Vv285 Recitation Class 5
Basic Topology & Derivative

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Outline

Equivalence of Norms - Exercise

Show directly that the following two norms are equivalent on $\boxed{\mathbb{R}^n:}$

$$\|(x_i)\|_2 = \sqrt{\sum_i x_i^2}$$

$$\|(x_i)\|_\infty = \max\{|x_i| : 1 \le i \le n\}$$

$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ denote $x_k := \max\{x_1, \cdots x_n\}$

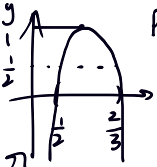
$\le \sqrt{\underbrace{x_k^2 + \cdots + x_k^2}_{n}} = \sqrt{n} \cdot x_k = \sqrt{n} \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_\infty$

$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} \ge \sqrt{x_k^2} = \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_\infty$

Judge the Followings

[1]Suppose X is a complete $\boxed{\text{normed}}$ space and $S_1, S_2, \dots, S_n$ is a finite collection of subsets of X. Judge T or F:

- T ➢ If S_i's are open, then $\bigcup_i S_i$ is open.
- T ➢ If S_i's are open, then $\bigcap_i S_i$ is open.
- T ➢ If S_i's are closed, then $\bigcup_i S_i$ is closed.
- T ➢ If S_i's are closed, then $\bigcap_i S_i$ is closed.
- F ➢ If Y is a normed space and $f : X \to Y$ is continuous, then $f(O)$ is open whenever $O \subseteq X$ is open.
- T ➢ Suppose $K \subseteq X$ is compact and X is a normed space. Any continuous functions $f : K \to X$ is uniformly continuous.



pre-img $(\frac{1}{2}, \frac{1}{3})$

img $(\frac{1}{2}, 1]$

[1]This question is provided by Leyang Zhang. Vielen Dank!

① take $e \in \bigcup_i S_i \Rightarrow e \in S_k$ (at least)

Since S_k open, $B_\delta(e) \subseteq S_k \subseteq \bigcup_i S_i$

$\bigcup_i S_i$

② take $e \in \bigcap_i S_i \Rightarrow (e \in S_1) \wedge (e \in S_2) \dots \wedge (e \in S_n)$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{B_{\delta_1}(e) \in S_1 \quad B_{\delta_2}(e) \in S_2 \ldots \quad B_{\delta_n}(e) \in S_n}$$

$\delta := \frac{\min\{\delta_1, \dots \delta_n\}}{2} \Rightarrow (B_\delta(e) \subseteq S_1) \wedge (B_\delta(e) \subseteq S_2) \dots \wedge (B_\delta(e) \subseteq S_n)$

$\Rightarrow B_\delta(e) \subseteq \bigcap_i S_i \Rightarrow$ open

If $n \to \infty$ $\delta_1 = \frac{1}{1}, \delta_2 = \frac{1}{2}, \dots \delta_n = \frac{1}{n}, \dots$ $\delta_n \xrightarrow{n\to\infty} 0$

③. S is closed and V is the whole space

$V \setminus S$ open

$$O_i^c = S_i$$

$$\boxed{\bigcup_i S_i} = (S_1 \cup S_2 \cdots \cup S_n) = (O_1^c \cup O_2^c \cdots \cup O_n^c)$$

→ infilite

$$= \boxed{(\underbrace{O_1 \cap O_2 \cdots \cap O_n}_{\text{open}})^c} \rightarrow \text{close}$$

$$\bigcap_i S_i = (S_1 \cap S_2 \cdots \cap S_2) = (O_1^c \cap O_2^c \cdots \cap O_2^c)$$

$$= \boxed{(\underbrace{O_1 \cup O_2 \cdots \cup O_n}_{\text{open}})^c} \rightarrow \text{close}$$

$=$

Open Set: O set, $\forall e \in O$, $B_\delta(e) \subseteq O$

close Set: $V \setminus O$, O open set

closeure: $\underset{\sim}{\bar{S}}$ S

compact set:

K is compact

$\Rightarrow$ K is closed and bound

Topology on space of linear maps

Let X be a complete normed vector space. Then the set:

endomorphism

$$\underline{\mathrm{GL}(X) := \{L \in \mathscr{L}(X,X)}: L^{-1} \text{exists}\}$$

Show that this set is open. Further more, show that if $\boxed{\|L\| \leq 1}$, then
operator norm
$\mathbb{I} - L \in \mathrm{GL}(X)$ and

$$\boxed{(\mathbb{I} - L)^{-1} = \boxed{\sum_{n=0}^{\infty} L^n}}$$

where $L^0 := \mathbb{I}$ which is the identity.

