OFFICE HOUR NOW!

I'm behind the screen,
just SCREEN the Mic

Vv285 Recitation Class 2

Yuxiang Chen

May 18, 2022

Outline

$f(x) = k \cdot x \quad \mathbb{R} \to \mathbb{R}$

Add. $L(u \oplus u') = Lu \boxplus Lu'$

$L \quad U \to V$, Homo.. $L(\underbrace{\lambda \odot u}_{\in U}) = \underbrace{\lambda \boxdot L(u)}_{\in V}$

Review for this week - Linear maps

U: $P(5)$, $\dim U = 6$

V: (x_1, x_2, x_3), $\dim V = 3$

1. Homogeneity and additivity. $L(\mathbb{R}^n, \mathbb{R})$
   - F ➢ Is $L(x) = c$, $x \in \mathbb{R}^n$, $c \neq 0$ a linear map?
   - ➢ Why the map $z \mapsto \bar{z}$ is not linear when $\mathbb{C}$ is a complex vector space?

     $\times$ homogeneity $\quad L(\lambda x) = \bar{\lambda} \cdot \bar{x} \quad \bar{\lambda} \neq \lambda$
2. Important linear maps
   - ➢ The orthogonal projection map P_U: U is a subspace of V, $P_U v = u$ ~~$+ w$~~ for every $v \in V$, in which $v = \underbrace{u}_{\in U} + \underbrace{w}_{\in U^\perp}$ s.t. $u \in U$ and $w \in U^\perp$ $\quad P_U v = u$

     projection Theo.
   - ➢ The inclusion map i: U is a subspace of V, $i(u) = u$ for all $u \in U$ (what's its range and kernel?) $\Rightarrow$ $\operatorname{ran} i \subset V$, $\ker i \subset U$

Verification: $v_1 \in V$, $v_2 \in V \Rightarrow v_1 = u_1 + w_1$, $v_2 = u_2 + w_2$

Additivity $\quad P_U(v_1 + v_2) = P_U(\underbrace{u_1 + u_2}_{\in U} + \underbrace{w_1 + w_2}_{\in U^\perp}) = u_1 + u_2 = P_U v_1 + P_U v_2$

Identity: $\mathbb{1} : \mathcal{L}(U, U)$

Q: What's $\ker i$?

i is injective / not surjective if $U \neq V$

P. a. $L \in \mathcal{L}(U, V)$

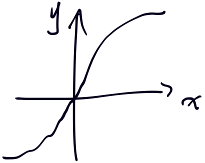
P b. L (function), L is <u>bijective</u>

<u>$a \Leftrightarrow b$</u>

① $a \not\Rightarrow b$

② $\Rightarrow$ injective × surjective

② $b \not\Rightarrow a$



Review for this week - Linear maps

1. Homogeneity and additivity.
   - Is $L(x) = c,\ x \in \mathbb{R}^n,\ c \neq 0$ a linear map?
   - Why the map $z \mapsto \bar{z}$ is not linear when $\mathbb{C}$ is a complex vector space?
2. Important linear maps
   - The orthogonal projection map P_U: U is a subspace of V, $P_U v = u + w$ for every $v \in V$, in which $v = u + w$ s.t. $u \in U$ and $w \in U^\perp$
   - The inclusion map i: U is a subspace of V, $i(u) = u$ for all $u \in U$ (what's its range and kernel?)
3. Theorem that uniquely defines a homomorphism (linear map): define what the basis in the original vector space is mapped into.
4. The space of linear map $\mathcal{L}(U, V)$ is a vector space. (what is the dimension of this space?)
   - The space of *linear functional*: $\mathcal{L}(V, \mathbb{F})$ is called the *dual space* of V, denoted by V^* or V' (notation in Axler)
5. Important: **range and kernel** of $L \in \mathcal{L}(U, V)$
   - In which space lies ran L (ker L) ?

