Vv285 Recitation Class 9
Potential & Vector Calculus

Yuxiang Chen

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Outline

Gradient and Directional Derivatives

$f: \mathbb{R}^n \to \mathbb{R}$ is called a **scalar function** or **potential**. Its first derivative is a map in the dual space of $\mathbb{R}^n$, i.e, $(\mathbb{R})^*$. We suppose f is differentiable.

1. The transpose of the Jacobian of f is called **gradient**:

$$\nabla f(x) := (J_f(x))^T = (\frac{\partial f}{\partial x_1}|_x, ..., \frac{\partial f}{\partial x_n}|_x)^T$$

   Note that the gradient is a vector in $\mathbb{R}^n$.

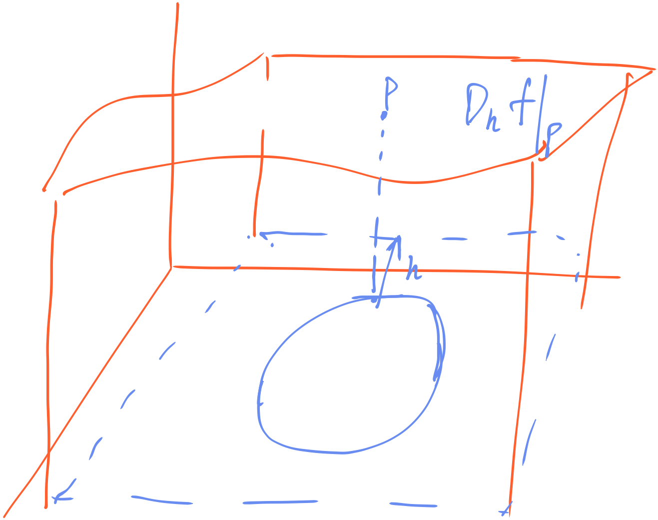
2. Suppose $h \in \mathbb{R}^n$ and $\|h\| = 1$. The directional derivative is defined as:

In $\mathbb{R}^2$, $h \in \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$

$$D_h f|_x := \boxed{\frac{d}{dt} f(x + th)}|_{t=0} = \langle \underbrace{\nabla f(x), h}_{\text{in } \mathbb{R}^2} \rangle$$

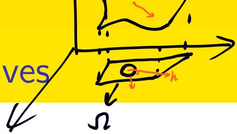
$h = \dfrac{\sum_{i=1}^{n} \lambda_i e_i}{\| h \|}$

3. Normal derivative is a special kind of directional derivative. Suppose $\mathcal{S}$ is a hypersurface in $\mathbb{R}^n$, $p \in \mathcal{S}$ and denote the normal vector at p as $N(p)$. The **normal derivative** is defined as $\frac{\partial f}{\partial n}|_p := D_{N(p)}|_p$



p
$D_h f|_p$
h

Exercise - Gradient and Directional Derivatives



TASK

1. $f(x, y) = xe^{xy} + y$. *Calculate the gradient at* (x_0, y_0) *and the directional derivative at* $\boxed{(2, 0)}$, *which is in direction of* $\theta = \frac{2\pi}{3}$.
2. *Calculate the normal derivative at* $p = (2, 2)$, *which is on the circle* $x^2 + y^2 = 4$.

$$\nabla f\Big|_{(x_0,y_0)} = \begin{pmatrix} (x_0y_0+1)e^{x_0y_0} \\ x_0y_0e^{x_0y_0}+1 \end{pmatrix} \leftarrow$$

$$n = \begin{pmatrix} \cos(\frac{2}{3}\pi) \\ \sin(\frac{2}{3}\pi) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$D_n f\Big|_{(2,0)} = \langle \nabla f\Big|_{(2,0)}, n \rangle = \langle \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \rangle = \frac{5\sqrt{3}-1}{2}$$