① $\sum_{n=0}^{\infty} L^n$ is absolutely convergent

$\sum_{n=0}^{\infty} \|L^n\| \le \sum_{n=0}^{\infty} \|L\|^n$ ($\|L_1 L_2\| \le \|L_1\| \cdot \|L_2\|$) $\underline{\|L\| < 1}$

$= \frac{1}{1-\|L\|} < C$ (bound for some C)

Show $\sum_{n=0}^{\infty} L^n$ have a limit (find limit)

$\mathbb{1} \leftarrow \boxed{\mathbb{1} - L^{n+1}} = (\mathbb{1} - L)(\mathbb{1} + L + \cdots + L^n)$

$\mathbb{1} - L^{n+1} = (\mathbb{1} + L + \cdots + L^n)(\mathbb{1} - L)$

want to $\mathbb{1} - L^{n+1} \xrightarrow{n\to\infty} \mathbb{1}$ $\qquad \|\mathbb{1} - L^{n+1}\| \xrightarrow{n\to\infty} 1$ ⊗

$\|(\mathbb{1} - L^{n+1}) - \mathbb{1}\| \xrightarrow{n\to\infty} 0 \iff \|-L^{n+1}\| \le \|L\|^{n+1} \xrightarrow{n\to\infty} 0$

($\|L\| < 1$)

So. $\mathbb{1} - L^{n+1} \xrightarrow{n\to\infty} \mathbb{1}$

$\Rightarrow \quad \mathbb{1} = (\mathbb{1} - L)\cdot\left(\sum_{n=0}^{\infty} L^n\right)$ (right & left both exist)

$\mathbb{1} = \left(\sum_{n \geq 0}^{\infty} L^n\right)(\mathbb{1} - L)$

$\Rightarrow \underbrace{(\mathbb{1} - L)^{-1} = \sum_{n \geq 0}^{\infty} L^n}$ $\quad (\|L\| < 1)$

2. Show $GL(X)$ is open

suppose $L_0 \in GL(X)$, there $\exists\, \delta > 0$, s.t.

$B_\delta(L_0) \subset GL(X)$ [What we want to show]

$\boxed{B_\delta(L_0) = \{ L \in \mathcal{L}(X, X) : \|L - L_0\| < \delta \}}$

Fix L_0 and we magically set $\delta \in (0, 1/\|L_0^{-1}\|)$

$\|LL_0^{-1} - \mathbb{1}\| = \|LL_0^{-1} - L_0 L_0^{-1}\| = \|(L - L_0)\cdot L_0^{-1}\|$

$\leq \|L - L_0\| \cdot \|L_0^{-1}\| < \delta \|L_0^{-1}\| < 1$

$$\underbrace{\quad}_{<\delta}$$

$$\underbrace{\|LL_0^{-1} - \mathbb{1}\| < 1} \longrightarrow \|\mathbb{1} - LL_0^{-1}\| < 1$$

$\underbrace{\mathbb{1} - (LL_0^{-1} - \mathbb{1})}$ has an inverse

$\mathbb{1} - (\mathbb{1} - LL_0^{-1}) = \underbrace{L}L_0^{-1}$ has an inverse

inverse

$\Rightarrow L$ is invertible $\Rightarrow L \in GL(X)$

$\Rightarrow B_\delta(L_0) \subseteq GL(X)$

$\Rightarrow GL(X)$ is open. $\square$.

