Vv285 Final Review Recitation Class
Manage Your Integral

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Outline

Managing Orientation of Curves

TASK

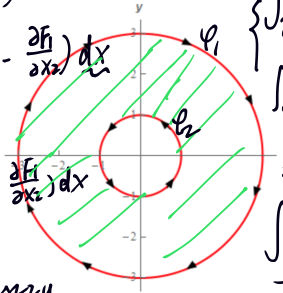
Evaluate $\int_{C^*} F\,d\vec{l}$, where $F(x,y) = \underline{(7x+y^2, -x^2+2y)^T}$. Where C^* is shown in fig(1).

① Green's Theorem: (in $\mathbb{R}^2$)

$$\int_{\partial R^{*(+)}} \vec{F}\cdot d\vec{l} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx$$

counter-clockwise

$$\int_{\partial R^{*(-)}} \vec{F}\cdot d\vec{l} = \boxed{-}\int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx$$



$$\begin{cases} \int_{\partial B_3(0)^-} \vec{F}\cdot d\vec{l} = -\int_{B_3(0)} \operatorname{rot} F\,dx \\ \int_{\partial B_1(0)^+} \vec{F}\cdot d\vec{l} = \int_{B_1(0)} \operatorname{rot} F\,dx \end{cases}$$

$\Rightarrow$

$$\int_{C^*} \vec{F}\cdot d\vec{l} = -\int_{B_3(0)\setminus B_1(0)} \operatorname{rot} F\,dA$$

$$\operatorname{rot} F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} = -2x-2y$$

Region: $B_3(0)\setminus B_1(0)$: (r,θ)

Figure: 1

$r \in [1,3], \quad \theta \in [0, 2\pi]$

$$-\int_0^{2\pi}\int_1^3 (-2r\cos\theta - 2r\sin\theta)\cdot r \, dr\, d\theta = 0$$

② $\varphi_1 : \varphi_1(\theta) = \begin{pmatrix} 3\cos\theta \\ -3\sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi]$

$$\int_{C^*} \vec{F} \cdot d\vec{l} = \int_I \langle F \circ \gamma, \gamma'(t) \rangle dt$$

$$\int_0^{2\pi} \left\langle \begin{pmatrix} 21\cos\theta + 9\sin^2\theta \\ -9\cos^2\theta + 6\sin\theta \end{pmatrix}, \begin{pmatrix} -3\sin\theta \\ -3\cos\theta \end{pmatrix} \right\rangle d\theta = 0$$

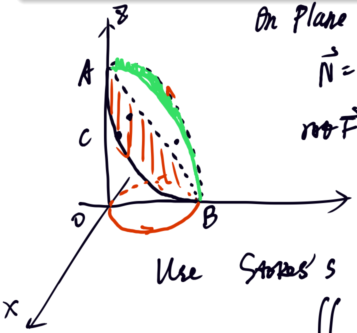
$\varphi_2 : \varphi_2(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \quad \theta \in [0, 2\pi]$

...

Managing Orientation of Surfaces

TASK

Calculate the circulation of field $F(x, y, z) = (z, x, y)^T$ *along the circle in* $\mathbb{R}^3$ *defined by* $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \; y + z = 1\}$.



On Plane ABC . $y+z=1 \Rightarrow \vec{n} = (0, 1, 1)$

$\vec{N} = \frac{\vec{n}}{\|\vec{n}\|} = (0, 1/\sqrt{2}, 1/\sqrt{2})$

$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial x & \partial y & \partial z \\ z & x & y \end{vmatrix} = 1 \cdot \vec{e}_1 - (-1)\vec{e}_2 + 1 \cdot \vec{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$d\vec{A} = \vec{N} \cdot dA$

Use Stokes's

$\int_{\partial S^*} \vec{F} \cdot d\vec{r} = \iint_S \langle \operatorname{rot} \vec{F}, d\vec{A} \rangle = \iint_S \langle \operatorname{rot} \vec{F}, \vec{N} \rangle dA$

$\iint_S \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \rangle dA = \frac{2}{\sqrt{2}} \iint_S 1 \cdot dA$

$= \frac{2}{\sqrt{2}} \cdot \pi \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{\sqrt{2}}$

Don't Calculate, Find the Easiest Way

TASK

Evaluate $\int_{C^*} \underbrace{(6x - 5y^2 + 2xy^3 - 10)}_{F_1}dx + \underbrace{(3x^2y^2 - 10xy)}_{F_2}dy$ *where* C^* *is shown in fig(2).*

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$= (6xy^2 - 10y) - (-10y + 6xy^2) = 0$

defined on $\mathbb{R}^2$

$\Rightarrow \vec{F}$ is potential field

$\vec{F} = \nabla U$

γ: $x \in [-1, 4]$. $\gamma(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$

$\int_{-1}^{4} \left\langle \begin{pmatrix} 6x-10 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle dx$

$= -5$

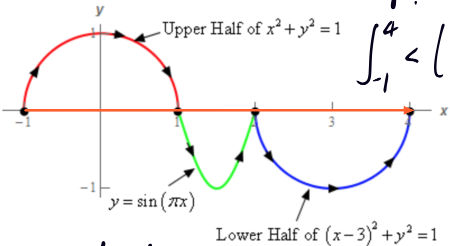


Figure: 2

Don't Calculate, Find the Easiest Way

TASK

Evaluate $\int_{\mathcal{C}^*} F\cdot d\vec{l}$ *where* $F(x,y,z) = (zx^3 - 2z, xz, yx)$ *and* $\mathcal{C}^*$ *is a three dimensional curve shown in fig(3), with vertices* $(2,0,0)$, $(0,2,0)$, $(0,0,4)$.

