Probability

(1) Trial and event:- Let an experiment be repeated under essentially the same conditions & let it result in any one of the several possible outcomes. Then, the experiment is called trial and the possible outcomes are known as events or cases.

E.g. Throwing a die is a trial and getting 1 or 2 or 3 or 4 or 5 is an event

or Performing a random experiment is trial & outcome is termed as event.

(2) Exhaustive events:- The total number of all possible outcomes in any trial is known as exhaustive events

E.g. In throwing a dice, there are 6 exhaustive cases, for any one of the six faces may turn up.

In general, in throwing n dice, the exhaustive cases are 6^n.

(3) Favourable events:- The cases which entail the happening of an event are said to be favourable events. It is the total number of possible outcomes in which the specified event happens.

E.g. In a throw of two dice, the number of favourable cases to getting the sum 6 is 5, ie.
(1,5); (5,1); (3,3); (2,4); (4,2).

(4) Sample Space:- Set of all possible outcomes of a random experiment is called sample space.

E.g. Sample space of random experiment of tossing 2 coins is $S = \{HH, TH, HT, TT\}$

(5) Mutually Exclusive outcomes!– Two or more ~~events~~ outcomes which cannot happen simultaneously

E.g.! In tossing a coin, the events head and tail are mutually exclusive.

(6) Equally likely events!– Each of the outcome has an equal chance of happening.

E.g.! In tossing a coin, the coming up of head or tail is equally likely.

(7) Independent & dependent events!– ~~If a card is drawn~~

Two or more events are said to be independent if the happening or non-happening of any one does not depend by the happening or non-happening of any other. Otherwise they are said to be dependent.

E.g.! If a card is drawn from a pack of well shuffled cards and replaced before drawing the second card, the result of the second draw is independent of the first draw. However, if the first card drawn is not replaced, then, the second draw is dependent on the first draw.

Classical Definition of Probability:-

If an event A can happen in m ways, & fails in n ways, all these ways being equally likely to occur, then the probability of the happening of A is

$$= \frac{\text{number of favourable cases}}{\text{Total no. of mutually exclusive \& equally likely cases}} = \frac{m}{m+n}$$

and that of its failing is defined as $\frac{n}{m+n}$

If the probability of the happening an event

i.e. $P(E) = p$

Not happening an event is ie $P(\bar{E}) = 1-p$ (say q)

then $p+q = \frac{m}{m+n} + \frac{n}{m+n} = \frac{m+n}{m+n} = 1$

$\Rightarrow p+q = 1$

* **Statistical or Empirical Defn of Probability:-** If an event A happens m times in n trials, then the probability of happening of an event A is given by

$$P(A) = \lim_{n\to\infty} \frac{m}{n}.$$

* **Random Experiments:-** There are experiments in which results may be altogether different even though they are performed under identical conditions. They are known as random experiments.

E.g. Tossing a coin or throwing a die is random experiment

Ques 1] Find the probability of throwing
(a) 5
(b) an even number
with an ordinary six faced die.

Soln (a) There are 6 possible ways in which the die can fall and there is only one way of throwing 5.

$$\text{Probability} = \frac{\text{No. of favourable ways}}{\text{Total no. of equally likely ways}} = \frac{1}{6}$$

(b) Total number of ways falling a die = 6
No. of ways falling 2, 4, 6 = 3
$\therefore$ Req. Prob = $\frac{3}{6} = \frac{1}{2}$.

Q 2]: Find the probability of throwing 9 with two dice.

Soln: Total no. of possible ways of throwing two dice = 6×6 = 36

No. of ways getting 9 ie (3+6); (6+3); (4+5); (5+4) = 4
$\therefore$ The req. prob. = $\frac{4}{36} = \frac{1}{9}$

Q3]:- From a pack of 52 cards, one card is drawn at random. Find the probability of getting a king.