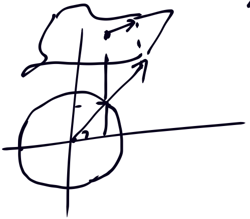
$$D_h f\Big|_{(2,0)} = \frac{d}{dt} f(x+th_1,\ y+th_2)\Big|_{t=0}$$

$$= \frac{d}{dt}\, f\left(2+\left(-\frac{t}{2}\right),\ \frac{\sqrt{3}}{2}\cdot t\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\left(2-\frac{t}{2}\right)e^{(2-t)\frac{\sqrt{3}}{2}t} + \frac{\sqrt{3}}{2}t\right)\Big|_{t=0}$$

$$= \left[-\frac{1}{2}e^{\sqrt{3}t-\frac{\sqrt{3}}{4}t^2} + \left(2-\frac{t}{2}\right)e^{\sqrt{3}t-\frac{\sqrt{3}}{4}t^2}\cdot\left(\sqrt{3}-\frac{\sqrt{3}}{2}t\right)+\frac{\sqrt{3}}{2}\right)\Big|_{t=0}$$

$$= \frac{5\sqrt{3}-1}{2}$$



$$h_2 = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) = \left(\cos\frac{\pi}{4},\ \sin\frac{\pi}{4}\right)$$

$$\frac{\partial f}{\partial n}\Big|_{(2,2)} = \left\langle \nabla f\big|_{(2,2)},\ h_2\right\rangle$$

$$= \frac{\sqrt{2}}{2}\left(9e^4+1\right)$$

Exercise - Gradient and Directional Derivatives

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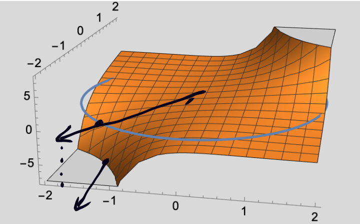


Figure: 1

Line Integral for Vector Fields

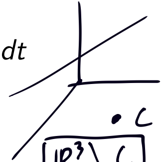
$\Omega \subset \mathbb{R}^n$

$F: \Omega \to \mathbb{R}^n$

1. Given $F: \Omega \to \mathbb{R}^n$, $\Omega \in \mathbb{R}^n$ a continuous vector fields and $\boxed{\mathcal{C}^*}$ an **oriented** open, smooth curve in $\mathbb{R}^n$, the line integral of F along $\mathcal{C}^*$ is given by:

   $\mathbb{R} \to \mathbb{R}^n$

$$\int_{\mathcal{C}^*} F d\vec{l} = \int_{\mathcal{C}^*} \langle F, T \rangle \, dl = \boxed{\int_I \langle F \circ \gamma(t), \gamma'(t) \rangle \, dt}$$

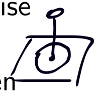


   where γ is a parameterization of curve $\mathcal{C}^*$.

$\cdot \, C$

$\boxed{\mathbb{R}^3 \setminus C}$

2. F is **conservative** if $\oint_{\mathcal{C}^*} F d\vec{l} = 0$ for any closed curve.
3. F is a **potential field** if there exists a differentiable potential function U s.t. $F(x) = \nabla U(x)$.
4. Potential fields are automatically conservative fields, but not vise versa.

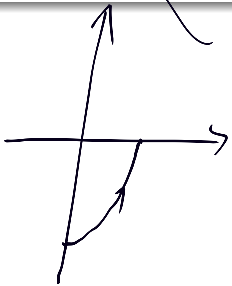


5. If F is a potential field defined on a **connected open set**, then $\boxed{\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}}$. The converse is true when F is defined on a **simply connected open set**.

Practice - Line Integral

TASK

Given $F(x,y) = (x+y, 1-x)^T$, $\mathcal{C}^*$ *is the portion of* $\frac{x^2}{4} + \frac{y^2}{9} = 1$ *that is in the forth quadrant with counter clockwise orientation. Evaluate* $\int_{\mathcal{C}^*} F \cdot d\vec{l}$.



$$\gamma(t) = \begin{pmatrix} 2\cos t \\ 3\sin t \end{pmatrix}, \quad t \in [-\tfrac{\pi}{2}, 0]$$

$$\gamma'(t) = \begin{pmatrix} -2\sin t \\ 3\cos t \end{pmatrix}$$

$$\int_{C^*} F \cdot d\vec{l} = \int_{-\frac{\pi}{2}}^{0} \underbrace{\begin{pmatrix} 2\cos\theta + 3\sin\theta \\ 1 - 2\cos\theta \end{pmatrix}}_{F} \cdot \begin{pmatrix} -2\sin\theta \\ 3\cos\theta \end{pmatrix} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} -4\sin\theta\cos\theta - 6 + 3\cos\theta \, d\theta$$

$= 5 \cdot 5 \| $

Practice - Line Integral

TASK

Given $F(x,y) = (x+y, 1-x)$, $\mathcal{C}^*$ *is the portion of* $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ *that is in the forth quadrant with counter clockwise orientation. Evaluate* $\displaystyle\int_{\mathcal{C}^*} F \cdot d\vec{l}$.