$L \in \mathcal{L}(U, V)$

basis of U $\quad B = \{e_1, \dots e_n\}$, L is unique

$$\begin{matrix} B = \{e_1, & \dots & e_n\} \\ L \downarrow & & \downarrow L \\ (v_1, & \dots & v_n) \end{matrix}$$

$\dim \mathcal{L}(U, V)$?

$\dim U = m$, $\dim V = n$

$\dim \mathcal{L}(U, V) = m \cdot n$

$\mathcal{L}(U, V) \cong \underline{Mat(n \times m ; \mathbb{R})}$

Prove $\mathcal{L}(U, V)$, $\dim U = m$, $\dim V = n$, $\Rightarrow$ $\dim \mathcal{L}(U,V) = m \cdot n$

Intu.. . find a basis for $\mathcal{L}(U,V)$

$B_U = \{e_1, \dots e_n\}$, $B_V = \{b_1, \dots b_m\}$, $\boxed{E_{j,k}(e_j) = b_k}$

$$E_{j,k}(u) = \begin{cases} b_k, & \text{if } u = e_j,\ j \in [1, n] \\ 0, & \text{otherwise} \end{cases} \quad u \in U$$

j: U, k: V

$u = \lambda_j e_j + \lambda_k e_k$

$E_{j,k}(u) = E_{j,k}(\lambda_j e_j + \lambda_k e_k)$

$= \lambda_j b_k + 0$

$E_{1,2}(e_1) = b_2$

① Spans the whole space

- $L \in \mathcal{L}(U,V)$ is defined by $(e_1, \dots e_n) \xrightarrow{L} (v_1, \dots v_n)$

$$\begin{cases} L(e_1) = v_1 = \sum_k \lambda_{1,k} \cdot b_k = \sum_k \lambda_{1,k} E_{1,k}(e_1) \\ \vdots \\ L(e_n) = v_n = \sum_k \lambda_{n,k} b_k = \sum_k \lambda_{n,k} E_{n,k}(e_n) \end{cases}$$

$\Updownarrow$

$$L = \sum_{j=1}^{n} \sum_{k=1}^{u} \lambda_{j,k} E_{j,k}$$

② Independent

$\boxed{E \quad (\quad) \quad b_k}$

$$\boxed{E_{j,k}(e_j) = b_k}$$

$$\sum_k \sum_j \lambda_{j,k} \cdot \underline{\underline{E_{j,k}}} = \underline{\underline{0}}$$

$$\sum_k \sum_j \lambda_{j,k} \underbrace{E_{j,k}(u)}_{\neq 0} = \sum_k \sum_j \lambda_{j,k} E_{j,k}\left(\sum_{i=1}^{n} \mu_i e_i\right)$$

$$= \sum_k \sum_j \lambda_{j,k} \left(\sum_{i=1}^{n} \mu_i E_{j,k}(e_i)\right)$$

$$\begin{cases} i \neq j, & 0 \\ i = j & \sum_k \sum_j \lambda_{j,k}(\mu_j E_{j,k}(e_j)) \end{cases}$$

$$= \sum_k \sum_j \underline{\lambda_{j,k} \cdot \mu_j} \, \underline{\underline{b_k}} = 0$$

$$\lambda_{j,k} \times \mu_j = 0$$

Exercise - Homogeneity or additivity ?

Q1 Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

Q2 Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ s.t.

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but φ is not linear ($\mathbb{C}$ is viewed as a complex vector space).

Exercise - Homogeneity or additivity ?

Q1 Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

Q2 Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ s.t.

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but φ is not linear ($\mathbb{C}$ is viewed as a complex vector space).

