The Natural Deduction Pack

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1 Using this pack

This pack consists of Natural Deduction problems, intended to be used alongside *The Logic Manual* by Volker Halbach. The pack covers Natural Deduction proofs in propositional logic ($\mathcal{L}_1$), predicate logic ($\mathcal{L}_2$) and predicate logic with identity ($\mathcal{L}_=$). The vast majority of these problems ask for the construction of a Natural Deduction proof; there are also worked examples explaining some connectives' proof strategies in more detail, and one or two more unusual questions about Natural Deduction.

The pack hopefully offers more questions to practice with than any student should need, but the sheer number of problems in the pack can be daunting. For this reason there is also a 'core' set of questions aimed at covering the most crucial skills needed to tackle a Natural Deduction proof.

Next to each problem is a number in brackets indicating the number of steps in my solution. This can be taken as a rough measure of the difficulty of the problem, although it should be emphasised that this is not always perfect: some long proofs can be methodical, while some short proofs can be counter-intuitive.

For each of these problems I provide a proof and an explanation of the strategy behind the proof. I use additional notation to annotate the Natural Deduction proofs in two ways. First, next to each horizontal line in a proof I label which rule has been applied. Where a connective has a pair of introduction rules (such as $\vee$Intro1 and $\vee$Intro2) or a pair of elimination rules (such as $\wedge$Intro1 and $\wedge$Intro2), I only distinguish between the first and second versions in the solutions for the earlier problems; in the later stages it should be clear which version has been applied. Second, for longer proofs, I sometimes use a number to mark both a discharged assumption and the point in the proof when that assumption is discharged. When an assumption has been discharged by $=$Intro I label the assumption with a superscript $^=$. These are not formal components of a proof, but they should help in explaining how the proof has been constructed.

The solutions I provide are never the only possible solutions. Usually I aim to provide the shortest possible solution, but in some cases I also present possible alternative proofs.

At the very end of the pack there are extra problems focused on ways in which Natural Deduction can be altered or extended, either by adding new rules or replacing existing rules.

If this pack is viewed as a PDF, it is possible to click on any problem to go directly to its solution. Clicking on the solution's header will take you back to the problem list. When using a printed copy of the document, these hyperlinks are unlikely to work as intended.

If you have any comments, questions or suggestions, please send them to me at alastair.carr@new.ox.ac.uk.

2 Summary of rules

Propositional logic $\mathcal{L}_1$

Conjunction

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \phi & \psi\end{array}}{\phi \wedge \psi}\wedge\text{Intro} \qquad \dfrac{\begin{array}{c}\vdots\\ \phi \wedge \psi\end{array}}{\phi}\wedge\text{Elim1} \qquad \dfrac{\begin{array}{c}\vdots\\ \phi \wedge \psi\end{array}}{\psi}\wedge\text{Elim2}$$

Implication

$$\dfrac{\begin{array}{c}[\phi]\\ \vdots\\ \psi\end{array}}{\phi \to \psi}\to\text{Intro} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \phi & \phi \to \psi\end{array}}{\psi}\to\text{Elim}$$

Disjunction

$$\dfrac{\begin{array}{c}\vdots\\ \phi\end{array}}{\phi \vee \psi}\vee\text{Intro1} \qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\phi \vee \psi}\vee\text{Intro2}$$

$$\dfrac{\begin{array}{ccc} & [\phi] & [\psi]\\ \vdots & \vdots & \vdots\\ \phi \vee \psi & \chi & \chi\end{array}}{\chi}\vee\text{Elim}$$

Biconditional

$$\dfrac{\begin{array}{cc}[\phi] & [\psi]\\ \vdots & \vdots\\ \psi & \phi\end{array}}{\phi \leftrightarrow \psi}\leftrightarrow\text{Intro}$$

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \phi & \phi \leftrightarrow \psi\end{array}}{\psi}\leftrightarrow\text{Elim1} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \psi & \phi \leftrightarrow \psi\end{array}}{\phi}\leftrightarrow\text{Elim2}$$

Negation

$$\dfrac{\begin{array}{cc}[\phi] & [\phi]\\ \vdots & \vdots\\ \psi & \neg\psi\end{array}}{\neg\phi}\neg\text{Intro} \qquad \dfrac{\begin{array}{cc}[\neg\phi] & [\neg\phi]\\ \vdots & \vdots\\ \psi & \neg\psi\end{array}}{\phi}\neg\text{Elim}$$

Predicate logic $\mathcal{L}_2$

Universal quantifier

$$\dfrac{\begin{matrix}\vdots \\ \phi[t/v]\end{matrix}}{\forall v\phi}\,\forall\text{Intro}$$

provided the constant t does not occur in ϕ or in any undischarged assumption in the proof of $\phi[t/v]$

$$\dfrac{\begin{matrix}\vdots \\ \forall v\phi\end{matrix}}{\phi[t/v]}\,\forall\text{Elim}$$

Existential quantifier

$$\dfrac{\begin{matrix}\vdots \\ \phi[t/v]\end{matrix}}{\exists v\phi}\,\exists\text{Intro}$$

$$\dfrac{\begin{matrix} & [\phi\,[t/v]] \\ \vdots & \vdots \\ \exists v\phi & \psi\end{matrix}}{\psi}\,\exists\text{Elim}$$

provided the constant t does not occur in $\exists v\phi$, or in ψ, or in any undischarged assumption other than $\phi\,[t/v]$ in the proof of ψ

Predicate logic with identity $\mathcal{L}_=$

$$\dfrac{[t=t]}{\vdots}\,{=}\text{Intro}$$

$$\dfrac{\begin{matrix}\vdots & \vdots \\ \phi[s/v] & s=t\end{matrix}}{\phi[t/v]}\,{=}\text{Elim} \qquad \dfrac{\begin{matrix}\vdots & \vdots \\ \phi[s/v] & t=s\end{matrix}}{\phi[t/v]}\,{=}\text{Elim}$$

3 Worked examples

3.1 Implication

We can use the rules for implication and conjunction to prove the following theorem:

$$\vdash (P \to Q) \to ((P \wedge R) \to (Q \wedge R))$$

The easiest way to start is by working from the bottom upwards, especially since we aren't given any premises to work from.

We know that the theorem we want to prove is an implication: it is a statement of the form $\phi \to \psi$. That means we can prove it by assuming ϕ, giving a proof of ψ and then applying $\to$Intro (discharging all of our assumptions of ϕ). Here, ϕ corresponds to $P \to Q$ and ψ corresponds to $(P \wedge R) \to (Q \wedge R)$, so our proof will look like this:

$$\dfrac{\begin{array}{c}[P \to Q] \\ \vdots \\ (P \wedge R) \to (Q \wedge R)\end{array}}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))}\ {\scriptstyle \to\text{Intro}}$$

What is shown above isn't a proof, but a way of helping us put the proof together. We know that we can assume $P \to Q$ as many times as we like, because our final $\to$Intro step will discharge all our assumptions of $P \to Q$.

$(P \wedge R) \to (Q \wedge R)$ is also an implication, so we can prove it by assuming $P \wedge R$ and proving $Q \wedge R$:

$$\dfrac{\dfrac{\begin{array}{c}[P \to Q], [P \wedge R] \\ \vdots \\ Q \wedge R\end{array}}{(P \wedge R) \to (Q \wedge R)}\ {\scriptstyle \to\text{Intro}}}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))}\ {\scriptstyle \to\text{Intro}}$$

We can provide a proof of $Q \wedge R$ by providing a proof of Q and a proof of R and then applying the $\wedge$Intro rule:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[P \to Q], [P \wedge R] \\ \vdots \\ Q\end{array} \quad \begin{array}{c}[P \to Q], [P \wedge R] \\ \vdots \\ R\end{array}}{Q \wedge R}\ {\scriptstyle \wedge\text{Intro}}}{(P \wedge R) \to (Q \wedge R)}\ {\scriptstyle \to\text{Intro}}}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))}\ {\scriptstyle \to\text{Intro}}$$

Now we have two branches to consider. Note that we can use our assumptions of $P \to Q$ and $P \land R$ in both branches: our applications of $\to$Intro discharge all occurrences of $P \to Q$ and $P \land R$ above them in the proof.

