浙江大学 2020-2021 学年 秋冬 学期

《离散数学》课程期末考试试卷

课程号： 21120401 开课学院： 计算机学院

考试试卷： ☑ A卷 ☐ B卷

考试形式： ☑ 闭卷 ☐ 开卷，允许带 ________入场

考试日期: 2021 年 1 月 26 日，考试时间: 120 分钟

诚信考试，沉着应考，杜绝违纪

考生姓名________________ 学号________________ 所属院系________________

题序	1	2	3	4	5	6	7	总分
得分								
评卷人								

ZHEJIANG UNIVERSITY

DISCRETE MATHEMATICS, FALL-WINTER 2020

FINAL EXAM

1. **(20 pts) Determine whether the following statements are true or false. If it is true fill a $\surd$ otherwise a $\times$ in the bracket before the statement.**

   (a) (✓) Let A, B and C be arbitrary sets. If $A - C \subseteq B - C$, then $A \cup C \subseteq B \cup C$.

   (b) (✓) Let A, B be two sets. If $\rho(A) \subseteq \rho(B)$, then $A \subseteq B$, where $\rho(X)$ is the power set of X.

   $(\neg \forall x P(x)) \wedge Q \equiv (\exists x \neg P(x)) \wedge Q$

   (c) (✗) Let $P(x)$ be a predicate, then $\forall x P(x) \to Q \Leftrightarrow \forall x(P(x) \to Q)$, where Q is independent of x.

   $\equiv \forall x(\neg P(x) \wedge Q)$

   (d) (✗) The poset $(\{1, 2, 4, 8, 12, 16, 32\}, |)$ is a lattice(格), where $x \mid y$ denote x divides y.

   ✗(e) ($\cancel{\checkmark}$) Let $(S, \preceq)$ be a partially ordered set, if there is unique maximal element a of S, then a is the greatest element of S. $A \cup N$

   +18

   (f) (✓) If the following assignments 000, 011 and 110 make the propositional formula φ false, then φ can be converted in full conjunctive normal form $\Pi(0, 3, 6)$.

   (g) (✗) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$ is countably infinite.

   (h) (✓) If there are 800 people in a room then at least 3 of them are guaranteed to have the same birthday.

   (i) (✗) All simple complete graphs with at least 3 vertices are Euler graphs.

   (j) (✓) In a binary tree with n vertices and l leaves, then $2 \cdot l \leq n + 1$.

   $n_0 + n_2 \leq n$ $n_0 = n_2 + 1$

   $2l - 1 \leq n$

   $2l \leq n + 1$

2. **(12 pts)** On mathematical logic

Construct arguments to prove that the following reasoning is valid.

Hypothesis: $\neg p \vee q \to r$, $s \vee \neg q$, $\neg t$, $p \to t$, $\neg p \wedge r \to \neg s$

Conclusion: $\neg q$

+12

① $\neg t$ (H)
② $p \to t$ (H)
③ $\neg p$ (Modus Tollens)
④ $\neg p \vee q \to r$ (H)
⑤ r (Modus Ponens)
⑥ $\neg p \wedge r$ (Rules of conjunction)
⑦ $\neg p \wedge r \to \neg s$ (H)
⑧ $\neg s$ (Modus Ponens)
⑨ $s \vee \neg q$ (H)
⑩ $\neg q$ (Disjuctive syllogism)

3. **(10 pts)** On Infinite sets

Let A be an arbitrary infinite set, B be a countably infinite set, and $A \cap B = \emptyset$. Prove that sets A and $A \cup B$ have the same cardinality.

若A为可数无穷集，则可同时对A、B集合的元素编号

然后构造一映射$f: A \longrightarrow A \cup B$，满足$f(a_{2k}) = a_k$, $f(a_{2k+1}) = b_k$

若A为不可数无穷集，则以一定方式取出一可数无穷集$C \subset A$，并对C作如上映射，对$A-C$作恒等映射

+10

4. **(12 pts)** On graph

Let G be a simple graph with n vertices and k connected components.

(a) What is the minimum possible number of edges of G? $n-k$

(b) What is the maximum possible number of edges of G? C_{n-k+1}^2

证明：平衡法

+12

5. **(24 pts)** ON SET AND RELATION

Let A be a set with n elements and $B = \{a, b, c\}$.

