Spatial predictions

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Review of last lecture

- Spatial linear regression model for univariate point-referenced spatial data
- Modeling unknown surfaces with Gaussian Processes
- MLE, choosing initial values of spatial parameters using the semivariogram
- Kriging: Predictions at new locations
- Out of sample prediction
- Model comparison: AIC, BIC, RMSPE, CP, CIW
- Analysis in R

Modeling with GPs

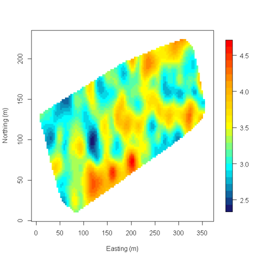
Spatial linear model

$$y(s) = x(s)'\beta + w(s) + \epsilon(s)$$

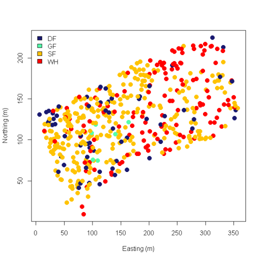
- Spatial random effects $w(s)$ modeled as $GP(0, C(\cdot \,|\, \theta))$ (usually without a nugget)
- $\epsilon(s) \overset{\text{iid}}{\sim} N(0, \tau^2)$ is the measurement error
- $y(s)$ becomes a GP with mean $x(s)'\beta$ and covariance function C_1 such that $C_1(s_i, s_j|\theta) = C(s_i, s_j|\theta) + \tau^2 Ind(s_i = s_j)$
- $y = (y(s_1), \ldots, y(s_n))' \sim N(X\beta, C(\theta) + \tau^2 I)$

Western Experimental Forestry (WEF) data

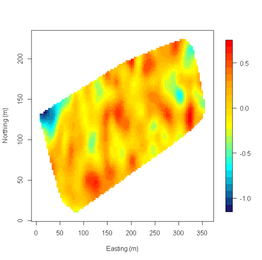
- Data consist of a census of all trees in a 10 ha. stand in Oregon
- Response of interest: log(Diameter at breast height), i.e., log(DBH)
- Covariate: Tree species (Categorical variable)



DBH

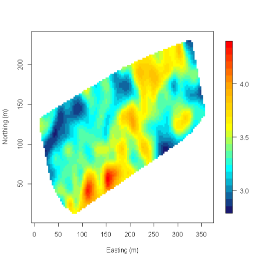


Species

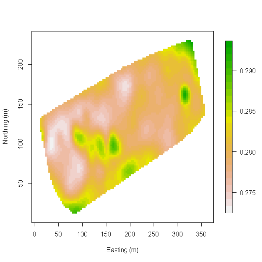


Lin. reg. Residuals

WEF data: Kriged surfaces after spatial regression



DBH Predictions



Variances

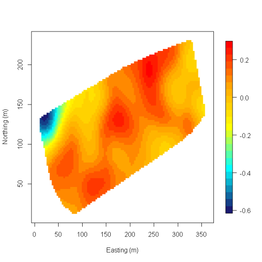
Recovering the spatial random effects

- The kriging used in last class was for predicting $y(s_0)|Y$
- The spatial random effects $w(s)$ was introduced to account for spatial variation not explained by the covariates
- Recovering the spatial surface $w(s)$ gives us an idea about what covariates may be missing
- Often of critical importance for scientists
- How do we predict $w(s_0)|Y$?

Recovering the spatial random effects

- Note that $Y = X\beta + w + \epsilon$
- $Cov(w(s_0), Y) = Cov(w(s_0), w) = (c(s_1, s_0|\theta), \ldots, c(s_n, s_0|\theta))'$
- Also, both Y and $w(s_0)$ are jointly normal
- $w(s_0)|Y \sim N(c'(C + \tau^2 I)^{-1}(Y - X\beta), c(s_0, s_0|\theta) - c'(C + \tau^2 I)^{-1} c)$
- Recall that when $s_0 \notin (s_1, s_2, \ldots, s_n)'$, then $y(s_0)|Y \sim N(x(s_0)'\beta + c'(C + \tau^2 I)^{-1}(Y - X\beta), c(s_0, s_0|\theta) + \tau^2 - c'(C + \tau^2 I)^{-1} c)$
- Setting $signal = TRUE$ and $trend.l = 0$ in $krige.conv$ will predict $w(s_0)$ instead of $y(s_0)$

Predicted $w(s)$ surface for WEF data



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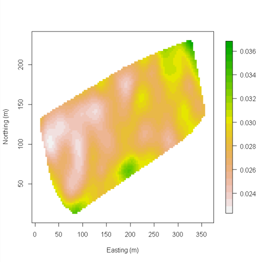
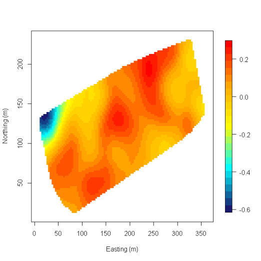


Variances of $w(s)$

Predicted $w(s)$ surface for WEF data



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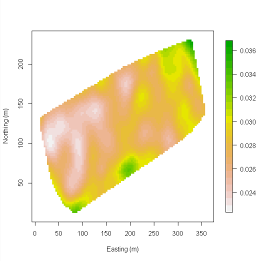
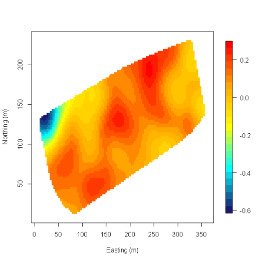


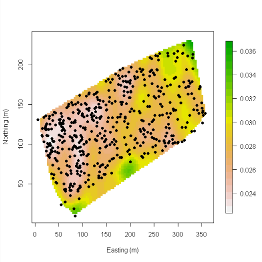
Variances of $w(s)$

Note the strong resemblance of the predicted surface with the residual surface from the non-spatial regression

Predicted $w(s)$ surface for WEF data



Predicted $w(s)$



Variances of $w(s)$ and data locations

Areas of high variance are usually located away from the data locations

The prediction variance

- Which part of the prediction variance in kriging comes from uncertainty in the covariates, i.e., from the linear regression?
- Which part is due to the random effects $w(s)$ and which is simply noise variance ?

The prediction variance

- Which part of the prediction variance in kriging comes from uncertainty in the covariates, i.e., from the linear regression?
- Which part is due to the random effects $w(s)$ and which is simply noise variance ?
- In linear regression (no spatial term), prediction at a new location is $\hat{y}_{new} = x'_{new}\hat{\beta} + \epsilon_{new}$
- So, $\widehat{var(\hat{y}_{new})} = \widehat{var(x'_{new}\hat{\beta})} + \widehat{var(\epsilon_{new})} = \hat{\tau}^2(x'_{new}(X'X)^{-1}x_{new} + 1)$
- The contributions from the covariates and noise are easily separable
- Can we do it for the spatial linear model?

The prediction variance

- $y(s_0)|Y \sim N(x(s_0)'\beta + c'(C(\theta) + \tau^2 I)^{-1}(Y - X\beta), c(s_0, s_0|\theta) + \tau^2 - c'(C + \tau^2 I)^{-1} c)$

The prediction variance

- $y(s_0)|Y \sim N(x(s_0)'\beta + c'(C(\theta) + \tau^2 I)^{-1}(Y - X\beta), c(s_0, s_0|\theta) + \tau^2 - c'(C + \tau^2 I)^{-1}c)$
- Rewrite this as:
  $y(s_0)|Y = x'\beta + E(w(s_0)|Y) + \eta(s_0) + \epsilon(s_0)$
  where $\eta(s_0) \sim N(0, c(s_0, s_0|\theta) - c'(C + \tau^2 I)^{-1}c)$ and
  $\epsilon(s_0) \sim N(0, \tau^2)$

The prediction variance

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- $\widehat{var}(y(s_0)|) = \widehat{var}(x(s_0)'\hat{\beta}) + \widehat{var}(E(w(s_0)|Y)) + c(s_0, s_0) - c'(C + \tau^2 I)^{-1}c + \tau^2$

The prediction variance

- $y(s_0)|Y \sim N(x(s_0)'\beta + c'(C(\theta) + \tau^2 I)^{-1}(Y - X\beta), c(s_0, s_0|\theta) + \tau^2 - c'(C + \tau^2 I)^{-1} c)$
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- $\widehat{var}(y(s_0)|) = \widehat{var}(x(s_0)'\hat{\beta}) + \widehat{var}(E(w(s_0)|Y)) + c(s_0, s_0) - c'(C + \tau^2 I)^{-1} c + \tau^2 + 2\widehat{cov}(x(s_0)'\hat{\beta}, E(w(s_0)|Y))$