$\mathbb{R}$

Hint

For 1, consider $f(x, y) = \sqrt{xy}$; for 2, consider $g(z) = \operatorname{Re} z$

Review for this week - Isomorphisms

invertible

1. Definition: $L \in \mathcal{L}(U, V)$ is a bijective linear map
   - Property: L maps a basis to another basis (is the reverse still valid ?)
   - $\dim U = \dim V$, i.e, U and V are isomorphic .
2. Important: **Dimension formula** $\dim \ker L + \dim \operatorname{ran} L = \dim U$

T **T** or **F** $L \in \mathcal{L}(U, V)$ is injective iff it's surjective. $\boxed{\dim U = \dim V}$

T **T** or **F** $L \in \mathcal{L}(U, V)$, then $U = (\ker L) \oplus (\operatorname{ran} L)$. $\in U$ $\in V$

T **T** or **F** $L \in \mathcal{L}(U, V)$, and $\dim U > \dim V$, then L can't be injective.

T **T** or **F** $L \in \mathcal{L}(U, V)$, and $\dim U < \dim V$, then L can't be surjective

T **T** or **F** $L \in \mathcal{L}(U, V)$ and $e_1, \ldots, e_n$ spans U, then $(Le_1, .., Le_n)$ spans ran L

$\forall v \in V$, $\quad v = \sum_i \lambda_i e_i$ (not unique)

$L(v) = \sum_i \lambda_i L(e_i) \subset \operatorname{span}(Le_1, \ldots Le_n)$

Review for this week - Isomorphisms

1. Definition: $L \in \mathcal{L}(U, V)$ is a bijective linear map
   - Property: L maps a basis to another basis (is the reverse still valid ?)
   - $\dim U = \dim V$, i.e, U and V are isomorphic .
2. Important: **Dimension formula**

T or **F** $L \in \mathcal{L}(U, V)$ is injective iff it's surjective.
T or **F** $L \in \mathcal{L}(U, V)$, then $U = (\ker L) \oplus (\operatorname{ran} L)$.
T or **F** $L \in \mathcal{L}(U, V)$, and $\dim U > \dim V$, then L can't be injective.
T or **F** $L \in \mathcal{L}(U, V)$, and $\dim U < \dim V$, then L can't be surjective
T or **F** $L \in \mathcal{L}(U, V)$ and $e_1, ..., e_n$ spans U, then $(Le_1, .., Le_n)$ spans $\operatorname{ran} L$

3. Quick proof: Suppose $T \in \mathcal{L}(\mathbb{F}^4, \mathbb{F}^2)$ such that $\ker T = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2,\ x_3 = 7x_4\}$. Show that T is surjective

$$\underbrace{\dim \ker T}_{=2} + \underbrace{\dim \operatorname{ran} T}_{=2} = 4$$

Exercise - range and kernel

Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{ran} S \subseteq \ker T$$

Prove that $T \circ S = 0$, where 0 denotes the 0 map.

$v \in V \quad T \circ S(v) = T(\underbrace{Sv}_{\in V \in \ker T}) = 0$

Exercise - Injectiveness and surjectiveness

pre-determined

Suppose $v_1, \ldots, v_m$ are distinct vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(\overset{\in \mathbb{F}}{z_1}, \ldots, z_m) = z_1 v_1 + \cdots z_m v_m \in \operatorname{span}\{v_1, \ldots, v_m\}$$

$$= 0 \iff z_1 = \cdots z_m = 0$$

(a) What property of T makes $v_1, \ldots, v_m$ span V?

(b) What property of T ensures the linear independence of $v_1, \ldots, v_m$?

Exercise - Injectiveness and surjectiveness

(a) T is surjective. This is because for any $w \in V$, we must find some $\{\lambda_1, ..., \lambda_m\} \subseteq \mathbb{F}$ such that

$$w = T(\lambda_1, ..., \lambda_m) = \lambda_1 v_1 + \cdots + \lambda_m v_m$$

Therefore, the list $\{v_1, ..., v_m\}$ spans V.
(b) T is injective. In this case, $T(z_1, ..., z_n) = 0$ implies that $z_1 = ... = z_n = 0$. Equivalently $\sum_{i=1}^{n} z_i v_i = 0$ implies that $z_1 = ... = z_n = 0$. Therefore, $v_1, .., v_n$ are linearly independent.

