Vv285 Mid 2 Recitation Class
Surface Integral

Yuxiang Chen

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Outline

Parameterization Using Different Coordinate System

TASK

Parameterize the following surfaces:

- *In* $\mathbb{R}^3$, *the cylinder* $y^2 + z^2 = 25$
- *In* $\mathbb{R}^3$, *the sphere* $x^2 + y^2 + z^2 = 30$
- *The elliptic paraboloid* $x = 5y^2 + 2z^2 - 10$ *which is in front of the yz-plane*

The Area of Surfaces

Property

1. *In* $\mathbb{R}^3$*, the area of given surface* $\mathcal{S}$*:*

$$A = \iint_{\Omega} \|t_1 \times t_2\| \circ \varphi(x) dx_1 dx_2 \tag{1}$$

Where t_1*,* t_2 *are the tangent vector at point* (x_1, x_2)*.*
Note that you don't need to normalize the tangent vector!

2. *At higher dimension, the infinitesimal area* $dA = \sqrt{g(x)} dx_1 dx_2 ... dx_m$ *(*x *is high dimensional vector, i.e,* $x = (x_1, .., x_n)$*.)* $g(x)$ *is calculated as:*

$$g(x) = \det \begin{pmatrix} \langle t_1, t_1 \rangle & ... & \langle t_1, t_m \rangle \\ ... & ... & ... \\ \langle t_m, t_1 \rangle & ... & \langle t_m, t_m \rangle \end{pmatrix}$$

Practice - Surface Area with Constraints

TASK

Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.

Practice - Surface Area with Constraints

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Determine the surface area of the portion of $z = 3 + 2y + \frac{1}{4}x^4$ that is above the region in the xy-plane bounded by $y = x^5$, $x = 1$ and the x-axis.

Surface Integral

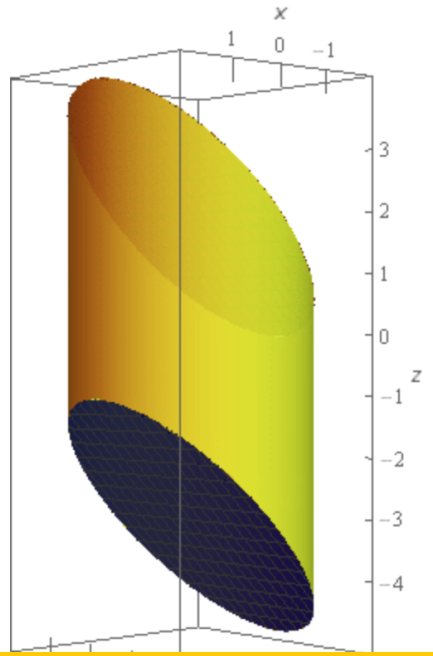
Reminder

1. *The Area of a given surface can be seen as a surface integral where the integrand is* $f(x) = 1$.
2. *General steps of calculating surface integral:*
   1. *Parameterize the surface, write out the domain* Ω *of your parameterization function* φ, *e.g,* $\Phi\colon [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3, \quad \Phi(\theta, \varphi) = (\Phi_1(\theta, \varphi), \Phi_2(\theta, \varphi), \Phi_2(\theta, \varphi))$. *In this step, you should clarify the boundary of your integration, i.e, the relation between* θ *and* φ, *their range and so on. Visualize! If necessary.*
   2. *Calculate tangent vectors, calculate dA using* $\|t_1 \times t_2\|$ *(in* $\mathbb{R}^3$*) or* $\sqrt{g(x)}$. *If you think vector product is more complicated to calculate, stick to* $\sqrt{g(x)}$ *even if you're in* $\mathbb{R}^3$.
   3. *You're almost finished! Integrate on* Ω, *remember to substitute dA with your parameterization variables!*

Practice - Surface Integral (Nothing difficult! Just be Patient and Careful!)

TASK

Evaluate $\iint_{\mathcal{S}} x - z dA$ *where* $\mathcal{S}$ *is the surface of the solid bounded by* $x^2 + y^2 = 4,\ z = x - 3,\ z = x + 2$. *Note that all three surfaces of this solid are included.*



Learn Well and Good Luck!

Tips in the end:

- Concepts! Concepts! Concepts! (Topology, continuous function on $\mathbb{R}^n$, Differentiation ...).
- Write out your process of solving a problems, don't think that any step is obvious!
- Integration skills and typical examples are shown in my RC6 & RC7 and this one. You need to know how to perform line integral, double(triple) integral, surface integral. If you need RC recordings to guide you through my slides, feel free to ask.
- Need more exercise? Go to Paul's Online Notes with URL in the next slides.
- In the end... **Be prudent and gook luck!**

References I

- VV285 slides from Horst Hohberger
- Paul's online note
  `https://tutorial.math.lamar.edu/Classes/CalcIII/ChangeOfVariables.aspx`