We'll consider the right branch first, because it's the more straightforward branch. We can easily obtain R by using our assumption of $P \land R$ and applying $\land$Elim; we don't even need to use our assumption of $P \to Q$.

$$[P \to Q], [P \land R]$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ Q\end{matrix} \qquad \dfrac{[P \land R]}{R}\,{\scriptstyle \land\mathrm{Elim}}}{Q \land R}\,{\scriptstyle \land\mathrm{Intro}}}{(P \land R) \to (Q \land R)}\,{\scriptstyle \to\mathrm{Intro}}}{(P \to Q) \to ((P \land R) \to (Q \land R))}\,{\scriptstyle \to\mathrm{Intro}}$$

The left branch requires two steps. Applying $\land$Elim on $P \land R$ gives us P. Using this P and our assumption of $P \to Q$ allows us to prove Q by $\to$Elim:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P \land R]}{P}\,{\scriptstyle \land\mathrm{Elim}} \qquad [P \to Q]}{Q}\,{\scriptstyle \to\mathrm{Elim}} \qquad \dfrac{[P \land R]}{R}\,{\scriptstyle \land\mathrm{Elim}}}{Q \land R}\,{\scriptstyle \land\mathrm{Intro}}}{(P \land R) \to (Q \land R)}\,{\scriptstyle \to\mathrm{Intro}}}{(P \to Q) \to ((P \land R) \to (Q \land R))}\,{\scriptstyle \to\mathrm{Intro}}$$

This gives us a complete proof.

3.2 Universal quantifier

Using the introduction and elimination rules for the universal quantifier we can construct a proof of the following:

$$\forall x \neg \forall y (Pxy \to Qxy) \vdash \forall x \neg \forall y \neg Pxy$$

Our conclusion is a universal statement, so we can prove it by applying the $\forall$Intro rule. In order to apply the $\forall$Intro rule we need to prove that $\forall x \neg \forall y (Pxy \to Qxy) \vdash \forall x \neg \forall y \neg Pxy$ is true when any arbitrary constant is substituted for x; we will choose a for our arbitrary constant, but we need to ensure that a appears in no undischarged assumptions when we apply the $\forall$Intro rule. This means our proof will look like this:

$$\begin{matrix}\forall x \neg \forall y (Pxy \to Qxy) \\ \vdots \\ \dfrac{\neg \forall y Pay}{\forall x \neg \forall y \neg Pxy}\,{\scriptstyle \forall\mathrm{Intro}}\end{matrix}$$

How do we get from $\forall x\neg\forall y(Pxy \to Qxy)$ to $\neg\forall yPay$? $\neg\forall yPay$ is a negated statement, so we can prove it by assuming $\forall yPay$ and showing it leads to a contradiction.

Where can we find a contradiction? Neither the assumption $\forall yPay$ nor our premise $\forall x\neg\forall y(Pxy \to Qxy)$ has a negation as their main connective, but from $\forall x\neg\forall y(Pxy \to Qxy)$ we can derive (by $\forall$Elim) $\neg\forall y(Pay \to Qay)$, which is a negated statement. This means we can obtain a contradiction if we can somehow derive $\forall y(Pay \to Qay)$ from our assumption of $\forall yPay$ and our premise $\forall x\neg\forall y(Pxy \to Qxy)$.

Note that the assumption $\forall yPay$ contains a, but if we can derive a contradiction from it and successfully apply $\neg$Intro this assumption will be discharged before we apply $\forall$Intro. If it is left undischarged by the time we reach the final step of the proof, we won't be able to apply $\forall$Intro.

$$\forall x\neg\forall y(Pxy \to Qxy), [\forall y\neg Pay]$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall y(Pay \to Qay)\end{array} \qquad \dfrac{\forall x\neg\forall y(Pxy \to Qxy)}{\neg\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Elim}}}{\neg\forall y\neg Pay}\,{\scriptstyle \neg\text{Intro}}}{\forall x\neg\forall y\neg Pxy}\,{\scriptstyle \forall\text{Intro}}$$

To prove $\forall y(Pay \to Qay)$ we need to apply $\forall$Intro, meaning we need to show that the statement is true for any arbitrary constant which could replace y. We cannot choose a as our arbitrary constant, because a appears in $\forall y(Pay \to Qay)$. Instead we will choose b: if we can derive $Pab \to Qab$ without b appearing in any undischarged assumptions, we can apply $\forall$Intro and derive $\forall y(Pay \to Qay)$:

$$\forall x\neg\forall y(Pxy \to Qxy), [\forall y\neg Pay]$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ Pab \to Qab\end{array}}{\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Intro}} \qquad \dfrac{\forall x\neg\forall y(Pxy \to Qxy)}{\neg\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Elim}}}{\neg\forall y\neg Pay}\,{\scriptstyle \neg\text{Intro}}}{\forall x\neg\forall y\neg Pxy}\,{\scriptstyle \forall\text{Intro}}$$

Proving $Pab \to Qab$ is a simple case of assuming Pab and proving Qab. Note that Pab contains b, but we plan to apply $\to$Intro and discharge it before we reach the $\forall$Intro step where we go from arbitrary b to universal y.

$$\forall x\neg\forall y(Pxy \to Qxy), [\forall y\neg Pay], [Pab]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ Qab\end{array}}{Pab \to Qab}\,{\scriptstyle \to\text{Intro}}}{\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Intro}} \qquad \dfrac{\forall x\neg\forall y(Pxy \to Qxy)}{\neg\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Elim}}}{\neg\forall y\neg Pay}\,{\scriptstyle \neg\text{Intro}}}{\forall x\neg\forall y\neg Pxy}\,{\scriptstyle \forall\text{Intro}}$$

We don't have a direct way of proving Qab, but the two assumptions we've made do give us a contradiction. From $\forall y \neg Pay$ we can derive $\neg Pab$, which contradicts our assumption of Pab. From this contradiction we can apply $\neg$Elim and derive Qab. It turns out that in this part of the proof we don't need to use our premise $\forall x \neg \forall y(Pxy \rightarrow Qxy)$ again.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[Pab] \quad \dfrac{[\forall y \neg Pay]}{\neg Pab}\,{\scriptstyle \forall\text{Elim}}}{Qab}\,{\scriptstyle \neg\text{Elim}}}{Pab \rightarrow Qab}\,{\scriptstyle \rightarrow\text{Intro}}}{\forall y(Pay \rightarrow Qay)}\,{\scriptstyle \forall\text{Intro}} \qquad \dfrac{\forall x \neg \forall y(Pxy \rightarrow Qxy)}{\neg \forall y(Pay \rightarrow Qay)}\,{\scriptstyle \forall\text{Elim}}}{\dfrac{\neg \forall y \neg Pay}{\forall x \neg \forall y \neg Pxy}\,{\scriptstyle \forall\text{Intro}}}\,{\scriptstyle \neg\text{Intro}}
$$

This gives us a complete proof, but it's worth verifying at this stage that both of our applications of $\forall$Intro are allowed.

The first time we apply $\forall$Intro, we move from $Pab \rightarrow Qab$ to $\forall y(Pay \rightarrow Qay)$. b doesn't appear in $\forall y(Pay \rightarrow Qay)$ or any undischarged assumptions in the proof of $\forall y(Pay \rightarrow Qay)$; Pab has already been discharged by this point.

The second time we apply $\forall$Intro, we move from $\neg \forall y \neg Pay$ to $\forall x \neg \forall y \neg Pxy$. a doesn't appear in $\forall x \neg \forall y \neg Pxy$ or any undischarged assumptions in the proof of $\forall x \neg \forall y \neg Pxy$; both Pab and $\forall y \neg Pay$ have been discharged by this point.

3.3 Existential quantifier

We can use the introduction and elimination rules for the existential quantifier to construct a proof of the following:

$$\exists x(Px \wedge Qx), \neg \exists x(Qx \wedge Rx) \vdash \exists x(Px \wedge \neg Rx)$$

The first thing to take note of is our existential premise $\exists x(Px \wedge Qx)$. In order to make use of it we need to apply $\exists$Elim at the end of the proof, discharging assumptions where x is instantiated with an arbitrary constant. We'll choose a as our arbitrary constant; we can use it because it doesn't appear in our premises ($\exists x(Px \wedge Qx)$ and $\neg \exists x(Qx \wedge Rx)$) or in the conclusion ($\exists x(Px \wedge \neg Rx)$). We'll also make sure that if a appears in any other assumptions, the assumptions are discharged by the time we apply $\exists$Elim.