Soln A king can be chosen in 4 ways
But a card can be drawn in 52 ways.
$\therefore$ The required Prob = $\frac{4}{52} = \frac{1}{13}$.

Q4]:- What is the probability that a leap year selected at random will contain 53 sundays.

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Ans:- In leap year, total no. of days = 366
& there are 52 full weeks (Hence 52 sundays)
and two extra days.
These two days:
(1) Monday, Tuesday (2) Tuesday, Wednesday
(3) Wednesday, Thursday (4) Thursday, Friday
(5) Friday, Saturday (6) Saturday, Sunday.
(7) Sunday, Monday.
Of these 7 cases, the last two are favourable.
Hence req. probability $= \frac{2}{7}$.

Axioms of Probability:-

Let S be sample space associated with a random experiment. If $A \subseteq S$ and $P(A)$ be prob. of event A, if following axioms are satisfied.

(i) $P(A) \geq 0$ ~~ie~~.

(ii) $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

where A_i's are pairwise disjoint sets
ie $A_i \cap A_j = \phi$ for $i \neq j$

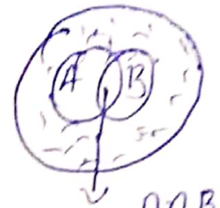
(iii) $P(S) = 1$

Note:- If $\bar{A}$ is complement of set A then $P(\bar{A}) = 1 - P(A)$

Addition Theorem where two events are not mutually exclusive

(i) If A, B are subsets of S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$A \cap B$

(ii) If A, B, C are subsets of S, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

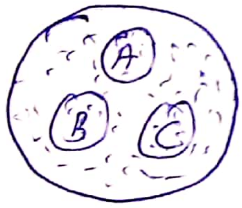
Statement:- If $P_1, P_2, P_3, --, P_n$ be separate probabilities of mutually exclusive events, then the probability P, that any of these events will happen is given by

$$P = P_1 + P_2 + P_3 + -- + P_n$$

Proof:- Let $A, B, C, --$ be the events where probabilities are respectively $P_1, P_2, --, P_n$.

Let n be the total no. of favourable cases to either A or B or C or ---

$= m_1 + m_2 + -- + m_n$



Hence $P(A+B+C+ --) = \dfrac{m_1 + m_2 + -- + m_n}{n}$

$= \dfrac{m_1}{n} + \dfrac{m_2}{n} + -- + \dfrac{m_n}{n}$

$= P(A) + P(B) + --$

$\Rightarrow P = P_1 + P_2 + -- + P_n$

Hence Proved.

Q. An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same colour. (4)

Soln: Probability of drawing two black balls $= \frac{^{10}C_2}{^{20}C_2}$

Probability of drawing two white balls $= \frac{^{10}C_2}{^{20}C_2}$

Probability of drawing two balls of the same colour $= \frac{^{10}C_2}{^{20}C_2} + \frac{^{10}C_2}{^{20}C_2} = 2 \cdot \frac{^{10}C_2}{^{20}C_2}$

$$= 2 \cdot \frac{\frac{10\times 9}{2\times 1}}{\frac{20\times 19}{2\times 1}} = \frac{2\times 10\times 9}{20\times 19} = \frac{9}{19}$$

Eg. 2] Three machines I, II & III manufacture resp. 0.4, 0.5 & 0.1 of the total production. The % of defective items produced by I, II and III is 2, 4, 1 percent respt. For an item chosen at random, what is the prob. it is defective?

Soln:- The defective item produced by machine I $= \frac{0.4\times 2}{100} = \frac{0.8}{100}$

" " " produced by machine II $= \frac{0.5\times 4}{100} = \frac{2}{100}$

" " " " " " III $= \frac{0.1\times 1}{100} = \frac{0.1}{100}$

The total defective items produced by machines I, II, III

$$= \frac{0.8}{100} + \frac{2}{100} + \frac{0.1}{100} = \frac{2.9}{100} = 0.029$$

$\therefore$ Req. Prob $= \frac{0.029}{1} = 0.029$.

