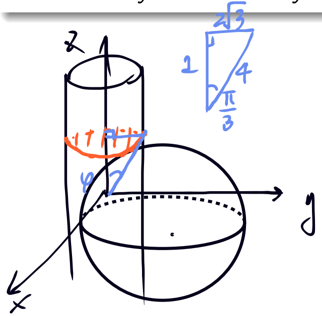
$$g(x) = \det \begin{pmatrix} \langle t_1, t_1\rangle & \ldots & \langle t_1, t_m\rangle \\ \ldots & \ldots & \ldots \\ \langle t_m, t_1\rangle & \ldots & \langle t_m, t_m\rangle \end{pmatrix}$$

$\det\begin{pmatrix} \langle t_1, t_1\rangle & \langle t_1, t_2\rangle \\ \langle t_2, t_1\rangle & \langle t_2, t_2\rangle \end{pmatrix}$

Practice - Surface Area with Constraints

TASK

Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.



$$\varphi(\theta, \varphi) = \begin{pmatrix} 4\cos\theta\sin\varphi \\ 4\sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix}, \quad \theta \in [0, 2\pi], \quad \varphi \in [0, \tfrac{\pi}{3}]$$

$$dA = r^2 \sin\varphi = 16\sin\varphi \qquad t_\theta,\ t_\varphi,\ \|t_\theta \times t_\varphi\|$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}} 1 \cdot 16\sin\varphi \, d\varphi \, d\theta$$

$$= 16\pi$$

Practice - Surface Area with Constraints

TASK

Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.

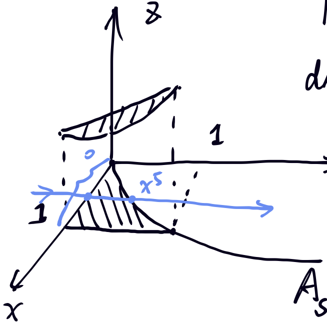
TASK

$z = f(x,y)$

Determine the surface area of the portion of $z = 3 + 2y + \frac{1}{4}x^4$ that is above the region in the xy-plane bounded by $y = x^5$, $x = 1$ and the x-axis.

$$\varphi(x,y) = \begin{pmatrix} x \\ y \\ 3+2y+\frac{1}{4}x^4 \end{pmatrix}$$

$$t_x = \begin{pmatrix} 1 \\ 0 \\ x^3 \end{pmatrix}, \quad t_y = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$



$\| t_x \times t_y \| \Rightarrow dA$

$$dA = \sqrt{g(x)}\, dx\, dy = \sqrt{\det\begin{pmatrix} x^6+1 & t_x \cdot t_y \\ 2x^3 & 5 \end{pmatrix}}$$

$$= \sqrt{x^6+5}\; dx\, dy$$

$$A_S = \int_0^1 \int_0^{x^5} 1 \cdot \underbrace{\sqrt{x^6+5}}\; \underbrace{dy}\, dx$$