(a) How many different symmetric relations on A? $2^{\frac{n(n+1)}{2}}$

(b) How many different anti-symmetric relations on A? $2^n \cdot 3^{\frac{n(n-1)}{2}}$

(c) How many both symmetric and antisymmetric binary relations on A are there? 2^n

(d) How many different equivalence relations are there on B ? 5

(e) How many different partial order relations are there on B? ~~16~~ 19

(f) Is there a binary relation R on B such that R is both an equivalence relation and a partial order? Either give an example, or show that no such R exist.

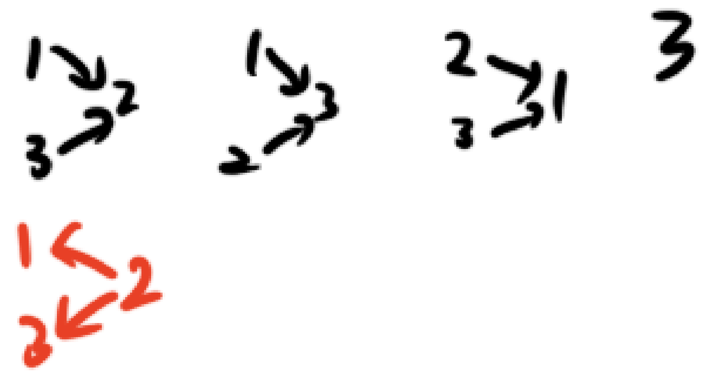
Justify your answer, but you don't need to give a formal proof.

+20

1 2 3   1

1→2 3
1→3 2
2→1 3
2→3 1
3→1 2
3→2 1   6

1→2→3 ×6



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. **(10 pts)** ON TREE

Suppose that T is a tree of 9 vertices with a vertex of 6 degrees.

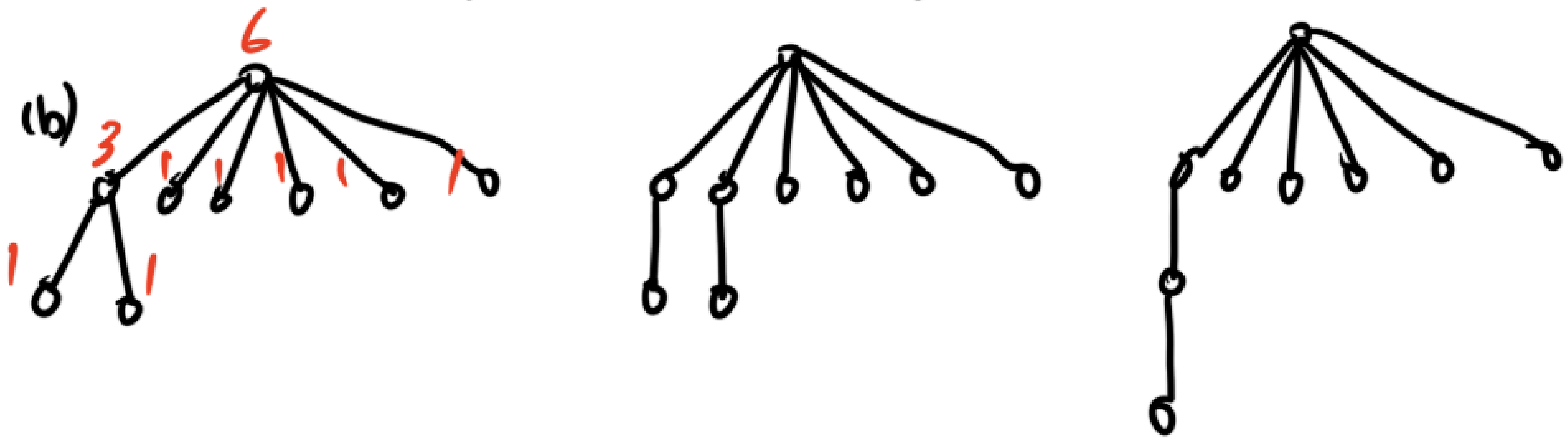
(a) What degree sequences can T have?