Ques A Card is drawn from a well shuffled pack of playing Cards. what is the probability that it is either a spade or an ace?

Solⁿ Let 'A' be an event of drawing a spade & 'B' be an event of drawing an ace.

As A & B are not mutually exclusive events.

$\Rightarrow$ $(A \cap B)$ be the event of drawing the ace of spades.

$$P(A) = \frac{13}{52}, \quad P(B) = \frac{4}{52}, \quad P(A \cap B) = \frac{1}{52}$$

$$\therefore P(A+B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$\Rightarrow \boxed{P(A+B) = \frac{4}{13}}$$

Ques: A Card is drawn from 52 cards at random. Find the prob. that card drawn is a heart or a face or an ace.

Solⁿ:- Let A: getting a heart Card.

B: " " face Card.

C: " an ace "

Req. Prob. $= P(A \cup B \cup C)$ or $P(A+B+C)$

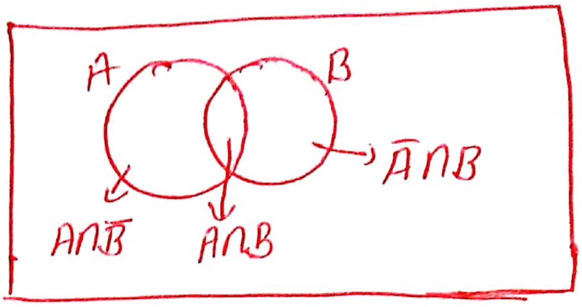
$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{13}{52} + \frac{12}{52} + \frac{4}{52} - \frac{3}{52} - \frac{0}{52} - \frac{1}{52} + \frac{0}{52}$$

$$= \frac{25}{52}$$

Statement:- If A and B are any two events, then
$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

Proof: If A and B are not mutually disjoint sets; then A and $\bar{A}\cap B$ are disjoint sets & their union is $A\cup B$.



$$\Rightarrow A\cup B = A\cup(\bar{A}\cap B)$$
$$\Rightarrow P(A\cup B) = P[A\cup(\bar{A}\cap B)] = P(A) + P(\bar{A}\cap B)$$
$$\Rightarrow P(A\cup B) = P(A) + [\underbrace{P(\bar{A}\cap B) + P(A\cap B)} - P(A\cap B)]$$
$$= P(A) + P[(\bar{A}\cap B)\cup \cancel{P}(A\cap B)] - P(A\cap B).$$
$$= P(A) + P(B) - P(A\cap B) \quad [\because \bar{A}\cap B \ \& \ \bar{A}\cap B \text{ are disjoint sets}]$$
$$\Rightarrow \boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

* By using this above result, we can proved the results for any three events, i.e.

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$$

OR

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).$$

Imp

Theorem 1: $\rightarrow$ Probability of the impossible event is zero i.e. $P(\phi)=0$

Proof: – Impossible event contains no sample point. As such, the sample S and the impossible event ϕ are mutually exclusive.

$\Rightarrow \quad S \cup \phi = S$

$\Rightarrow \quad P(S\cup\phi) = P(S) \quad \Rightarrow P(S)+P(\phi) = P(S)$

$\Rightarrow \quad P(\phi)=0$

Theorem: – Probability of the Complementary Event $\bar{A}$ of A is Given by $P(\bar{A}) = 1-P(A)$.

