Vv285 Recitation Class for Mid 1

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Outline

Important Properties

1. Additivity and homogeneity. $L(\lambda x + \mu y) = \lambda L x + \mu L y$
2. Determine a unique linear map $L \in \mathcal{L}(U, V)$ using a basis in U and a ordered list of vectors in V. $(b_1, \dots b_n) \xrightarrow{L} (v_1, \dots v_n)$
3. Dimension formula $\dim \ker L + \dim \operatorname{ran} L = \dim U$ for $L \in \mathcal{L}(U, V)$
4. Isomorphisms: basis to basis; injective and surjective.
5. Linear maps on finite dimensional vector space are bounded.
6. ~~Induced~~ operator norm.

$L \in \mathcal{L}(X, X)$

$$\begin{cases} \|Lx\|_X \le |||L|||\,\|x\|_X \\ |||\underbrace{L_1 L_2}_{\text{operator}}||| \le |||L_1||| \cdot |||L_2||| \end{cases}$$

$|L|_\infty = |\sup A_{ij}|$

$|L|_\infty = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix} = 3$

Dual Space and $\boxed{\text{Dual Map}}$ Ajoint A^*

$$V \to \boxed{V^* = \mathcal{L}(V;F)}$$

Definition

If $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T^* \in \mathcal{L}(W^*, V^*)$ such that $\underbrace{T^*(\varphi)}_{W^*} = \boxed{\varphi \circ T}$ for $\varphi \in W^*$

$\varphi \in \mathcal{L}(W; \mathbb{F})$
$T \in \mathcal{L}(V, W)$
$\boxed{\varphi \circ T(v)} \in \mathcal{L}(V; \mathbb{F}) = V^*$

TASK

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbb{R}$
(a) Suppose $\varphi \in \mathcal{P}^*(\mathbb{R})$ is defined by $\varphi(p) = p'(4)$, i.e, $\varphi(p(x)) = \boxed{\frac{d}{dx} p|_{x=4}.}$ Describe the linear functional $\boxed{T^*(\varphi)} \in \mathcal{P}^*(\mathbb{R})$ on $\mathcal{P}(\mathbb{R})$.
(b) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.

(a) $\underbrace{T^*(\varphi)}_{\in P^*(\mathbb{R})}(p(x)) = \underbrace{(\varphi \circ T)}_{\in P^*(\mathbb{R})}(p(x)) = \varphi(T(p(x)))$

$= \varphi(\underbrace{x^2 p(x) + p''(x)})$

$= 2xp(x) + x^2 p'(x) + p'''(x) \Big|_{x=4}$

$= 8\underline{p(4)} + 16\underline{p'(4)} + \underline{p'''(4)} \in \mathbb{F}$

(b) $\underbrace{T^*(\varphi)}(x^3) = (\varphi \circ T)(x^3) = \varphi(T(x^3)) = \varphi(x^5 + 6x)$

$= \int_0^1 (x^5 + 6x)\, dx = \underbrace{\frac{19}{6}} \in \mathbb{F}$

Dual Space and Dual Map

① 0 map

② $\varphi_1, \varphi_2 \in U^0$, $(\varphi_1+\varphi_2)(u) = 0$ ③ $\lambda\varphi_1(u)=0$

Definition

For $U \subseteq V$, the **annihilator** of U, denoted as U^0, is defined by

subspace

$$U^0 = \{\varphi \in V^* : \varphi(u) = 0 \text{ for all } u \in U\} \subset U^*$$

$\varphi(U)=0$

This clear that U^0 is a subspace of V^*

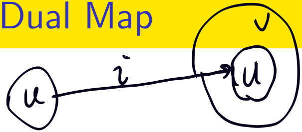
$\mathcal{L}(V; \mathbb{F})$

TASK

Suppose V is finite dimensional and $U \subseteq V$ is a subspace, prove:

$$\boxed{\dim U + \dim U^0 = \dim V}$$

Dual Space and Dual Map



Proof.

Proof using dual map: Let $i \in \mathcal{L}(U, V)$ to be the inclusion map, i.e, $i(u) = u$ for $u \in U$. Then $i^* \in \mathcal{L}(V^*, U^*)$. Apply the dimension formula on i:

$$\dim \operatorname{ran} i^* + \dim \ker i^* = \dim V^* = \dim V$$

Now, consider the kernel of i^*. $i^*(\varphi) = \varphi \circ i$ $(\varphi \in V^*$

$\in U^*$

$\varphi \circ i \Rightarrow \boxed{L \in \mathcal{L}(U; \mathbb{F})}$

$V \to \mathbb{F}$ $U \to V$ $\varphi \circ i$

$$\varphi \in \ker i^* \Leftrightarrow \varphi \circ i = 0$$

$$\forall u \in U, \ \varphi \circ i(u) = 0 \Leftrightarrow \varphi(u) = 0 \in \mathbb{F}$$

$$\Leftrightarrow \varphi \in U^0$$

$\dim \ker i^* = \dim U^0$

□

Dual Space and Dual Map

$i^* \in \mathcal{L}(V^*, U^*)$

$\varphi \in V^*$, $l = i^*(\varphi) = \underline{\varphi \circ i} \in \operatorname{ran} i^* \overset{?}{=} U^*$

$\in U^* = \mathcal{L}(U; \mathbb{F})$ $(\dim U^* = \dim U)$

Proof.