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial x & \partial y & \partial z \\ zx^3-2z & xz & yx \end{vmatrix} = \begin{pmatrix} 0 \\ x^3-y+z \\ 0 \end{pmatrix}$$

$P=(2,0,0)$

$Q=(0,2,0)$

$M=(0,0,4)$

$\vec{PM} = (-2,0,4)$

$\vec{PQ} = (0,-2,4)$

$$\vec{PM}\times\vec{PQ} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2 & 0 & 4 \\ 0 & -2 & 4 \end{vmatrix} = \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix}$$

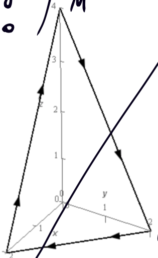


Figure: 3

$$\vec{n}_0 = -\left(\frac{2}{\sqrt5}, \frac{2}{\sqrt5}, \frac{1}{\sqrt5}\right)$$

$$\int_{\partial P^*} \vec{F}\cdot d\vec{l} = \iint_P \langle \operatorname{rot}\vec{F}, \vec{n}_0\rangle\, dA$$

$$= \iint_P \left\langle \begin{pmatrix} 0 \\ x^3-y+z \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt5} \\ -\frac{2}{\sqrt5} \\ -\frac{1}{\sqrt5} \end{pmatrix} \right\rangle dA$$

$$= \underline{\iint_P \frac{-2}{\sqrt5}(x^3-y+z)\, dA}$$

Parameterization of plane P $\quad 8(x-2)+8y+4z=0$

$$\varphi(x,y)=\begin{pmatrix} x \\ y \\ \underset{\sim\sim\sim}{4-2x-2y} \end{pmatrix}, \quad \varphi_x=\begin{pmatrix}1\\0\\-2\end{pmatrix}, \quad \varphi_y=\begin{pmatrix}0\\1\\-2\end{pmatrix}$$

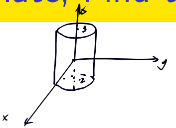
$$dA=\sqrt{\begin{vmatrix} \langle\varphi_x,\varphi_x\rangle & \langle\varphi_x,\varphi_y\rangle \\ \langle\varphi_y,\varphi_x\rangle & \langle\varphi_y,\varphi_y\rangle \end{vmatrix}} = \sqrt{\det\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}} = 3\,dx\,dy$$

$x \in [0,2]$

$0 \le y \le 2-x$

$$\int_0^2 \int_0^{2-x} \frac{-2}{\sqrt{5}}\left(x^3-y+4-2x-2y\right)\cdot 3\,dy\,dx = -\frac{64}{15}$$

Don't Calculate, Find the Easiest Way



TASK

Evaluate $\iint_{\mathcal{S}^*} F \cdot d\vec{S}$ *where* $F(x,y,z) = (4x - z^2, x + 3z, 6 - z)$ *and* $\mathcal{S}$ *is the surface of the solid bounded by the cylinder* $x^2 + y^2 = 36$ *and planes* $z = -2$ *and* $z = 3$.

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 4 + 0 - 1 = 3$$

$$\iint_{S^*} \vec{F} \cdot d\vec{A} = \iiint_R \operatorname{div} F \cdot dV = 3 \boxed{\iiint_R 1 \cdot dV}$$

$$= 3 \cdot (\pi R^2 \cdot h)$$

$$= 3 \cdot \pi \cdot 36 \cdot 5$$

$$= 540\pi$$

Warm Reminders - Concepts, Concepts, Concepts!

$\Delta = \langle \nabla, \nabla \rangle$ $\nabla \times \nabla = \sim$

- Look up Laplacian which is related to fluid dynamics. This is a major application of vector calculus.
- Review concepts of gradient and first derivative, relate to the second derivative. You should know how to calculate directional derivative as well as normal derivative. $\Rightarrow$ Green's Identity
- Review chapter two: property of continuous function from $\mathbb{R}^m$ to $\mathbb{R}^n$. Enhance your skill on line integral, surface integral, double integral and triple integral.
- Parametrization is important! Give a review on change of variables (a det of Jacobian will be inserted to the integral).
- At last, it's lucky to be your TA. Though it's a bit unlucky that I can't meet you in person, but thank you for your company!
  Gook luck and brace yourself for challenges!

References I

- VV285 slides from Horst Hohberger
- VV285 Sample exam 3 from Horst Hohberger
- Paul's online note
  https://tutorial.math.lamar.edu/