Vv285 Recitation Class 3

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Outline

Calculus of Linear Maps - Matrices

1. The column i of a matrix $A \in \mathrm{Mat}(n \times m; \mathbb{F})$ is the **image** of the standard basis b_i in $\mathbb{R}^n$. You may turn every linear maps to a equivalent map from $\mathbb{R}^m$ to $\mathbb{R}^n$.
   - ? Suppose you have a linear map $L \in \mathcal{L}(U, V)$, you want to solve some properties of this map.
   - ➢ Map domain and codomain of a map to $\mathbb{F}^n$. This will be a linear process, you can define $\varphi_1 \in \mathcal{L}(U, X)$ and $\varphi_2 \in \mathcal{L}(V, Y)$, where X and Y are subspace of $\mathbb{R}^n$ and $\mathbb{R}^m$.
   - ➢ turn everything into basis and determine the image of basis in U. Write down the matrix A.
   - ➢ Analyse A instead of L. Then translate the properties of A back to L.

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   - ➢ Analyse A instead of L. Then translate the properties of A back to L.
2. Matrix multiplication: row by column (practice this!)

Recap for this week

1. Fredholm Alternative: equivalent to
   - Either $\ker L = \{0\}$, i.e, L is injective.
   - Or $\ker L$ is not trivial, i.e, L is not injective.
2. Matrix Rank: $\mathrm{rank}\ A =$ row $\mathrm{rank}\ A =$ col $\mathrm{rank}\ A$ (use the Adjoint of a matrix to prove it)

Short Comment - Matrix Ajoint (HW2)

Definition

V is finite dimensional with inner product and $A \in \mathcal{L}(V, V)$. The adjoint $A^* \in \mathcal{L}(V, V)$ is defined by:

$$\langle u, Av \rangle = \langle A^* u, v \rangle \quad \text{for all } u, v \in V$$

Reminder

We don't need L to be endomorphism, we can also define the adjoint of T if $T \in \mathcal{L}(V, W)$. Then $T^* \in \mathcal{L}(W, V)$ and $\langle w, Tv \rangle = \langle T^* w, v \rangle$ for all $v \in V$ and $w \in W$

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TASK

- *Show A^* is well-defined.*
- *If $V = \mathbb{C}^n$, prove $A^* = \overline{A^T}$.*
- *Show* $(\operatorname{ran} A)^\perp = \ker A^*$ *and* $(\ker A)^\perp = \operatorname{ran} A^*$

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2. Matrix Rank: $\mathrm{rank}\ A =$ row $\mathrm{rank}\ A =$ col $\mathrm{rank}\ A$
   - $\mathrm{Mat}(m \times n; \mathbb{F})$ need to be a real vector space (why?)
   1. Recap of this proof

Practice on calculating linear maps

TASK

Let $A \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ *which satisfies* $A = A^*$ *and* $\{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$ *is in the range of* A*, such that* $A(1, 0, 1) = (1, 0, 1)$

(a) *Suppose* $\dim \ker A = 2$*. Determine the matrix of* A *w.r.t standard basis on* $\mathbb{R}^3$

(b) *Suppose that* $V \subseteq \mathbb{R}^3$ *is a subspace such that* $Ah = 2h$ *for all* $h \in V$*. Show that* $V \perp \{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$

(a) *Let* $\{(x, 0, -x) \in \mathbb{R}^3 : x \in \mathbb{R}\} \subseteq V$ *and* $Ah = 2h$ *for all* $h \in V$*. Give all possible representations of* A *(w.r.t standard basis on* $\mathbb{R}^3$*).*

Invertible Matrices & Linear Maps

1. To verify whether a map (matrix) A is invertible or not, we need to verify that there exists a map B that $AB = \mathbb{I}$ and $BA = \mathbb{I}$ (in infinite dimensional situation, we must show the left inverse and right inverse both exists!)
   - However, if A is on finite dimensional spaces, we only need to show $BA = \mathbb{I}$ or $AB = \mathbb{I}$
   - In finite dimensional spaces, $L \in \mathcal{L}(U, V)$ and $\dim U = \dim V$, L's injectiveness $\Leftrightarrow$ L's surjectiveness $\Leftrightarrow$ L's bijective
   - If A is a square matrix, i.e, $A \in \mathrm{Mat}(n; \mathbb{F})$. Then A is invertible iff $\operatorname{rank} A = n$

Practice on Invertibility

TASK

Suppose V is finite dimensional, U is a subspace of V, i.e, $U \subseteq V$, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible operator $T \in \mathcal{L}(V, V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Practice on Invertibility

