Vv285 Recitation Class 10
Fundamental Theorems of Vector Calculus

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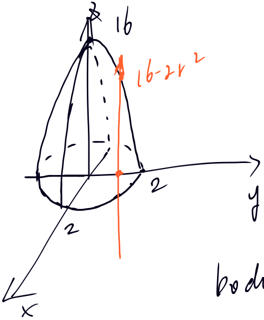


$$z = 16 - 2x^2 - 2y^2$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ \underbrace{z = 16 - 2r^2} \end{cases} \qquad E(r, \theta, z)$$

body : $r \in [0, 2]$, $\theta \in [0, 2\pi]$, $z \in [0, 16 - 2r^2]$

$$|V| = \int_0^{2\pi} \int_0^{2} \int_0^{16-2r^2} \underline{r} \, dz \, dr \, d\theta$$

Outline

Admissible Regions & Admissible Hypersurfaces in $\mathbb{R}^3$

Suppose $R \subseteq \mathbb{R}^n$

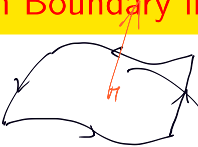
1. R is a **region** if R is open and connected.
2. R is **admissible** if it's bounded and its boundary is a union of finite number of parametrized hypersurfaces whose normal vector point outward.

➢ An admissible region is just a body with volume (in $\mathbb{R}^2$, area).

A hypersurface $\mathcal{S} \subseteq \mathbb{R}^3$ with parametrization $\varphi\colon R \to \mathcal{S}$ is admissible if

1. R is closed
2. the interior of R is an admissible region in $\mathbb{R}^2$ with oriented boundary curve ∂R^*.

Stokes' Theorem on Surface with Boundary in $\mathbb{R}^3$



Condition when using Stokes's Theorem:

1. $\mathcal{S} \subseteq \Omega \subseteq \mathbb{R}^3$ a parametrized, **admissible** region with **boundary**
2. $F\colon \Omega \to \mathbb{R}^3$ a continuously differentiable vector field

Then, the **rotation vector**, i.e, circulation density, over the whole **oriented** surface is the same as the line integral of F along the boundary of the surface $\mathcal{S}$.

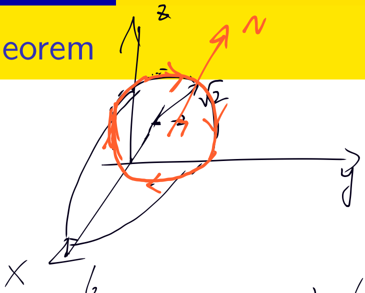
$$\langle \vec{F}, \vec{T} \rangle \quad \int_{\partial\mathcal{S}^*} F d\vec{l} = \iint_{\mathcal{S}^*} \mathrm{rot} F d\vec{A}. \quad \langle \mathrm{rot}\vec{F}, \vec{n} \rangle$$

The orientation of the surface satisfies the **right hand rule** (demo)

curve

Practice - Stokes's Theorem

$$\boxed{F = \begin{pmatrix} z^2-1 \\ z+xy^3 \\ 6 \end{pmatrix}}$$



TASK

Evaluate the integral

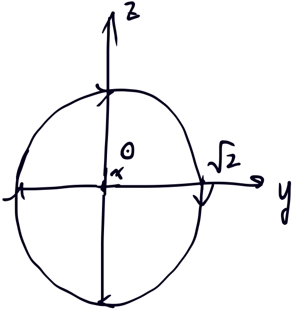
$$\iint_{\mathcal{S}} \text{rot}F \cdot d\vec{S}$$

$-2 = 6 - 4y^2 - 4z^2$

$y^2 + z^2 = 2$

$\in \mathbb{R}^3$

Where $F = (z^2 - 1, z + xy^3, 6)^T$ *and* $\mathcal{S}$ *is the portion of* $x = 6 - 4y^2 - 4z^2$ *in front of* $x = -2$ *with orientation in the negative x-axis direction.*



$\theta \in [0, 2\pi]$

$$\gamma(\theta) = \begin{pmatrix} -2 \\ \cos(-\theta) \\ \sin(-\theta) \end{pmatrix} = \begin{pmatrix} -2 \\ \cos\theta \\ -\sin\theta \end{pmatrix}, \quad \gamma'(\theta) = \begin{pmatrix} 0 \\ -\sin\theta \\ -\cos\theta \end{pmatrix}$$

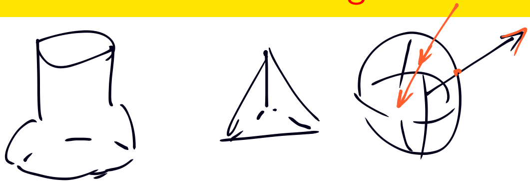
$$\iint_{S^*} \overrightarrow{\mathrm{rot} F} \cdot d\vec{A} = \int_0^{2\pi} \langle \underbrace{F \circ \gamma}, \gamma'(\theta) \rangle \, d\theta$$

$$= \int_0^{2\pi} \left\langle \begin{pmatrix} 2\cos^2\theta - 1 \\ \sqrt{2}\cos\theta - 4\sqrt{2}\sin^3\theta \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ -\sin\theta \\ -\cos\theta \end{pmatrix} \right\rangle d\theta$$

$$= 2\pi$$

Gauss's Theorem over Admissible Region in $\mathbb{R}^3$

B



Condition using Gauss' Theorem

1. $R \subseteq \mathbb{R}^3$ an $\boxed{\text{admissible region}}$ (also can be generalized into $\mathbb{R}^n$)
2. $F : \overline{R} \to \mathbb{R}^3$ a continuously differentiable vector field.

$$\iiint_R \operatorname{div} F dx = \iint_{\partial R^*} F d\vec{A}.$$

Practice - Gauss's Theorem

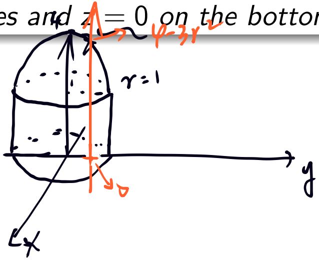
$$F = \begin{pmatrix} xy \\ -\frac{1}{2}y^2 \\ z \end{pmatrix}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \cdots \frac{\partial F_n}{\partial x_n} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= y - y + 1 = 1$$

TASK

Evaluate $\iint_S F \cdot \vec{S}$ *where* $F = (xy, -\frac{1}{2}y^2, z)$ *and the surface consists of three surfaces:* $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$; $x^2 + y^2 = 1$, $0 \le z \le 1$ *on the sides and* $z = 0$ *on the bottom.*



$$\iint_S F \cdot d\vec{S} = \iiint_R (\operatorname{div} F = 1)\, dx$$

$$= \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} 1 \cdot r \cdot dz\,dr\,d\theta$$

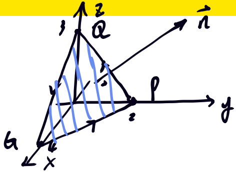
$$= \frac{5}{2}\pi$$

Summary of Green's, Stokes's and Gauss's

Theorem	Dimension	Domain	Characteristic
Green's	$\mathbb{R}^2$	bounded, simple region	$\int_{\partial R} \vec{F} d\vec{l} \Leftrightarrow \iint_R \mathrm{rot}\vec{F} dx$
Stokes's	$\mathbb{R}^3$	surface with boundary	$\iint_{\mathcal{S}} \vec{F} d\vec{S} \Leftrightarrow \int_{\partial\mathcal{S}} \vec{F} d\vec{l}$
Gauss's	$\mathbb{R}^3$	admissible region	$\underline{\iiint_R \mathrm{div}\vec{F} dx} \Leftrightarrow \underline{\iint_{\partial R} \vec{F} d}\vec{A}$

Table: Summary

Practice - Choose the right theorem!



$$\operatorname{rot} F = \begin{vmatrix} \vec e_1 & \vec e_2 & \vec e_3 \\ \partial x & \partial y & \partial z \\ 3yx^2+z^3 & y^2 & 4yx^2 \end{vmatrix} = 4x^2 \vec e_1 - (8xy - 3z^2)\vec e_2 + -3x^2 \vec e_3 = \boxed{\begin{pmatrix} 4x^2 \\ 3z^2 - 8xy \\ -3x^2 \end{pmatrix}}$$

TASK

Evaluate $\int_C \vec F d\vec r$ *where* $F = (3yx^2 + z^3, y^2, 4yx^2)^T$ *and* C *is a triangle with vertices* $(0,0,3)$, $(0,2,0)$, $(4,0,0)$, *which has counter clockwise orientation when looking down towards* $x - o - y$ *plane.*

$\vec{GP} = (-4, 2, 0)$ $\quad \vec N = \vec{GP} \times \vec{GQ} = (-6, -12, -8)$

$\vec{GQ} = (-4, 0, 3)$ $\quad W = (x, y, z) \in$ Plane

$\vec{GW} = (x-4, y, z)$. $\quad \vec{GW} \perp \vec N \Rightarrow \langle \vec{GW}, \vec N \rangle = 0 \Rightarrow \left\langle \begin{pmatrix} x-4 \\ y \\ z \end{pmatrix}, \begin{pmatrix} -6 \\ -12 \\ -8 \end{pmatrix} \right\rangle = 0$

$\Rightarrow \boxed{z = 3 - \frac{3}{4}x - \frac{3}{2}y} = g(x,y)$

$f = z - g(x,y) = z + \frac{3}{4}x + \frac{3}{4}y - 3$

$\nabla f = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{2} \\ 1 \end{pmatrix}$, $\quad \iint_C \vec F \cdot d\vec T = \int_0^4 \int_0^{2-\frac{1}{2}x} \left\langle \begin{pmatrix} \\ \\ \end{pmatrix}, \begin{pmatrix} \frac{3}{4} \\ \frac{3}{2} \\ 1 \end{pmatrix} \right\rangle dxdy$

$= -5$

Surface S. $r(x,y) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$

$f(x,y,z) = z - g(x,y)$

$\nabla f = \begin{pmatrix} -g_x \\ -g_y \\ 1 \end{pmatrix}$

$\frac{\nabla f}{\|\nabla f\|}$ is the normal vector of surface S

$r(x,y) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$

$t_x = \begin{pmatrix} 1 \\ 0 \\ g_x \end{pmatrix}, \quad t_y = \begin{pmatrix} 0 \\ 1 \\ g_y \end{pmatrix}$

$\langle \nabla f, t_x \rangle = -g_x + g_x = 0$

$\langle \nabla f, t_y \rangle = 0$

$$\iint_S \vec{h} \cdot d\vec{A} = \iint \langle h, \vec{n} \rangle \, dA$$

$$= \iint \left\langle h, \frac{\nabla f}{\|\nabla f\|} \right\rangle \|\nabla f\| \, dx\,dy$$

$$= \boxed{\iint_{x\,y} \langle h, \nabla f \rangle \, dx\,dy}$$

$$dA = \sqrt{\det \begin{pmatrix} \langle t_x, t_x \rangle & \langle t_x, t_y \rangle \\ \langle t_y, t_x \rangle & \langle t_y, t_y \rangle \end{pmatrix}}$$

$$= \sqrt{1 + g_x^2 + g_y^2}$$

$$\|\nabla f\| = \sqrt{1 + g_x^2 + g_y^2} = dA$$

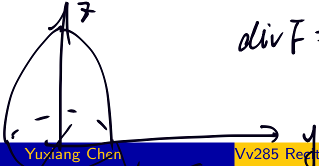
Practice - Choose the right theorem!

