Vv285 Recitation Class 5
Basic Topology & Derivative

Yuxiang Chen

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Outline

Equivalence of Norms - Exercise

Show directly that the following two norms are equivalent on $\mathbb{R}^n$:

$$\|(x_i)\|_2 = \sqrt{\sum_i x_i^2}$$

$$\|(x_i)\|_\infty = \max\{|x_i| \colon 1 \leq i \leq n\}$$

Judge the Followings

[1]Suppose X is a complete normed space and $S_1, S_2, ..., S_n$ is a finite collection of subsets of X. Judge T or F:

- If S_i's are open, then $\bigcup_i S_i$ is open.
- If S_i's are open, then $\bigcap_i S_i$ is open.
- If S_i's are closed, then $\bigcup_i S_i$ is closed.
- If S_i's are closed, then $\bigcap_i S_i$ is closed.
- If Y is a normed space and $f: X \to Y$ is continuous, then $f(O)$ is open whenever $O \subseteq X$ is open.
- Suppose $K \subseteq X$ is compact and X is a normed space. Any continuous functions $f: K \to X$ is uniformly continuous.

[1]This question is provided by Leyang Zhang. Vielen Dank!

Topology on space of linear maps

Let X be a complete normed vector space. Then the set:

$$\mathrm{GL}(X) := \{L \in \mathscr{L}(X, X) \colon L^{-1} \text{exists}\}$$

Show that this set is open. Further more, show that if $\|L\| \leq 1$, then $\mathbb{I} - L \in \mathrm{GL}(X)$ and

$$(\mathbb{I} - L)^{-1} = \sum_{n=0}^{\infty} L^n$$

where $L^0 := \mathbb{I}$ which is the identity.

Judge the Followings about the First Derivative

Suppose X, V are finite-dimensional vector spaces and $\Omega \in X$ an open set. $f : X \to V$ is any function:

- $Df|_x$ is a linear map.
- $Df : x \mapsto Df_x$ is a linear map.
- $D\colon C^1(\Omega, V) \to C(\Omega, \mathcal{L}(X, V)), \quad f \mapsto Df$ is a linear map

Calculate Derivatives

[2]Suppose $A \in \mathrm{Mat}(n \times n\colon \mathbb{F})$, calculate their first derivatives:

1. $f(x) = \langle x, Ax \rangle$
2. $g(A) = \mathrm{tr}(A^2)$
3. $p(A) = A^2$

[2]This question is from Leyang Zhang. Vielen Dank!

References I

- Practice questions from Leyang Zhang
- VV285 slides from Horst Hohberger
- VV286 slides from Horst Hohberger