Solution

[1] *If S is not injective, any $T \in \mathcal{L}(V, V)$ such that $Tu = Su$ would have non-trivial kernel, whence it cannot be invertible. Conversely, suppose that S is injective. Then the dimension formula says that $\dim \operatorname{ran} S = \dim U$. Suppose that $\dim V = n$ and $\dim U = m$. Let $\mathcal{U} = \{u_1, \ldots, u_m\}$ be a basis for u and $\mathcal{W} = \{w_1, \ldots, w_m\}$ be a basis for $\operatorname{ran} S$. Extend $\mathcal{U}$ to a basis $\{u_1, \ldots, u_n\}$ of V and $\mathcal{W}$ to a basis $\{w_1, \ldots, w_n\}$ of V. Define*

$$T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = S\left(\sum_{i=1}^{m} \alpha_i u_i\right) + \sum_{i=m+1}^{n} \alpha_i w_i.$$

Then $\ker T = \{0\}$ (why?), so T is injective and thus is bijective. This completes our proof.

[1] This proof is from Leyang Zhang

Practice on Invertibility

TASK

Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\ker T_1 = \ker T_2$ *if and only if there exists an invertible operator* $S \in \mathcal{L}(W, W)$ *s.t.* $T_1 = S \circ T_2$.

Practice on Invertibility

Solution

We first prove ($\Rightarrow$)*:*
Then we want to find a linear map s.t. $T_1 e = S(T_2 e)$ *for all* $e \in (V \setminus \ker T_1)$*. Clearly,* $S \in \mathcal{L}(\text{ran } T_2, \text{ran } T_1)$*. Since we have:*

$$\begin{aligned} \ker T_1 &= \ker T_2 \\ (\Leftrightarrow) \dim \ker T_1 &= \dim \ker T_2 \\ (\Leftrightarrow) \dim V - \dim \ker T_1 &= \dim V - \dim \ker T_2 \\ (\Leftrightarrow) \dim \text{ran } T_1 &= \dim \text{ran } T_2 \end{aligned}$$

Also, $\ker S = \{0\}$ *(If there is some* $v \in V \setminus \ker T_1$ *s.t.* $0 = S(T_2 v) = T_1 v \neq 0$*, yielding a contradiction. Therefore,* $\ker S = \{0\}$*). Hence,* S *is injective thus invertible on* $\mathcal{L}(\text{ran } T_2, \text{ran } T_1)$*. By the conclusion from the last proof, we can extend* S *to an invertible linear map* $\tilde{S}$ *on* $\mathcal{L}(W, W)$ *s.t.* $\tilde{S}$ *equals to* S *on the latter one's domain.*

Practice on Invertibility

The proof for ($\Leftarrow$) is relatively easy. Since $T_1 e = S(T_2 e)$ for all $e \in V$, thus is also true for $e \in \ker T_1$. Then $0 = T_1 e = S(T_2 e)$. Since $\ker S = \{0\}$, this requires $T_2 e = 0 \Rightarrow e \in \ker T_2$. Therefore, $\ker T_1 = \ker T_2$.
If there is some $S \in \mathcal{L}(W, W)$ with $T_1 = S \circ T_2$, then $T_1 h = 0$ if and only if $T_2 h = 0$, if and only if $h \in \ker T_2$. Thus, $\ker T_1 = \ker T_2$. Conversely, suppose that $\ker T_1 = \ker T_2$. Let U be any subspace of V with $U \oplus \ker T_1 = V$. Let $\{u_1, \ldots, u_n\}$ be a basis of U. Note that T_1, T_2 restricted to U are invertible. Define

$$\tilde{S} : span\{T_2(u_1), \ldots, T_2(u_n)\} \to span\{T_1(u_1), \ldots, T_1(u_n)\}, \tilde{S}\left(\sum_{i=1}^{n} \alpha_i T_2(u_i)\right)$$

It is clear that $\tilde{S}$ is injective. Extend $\tilde{S}$ to an invertible S on W as we do in the last question and we are done.

Practice on Invertibility

TASK

[2] *Suppose A, B are linear maps on finite dimensional vector spaces, prove that:*

$$\mathbb{I} - AB \textit{ invertible} \quad \Leftrightarrow \quad \mathbb{I} - BA \textit{ invertible}$$

[2] The solution is provided by Yiwen Tu

References I

- VV285 slides from Horst Hohberger
- Linear Algebra Done Right from Axler
- Answer from Leyang Zhang
- Answer from Yiwen Tu