Exercise - Riesz representation

linear functional : $f \in \mathcal{L}(V, \mathbb{F})$

Example

① $\mathcal{P}(\mathbb{R})$. $\varphi(p) = \int_0^1 p(x)\,dx$, $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$

② In $\mathbb{R}^2$, $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x_1, x_2) = x_1$

③ Dual Space. $\mathcal{L}(V, \mathbb{F}) := V^*$ (V')

(?) $\dim V^* = \dim V$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ D.I.Y

$T \in \mathcal{L}(U, V)$ $T' \in \mathcal{L}(V^*, U^*)$, $T'(\varphi) = T \circ \varphi$

Prove that suppose V is finite-dimensional inner product space and φ is a linear functional on V. Then there is a unique vector $u \in V$ such that

$$\varphi(v) = \langle u, v \rangle \qquad B_V = \{e_1, \dots e_n\}$$

for every v.

$$v = \sum_i \langle e_i, v \rangle e_i$$

Exercise - Riesz representation

- $\boxed{\text{Existence:}}$
  let $\{e_1, .., e_n\}$ be an orthonormal basis for V, then we have

$$\varphi(v) = \varphi(\sum_i \langle e_i, v\rangle\, e_i) = \sum_i \langle e_i, v\rangle\, \varphi(e_i)$$

$$= \left\langle \sum_i \overline{\varphi(e_i)} e_i, v \right\rangle \Rightarrow u := \sum_i \overline{\varphi(e_i)} e_i$$

- Uniqueness:
  Suppose $w \neq u$, and $\langle v, w\rangle = \varphi(v) = \langle v, u\rangle$, then $\langle v, u - w\rangle = 0$ for all v, which implies that $u - w = 0$, i.e, $u = w$.

Operator Norm

$\{ L : L \text{ is invertible} \}$ → open? close? bounded?

$\|L_1\| \quad \|L_2\|$?

Different

Properties (you need to remember this!), suppose $T \in \mathcal{L}(U, V)$

1. $\|Tu\|_V \le \|T\| \|u\|_U$ for $u \in U$ ②
2. $\|L_2 \circ L_1\| \le \|L_2\| \cdot \|L_1\|$

Def. $\|T\| := \sup\limits_{u \in U} \dfrac{\|Tu\|_V}{\|u\|_U}$ ← norm in V, ← norm in U

② $L_1 \in \mathcal{L}(X, W)$, $L_2 \in \mathcal{L}(W, Y)$

$L_2 \circ L_1 \in \mathcal{L}(X, Y)$

$\exists\, x \in X$ s.t. $\|(L_2 L_1) x\|_Y = \cancel{\|L_2 L_1\| \|x\|_X}$

$= \|L_2 (\underbrace{L_1 x}_{\in W})\| \xrightarrow{②} \le \|L_2\| \|L_1 x\|_W \le \|L_1\| \|L_2\| \cancel{\|x\|_X}$

$L \in \mathcal{L}(U, V)$ is bounded when $\dim U < \infty$

Theorem (1.1)

(fine with $\dim V = \infty$)

Suppose U and V are normed vector space, $\dim U < \infty$ and $L \in \mathcal{L}(U, V)$. Define $\|L\| = \sup\{\|Le\| : \|e\| = 1\}$ for $e \in U$, then L is **continuous** *and $\|L\|$ is* **bounded**.

Theorem (1.2)

Suppose U and V are normed vector space and $L \in \mathcal{L}(U, V)$, if L is continuous, then $\|L\|$ is bounded. (we do not require either U or V is finite dimensional)

Reminder

You're encouraged to learn the proof of these two, but it's not **necessary**. *As long as you remember this conclusion, it'll be helpful to your exams and understandings of behavior of linear maps.*

Proof for Theorem (1.1)

Proof.