Proof: $\bar{A}$ & A are disjoint events. Also $A\cup\bar{A}=S$

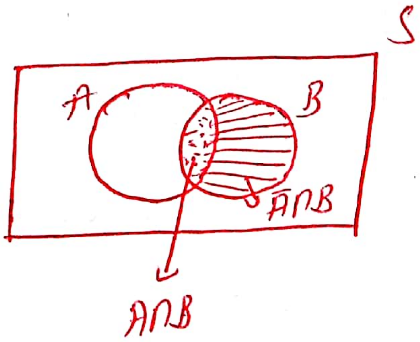
$\therefore \quad P(A\cup\bar{A}) = P(S)$

$\Rightarrow \quad P(A)+P(\bar{A}) = 1$

$\Rightarrow \quad \boxed{P(\bar{A}) = 1-P(A)}$

Theorem: – For any two events A & B, $P(\bar{A}\cap B) = P(B) - P(A\cap B)$

Proof: – As $\bar{A}\cap B = \{p : p\in B \ \& \ p\notin A\}$



Now $\bar{A}\cap B$ & $A\cap B$ are disjoint sets

& $(\bar{A}\cap B)\cup(A\cap B) = B$

$\Rightarrow \quad P[(\bar{A}\cap B)\cup(A\cap B)] = P(B)$

$\Rightarrow \quad P(\bar{A}\cap B) + P(A\cap B) = P(B)$

$\Rightarrow \quad \boxed{P(\bar{A}\cap B) = P(B) - P(A\cap B)}$

Note: – Similarly, we can proved that $P(A\cap\bar{B}) = P(A) - P(A\cap B)$

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Conditional Probability:-

The probability of the happening of an event A when the another event B is known to have already happened is called Conditional Probability & is denoted by $P(A/B)$.

* If ~~the~~ an event A is ~~indepent~~ independent from an event B then $P(A/B) = P(A)$

* Multiplication law of probability (or Theorem of Compound probability).

Statement:- The probability of simultaneous occurence of two events is equal to the probability of one of the events multiplied by the conditional probability of other.

For two events A and B, we have:

$$P(A \cap B) = P(A) \times \underbrace{P(B/A)}$$

represents cond. prob. of occurrence of B when the event A has already happened.

Proof:- Let 'm' be favouriable cases to the happening of an event A in 'n' trials, mutually exclusive & equally likely events.

$$\therefore \quad P(A) = \frac{m}{n} \quad \text{——} \; (1)$$

But out of m outcomes favourable to the happening of A, let m_1 be favourable to the happening of the event B.

$\therefore$ Conditional prob. of B, given that A has happened $= P(B/A) = \frac{m_1}{m}$ — (2)

Now out of n trials, m_1 are favourable to the happening of 'A & B'.

$\therefore$ Prob. of simultaneous occurence of A & B

i.e $P(A \cap B) = \frac{m_1}{n} = \frac{m_1}{m} \times \frac{m}{n}$

$= \frac{m}{n} \times \frac{m_1}{m} = P(A) \times P(B/A)$ [from (1) & (2)]

$\Rightarrow \boxed{P(A \cap B) = P(A) \times P(B/A)}$

Cor:- If A & B are independent events, then

$P(B/A) = P(B)$

$\therefore \boxed{P(A \cap B) = P(A) \times P(B)}$

Example:- 'A' ~~gun~~ can hit a target 4 times in 5 shots; B can hit 3 times in 4 shorts; C 2 times in 3 shots. what is prob. that atleast two shots hit?

Soln: Probability of A's hitting the target $= \frac{4}{5}$

" " B's " " " $= \frac{3}{4}$

" " C's " " " $= \frac{2}{3}$

For atleast two hits, we may have

(i) A, B, C all hit the target, the prob. of which is

$\frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{24}{60}$

(ii) A, B hit the target & C misses it, the prob. for which is $\frac{4}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{3}\right) = \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{12}{60}$

A, C hit the target & B misses it, the prob. for which is

$$\frac{4}{5}\times\left(1-\frac{3}{4}\right)\times\frac{2}{3}=\frac{8}{60}$$

(iv) B, C hit the target & A misses it, the prob. for which is

$$\left(1-\frac{4}{5}\right)\times\left(\frac{3}{4}\right)\times\frac{2}{3}=\frac{6}{60}$$

Since all these events are mutually exclusive events, req. prob.

$$=\frac{24}{60}+\frac{12}{60}+\frac{8}{60}+\frac{6}{60}=\frac{50}{60}=\frac{5}{6}$$

Example: Three groups of children contain resp. 3 girls & 1 boy; 2 girls & 2 boys; 1 girl & 3 boys. One child is selected at random from each group. Find the chance of selecting 1 girl & 2 boys.