The prediction variance

- $y(s_0)|Y \sim N(x(s_0)'\beta + c'(C(\theta) + \tau^2 I)^{-1}(Y - X\beta), c(s_0, s_0|\theta) + \tau^2 - c'(C + \tau^2 I)^{-1}c)$
- Rewrite this as:
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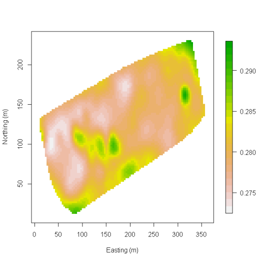
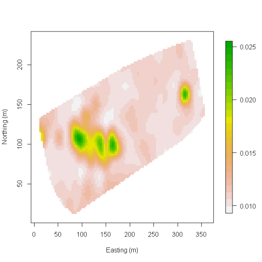
  where $\eta(s_0) \sim N(0, c(s_0, s_0|\theta) - c'(C + \tau^2 I)^{-1}c)$ and $\epsilon(s_0) \sim N(0, \tau^2)$
- $\widehat{var}(y(s_0)|) = \widehat{var}(x(s_0)'\hat{\beta}) + \widehat{var}(E(w(s_0)|Y)) + c(s_0, s_0) - c'(C + \tau^2 I)^{-1}c + \tau^2$
  $+2\widehat{cov}(x(s_0)'\hat{\beta}, E(w(s_0)|Y))$
- The blue, green and red components are used to understand variation due to covariates, spatial effects and noise

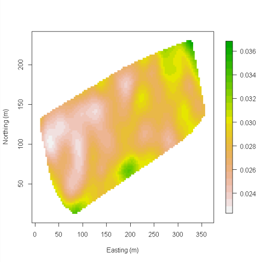
Breaking down prediction variances for WEF data



total

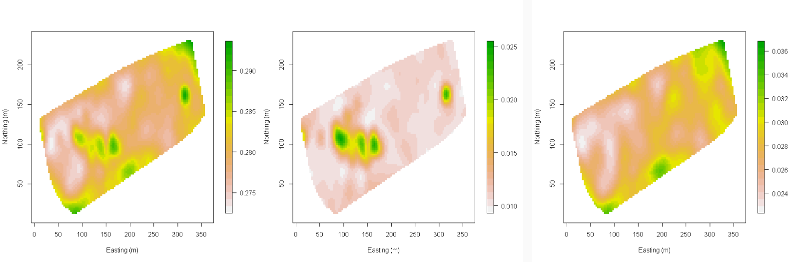


covariates



$w(s)$

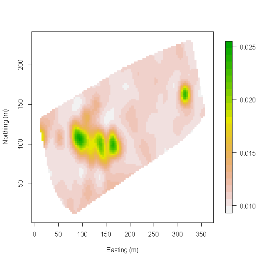
Breaking down prediction variances for WEF data

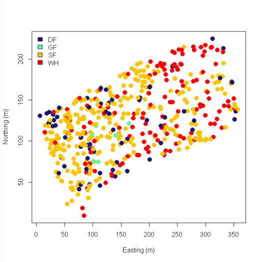


total covariates $w(s)$

- Noise part is simply τ^2 for every location
- What's going on in the patches for the variance from the covariates?

Breaking down prediction variances for WEF data

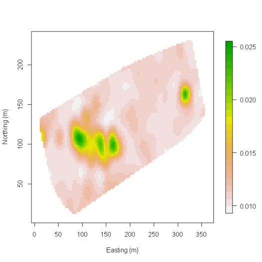




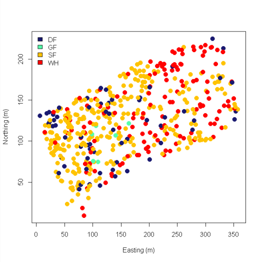
covariates

Species

Breaking down prediction variances for WEF data



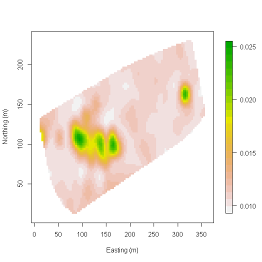
covariates



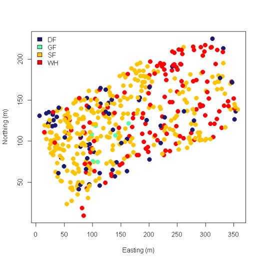
Species

- The in-sample data contains only a handful (6 out of 500) of datapoints for species GF
- Small subsample size $\Rightarrow$ high variance for the regression coefficient corresponding to GF

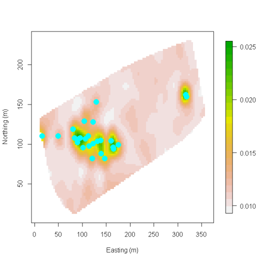
Breaking down prediction variances for WEF data



covariates



Species



locations of GF

- The in-sample data contains only a handful (6 out of 500) of datapoints for species GF
- Small subsample size ⇒ high variance for the regression coefficient corresponding to GF
- Areas of high variance coincide with spots having GF species