Judge the Followings about the First Derivative

$g(x)=x^3$, $g(x+h)=(x+h)^3=x^3+3x^2h+3xh^2+h^3$

$=g(x)+3x^2h+3xh^2+o(h)$

$\neq 3x_1^2h+3x_2^2h$

$Df|_x(h)=3x^2h$ $\qquad Df|_x(h)$ $\qquad Df(x_1+x_2)(h)=3(x_1^2+x_2^2)\cdot h$

$Df: x\mapsto Df|_x$ $\quad Df(x_1)h=Df|_{x_1}h=3x_1^2h$ $\quad Df(x_2)(h)=Df|_{x_2}h=3x_2^2h$

Suppose X, V are finite-dimensional vector spaces and $\Omega\in X$ an open set. $f: X\to V$ is any function:

$\boxed{Df}$ $x\mapsto Df|_x$

- T ➢ $Df|_x$ is a linear map. $\quad f(x+h)=f(x)+Df|_x(h)+o(h)$
- F ➢ $Df: x\mapsto Df|_x$ is a linear map. $\quad Df:\Omega\to\mathcal{L}(X,V)$ $\quad D^2f|_x\in\mathcal{L}(X,\mathcal{L}(X,V))$
- T ➢ $D: C^1(\Omega,V)\to C(\Omega,\mathcal{L}(X,V))$, $\quad f\mapsto Df$ is a linear map

$\mathbb{R}\to\mathbb{R}$. $f(x)=x^2$. $\boxed{f'(x)=2x}$ $\qquad f(x)=e^x$, e^x $\qquad 1-$

$f(x+h)=(x+h)^2=x^2+2xh+h^2$ $\qquad Df|_x(h)=2xh$

$=f(x)+Df|_x(h)+o(h)$ $\qquad =(2x)\cdot h$

$Mat(3\times3:\mathbb{R})\to Mat(3\times3:\mathbb{R})$ $\qquad D(f_1+f_2)=Df_1+Df_2$

$D(\lambda\cdot f_1)=\lambda\cdot Df_1$

Calculate Derivatives

$f_1 = x^2. \quad f_2 = e^x$

x^2

$(f_1+f_2)' = f_1' + f_2'$

$Df_1 \quad Df_2.$

[2]Suppose $A \in \mathrm{Mat}(n \times n\colon \mathbb{F})$, calculate their first derivatives:

1. $f(x) = \langle x, Ax\rangle$
2. $g(A) = \mathrm{tr}(A^2)$
3. $\cancel{p(x) = (Ax)}^2$

$p(A) = A^2$

$f: \mathbb{R}^n \to \mathbb{F}$

$g: \mathrm{Mat}(n\times n ; \mathbb{F}) \to \mathbb{F}$

$p: \mathrm{Mat}(n\times n ; \mathbb{F}) \to \mathrm{Mat}(n\times n , \mathbb{F})$

$\langle \frac{h}{\|h\|}, Ax\rangle$

[2]This question is from Leyang Zhang. Vielen Dank!

$$f(x+h) = \langle x+h, A(x+h)\rangle = \langle x+h, Ax+Ah\rangle = \langle x+h, Ax\rangle + \langle x+h, Ah\rangle$$

$$= \langle x, Ax\rangle + \langle h, Ax\rangle + \langle x, Ah\rangle + \langle h, Ah\rangle$$

$$= f(x) + \langle h, Ax\rangle + \langle x, Ah\rangle + \underbrace{\langle h, Ah\rangle}_{o(h)}$$

$$\frac{\langle h, Ah\rangle}{\sqrt{\langle h, h\rangle}} = \frac{\langle h, Ah\rangle}{\|h\|} = \langle h, A\frac{h}{\|h\|}\rangle = \langle h, \underline{A\overset{1}{e}}\rangle \xrightarrow{h\to 0} 0$$

$$\begin{cases} Df|_x = \langle (\cdot), Ax\rangle + \langle x, A(\cdot)\rangle \\ Df|_x h = \langle h, Ax\rangle + \langle x, Ah\rangle \end{cases}$$

$$g(x+h) = tr(A+h)^2 = tr[(A+h)(A+h)]$$

$$= tr[A^2 + Ah + hA + h^2]$$

$$= trA^2 + \underbrace{tr(Ah) + tr(hA)}_{Df|_A(h)} + \underbrace{tr\,h^2}_{o(h)}$$

$$Df|_A h = \boxed{2\,tr(Ah)}$$

References I

- Practice questions from Leyang Zhang
- VV285 slides from Horst Hohberger
- VV286 slides from Horst Hohberger