This means our proof will take the following form:

$$
\dfrac{\exists x(Px \wedge Qx) \qquad \begin{array}{c}[Pa \wedge Qa], \neg \exists x(Qx \wedge Rx) \\ \vdots \\ \exists x(Px \wedge \neg Rx)\end{array}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{Elim}}
$$

Our conclusion is also an existential statement, so we can prove it by applying $\exists$Intro. There is an infinite number of different statements which we could derive $\exists x(Px \land \neg Rx)$ from ($Pb \land \neg Rb$ and $Pc_{169} \land \neg Rc_{169}$ are two examples) but a is the only constant we have any assumptions about, so it seems likely that we will derive $\exists x(Px \land \neg Rx)$ from $Pa \land \neg Ra$:

$$\dfrac{\exists x(Px \land Qx) \qquad \dfrac{\begin{matrix}[Pa \land Qa], \neg\exists x(Qx \land Rx) \\ \vdots \\ Pa \land \neg Ra\end{matrix}}{\exists x(Px \land \neg Rx)}\,\exists\text{Intro}}{\exists x(Px \land \neg Rx)}\,\exists\text{Elim}$$

This is a conjunction, so using our premise and our assumption we need to provide a proof of Pa and a proof of $\neg Ra$:

$$\dfrac{\exists x(Px \land Qx) \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[Pa \land Qa], \\ \neg\exists x(Qx \land Rx) \\ \vdots \\ Pa\end{matrix} \qquad \begin{matrix}[Pa \land Qa], \\ \neg\exists x(Qx \land Rx) \\ \vdots \\ \neg Ra\end{matrix}}{Pa \land \neg Ra}\,\land\text{Intro}}{\exists x(Px \land \neg Rx)}\,\exists\text{Intro}}{}}{\exists x(Px \land \neg Rx)}\,\exists\text{Elim}$$

On the left-hand side, Pa is very easy to prove: it can be derived from our assumption of $Pa \land Qa$ by $\land$Elim:

$$\dfrac{\exists x(Px \land Qx) \qquad \dfrac{\dfrac{\dfrac{[Pa \land Qa]}{Pa}\,\land\text{Elim} \qquad \begin{matrix}[Pa \land Qa], \\ \neg\exists x(Qx \land Rx) \\ \vdots \\ \neg Ra\end{matrix}}{Pa \land \neg Ra}\,\land\text{Intro}}{\exists x(Px \land \neg Rx)}\,\exists\text{Intro}}{\exists x(Px \land \neg Rx)}\,\exists\text{Elim}$$

On the right-hand side, $\neg Ra$ is a negated statement. This means we can prove it by assuming Ra and deriving a contradiction.

Our other premise $\neg\exists x(Qx \land Rx)$ is a negated statement, so if we can prove $\exists x(Qx \land Rx)$ we have the contradiction we need:

$$\dfrac{\exists x(Px \wedge Qx) \qquad \dfrac{\dfrac{\dfrac{[Pa \wedge Qa]}{Pa}\,{\scriptstyle \wedge\text{E}} \qquad \dfrac{\begin{matrix}[Pa \wedge Qa], [Ra],\\ \neg\exists x(Qx \wedge Rx)\\ \vdots\\ \exists x(Qx \wedge Rx)\end{matrix} \qquad \neg\exists x(Qx \wedge Rx)}{\neg Ra}\,{\scriptstyle \neg\text{I}}}{Pa \wedge \neg Ra}\,{\scriptstyle \wedge\text{I}}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{I}}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{E}}$$

$\exists x(Qx \wedge Rx)$ can be derived from $Qa \wedge Ra$ by $\exists$Intro:

$$\dfrac{\exists x(Px \wedge Qx) \qquad \dfrac{\dfrac{\dfrac{[Pa \wedge Qa]}{Pa}\,{\scriptstyle \wedge\text{E}} \qquad \dfrac{\dfrac{\begin{matrix}[Pa \wedge Qa], [Ra],\\ \neg\exists x(Qx \wedge Rx)\\ \vdots\\ Qa \wedge Ra\end{matrix}}{\exists x(Qx \wedge Rx)}\,{\scriptstyle \exists\text{I}} \qquad \neg\exists x(Qx \wedge Rx)}{\neg Ra}\,{\scriptstyle \neg\text{I}}}{Pa \wedge \neg Ra}\,{\scriptstyle \wedge\text{I}}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{I}}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{E}}$$

$Qa \wedge Ra$ is a conjunction, so we need to provide a proof of Qa and a proof of Ra. It turns out we don't need to use our premise $\neg\exists x(Qx \wedge Rx)$ again. We can obtain Qa from our assumption of $Pa \wedge Qa$, and we have Ra because we have assumed it in order to derive $\neg Ra$:

$$\dfrac{\exists x(Px \wedge Qx) \qquad \dfrac{\dfrac{\dfrac{[Pa \wedge Qa]}{Pa}\,{\scriptstyle \wedge\text{E}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Pa \wedge Qa]}{Qa}\,{\scriptstyle \wedge\text{E}} \qquad [Ra]}{Qa \wedge Ra}\,{\scriptstyle \wedge\text{I}}}{\exists x(Qx \wedge Rx)}\,{\scriptstyle \exists\text{I}} \qquad \neg\exists x(Qx \wedge Rx)}{\neg Ra}\,{\scriptstyle \neg\text{I}}}{Pa \wedge \neg Ra}\,{\scriptstyle \wedge\text{I}}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{I}}}{\exists x(Px \wedge \neg Rx)}\,{\scriptstyle \exists\text{E}}$$

This gives us a complete proof. Our assumption of Ra is discharged when we apply $\neg$Intro and derive $\neg Ra$. This means that by the time we apply $\exists$Elim, the only undischarged assumption left in the proof which involves a is $Pa \wedge Ra$. This means that at the end of the proof we are allowed to apply $\exists$Elim and discharge our two assumptions of $Pa \wedge Qa$.

4 Practice problems

4.1 Core

Knowing the rules

These seven proofs cover all of the Natural Deduction rules, and can be used to diagnose how familiar you are with the rules themselves and the strategies which correspond to them.

Learning the rules by heart is dull, but the best way is through practice. The later sections of the pack should have enough problems for each connective to establish a familiarity with them. If the seven proofs below prove to be straightforward, the later sections also have more challenging problems for each connective.

1. $(P \to P) \leftrightarrow Q \vdash Q \vee R$ (3)
2. $\forall x(Px \vee Px) \vdash \forall x Px$ (3)
3. $\exists x Px \vdash \exists x(Px \vee Qx)$ (3)
4. $\neg P_1 \vdash \neg((P_1 \wedge P_2) \wedge P_3)$ (3)
5. $\neg P \vee Q \vdash P \to Q$ (3)
6. $Qa, \neg Qb \vdash a = a \wedge \neg a = b$ (4)
7. $\vdash (P \wedge Q) \leftrightarrow (Q \wedge P)$ (7)

Making substitutions

The usual strategy for a proof involving quantifiers is to use the elimination rules to turn quantified statements into statements involving constants, manipulate those statements using the connective rules, and then turn those statements back into quantified statements with the introduction rules.

Once you've cracked how the quantifier rules work (including the nasty $\exists$Elim), the challenge becomes knowing which constants to substitute. Sometimes you will need to use your premises multiple times, making different substitutions each time. The proofs below test this; many more can be found in the quantifier sections of the pack.

8. $\forall x \forall y(Rxy \to Ryx) \vdash \forall x \forall y(Rxy \leftrightarrow Ryx)$ (7)
9. $\forall x \exists y Rxy \vdash \forall x \exists y \exists z(Rxy \wedge Ryz)$ (8)
10. $\forall x \forall y \forall z((Rxy \wedge Rxz) \to Ryz), \forall x Rxx \vdash \forall x \forall y(Rxy \to Ryx)$ (9)

Indirect proofs

Sometimes you find that, no matter how hard you try, you can't obtain the proof you want. This might be because you need an indirect proof: you prove a sentence ϕ by assuming $\neg\phi$ and showing that it leads to a contradiction.

Indirect kinds of proofs have often appeared in past papers. Part of the challenge is spotting them in the first place. Having to derive a disjunction without having any disjunctive premises is often a hint ($\vdash P \vee \neg P$ is the classic example). Similarly, you're likely to need an indirect proof to derive an existential statement from premises with no existential quantifiers (such as $\neg\forall x Px \vdash \exists x \neg Px$). Sometimes they can be harder to spot, which means it can be a good idea just to try an indirect proof if nothing else seems to be working.

There is a kind of indirect proof which is especially common. You prove ϕ by constructing a proof of the following shape:

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}[\psi]^1\\ \vdots\\ \phi\end{array}\quad [\neg\phi]^2}{\neg\psi}\,\neg\text{Intro}^1\\ \vdots\\ \phi\end{array}\qquad\qquad [\neg\phi]^2}{\phi}\,\neg\text{Elim}^2$$

Instead of simply proving ϕ, you show that ϕ follows from ψ and then from $\neg\psi$. The tricky part now is knowing which ψ to assume: usually you should look for a ψ which ϕ very easily follows from, but often there are many ψs which result in proofs which work.

The examples below are all indirect proofs, including at least one with the special shape above. Many more indirect proofs are located in later sections of the pack, not always in obvious places.