TASK

Evaluate $\int_{\mathcal{C}^*} F \cdot d\vec{l}$ *where* $F(x,y) = (\boxed{2x^3y^4 + x}, 2x^4y^3 + y)$ *and* $\mathcal{C}^*$ *is given by a parameterization* $\gamma(t) = (t\cos(\pi t) - 1, \sin(\frac{\pi t}{2}))$, $t \in [0,1]$

$\dfrac{\partial F_1}{\partial y} = 8x^3y^2 = \dfrac{\partial F_2}{\partial x} = 8x^3y^2 \Rightarrow \underline{\underline{F}}$ is a potential field start $\gamma(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, end $\gamma(1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\nabla u = F \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = F_1 = 2x^3y^4 + x \\[2ex] \dfrac{\partial u}{\partial y} = F_2 = 2x^4y^3 + y \end{cases}$$

$$\int \frac{\partial u}{\partial x} = u = \int 2x^3y^4\,dx = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + C(y)$$

$$\int \frac{\partial u}{\partial y} = u = \int (2x^4y^3 + y)\,dy = \frac{1}{2}x^4y^4 + \frac{1}{2}y^2 + C(x)$$

$$u = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$$

$$\int_{C^*} F \cdot d\vec{r} = u(\gamma(1)) - u(\gamma(0))$$

$$= 10$$

C^*, F

~~Circulation~~ and Divergence

Rotation

Circulation: $\int_{C^*} \langle F, T \rangle \, d\ell$

Flux: $\int_{C^*} \langle F, N \rangle \, dA$

$\Omega \in \mathbb{R}^n$ and $F: \Omega \to \mathbb{R}^n$ is a continuously differentiable vector field.

1. Divergence of F: $\Omega \to \mathbb{R}$, $\underline{\operatorname{div} F} := \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \langle \nabla, F \rangle$, which represents the flux density.
2. ~~Circulation~~ of F (in 3 dimension):

   rotation

   $A^T = -A$

   $\mathbb{R}^2$, 1-d

   $\mathbb{R}^3$, 3-d

   $\mathbb{R}^4$, 6-d

$$\operatorname{rot} F = \nabla \times F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

which represents the circulation density. For 2-D, $\operatorname{rot} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Green's Theorem

Only in $\mathbb{R}^2$, $R \subseteq \mathbb{R}^2$ be a bounded and simple region, then

$$\boxed{\int_{\partial R^*} F d\vec{l}} = \boxed{\int_R \boxed{\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)} dx}$$

where ∂R^* denotes the boundary of R in **counter-clockwise** orientation.

parametrize $\partial R^* \rightarrow \gamma(t)$

$$\int_I \langle F \circ \gamma, \gamma'(t) \rangle dt$$

Green's Theorem

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$$\int_{\partial R^*} F d\vec{l} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx$$

where ∂R^* denotes the boundary of R in **counter-clockwise** orientation.

TASK

Evaluate $\oint_{\mathcal{C}^*} xy\ dx + x^2y^3\ dy$ *where* $\mathcal{C}$ *is the triangle with vertices* $(0,0)$, $(1,0)$, $(1,2)$ *with positive orientation.*

Reminder

Similarly, we can calculate flux using the same way:

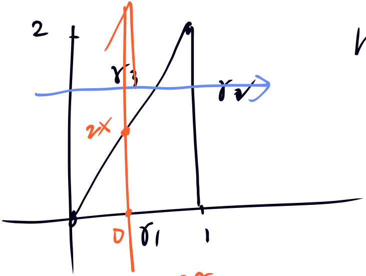
$$\int_{\partial R^*} \langle F, N \rangle\, dl = \int_R \operatorname{div} F dx$$

$$\oint_{C^*} xy\,dx + x^2y^3\,dy$$

$$\bar{F} = \begin{pmatrix} xy \\ x^2y^3 \end{pmatrix}$$

$$\text{rot } \bar{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= 2xy^3 - x$$



$0 \le y \le 2x$

$$\int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$

$$= \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx$$

$$= \int_0^1 8x^5 - 2x^2\,dx = \frac{2}{3}$$

References I

- VV285 slides from Horst Hohberger
- Paul's online note
  `https://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx`