Vv285 Recitation Class for Mid 1

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Outline

1. Linear Map
2. Matrices
3. Reference

Important Properties

1. Additivity and homogeneity.
2. Determine a unique linear map $L \in \mathcal{L}(U, V)$ using a basis in U and a ordered list of vectors in V.
3. Dimension formula $\dim \ker L + \dim \operatorname{ran} L = \dim U$ for $L \in \mathcal{L}(U, V)$
4. Isomorphisms: basis to basis; injective and surjective.
5. Linear maps on finite dimensional vector space are bounded.
6. Induced norm.

Dual Space and Dual Map

Definition

If $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T^* \in \mathcal{L}(W^*, V^*)$ such that $T^*(\varphi) = \varphi \circ T$ for $\varphi \in W^*$

TASK

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ *by* $(Tp)(x) = x^2 p(x) + p''(x)$ *for* $x \in \mathbb{R}$

(a) Suppose $\varphi \in \mathcal{P}^*(\mathbb{R})$ *is defined by* $\varphi(p) = p'(4)$*, i.e,* $\varphi(p(x)) = \dfrac{d}{dx} p|_{x=4}$*. Describe the linear functional* $T'(\varphi)$ *on* $\mathcal{P}(\mathbb{R})$*.*

(b) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ *is defined by* $\varphi(p) = \int_0^1 p(x)dx$*. Evaluate* $(T'(\varphi))(x^3)$*.*

Dual Space and Dual Map

Definition

For $U \subseteq V$, the **annihilator** of U, denoted as U^0, is defined by

$$U^0 = \{\varphi \in V^* \colon \varphi(u) = 0 \text{ for all } u \in U\}$$

This clear that U^0 is a subspace of V^*

TASK

Suppose V is finite dimensional and $U \subseteq V$ is a subspace, prove:

$$\dim U + \dim U^0 = \dim V$$

Dual Space and Dual Map

Proof.

Proof using dual map: Let $i \in \mathcal{L}(U, V)$ to be the inclusion map, i.e, $i(u) = u$ for $u \in U$. Then $i^* \in \mathcal{L}(V^*, U^*)$. Apply the dimension formula on i:

$$\dim \operatorname{ran} i^* + \dim \ker i^* = \dim V^* = \dim V$$

Now, consider the kernel of i^*.

$$\begin{aligned} \varphi \in \ker i^* &\Leftrightarrow \varphi \circ i = 0 \\ \forall u \in U,\ \varphi \circ i(u) = 0 &\Leftrightarrow \varphi(u) = 0 \\ &\Leftrightarrow \varphi \in U^0 \end{aligned}$$

$\square$

Dual Space and Dual Map

Proof.

(continued) Therefore, $\ker i^* = U^0$. Similarly, the range of i^* consists all the linear functional in U^* such that $l = \varphi \circ i$ for $l \in U^*$ and $\varphi \in V^*$. Obviously, l is a restriction of φ on to U and any linear functional in U^* can be represented by $\varphi \circ i$ for some specific $\varphi \in V^*$(why?). Therefore, $\operatorname{ran} i^* = U^*$. We can conclude that:

$$\dim \operatorname{ran} i^* + \dim \ker i^* = \dim U^* + \dim \ker i^* = \dim U + \dim U^0 = \dim V$$

$\square$

Exercise - Projection Matrix (do you really know about projection?)

[1]Let $A \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be a projection onto the plane $\{(x, y, z) \in \mathbb{R}^3 \colon x = z\}$

(a) If $\ker A$ is the x-axis, determine the matrix M_1 representing A (w.r.t standard basis in $\mathbb{R}^3$).

(b) If A is self-adjoint, determine the matrix M_3 representing A (w.r.t standard basis in $\mathbb{R}^3$).

(c) Suppose that $B \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ is another projection whose range is the x-axis and $\ker B = \{(x, y, z) \in \mathbb{R}^3 \colon x = z\}$. Let M_3 be the representing matrix of B (w.r.t standard basis in $\mathbb{R}^3$). Calculate $(M_1 + M_3)M_1$.

[1]This exercise is from Leyang Zhang. Vielen Dank!

References I

- Linear Algebra Done Right by Axler
- Practice questions from Leyang Zhang