11. $\neg P \to Q \vdash P \vee Q$ (5)

12. $\vdash ((Q \to R) \to Q) \to Q$ (5)

13. $\neg\forall x \forall y Rxy \vdash \exists x \neg\forall y Rxy$ (7)

14. $Pa, Qb \vdash \exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$ (9)

4.2 Conjunction

1. $P, Q \vdash P \wedge Q$ (1)
2. $(P_1 \wedge P_2) \wedge P_3 \vdash P_2$ (2)
3. $P \wedge Q \vdash Q \wedge P$ (3)
4. $Q \wedge P, R \vdash P \wedge (R \wedge Q)$ (4)
5. $P_1 \wedge P_2, (Q_1 \wedge Q_2) \wedge R \vdash (P_1 \wedge Q_2) \wedge R$ (6)
6. $P \wedge (Q \wedge R) \vdash (R \wedge P) \wedge Q$ (7)

4.3 Implication

1. $\vdash P \to P$ (1)
2. $\vdash P \to (Q \to P)$ (2)
3. $P \to Q, Q \to R \vdash P \to R$ (3)
4. $\vdash P \to ((P \to Q) \to Q)$ (3)
5. $(P \to Q) \to (P \to R) \vdash Q \to (P \to R)$ (3)
6. $(P \to Q) \to P \vdash Q \to P$ (3)
7. $P \to (Q \to R) \vdash Q \to (P \to R)$ (4)
8. $P \to (Q \to R), P \to Q \vdash P \to R$ (4)
9. $(P \to P) \to Q \vdash (Q \to R) \to R$ (4)
10. $\vdash (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ (6)

Mixed problems with conjunction

11. $P \land Q \vdash P \to Q$ (2)
12. $\vdash P \land Q \to P$ (2)
13. $P \to (Q \land R) \vdash P \to Q$ (3)
14. $((P \land Q) \to Q) \to (Q \to P) \vdash Q \to P$ (3)
15. $(P \land Q) \to R \vdash P \to (Q \to R)$ (4)
16. $(P \to Q) \land (P \to R) \vdash P \to (Q \land R)$ (6)
17. $P \to (Q \land R) \vdash (P \to Q) \land (P \to R)$ (7)

Bonus challenge

Provide a Natural Deduction proof of the following which consists of no more than eight steps:

$$(((P_1 \land P_2) \land P_3) \land P_4) \land P_5 \vdash P_1 \land P_1$$

(Here, a 'step' is considred to be any application of any rule, so the number of steps is equivalent to the number of times a horizontal line is drawn.)

4.4 Disjunction

1. $P \vee Q \vdash Q \vee P$ (3)
2. $P \vee Q \vdash P \vee (Q \vee R)$ (4)
3. $(P \vee Q) \vee R) \vdash P \vee (Q \vee R)$ (7)
4. $(P \vee Q) \vee (R \vee P_1) \vdash (P \vee P_1) \vee (R \vee Q)$ (11)

Mixed problems with conjunction

5. $P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)$ (8)
6. $(P \vee Q) \wedge (P \vee R) \vdash P \vee (Q \wedge R)$ (8)
7. $(P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)$ (9)
8. $P \vee (Q \wedge R) \vdash (P \vee Q) \wedge (P \vee R)$ (9)

Mixed problems with implication

9. $(P \to Q) \vee Q \vdash P \to Q$ (2)
10. $P \vee Q \vdash (P \to Q) \to Q$ (3)
11. $(P \to Q) \to (P \to R) \vdash (P \vee R) \to (Q \to R)$ (6)
12. $(P \to Q) \vee (P \to R) \vdash P \to (Q \vee R)$ (7)

Mixed problems with conjunction and implication

13. $(P \to Q) \wedge (Q \to P) \vdash (P \vee Q) \to (P \wedge Q)$ (8)
14. $(P \vee Q) \to (P \wedge Q) \vdash (P \to Q) \wedge (Q \to P)$ (9)
15. $(Q \to R) \wedge (Q \vee P) \vdash (P \to Q) \to (R \wedge Q)$ (10)

4.5 Biconditional

1. $P \leftrightarrow Q \vdash Q \leftrightarrow P$ (3)
2. $P, (P \leftrightarrow Q) \leftrightarrow R \vdash Q \leftrightarrow R$ (5)
3. $\vdash (P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)$ (7)

Mixed problems

4. $(P \vee Q) \leftrightarrow Q \vdash P \rightarrow Q$ (3)
5. $(P \wedge Q) \leftrightarrow P \vdash P \rightarrow Q$ (3)
6. $P \rightarrow Q \vdash (P \vee Q) \leftrightarrow Q$ (4)
7. $P \rightarrow Q \vdash (P \wedge Q) \leftrightarrow P$ (4)
8. $(P \rightarrow Q) \wedge (Q \rightarrow P) \vdash P \leftrightarrow Q$ (5)
9. $\vdash (P \wedge Q) \rightarrow ((P \rightarrow Q) \rightarrow P)$ (5)
10. $\vdash ((P \rightarrow Q) \leftrightarrow P) \rightarrow (P \leftrightarrow Q)$ (6)
11. $((P \vee Q) \leftrightarrow Q) \leftrightarrow P \vdash P \leftrightarrow Q$ (7)
12. $P \rightarrow (Q \leftrightarrow R) \vdash (P \wedge Q) \leftrightarrow (P \wedge R)$ (13)
13. $\vdash (P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))$ (18)

4.6 Negation

Negation introduction

1. $P \vdash \neg\neg P$ (1)
2. $\neg P \vdash \neg(P \wedge Q)$ (2)
3. $P \to \neg P \vdash \neg P$ (2)
4. $\neg(P \to Q) \vdash \neg Q$ (2)
5. $\neg(P \wedge Q) \vdash P \to \neg Q$ (3)
6. $P \to Q \vdash \neg Q \to \neg P$ (3)
7. $\vdash \neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ (4)
8. $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$ (5)
9. $\neg P \vee \neg Q \vdash \neg(P \wedge Q)$ (5)

Ex falso quodlibet

10. $\neg P \vdash P \to Q$ (2)
11. $P \wedge \neg P \vdash Q$ (3)
12. $P \vee Q \vdash \neg P \to Q$ (3)
13. $P \to Q, P \wedge \neg Q \vdash R$ (4)
14. $P \vee Q, P \leftrightarrow Q, \neg(P \wedge Q) \vdash R$ (6)

Indirect proofs

15. $\neg\neg P \vdash P$ (1)
16. $\vdash P \vee \neg P$ (4)
17. $\neg(\neg P \vee \neg Q) \vdash P \wedge Q$ (5)
18. $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$ (6)

Mixed problems

19. $\neg(P \to Q) \vdash P$ (3)
20. $(P \to Q) \to P \vdash P$ (4)
21. $P \leftrightarrow \neg\neg Q \vdash P \leftrightarrow Q$ (5)
22. $(P \to Q) \to Q \vdash \neg Q \to P$ (5)

23. $\neg P \wedge \neg Q \vdash \neg(P \vee Q)$ (6)

24. $\vdash P \vee (P \to Q)$ (6)

25. $\vdash (P \to Q) \vee (Q \to R)$ (7)

26. $\neg P \to Q, R \vee \neg Q, P \to (Q_1 \vee Q_2), \neg R \wedge \neg Q_2 \vdash Q_1$ (9)

27. $P \to (Q \vee R) \vdash (P \to Q) \vee (P \to R)$ (11)

28. $\vdash \neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$ (12)

Bonus challenge 1

First, provide a proof of the following without using ¬Elim:

$$\neg\neg\neg P \vdash \neg P$$

Second, provide a proof of the following without using ¬Intro:

$$P \vdash \neg\neg P$$

Bonus challenge 2

Provide two different proofs of the following:

$$\neg\neg P \wedge \neg\neg Q \vdash P \wedge Q$$

The first proof should consist only of five steps (five applications of Natural Deduction rules).

In the second proof, you may only discharge assumptions using ¬Elim in the final step of the proof. In other words, you may make any number of applications of ¬Elim which don't discharge assumptions, but an application of ¬Elim may only discharge assumptions if no rules are applied below it.