(b) Draw all non-isomorphic trees of 9 vertices with a vertex of 6 degrees.

+10

(a)度数之和为16 1+1+1+1+1+1+1+3+6
1+1+1+1+1+1+2+2+6

(b)



7. **(12 pts)** On counting

Let b_n denote the number of binary strings of length n that contain 101 as a substring and $B(x) = \sum_{n=1}^{\infty} b_n x^n$.

(a) Determine the value of b_1, b_2, b_3, b_4, b_5.

(b) Derive an explicit closed-form expression for $B(x)$.

Hint: You might want to set up recurrence relation for the appropriate sequences.

(a) f_{ij}：长度为i的串与101有j位匹配且未匹配成功，特别地：$f_{i,3}$表示长度为i的串与101有且仅有一次匹配成功，且出现在末尾

则 $f_{i+1,0} = f_{i,0} + f_{i,2}$

$f_{i+1,1} = f_{i,0} + f_{i,1}$

$f_{i+1,2} = f_{i,1}$

$f_{i+1,3} = f_{i,2}$

矩阵为 $\begin{bmatrix} f_{i+1,0} \\ f_{i+1,1} \\ f_{i+1,2} \\ f_{i+1,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{i,0} \\ f_{i,1} \\ f_{i,2} \\ f_{i,3} \end{bmatrix}$

$b_n = \sum_{k=1}^{n} f_{k,3} \times 2^{n-k}$

+6

则

j \ i	0	1	2	3	4	5	6	7	8
0	1	1	1	2	4	7	12	21	37
1	0	1	2	3	5	9	16	28	49
2	0	0	1	2	3	5	9	16	28
3	0	0	0	1	2	3	5	9	16

即 $b_1 = 0$

$b_2 = 0$

$b_3 = 1$

$b_4 = 4$

$b_5 = 11$

(b) $b_n = 2b_{n-1} - b_{n-2} + b_{n-3} + 2^{n-3}$, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$ (OEIS出来的)

$b_n - 2^n = 2(b_{n-1} - 2^{n-1}) - (b_{n-2} - 2^{n-2}) + (b_{n-3} - 2^{n-3})$

令 $a_n = b_n - 2^n$，则 $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$

特征方程 $r^3 - 2r^2 + r - 1 = 0$

d_n：长度为n，以0打头

e_n：长度为n，以1打头

$d_1 = e_1 = d_2 = e_2 = 0$

$d_n = \overset{00}{d_{n-1}} + \overset{01}{e_{n-1}}$

$e_n = \underset{11}{e_{n-1}} + \underset{100}{d_{n-2}} + \underset{101}{2^{n-3}}$

则 $b_n = d_{n+1}$

$D(x) = \sum_{n\ge3} d_n x^n \quad E(x) = \sum_{n\ge3} e_n x^n \quad B(x) = \frac{1}{x} D(x)$

$\sum_{n\ge3} d_n x^n = \sum_{n\ge3} d_{n-1} x^n + \sum_{n\ge3} e_{n-1} x^n$

$D(x) = xD(x) + xE(x) = xD(x) + \frac{x}{1-x}\left[x^2 D(x) + \frac{x^3}{1-2x}\right] = xD(x) + \frac{x^3}{1-x} D(x) + \frac{x^3}{1-2x}$

$E(x) = xE(x) + x^2 D(x) + \sum_{n\ge3} 2^{n-3} x^n$

$= xE(x) + x^2 D(x) + x^3 \sum_{n\ge0} (2x)^n$

$= xE(x) + x^2 D(x) + \frac{x^3}{1-2x}$

$E(x) = \frac{1}{1-x}\left[x^2 D(x) + \frac{x^3}{1-2x}\right]$

$1 - x - \frac{x^3}{1-x} = \frac{x^2 - 2x + 1 - x^3}{1-x}$

$\frac{1 - 2x + x^2 - x^3}{1-x} D(x) = \frac{x^3}{1-2x}$

$D(x) = \frac{x^3(1-x)}{(1-2x+x^2+x^3)(1-2x)}$

Enjoy Your Spring Festival!

$B(x) = \frac{x^2(1-x)}{(1-2x+x^2+x^3)(1-2x)}$