$$= \int_0^1 y\sqrt{x^6+5}\,\Big|_0^{x^5}\, dx$$

$$= \int_0^1 x^5\sqrt{x^6+5}\; dx$$

$$= \frac{1}{6}\int_0^1 \sqrt{x^6+5}\; dx^6$$

$$= \frac{1}{9}\left(6^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

Surface Integral

$$\iint_S f(x,y)\, dA$$

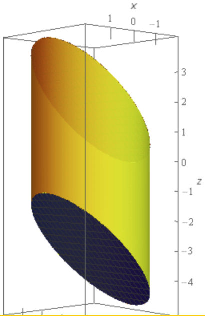
Reminder

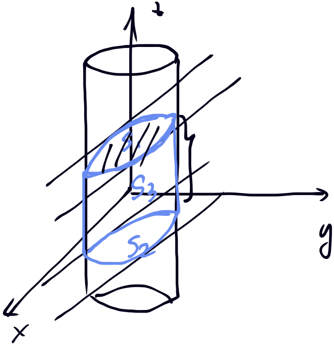
1. *The Area of a given surface can be seen as a surface integral where the integrand is $f(x) = 1$.*
2. *General steps of calculating surface integral:*
   1. 50% *Parameterize the surface, write out the domain Ω of your parameterization function φ, e.g,* $\Phi : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3, \quad \Phi(\theta, \varphi) = (\Phi_1(\theta, \varphi), \Phi_2(\theta, \varphi), \Phi_2(\theta, \varphi))$. *In this step, you should clarify the boundary of your integration, i.e, the relation between θ and φ, their range and so on.* *Visualize! If necessary.*
   2. 30% - 40% *Calculate tangent vectors, calculate $\boxed{dA}$ using $\|t_1 \times t_2\|$ (in $\mathbb{R}^3$) or $\sqrt{g(x)}$. If you think vector product is more complicated to calculate, stick to $\sqrt{g(x)}$ even if you're in $\mathbb{R}^3$.*
   3. 20% - 30% *You're almost finished! Integrate on Ω, remember to substitute dA with your parameterization variables!*

Practice - Surface Integral (Nothing difficult! Just be Patient and Careful!)

TASK

Evaluate $\iint_{\mathcal{S}} x - z dA$ *where* $\mathcal{S}$ *is the surface of the solid bounded by* $x^2 + y^2 = 4$, $z = x - 3$, $z = x + 2$. *Note that all three surfaces of this solid are included.*





For S_3: $x^2+y^2=4$

$$\phi_1(\theta, z) = \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ z \end{pmatrix}$$

$$\begin{cases} z \le x+2 \\ z \ge x-3 \end{cases} \Rightarrow \begin{cases} z \le 2\cos\theta + 2 \\ z \ge 2\cos\theta - 3 \end{cases}$$

$\theta \in [0, 2\pi]$. $\quad z \in [\boxed{2\cos\theta+2}, \boxed{2\cos\theta-3}]$

$$t_\theta = \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{pmatrix}, \quad t_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$dA = \sqrt{g(x)}\, d\theta\, dz = \sqrt{\det\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= 2\, d\theta\, dz$$

$$\iint_{S_3} x - z \, dA = \int_0^{2\pi} \int_{2\cos\theta-3}^{2\cos\theta+2} (2\cos\theta - z)\, dz\, d\theta$$

$$= 10\pi$$

For S_1. $\phi_2(x, y) = \begin{pmatrix} x \\ y \\ x+2 \end{pmatrix} \Rightarrow$ cylindrical coordinate.

$x^2+y^2 \le 4$

$$\phi(\ ,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$ $\to$ $0 \le r \le 2$, $\theta \in [0, 2\pi]$

$\varphi_2(r,\theta) = \begin{pmatrix} r\sin\theta \\ r\cos\theta + 2 \end{pmatrix}$

$\to t_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \cos\theta \end{pmatrix}, \quad t_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ -r\sin\theta \end{pmatrix}$

$$dA = \sqrt{g(x)}\, dr\, d\theta = \sqrt{\det\begin{pmatrix} 1+\cos^2\theta & -r\sin\theta\cos\theta \\ -r\cos\theta\sin\theta & r^2(1+\sin^2\theta) \end{pmatrix}}$$

$$= \sqrt{r^2 \cdot 2} = \sqrt{2} \cdot r\, dr\, d\theta$$

$$\iint_{S_1} x - z\, dA = \int_0^{2\pi}\int_0^2 [r\cos\theta - (r\cos\theta + 2)] \cdot \sqrt{2}\, r\, dr\, d\theta$$

$$= -8\sqrt{2}\pi$$

For S_3: $\iint_{S_3} x - z\, dA = 12\sqrt{2}\pi$

$$\iint_S x - z\, dA = (10 + 4\sqrt{2})\pi$$

Learn Well and Good Luck!

Tips in the end:

- Concepts! Concepts! Concepts! (Topology, continuous function on $\mathbb{R}^n$, Differentiation ...).
- Write out your process of solving a problems, don't think that any step is obvious!
- Integration skills and typical examples are shown in my RC6 & RC7 and this one. You need to know how to perform line integral, double(triple) integral, surface integral. If you need RC recordings to guide you through my slides, feel free to ask.
- Need more exercise? Go to Paul's Online Notes with URL in the next slides.
- In the end... **Be prudent and gook luck!**

References I

- VV285 slides from Horst Hohberger
- Paul's online note https://tutorial.math.lamar.edu/Classes/CalcIII/ChangeOfVariables.aspx