Soln: There are 3 ways of selecting 1 girl & 2 boys.

1st way: Girl is selected from 1st group, 1st boy from ~~2nd grp & 2nd group boys~~ 2nd group & 2nd boy from 3rd group.

Prob. of selecting of (Girl, Boy, Boy) $=\frac{3}{4}\times\frac{2}{4}\times\frac{3}{4}=\frac{18}{64}$

2nd way: Boy is selected from 1st group, girl from second group & second boy from 3rd group.

Prob. of selecting (Boy, Girl, Boy) $=\frac{1}{4}\times\frac{2}{4}\times\frac{3}{4}=\frac{6}{64}$

3rd way: Boy is selected from 1st group, 2nd boy from 2nd group

& the girl from the third group

Probability of selecting (Boy, Boy, Girl)

$= \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{2}{64}$

All these events are mutually exclusive events,

Required Probability $= \frac{18}{64} + \frac{6}{64} + \frac{2}{64} = \frac{26}{64} = \frac{13}{32}$.

Baye's Theorem:-

If $B_1, B_2, --, B_n$ are mutually exclusive & exhaustive events with $P(B_i) \neq 0$ $(i=1,2,3,--,n)$ of a random experiment then for any arbitary event A of the sample space with $P(A) > 0$, we have

$$P(B_i/A) = \frac{P(B_i)\, P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A/B_i)}$$

Proof:- Let S be a sample space of the random experiment.

The events $B_1, B_2, --, B_n$ being exhaustive events.

$$S = B_1 \cup B_2 \cup -- \cup B_n$$

As $A \subseteq S$ $\Rightarrow$ $A = A \cap S$

$= A \cap (B_1 \cup B_2 \cup -- \cup B_n)$

$= (A \cap B_1) \cup (A \cap B_2) \cup -- \cup (A \cap B_n)$.

$\Rightarrow P(A) = P(A \cap B_1) + P(A \cap B_2) + -- + P(A \cap B_n)$

$= P(B_1) P(A/B_1) + P(B_2) P(A/B_2) + -- + P(B_n) P(A/B_n)$

$= \sum_{i=1}^{n} P(B_i)\, P(A/B_i)$ —— (1)

Example:- A candidate has to reach exam centre in time. Prob. of his going by bus or scooter or by other means of transport is $\frac{3}{10}, \frac{1}{10}, \frac{3}{5}$ resp. The prob. of his getting late are $\frac{1}{4}$ & $\frac{1}{3}$ resp, if he travels by bus or scooter. But he reaches in time, if he uses any other mode of transportation. If he reached late at the centre. Find the Prob. that he travelled by bus.

Soln: Let E_1: Candidate travels by bus

E_2: " " " scooter.

E_3: Candidate " other mode of transport.

A: candidate reaches late.

Then, $P(E_1) = \frac{3}{10}$, $P(E_2) = \frac{1}{10}$, $P(E_3) = \frac{3}{5}$

$P(A|E_1) = \frac{1}{4}$, $P(A|E_2) = \frac{1}{3}$, $P(A|E_3) = 0$

$\therefore$ Req. Prob $= P(E_1|A)$

$$= \frac{P(E_1)\,P(A|E_1)}{\sum_{i=1}^{3} P(E_i)\,P(A|E_i)}$$

$$= \frac{\left(\frac{3}{10}\right)\left(\frac{1}{4}\right)}{\left(\frac{3}{10}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{10}\right)\left(\frac{1}{3}\right) + 0} = \frac{9}{13}.$$