First, we select a norm for U. Suppose $\boxed{\{e_1, .., e_n\}}$ is a basis for U and $u = \sum_i \lambda_i e_i$, then $\|u\|_1 := \sum_i |\lambda_i|$. Using the same u, we have:

$$\underbrace{\|Lu\|_V}_{\in V} = \left\| L\left(\sum_i \lambda_i e_i\right) \right\|_V = \sum_i |\lambda_i| \underbrace{\|Le_i\|_V}_{\in V} \le \boxed{\sum_i |\lambda_i|} \cdot C = C \|u\|_1$$

$$C := \max\{\|Le_1\|_V, ..., \|Le_n\|_V\}$$

$|\lambda_1| \cdot \|Le_1\| + |\lambda_2| \cdot \|Le_2\| \cdots$

which proves L is bounded when U is endowed with $\|\cdot\|_1$. However, since $\dim U < \infty$, by the equivalence of norms (will be taught in chapter 2), there exist $D > 0$ such that $\boxed{\|u\|_1 \le D \|u\|_U.}$ Therefore, $\|Lu\|_V \le C \cdot D \|u\|_U$, showing L is bounded.

$\dim \operatorname{ran} L + \dim \ker L = \dim U < \infty$

Some Preliminaries

restrict $\underline{\underline{L_r}} \in \mathcal{L}(\mathcal{U}, \operatorname{ran} L)$

$\underline{L_r}(u) = \underline{Lu}$

Lemma (2.1)

Suppose $T \in \mathcal{L}(X, Y)$, then T is continuous at 0 implies continuity everywhere on X.

2.1.

T continuous at 0 $\Leftrightarrow \forall \epsilon$, $\exists \delta$ such that $\|\boxed{(x-y)}^{\in \mathcal{U}} - 0\| = \boxed{\|x-y\| \leq \delta}$ implies $\|T(x \overset{u}{-} y) - T(0)\| = \boxed{\|Tx - Ty\| \leq \epsilon}$, which reads off the continuity at any $x \in X$. □

Proof for Theorem (1.2)

$\|Tx\| \leq \cancel{\frac{\epsilon}{\delta}}\ \varepsilon \cdot \delta$

Theorem (1.2)

Suppose $T \in \mathcal{L}(X, Y)$, then T is continuous if and only if T is bounded.

$\|T\frac{x}{\delta}\| \leq \varepsilon \quad \Longleftarrow \quad \|\frac{x}{\delta}\| \leq 1$

1.2.

($\Rightarrow$) First, suppose T is continuous, then at the origin

$$\|Tx\| = \|T(x) - \overset{0}{T(0)}\| \leq \epsilon \quad \text{for} \quad \|x\| = \|x - 0\| \leq \delta$$

$\epsilon \cdot \delta$

We simply re-scale x so that $\|Tx\| \leq \frac{\epsilon}{\delta}$ for $\|x\| \leq 1$, then T is bounded.

($\Leftarrow$) Then, suppose T is bounded, then $\|Tx\|_Y \leq \|T\| \|x\| \to 0$ as $\|x\| \to 0$, this shows T's continuity at origin. By lemma (2.1), we know T is continuous. $\square$

$\|Tx\|_Y \leq \|T\| \cdot \|x\|_X$

Don't let your hands get cold !

$$\int \sqrt{x^2 + a^2} dx \quad \text{where } a \in \mathbb{R}$$

References I

- VV186 slides from Horst Hohberger
- VV285 slides from Horst Hohberger
- Linear Algebra Done Right from Axler Homepage Vv 285
- StackExchange https://math.stackexchange.com/questions/365900/if-u-is-finite-dimensional-then-operator-norm-is-finite
- Linear Functional Analysis for Scientists and Engineers by Balmohan V.Limaye. Access link from Springer: https://link.springer.com/chapter/10.1007/978-981-10-0972-3_3