(continued) Therefore, $\ker i^* = U^0$. Similarly, the range of i^* consists all the linear functional in U^* such that $l = \varphi \circ i$ for $l \in U^*$ and $\varphi \in V^*$. Obviously, l is a restriction of φ on to U and any linear functional in $\boxed{U^*}$ can be represented by $\underset{\in V^*}{\varphi} \circ i$ for some specific $\varphi \in V^*$(why?). Therefore, $\operatorname{ran} i^* = U^*$. We can conclude that: $\forall\, l \in U^*$, we can find $\varphi \in V^*$ s.t. $l = \varphi \circ i$

$$\underline{\dim \operatorname{ran} i^*} + \dim \ker i^* = \underline{\dim U^*} + \dim \ker i^* = \underline{\dim} U + \dim U^0 = \dim V$$

$l(u) = \varphi \circ i(u)$
$= \varphi(u)$

② Consider dual basis for V^* $\{b_1^*, \dots, b_n^*\}$

$$b_i^*(v) = \begin{cases} 1, & v = b_i \\ 0, & \text{otherwise} \end{cases} \qquad V^* \updownarrow V \qquad \underbrace{\{b_1^*, \dots, b_n^*\}}_{} \updownarrow \{b_1, \dots b_n\}$$

$v \in V$

$$b_k^*(v) = \begin{cases} 1, & v = b_k \\ 0, & \text{otherwise} \end{cases}$$

Since $U \subset V$, suppose $U = \text{span}\{\underline{b_1, b_2, \dots b_m}\}$ $\dim U = m$

Next, show $M = \text{span}\{\underbrace{\boxed{b_{m+1}^*, \dots b_n^*}}\} = U^\circ \Rightarrow \dim U^\circ = n - m$

$+$

$= \dim V = n$

① $\forall \varphi \in M$, $\varphi = \sum_{i=m+1}^{n} b_i^*$

$$\underset{\in U}{\varphi(u)} = \varphi\left(\sum_{j=1}^{m} b_j\right) = \sum_{\underline{i=m+1}}^{n} b_i^*\left(\boxed{\sum_{j=1}^{m} b_j}\right) = 0$$

$\varphi \in U^\circ$, $M \subset U^\circ$

② If $\dim U^\circ > \dim \underline{M}$

U° must contain vectors in $\text{span}\{b_1^*, \dots b_m^*\}$

□

Suppose b_k^* ($k \in [1, m]$) in U

$$b_k^*(b_k) = 1 \Rightarrow b_k^* \notin U^0$$

$\Rightarrow \dim U^0 = \dim M$

$\Rightarrow U^0 = M$

Exercise - Projection Matrix (do you really know about projection?)

orthogonal projection A : $\boxed{A \cdot A = A}$, $\ker A \perp U$

Let[1] $A \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be a projection onto the plane $\{(x, y, z) \in \mathbb{R}^3 : x = z\}$

$\hookrightarrow A \cdot A = A$ $\quad A^2 = A$ $\quad (\alpha, \beta, \alpha)$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(a) If $\ker A$ is the x-axis, determine the matrix M_1 representing A (w.r.t standard basis in $\mathbb{R}^3$). $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(b) If A is self-adjoint, determine the matrix M_3 representing A (w.r.t standard basis in $\mathbb{R}^3$). $A = A^*$

(c) Suppose that $B \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ is another projection whose range is the x-axis and $\ker B = \{(x, y, z) \in \mathbb{R}^3 : x = z\}$. Let M_3 be the representing matrix of B (w.r.t standard basis in $\mathbb{R}^3$). Calculate $(M_1 + M_3)M_1$.

[1] This exercise is from Leyang Zhang. Vielen Dank!

(b) $A = A^*$ $\quad (\operatorname{ran} A^*)^\perp = \underbrace{\ker A} = (\operatorname{ran} A)^\perp$

$\operatorname{ran} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\ker A = \operatorname{span}^\perp \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

$A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

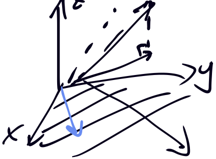
$M_2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

(c) $M_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $\int$ $(M_1 + M_3) M_1$ $\qquad \operatorname{ran} M_1 = \ker M_3$

$= M_1 \cdot M_1 + M_3 M_1$

$= M_1 + 0 = M_1$

(a) $A \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$



$\ker A = \left\{ \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\operatorname{ran} A = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to ? \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

References I

- Linear Algebra Done Right by Axler
- Practice questions from Leyang Zhang