4.7 Universal quantifier

Unary predicates

1. $\forall x Px \vdash \forall y Py$ (2)
2. $\vdash \forall x(Px \to Px)$ (2)
3. $\forall x(Pa \to Qx) \vdash Pa \to \forall z Qz$ (4)
4. $\forall x Px \land \forall y Qy \vdash \forall z(Pz \land Qz)$ (5)
5. $\forall x(Px \to Qx) \vdash \forall y_1 Py_1 \to \forall y_2 Qy_2$ (5)
6. $\forall z(Pz \land Qz) \vdash \forall y Py \land \forall y Qy$ (5)
7. $\forall x(Px \to Qx), \forall x \neg Qx \vdash \forall x \neg Px$ (5)
8. $\forall x_1 Px_1 \lor \forall x_2 Qx_2 \vdash \forall x(Px \lor Qx)$ (6)
9. $\forall x \forall y(Px \to Qy) \vdash \forall x(Px \to \forall z Qz)$ (6)
10. $\forall x(Px \to Qx), \forall x(Qx \to Rx) \vdash \forall x(Px \to Rx)$ (6)
11. $\forall x(Px \lor Qx), \neg \forall x Px \vdash \neg \forall x \neg Qx$ (6)
12. $\forall x(Px \land Qx) \vdash \forall x \forall y(Px \land Qy)$ (7)

Binary predicates

13. $\forall x \forall y Rxy \vdash \forall x Rxx$ (3)
14. $\forall x \neg \forall y Rxy \vdash \neg \forall x \forall y Rxy$ (3)
15. $\forall x Rxx \vdash \forall x \neg \forall y \neg Rxy$ (4)
16. $\forall x \neg Rxx \vdash \neg \forall x \forall y(Rxy \lor Ryx)$ (5)
17. $\neg \forall x \neg \forall y Ryx \vdash \forall x \neg \forall y \neg Rxy$ (6)
18. $\forall x Rxx \vdash \forall x \forall y(Rxy \to \neg \forall z \neg(Rxz \land Rzy))$ (6)
19. $\forall x \forall y Rxy \vdash \forall x(Rxx \land \forall y Ryx)$ (7)
20. $\forall x \forall y Rxy \vdash \forall x \forall y(Rxy \land Ryx)$ (7)
21. $\forall x \forall y(Rxy \to Ryx), \forall x \forall y \neg(Rxy \land Ryx) \vdash \forall x \forall y \neg Rxy$ (9)
22. $\forall x \forall y(Qxy \to Qyx), \forall x \forall y(\neg Qxy \lor \neg Qyx) \vdash \forall x \forall y \neg Qxy$ (9)
23. $\forall x \neg \forall y \neg Rxy \vdash \forall x \neg \forall y \forall z \neg(Rxy \land Ryz)$ (11)
24. $\forall x \forall y(Rxy \to Ryx), \forall x \forall y \forall z((Rxy \land Ryz) \to Rxz), \forall x \neg \forall y \neg Rxy$
$\vdash \forall x Rxx$ (13)

25. $\forall x \neg Rxx, \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$
$$\vdash \forall x \forall y \forall z \neg((Rxy \wedge Ryz) \wedge Rzx) \tag{16}$$

26. $\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow Ryz), \forall x Rxx$
$$\vdash \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz) \tag{17}$$

4.8 Existential quantifier

Unary predicates

1. $\exists x Px \vdash \exists y Py$ (2)
2. $\neg\exists x Px \vdash \exists x \neg Px$ (3)
3. $\exists x_1(Pa \to Qx_1) \vdash Pa \to \exists x_2 Qx_2$ (4)
4. $\exists x(Px \wedge Qx) \vdash \exists y Py \wedge \exists z Qz$ (6)
5. $\exists x(Px \vee Qx) \vdash \exists y Py \vee \exists z Qz$ (6)
6. $\exists x Px \vee \exists y Qy \vdash \exists z(Pz \vee Qz)$ (7)
7. $Pa \to \exists x Qx \vdash \exists x(Pa \to Qx)$ (9)

Binary predicates

8. $\vdash \exists x \exists y(Rxy \to Ryx)$ (3)
9. $\exists x \exists y Rxy \vdash \exists x \exists y Ryx$ (4)
10. $\exists x Rxx \vdash \exists x \exists y(Rxy \wedge Ryx)$ (4)
11. $\neg\exists x \exists y Rxy \vdash \neg\exists y Ryy$ (4)
12. $\vdash \neg\exists x \exists y(Rxy \wedge \neg Rxy)$ (6)
13. $\vdash \exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$ (8)
14. $Rab \wedge Rbc, \neg Qa, Qc \vdash \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ (12)

Ternary predicates

15. $\exists x \exists y \neg\exists z \neg Pxyz \vdash \neg\exists x \neg\exists y \exists z Pyzx$ (8)

Mixed quantifier problems

16. $\neg\exists x Px \vdash \forall x \neg Px$ (3)
17. $\exists x \neg Px \vdash \neg\forall x Px$ (3)
18. $\neg\forall x Px \vdash \exists x \neg Px$ (4)
19. $\forall x \neg Px \vdash \neg\exists x Px$ (4)
20. $\forall x(\exists y Py \to Qx) \vdash \forall x \exists y(Py \to Qx)$ (6)
21. $\forall x \neg\forall y(Pxy \to Qxy) \vdash \forall x \exists y Pxy$ (7)
22. $\forall x(Pxx \vee \forall y Qxy) \vdash \forall x(\exists y Pxy \vee Qxx)$ (7)

23. $\exists x(Pxx \land \forall y Qxy) \vdash \exists x(\exists y Pxy \land Qxx)$ (7)

24. $\vdash \forall x \exists y Rxy \lor \neg \forall x Rxx$ (7)

25. $\forall x \exists y Rxy \to \neg \exists x Rxx, \exists x \forall y Ryx \vdash \forall x \neg Rxx$ (8)

26. $\forall x(Px \to \exists y Ryx) \vdash \exists z Rza \lor \neg \forall x Px$ (10)

Bonus challenge

Construct a Natural Deduction proof with the following two features:

- The proof consists of a single application of ∃Elim, and no applications of any other rules.
- By the end of the proof, one assumption is discharged and one is left undischarged.

4.9 Identity

1. $\vdash \exists x \exists y x = y$ (3)
2. $\vdash \exists x \forall y x = y \leftrightarrow \forall x \forall y x = y$ (9)

Unary predicates

3. $Pa, \neg Pb \vdash \neg a = b$ (2)
4. $Pb \wedge Qb, \forall x(Px \to x = a) \vdash Qa$ (5)
5. $\exists x \forall y(Py \leftrightarrow x = y) \vdash \exists x \forall y(Py \to x = y)$ (6)
6. $\forall x(x = a \vee x = b), \exists x Px \vdash \neg Pa \to Pb$ (7)
7. $\exists x(Px \wedge Qx \wedge Rax), Pb \wedge Qb, \forall x \forall y((Px \wedge Qx) \wedge (Py \wedge Qy) \to x = y)$
   $\vdash Rab$ (8)
8. $\forall x \forall y(Px \wedge x = y \to \neg Qy) \vdash \forall z \neg(Pz \wedge Qz)$ (9)

Binary predicates

9. $\forall x \forall y(Rxy \leftrightarrow x = y) \vdash \forall x Rxx$ (5)
10. $\forall x \neg Rxx, Rab \vdash \exists x \exists y \neg x = y$ (5)
11. $\exists x \exists y Rxy, \exists x \forall y x = y \vdash \forall x \forall y Rxy$ (9)
12. $\vdash \forall x Pxx \leftrightarrow \forall x \forall y(\neg Pxy \to \neg x = y)$ (13)
13. $\forall x \exists y(Rxy \wedge Py), \forall x \neg Rxx \vdash \neg \forall x \forall y(Px \to (Py \to x = y))$ (14)

4.10 Additional challenges

Admissible rules

In this question we consider alternative rules which could be added to the system of Natural Deduction.

We say a rule is 'admissible' if we can add it to the system of Natural Deduction without changing which conclusions we can derive from which premises. In other words, a rule is *not* admissible if there is a set of sentences Γ and a sentence ϕ such that ϕ is provable from Γ with the rule, but ϕ is not provable from Γ in the original unaugmented system of Natural Deduction.

Which of the rules below are admissible? In each case, justify your answer: provide a proof of ϕ from Γ which would not be possible in the unaugmented system of Natural Deduction; or show that any proof of ϕ from Γ using the new rule can be rewritten using only the original Natural Deduction rules.

1. $$\dfrac{\begin{matrix} \\ \vdots \\ \phi \wedge \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix}}{\chi} \star 1$$

2. $$\dfrac{\begin{matrix} \vdots \\ \neg\phi \end{matrix} \qquad \begin{matrix} \vdots \\ \phi \to \psi \end{matrix}}{\neg\psi} \star 2$$

3. $$\dfrac{\begin{matrix} [\psi \to \phi] \\ \vdots \\ \phi \end{matrix}}{\phi} \star 3$$

4. $$\dfrac{\begin{matrix} [\neg\phi] \\ \vdots \\ \psi \end{matrix}}{\phi \vee \psi} \star 4$$

5. $$\dfrac{\begin{matrix} [\phi \to \psi] \\ \vdots \\ \phi \end{matrix}}{\phi} \star 5$$

Contraposition

Consider the rule with the following graphical representation:

$$\dfrac{\begin{array}{c}\vdots \\ \neg\psi \to \neg\phi\end{array}}{\phi \to \psi}\ \text{C}$$

Write $\Gamma \vdash_C \phi$ if there is a proof of ϕ from Γ in an alternate Natural Deduction system where the rule above (called contraposition) can be used, and all original Natural Deduction rules except ¬Intro and ¬Elim can be used. We continue to write $\Gamma \vdash \phi$ if there is a proof of ϕ from Γ in the original system of Natural Deduction (i.e. with ¬Intro and ¬Elim but without contraposition).