TASK

Evaluate $\int_{\mathcal{C}} \vec{F} d\vec{r}$ *where* $F = (3yx^2 + z^3, y^2, 4yx^2)^T$ *and* $\mathcal{C}$ *is a triangle with vertices* $(0,0,3),\ (0,2,0),\ (4,0,0)$, *which has counter clockwise orientation when looking down towards* $x - o - y$ *plane.*

TASK

Evaluate $\iint_{\mathcal{S}} \vec{F} d\vec{S}$ *where* $F = (2xz, 1 - 4xy^2, 2z - z^2)^T$ *and* $\mathcal{S}$ *is the surface of the solid bounded by* $z = 6 - 2x^2 - 2y^2$ *and plane* $z = 0$



$\operatorname{div} F = 2 - 8xy$

$\iint_S \vec{F} d\vec{S} = \iiint (2-8xy)\, dA$

$r \in [0, \sqrt{3}]$, $\theta \in [0, 2\pi]$

$z \in [0, 6-2r^2]$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} \left(2 - 8r\cos\theta \, r\sin\theta\right) \cdot r \, dz \, dr \, d\theta$$

$$= 18\pi$$

Green's Identity

Remember the following relation:

$$\int_R \langle \nabla u, \nabla v \rangle \, dx = -\int_R u \cdot \Delta v dx + \int_{\partial R^*} u \frac{\partial v}{\partial n} dA$$

where R is an admissible region in $\mathbb{R}^n$ and u, v are both continuously differentiable potential functions in $\mathbb{R}^n$.

TASK u, v potentials

Prove $\operatorname{div}(u \nabla v) = u \Delta v + \langle \nabla u, \nabla v \rangle$

$u \quad \nabla v$

$\mathbb{R}^n \to \mathbb{R} \quad \mathbb{R}^n \to \mathbb{R}^n$

$$u \nabla v = \begin{pmatrix} u \frac{\partial v}{\partial x_1} \\ u \frac{\partial v}{\partial x_2} \\ \vdots \\ u \frac{\partial v}{\partial x_n} \end{pmatrix} = F$$

$$\operatorname{div}(u \nabla v) = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} = \frac{\partial (u \cdot \frac{\partial v}{\partial x_1})}{\partial x_1} + \cdots + \frac{\partial (u \cdot \frac{\partial v}{\partial x_n})}{\partial x_n}$$

$$= \boxed{\frac{\partial u}{\partial x_1} \cdot \frac{\partial v}{\partial x_1}} + \boxed{u \frac{\partial^2 v}{\partial x_1^2}} + \cdots + \boxed{\frac{\partial u}{\partial x_n} \cdot \frac{\partial v}{\partial x_n}} + \boxed{u \cdot \frac{\partial^2 v}{\partial x_n^2}}$$

$$= \sum_{i=1}^n \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} + u \cdot \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}$$

$$= \langle \nabla u, \nabla v \rangle + u \cdot \Delta v$$

[1] This practice is from Leyang Zhang. Vielen Dank!

Green's Identity

Remember the following relation:

$$\int_R \langle \nabla u, \nabla v \rangle \, dx = -\int_R u \cdot \Delta v dx + \int_{\partial R^*} u \boxed{\frac{\partial v}{\partial n}} dA$$

where R is an admissible region in $\mathbb{R}^n$ and u, v are both continuously differentiable potential functions in $\mathbb{R}^n$.

$$\int_R \langle \nabla u, \nabla u \rangle \, dx > 0$$

TASK

Prove $\operatorname{div}(u\nabla v) = u\Delta v + \langle \nabla u, \nabla v \rangle$

∇u

$= 0$

$\|\nabla u\|^2$

$$\int_R \langle \nabla u, \nabla u \rangle \, dx = -\int_{\substack{R \\ B_1(0)}} u \cdot \Delta u \, dx + \boxed{\int_{\substack{\partial R^* \\ \partial B_1(0)}} u \cdot \frac{\partial u}{\partial n} \, dA}$$

TASK

[1] *Suppose u is a non-constant C^2 function on some open set U containing $\overline{B_1(0)}$. Furthermore, suppose $u = 0$ on $\partial B_1(0)$. Show that* $\int_{B_1(0)} u\Delta u < 0.$

[1] This practice is from Leyang Zhang. Vielen Dank!

References I

- VV285 slides from Horst Hohberger
- Paul's online note
  https://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx
- Vv285 Review 11 from Leyang Zhang