Vv285 Recitation Class 10
Fundamental Theorems of Vector Calculus

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Outline

Admissible Regions & Admissible Hypersurfaces in $\mathbb{R}^3$

Suppose $R \subseteq \mathbb{R}^n$

1. R is a **region** if R is open and connected.
2. R is **admissible** if it's bounded and its boundary is a union of finite number of parametrized hypersurfaces whose normal vector point outward.

➢ An admissible region is just a body with volume (in $\mathbb{R}^2$, area).

A hypersurface $\mathcal{S} \subseteq \mathbb{R}^3$ with parametrization $\varphi\colon R \to \mathcal{S}$ is admissible if

1. R is closed
2. the interior of R is an admissible region in $\mathbb{R}^2$ with oriented boundary curve ∂R^*.

Stokes' Theorem on Surface with Boundary in $\mathbb{R}^3$

Condition when using Stokes's Theorem:

1. $\mathcal{S} \subseteq \Omega \subseteq \mathbb{R}^3$ a parametrized, **admissible** region with **boundary**
2. $F\colon \Omega \to \mathbb{R}^3$ a continuously differentiable vector field

Then, the **rotation vector**, i.e, circulation density, over the whole **oriented** surface is the same as the line integral of F along the boundary of the surface $\mathcal{S}$.

$$\iint_{\partial \mathcal{S}^*} F d\vec{l} = \int_{\mathcal{S}^*} \mathrm{rot} F d\vec{A}.$$

The orientation of the surface satisfies the **right hand rule** (demo)

Practice - Stokes's Theorem

TASK

Evaluate the integral

$$\iint_{\mathcal{S}} rotF \cdot d\vec{S}$$

Where $F = (z^2 - 1, z + xy^3, 6)^T$ *and* $\mathcal{S}$ *is the portion of* $x = 6 - 4y^2 - 4z^2$ *in front of* $x = -2$ *with orientation in the negative* x*-axis direction.*

Gauss's Theorem over Admissible Region in $\mathbb{R}^3$

Condition using Gauss' Theorem

1. $R \subseteq \mathbb{R}^3$ an admissible region (also can be generalized into $\mathbb{R}^n$)
2. $F : \overline{R} \to \mathbb{R}^3$ a continuously differentiable vector field.

$$\iiint_R \operatorname{div} F dx = \iint_{\partial R^*} F d\vec{A}.$$

Practice - Gauss's Theorem

TASK

Evaluate $\iint_{\mathcal{S}} F \cdot \vec{S}$ *where* $F = (xy, -\frac{1}{2}y^2, z)$ *and the surface consists of three surfaces:* $z = 4 - 3x^2 - 3y^2$, $1 \leq z \leq 4$; $x^2 + y^2 = 1$, $0 \leq z \leq 1$ *on the sides and* $z = 0$ *on the bottom.*

Summary of Green's, Stokes's and Gauss's

Theorem	Dimension	Domain	Characteristic
Green's	$\mathbb{R}^2$	bounded, simple region	$\int_{\partial R} \vec{F} d\vec{l} \Leftrightarrow \iint_R \text{rot}\vec{F} dx$
Stokes's	$\mathbb{R}^3$	surface with boundary	$\iint_{\mathcal{S}} \vec{F} d\vec{S} \Leftrightarrow \int_{\partial \mathcal{S}} \vec{F} d\vec{l}$
Gauss's	$\mathbb{R}^3$	admissible region	$\iiint_R \text{div}\vec{F} dx \Leftrightarrow \iint_{\partial R} \vec{F} d\vec{A}$

Table: Summary

Practice - Choose the right theorem!

TASK

Evaluate $\int_{\mathcal{C}} \vec{F} d\vec{r}$ *where* $F = (3yx^2 + z^3, y^2, 4yx^2)^T$ *and* $\mathcal{C}$ *is a triangle with vertices* $(0,0,3),\ (0,2,0),\ (4,0,0)$, *which has counter clockwise orientation when looking down towards* $x - o - y$ *plane.*

Practice - Choose the right theorem!

TASK

Evaluate $\int_{\mathcal{C}} \vec{F} d\vec{r}$ *where* $F = (3yx^2 + z^3, y^2, 4yx^2)^T$ *and* $\mathcal{C}$ *is a triangle with vertices* $(0,0,3),\ (0,2,0),\ (4,0,0)$*, which has counter clockwise orientation when looking down towards* $x-o-y$ *plane.*

TASK

Evaluate $\iint_{\mathcal{S}} \vec{F} d\vec{S}$ *where* $F = (2xz, 1-4xy^2, 2z-z^2)^T$ *and* $\mathcal{S}$ *is the surface of the solid bounded by* $z = 6 - 2x^2 - 2y^2$ *and plane* $z = 0$

Green's Identity

Remember the following relation:

$$\int_R \langle \nabla u, \nabla v \rangle \, dx = -\int_R u \cdot \Delta v dx + \int_{\partial R^*} u \frac{\partial v}{\partial n} dA$$

where R is an admissible region in $\mathbb{R}^n$ and u, v are both continuously differentiable potential functions in $\mathbb{R}^n$.

TASK

Prove $\operatorname{div}(u\nabla v) = u\Delta v + \langle \nabla u, \nabla v \rangle$

[1]This practice is from Leyang Zhang. Vielen Dank!

Green's Identity

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TASK

Prove $\operatorname{div}(u \nabla v) = u \Delta v + \langle \nabla u, \nabla v \rangle$

TASK

[1] *Suppose u is a non-constant C^2 function on some open set U containing $\overline{B_1(0)}$. Furthermore, suppose $u = 0$ on $\partial B_1(0)$. Show that* $\int_{B_1(0)} u \Delta u < 0$.

[1] This practice is from Leyang Zhang. Vielen Dank!

References I

- VV285 slides from Horst Hohberger
- Paul's online note
  `https://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx`
- Vv285 Review 11 from Leyang Zhang