Show that:

1. $\neg P \vdash_C P \to Q$ (2)
2. $P \vdash_C \neg P \to Q$ (4)
3. $\neg\neg P \vdash_C P$ (4)
4. If $\Gamma \vdash_C \phi$ then $\Gamma \vdash \phi$

If you're looking for a challenge, show that:

5. If $\Gamma \vdash \phi$ then $\Gamma \vdash_C \phi$

(Hint: First show that every ¬Intro step can be replaced by ¬Elim steps. Then show how every ¬Elim step can be replaced by contraposition, →Intro and →Elim.)

5 Solutions

5.1 Core

Core 1

$$\cfrac{\cfrac{\cfrac{[P]}{P \to P}\ {\scriptstyle \to\mathrm{Intro}} \quad (P \to P) \leftrightarrow Q}{Q}\ {\scriptstyle \leftrightarrow\mathrm{Elim}}}{Q \vee R}\ {\scriptstyle \vee\mathrm{Intro}}$$

Our conclusion $Q \vee R$ is a disjunction, so we can either prove it by proving Q or by proving R. We're probably not going to be able to prove R since it doesn't appear in our premise $(P \to P) \leftrightarrow Q$, so instead we'll try and prove Q.

Our premise $(P \to P) \leftrightarrow Q$ is a biconditional, which means that if we can prove $P \to P$ we can use $\leftrightarrow$Elim to derive Q. By assuming P, and then using $\to$Intro to discharge this assumption of P, we can provide this proof of $P \to P$.

Core 2

$$\cfrac{\cfrac{\cfrac{\forall x(Px \vee Px)}{Pa \vee Pa}\ {\scriptstyle \forall\mathrm{Elim}} \quad [Pa] \quad [Pa]}{Pa}\ {\scriptstyle \vee\mathrm{Elim}}}{\forall x Px}\ {\scriptstyle \forall\mathrm{Intro}}$$

Our conclusion $\forall x Px$ is a universal statement. We can derive it from Pa using $\forall$Intro, as long as the constant a appears in no undischarged assumptions by the time we apply $\forall$Intro.

Our premise $\forall x(Px \vee Px)$ lets us derive $Pa \vee Pa$, a disjunction. Deriving Pa from $Pa \vee Pa$ requires a slightly bizarre use of $\vee$Elim: we make two assumptions of Pa and then immediately discharge them with $\vee$Elim to derive Pa in one step. Because these assumptions of Pa have been discharged, we are free to apply $\forall$Intro and derive the conclusion $\forall x Px$.

Core 3

$$\dfrac{\exists x Px \qquad \dfrac{\dfrac{[Pa]}{Pa \lor Qa}\,{\scriptstyle \lor\mathrm{Intro}}}{\exists x(Px \lor Qx)}\,{\scriptstyle \exists\mathrm{Intro}}}{\exists x(Px \lor Qx)}\,{\scriptstyle \exists\mathrm{Elim}}$$

Our premise $\exists x Px$ is an existential statement, which means we need to use the dreaded $\exists$Elim rule. We will use this rule at the very end of the proof to discharge an assumption of Pa.

From this assumption of Pa we can use $\lor$Intro to derive $Pa \lor Qa$, and then $\exists$Intro to derive the conclusion $\exists x(Px \lor Qx)$.

Our use of $\exists$Elim at the end of the proof is allowed because the constant a doesn't appear in our premise $\exists x Px$, our conclusion $\exists x(Px \lor Qx)$ or any other assumptions we made when proving $\exists x(Px \lor Qx)$.

Core 4

$$\dfrac{\dfrac{\dfrac{[(P_1 \land P_2) \land P_3]}{P_1 \land P_2}\,{\scriptstyle \land\mathrm{Elim}}}{P_1}\,{\scriptstyle \land\mathrm{Elim}} \qquad \neg P_1}{\neg((P_1 \land P_2) \land P_3)}\,{\scriptstyle \neg\mathrm{Intro}}$$

Our conclusion $\neg((P_1 \land P_2) \land P_3)$ is a negation, so we prove it by assuming $(P_1 \land P_2) \land P_3$ and deriving a contradiction. Our premise $\neg P_1$ is a negated statement, so we have a contradiction if we have a proof of P_1.

We can prove P_1 from our assumption of $(P_1 \land P_2) \land P_3$ by using $\land$Elim twice. Then with our premise $\neg P_1$ we can apply $\neg$Intro, discharge our assumption of $(P_1 \land P_2) \land P_3$ and derive $\neg((P_1 \land P_2) \land P_3)$.

Core 5

$$\dfrac{\dfrac{\neg P \lor Q \qquad \dfrac{[P]^2 \quad [\neg P]^1}{Q}\,{\scriptstyle \neg\mathrm{Elim}} \qquad [Q]^1}{Q}\,{\scriptstyle \lor\mathrm{Elim}^1}}{P \to Q}\,{\scriptstyle \to\mathrm{Intro}^2}$$

Our conclusion $P \to Q$ is an implication, so we prove it by assuming P and deriving Q from it. Our premise $\neg P \lor Q$ is a disjunction, so we can use $\lor$Elim to split the proof into a case where we can assume $\neg P$ and a case where we can assume Q.

The right-hand case is easy. We want to prove Q and we have an assumption of Q, so we don't need to do anything else. In the left-hand case, we can use our assumptions of P and $\neg P$ to derive Q by $\neg$Elim. Now we have established Q is true in both cases, we can use $\to$Intro to derive $P \to Q$.

Core 6

$$\dfrac{[a = a]^{=} \qquad \dfrac{\dfrac{Qa \quad [a = b]}{Qb}\,{\scriptstyle =\text{Elim}} \qquad \neg Qb}{\neg a = b}\,{\scriptstyle \neg\text{Intro}}}{a = a \wedge \neg a = b}\,{\scriptstyle \wedge\text{Intro}}$$

Our conclusion $a = a \wedge \neg a = b$ is a conjunction, so we need to provide two proofs: a proof of $a = a$ and a proof of $\neg a = b$.

The left-hand proof is easy. We can assume $a = a$ and immediately discharge it using =Intro.

The right-hand proof is a proof of a negated statement, so we assume $a = b$ and try to derive a contradiction from it. Since one of our premises $\neg Pb$ is a negated statement, we have the contradiction we need if we can prove Pb. We prove Pb by using =Elim together with our other premise Pa and our assumption of $a = b$.

Core 7

$$\dfrac{\dfrac{\dfrac{[P \wedge Q]}{Q}\,{\scriptstyle \wedge\text{Elim}} \quad \dfrac{[P \wedge Q]}{P}\,{\scriptstyle \wedge\text{Elim}}}{Q \wedge P}\,{\scriptstyle \wedge\text{Intro}} \qquad \dfrac{\dfrac{[Q \wedge P]}{P}\,{\scriptstyle \wedge\text{Elim}} \quad \dfrac{[Q \wedge P]}{Q}\,{\scriptstyle \wedge\text{Elim}}}{P \wedge Q}\,{\scriptstyle \wedge\text{Intro}}}{(P \wedge Q) \leftrightarrow (Q \wedge P)}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

The conclusion $(P \wedge Q) \leftrightarrow (Q \wedge P)$ is a biconditional, so we prove it by providing a proof of $Q \wedge P$ from assumptions of $P \wedge Q$ and a poof of $P \wedge Q$ from assumptions of $Q \wedge P$.

Both of the sides of the proof work similarly. We use $\wedge$Elim with our assumption to obtain P and Q, and then we use $\wedge$Intro to derive $Q \wedge P$ on the left and $P \wedge Q$ on the right.

Core 8

$$
\begin{prooftree}
\AxiomC{$\forall x \forall y (Rxy \to Ryx)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y (Ray \to Rya)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Rab \to Rba$}
\AxiomC{$[Rab]$}
\RightLabel{$\to$E}
\BinaryInfC{Rba}
\AxiomC{$\forall x \forall y (Rxy \to Ryx)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y (Rby \to Ryb)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Rba \to Rab$}
\AxiomC{$[Rba]$}
\RightLabel{$\to$E}
\BinaryInfC{Rab}
\RightLabel{$\leftrightarrow$I}
\BinaryInfC{$Rab \leftrightarrow Rba$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall y (Ray \leftrightarrow Rya)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y (Rxy \leftrightarrow Ryx)$}
\end{prooftree}
$$

This is a past paper question from 2015. Our conclusion $\forall x \forall y (Rxy \leftrightarrow Ryx)$ is a universal statement, which we can derive from $\forall y (Ray \leftrightarrow Rya)$. This in turn we can derive from $Rab \leftrightarrow Rba$. In order to prove this, we need to provide a proof of Rba from Rab and of Rab from Rba.

On the left-hand side we assume Rab and derive $Rab \to Rba$ from our premise $\forall x \forall y (Rxy \to Ryx)$, which gives us Rba. The right-hand side is similar: we assume Rba and derive $Rba \to Rab$ from the premise, which gives us Rab. Both of these assumptions are discharged when we apply $\leftrightarrow$Intro, so we are free to apply $\forall$Intro.

Core 9

$$
\begin{prooftree}
\AxiomC{$\forall x \exists y Rxy$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\exists y Ray$}
\AxiomC{$\forall x \exists y Rxy$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\exists y Rby$}
\AxiomC{$[Rab]^2$}
\AxiomC{$[Rbc]^1$}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$Rab \wedge Rbc$}
\RightLabel{$\exists$Intro}
\UnaryInfC{$\exists z (Rab \wedge Rbz)$}
\RightLabel{$\exists$Intro}
\UnaryInfC{$\exists y \exists z (Ray \wedge Ryz)$}
\RightLabel{$\exists$Elim^1}
\BinaryInfC{$\exists y \exists z (Ray \wedge Ryz)$}
\RightLabel{$\exists$Elim^2}
\BinaryInfC{$\exists y \exists z (Ray \wedge Ryz)$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall x \exists y \exists z (Rxy \wedge Ryz)$}
\end{prooftree}
$$

This is adapted from a past paper question from 2012. Our conclusion $\forall x \exists y \exists z (Rxy \wedge Ryz)$ is a universal statement, so we can prove it by proving $\exists y \exists z (Ray \wedge Ryz)$ as long as a doesn't appear in any undischarged assumptions by the time we apply $\forall$Intro.

From our premise $\forall x \exists y Rxy$ we can derive the existential statement $\exists y Ray$, which lets us discharge an assumption of Rab by $\exists$Elim (as long as b doesn't appear in any other undischarged assumptions when we apply $\exists$Elim). However, Rab alone isn't enough to derive $\exists y \exists z (Ray \wedge Ryz)$. We need to use the premise a second time to derive $\exists y Rby$, which lets us discharge an assumption of Rbc. With these assumptions of Rab and Rbc we can derive $\exists y \exists z (Ray \wedge Ryz)$.

We need to be careful about the order in we apply the $\exists$Elim steps. If we tried to discharge Rab before Rbc, we wouldn't be allowed to, because at that stage b still appears in an undischarged assumption. Instead we use $\exists y Ray$ first to discharge Rbc, and then use $\exists y Ray$ to discharge Rab.

Core 10

$$
\begin{prooftree}
\AxiomC{$[Rab]$}
\AxiomC{$\forall x Rxx$}
\RightLabel{$\forall$Elim}
\UnaryInfC{Raa}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$Rab \wedge Raa$}
\AxiomC{$\forall x \forall y \forall z((Rxy \wedge Rxz) \to Ryz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall y \forall z((Ray \wedge Raz) \to Ryz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall z((Rab \wedge Raz) \to Rbz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$(Rab \wedge Raa) \to Rba$}
\RightLabel{$\to$Elim}
\BinaryInfC{Rba}
\RightLabel{$\to$Intro}
\UnaryInfC{$Rab \to Rba$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall y(Ray \to Rya)$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall x \forall y(Rxy \to Ryx)$}
\end{prooftree}
$$

Our conclusion $\forall x \forall y(Rxy \to Ryx)$ features two universal quantifiers; we can prove it by proving $Rab \to Rba$ without a or b appearing in any undischarged assumptions. To do this we assume Rab and prove Rba.

We can't obtain Rba from Rab alone, but from one of our premises we can derive Raa, which allows us to derive $Rab \wedge Rba$. From our other premise we can obtain $(Rab \wedge Rba) \to Rba$, giving us the proof of Rba we need. Our assumption of Rab is discharged when we apply $\to$Intro, so we are free to apply $\forall$Intro and derive our conclusion.

Core 11

$$
\begin{prooftree}
\AxiomC{$[P]^1$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$P \vee Q$}
\AxiomC{$[\neg(P \vee Q)]^2$}
\RightLabel{$\neg$Intro^1}
\BinaryInfC{$\neg P$}
\AxiomC{$\neg P \to Q$}
\RightLabel{$\to$Elim}
\BinaryInfC{Q}
\RightLabel{$\vee$Intro}
\UnaryInfC{$P \vee Q$}
\AxiomC{$[\neg(P \vee Q)]^2$}
\RightLabel{$\neg$Elim^2}
\BinaryInfC{$P \vee Q$}
\end{prooftree}
$$

We want to try to prove $P \vee Q$ but our premise $\neg P \to Q$ doesn't give us a direct proof of P or a direct proof of Q. We will need to assume $\neg(P \vee Q)$ and show that it leads to a contradiction.

We start by assuming P. From this, we can derive $P \vee Q$, which contradicts our assumption of $\neg(P \vee Q)$. Even though we have a contradiction, we can't immediately conclude $P \vee Q$, because our assumption of P is undischarged. Instead, we apply $\neg$Intro to discharge P and prove $\neg P$.

This is where our premise comes in. We have proved $\neg P$, and our premise $\neg P \to Q$ allows us to derive Q. From this, we can derive $P \vee Q$ again, so assuming $\neg(P \vee Q)$ again leads to a contradiction. From this contradiction we can discharge our assumptions of $\neg(P \vee Q)$ and derive $P \vee Q$.

Core 12

$$
\begin{prooftree}
\AxiomC{$[Q]^1$}
\AxiomC{$[\neg Q]^2$}
\RightLabel{\scriptsize $\neg$Elim}
\BinaryInfC{R}
\RightLabel{\scriptsize $\to$Intro^1}
\UnaryInfC{$Q \to R$}
\AxiomC{$[(Q \to R) \to Q]^3$}
\RightLabel{\scriptsize $\to$Elim}
\BinaryInfC{Q}
\AxiomC{$[\neg Q]^2$}
\RightLabel{\scriptsize $\neg$Elim^2}
\BinaryInfC{Q}
\RightLabel{\scriptsize $\to$Intro^3}
\UnaryInfC{$((Q \to R) \to Q) \to Q$}
\end{prooftree}
$$

There are many ways of carrying out this proof; shown above is one of the shortest possible methods. To prove $((Q \to R) \to Q) \to Q$, an implication, we assume $((Q \to R) \to Q)$ and provide a proof of Q. We can't provide a direct proof of Q, so instead we assume $\neg Q$ and show that it leads to a contradiction.

Since $\neg Q$ is itself a negated statement, we can show that it leads to a contradiction if we can show it leads to Q. We have assumed $(Q \to R) \to Q$, which is an implication with Q as its consequent. This means that if we can derive $Q \to R$ from $\neg Q$, we will be able to obtain Q and hence a contradiction.

Deriving $Q \to R$ from $\neg Q$ isn't too tricky. Since $Q \to R$ is an implication, we can assume Q, and Q and $\neg Q$ together give us R by $\neg$Elim.

We could have tried to prove $((Q \to R) \to Q) \to Q$ by assuming $\neg(((Q \to R) \to Q) \to Q)$ and deriving a contradiction from that. Doing this will result in a longer proof, but it is still possible. Two examples below illustrate this approach.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[Q]^2 \qquad \dfrac{\dfrac{[Q]^1}{((Q \to R) \to Q) \to Q}\,{\scriptstyle \to\mathrm{I}} \qquad [\neg(((Q \to R) \to Q) \to Q)]^4}{\neg Q}\,{\scriptstyle \neg\mathrm{Intro}^1}}{R}\,{\scriptstyle \neg\mathrm{E}}}{Q \to R}\,{\scriptstyle \to\mathrm{I}^2} \qquad [((Q \to R) \to Q)]^3}{Q}\,{\scriptstyle \to\mathrm{E}}}{((Q \to R) \to Q) \to Q}\,{\scriptstyle \to\mathrm{I}^3} \qquad [\neg(((Q \to R) \to Q) \to Q)]^4}{((Q \to R) \to Q) \to Q}\,{\scriptstyle \neg\mathrm{E}^4}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Q]^1 \qquad [\neg Q]^4}{R}\,{\scriptstyle \neg\mathrm{E}}}{Q \to R}\,{\scriptstyle \to\mathrm{I}^1} \qquad \dfrac{\dfrac{\dfrac{[Q \to R]^3 \qquad [((Q \to R) \to Q)]^2}{Q}\,{\scriptstyle \to\mathrm{E}}}{((Q \to R) \to Q) \to Q}\,{\scriptstyle \to\mathrm{I}^2} \qquad [\neg(((Q \to R) \to Q) \to Q)]^5}{\neg(Q \to R)}\,{\scriptstyle \neg\mathrm{I}^3}}{Q}\,{\scriptstyle \neg\mathrm{E}^4}}{((Q \to R) \to Q) \to Q}\,{\scriptstyle \to\mathrm{I}} \qquad [\neg(((Q \to R) \to Q) \to Q)]^5}{((Q \to R) \to Q) \to Q}\,{\scriptstyle \neg\mathrm{E}^5}
$$

Core 13

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall y Ray]^1}{Rab}{\scriptstyle\forall\text{Elim}} \qquad [\neg Rab]^2}{\neg\forall y Ray}{\scriptstyle\neg\text{Intro}^1}}{\exists x\neg\forall y Rxy}{\scriptstyle\exists\text{Intro}} \qquad [\neg\exists x\neg\forall y Rxy]^3}{Rab}{\scriptstyle\neg\text{Elim}^2}}{\forall y Ray}{\scriptstyle\forall\text{Intro}}}{\forall x\forall y Rxy}{\scriptstyle\forall\text{Intro}} \qquad \neg\forall x\forall y Rxy}{\exists x\neg\forall y Rxy}{\scriptstyle\neg\text{Elim}^3}
$$

This is a very awkward proof. Our conclusion is $\exists x\neg\forall y Rxy$, but we cannot prove it directly by $\exists$Intro. Instead we assume $\neg\exists x\neg\forall y Rxy$ and derive a contradiction from it. Since our premise $\neg\forall x\forall y Rxy$ is a negated statement, we have a contradiction if we can derive $\forall x\forall y Rxy$ from $\neg\exists x\neg\forall y Rxy$.

We can derive $\forall x\forall y Rxy$ from Rab, provided that neither a nor b appears in any undischarged assumptions by the time we've derived Rab. But we have no way of proving Rab directly either; instead we need to make another indirect proof, assuming $\neg Rab$ and deriving a contradiction from that.

Since $\neg\exists x\neg\forall y Rxy$ is a negated statement, we have a contradiction if we can derive $\exists x\neg\forall y Rxy$ from $\neg Rab$. This time we do have a way of proving $\exists x\neg\forall y Rxy$ from $\neg Rab$ directly: it follows from $\neg\forall y Ray$. This in turn is a negated statement, so we can derive it if we assume $\forall y Ray$ and derive a contradiction from it. We do this by deriving Rab, which contradicts our assumption of $\neg Rab$.

Core 14

$$
\begin{prooftree}
\AxiomC{Pa}
\AxiomC{Qb}
\AxiomC{$[a = b]$}
\RightLabel{\scriptsize =Elim}
\BinaryInfC{Qa}
\RightLabel{\scriptsize $\wedge$Intro}
\BinaryInfC{$Pa \wedge Qa$}
\RightLabel{\scriptsize $\exists$Intro}
\UnaryInfC{$\exists x(Px \wedge Qx)$}
\RightLabel{\scriptsize $\vee$Intro}
\UnaryInfC{$\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$}
\AxiomC{$[\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)]$}
\RightLabel{\scriptsize $\neg$Intro}
\BinaryInfC{$\neg a = b$}
\RightLabel{\scriptsize $\exists$Intro}
\UnaryInfC{$\exists y \neg a = y$}
\RightLabel{\scriptsize $\exists$Intro}
\UnaryInfC{$\exists x \exists y \neg x = y$}
\RightLabel{\scriptsize $\vee$Intro}
\UnaryInfC{$\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$}
\AxiomC{$[\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)]$}
\RightLabel{\scriptsize $\neg$Elim}
\BinaryInfC{$\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$}
\end{prooftree}
$$

Our conclusion $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$ is a disjunction and we have no disjunctive premises. This means we probably need to prove $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$ indirectly: we assume $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$ and show that it leads to a contradiction.

We start by assuming $a = b$, since this lets us easily prove Qa, $Pa \wedge Qa$ and $\exists x(Px \wedge Qx)$. From there we derive $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$, which contradicts our assumption of $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$. With this contradiction we use $\neg$Intro to discharge $a = b$ and derive $\neg a = b$.

From $\neg a = b$ we can also obtain our conclusion. By applying $\exists$Intro twice we can derive $\exists x \exists y \neg x = y$, and from there we can derive $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$. Finally we assume $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$ again and apply $\neg$Elim, discharging both assumptions of $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$ and giving us the conclusion $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$.

5.2 Conjunction

Conjunction 1

$$\dfrac{P \qquad Q}{P \wedge Q}\,{\scriptstyle \wedge\mathrm{Intro}}$$

Conjunction 2

$$\dfrac{\dfrac{(P_1 \wedge P_2) \wedge P_3}{P_1 \wedge P_2}\,{\scriptstyle \wedge\mathrm{Elim1}}}{P_2}\,{\scriptstyle \wedge\mathrm{Elim2}}$$

Conjunction 3

$$\dfrac{\dfrac{P \wedge Q}{Q}\,{\scriptstyle \wedge\mathrm{Elim2}} \quad \dfrac{P \wedge Q}{P}\,{\scriptstyle \wedge\mathrm{Elim1}}}{Q \wedge P}\,{\scriptstyle \wedge\mathrm{Intro}}$$

Conjunction 4

$$\dfrac{\dfrac{Q \wedge P}{P}\,{\scriptstyle \wedge\mathrm{Elim2}} \quad \dfrac{R \quad \dfrac{Q \wedge P}{Q}\,{\scriptstyle \wedge\mathrm{Elim1}}}{R \wedge Q}\,{\scriptstyle \wedge\mathrm{Intro}}}{P \wedge (R \wedge Q)}\,{\scriptstyle \wedge\mathrm{Intro}}$$

Conjunction 5

$$\dfrac{\dfrac{\dfrac{P_1 \wedge P_2}{P_1}\,{\scriptstyle \wedge\mathrm{Elim1}} \quad \dfrac{\dfrac{(Q_1 \wedge Q_2) \wedge R}{Q_1 \wedge Q_2}\,{\scriptstyle \wedge\mathrm{Elim1}}}{Q_2}\,{\scriptstyle \wedge\mathrm{Elim2}}}{P_1 \wedge Q_1}\,{\scriptstyle \wedge\mathrm{Intro}} \quad \dfrac{(Q_1 \wedge Q_2) \wedge R}{R}\,{\scriptstyle \wedge\mathrm{Elim2}}}{(P_1 \wedge Q_2) \wedge R}\,{\scriptstyle \wedge\mathrm{Intro}}$$

Conjunction 6

$$\dfrac{\dfrac{\dfrac{\dfrac{P \wedge (Q \wedge R)}{Q \wedge R}\,{\scriptstyle \wedge\mathrm{Elim2}}}{R}\,{\scriptstyle \wedge\mathrm{Elim2}} \quad \dfrac{P \wedge (Q \wedge R)}{P}\,{\scriptstyle \wedge\mathrm{Elim1}}}{R \wedge P}\,{\scriptstyle \wedge\mathrm{Intro}} \quad \dfrac{\dfrac{P \wedge (Q \wedge R)}{Q \wedge R}\,{\scriptstyle \wedge\mathrm{Elim2}}}{Q}\,{\scriptstyle \wedge\mathrm{Elim1}}}{(R \wedge P) \wedge Q}\,{\scriptstyle \wedge\mathrm{Intro}}$$

5.3 Implication

Implication 1

$$\dfrac{[P]}{P \rightarrow P}\,{\scriptstyle \rightarrow\text{Intro}}$$

This proof relies on a special case of the $\rightarrow$Intro rule: both ϕ and ψ are the same. That means when we apply the rule we discharge P and put P on both sides of the arrow.

Implication 2

$$\dfrac{\dfrac{[P]}{Q \rightarrow P}\,{\scriptstyle \rightarrow\text{Intro}}}{P \rightarrow (Q \rightarrow P)}\,{\scriptstyle \rightarrow\text{Intro}}$$

In the first step of this proof, we discharge all assumptions of Q, but don't actually discharge any assumptions. We can go straight from P to $Q \rightarrow P$. It's in the second step that we discharge our original assumption of P.

Implication 3

$$\dfrac{\dfrac{\dfrac{[P] \quad P \rightarrow Q}{Q}\,{\scriptstyle \rightarrow\text{Elim}} \qquad Q \rightarrow R}{R}\,{\scriptstyle \rightarrow\text{Elim}}}{P \rightarrow R}\,{\scriptstyle \rightarrow\text{Intro}}$$

Here we can freely assume P, and we want to try to get to R. This is nice and easy P takes us to Q because we have $P \rightarrow Q$ as a premise, and then Q takes us to R because we have $Q \rightarrow R$ as a premise. The final application of $\rightarrow$Intro discharges our assumption of P.

Implication 4

$$\dfrac{\dfrac{\dfrac{[P]^2 \quad [P \rightarrow Q]^1}{Q}\,{\scriptstyle \rightarrow\text{Elim}}}{(P \rightarrow Q) \rightarrow Q}\,{\scriptstyle \rightarrow\text{Intro}^1}}{P \rightarrow ((P \rightarrow Q) \rightarrow Q}\,{\scriptstyle \rightarrow\text{Intro}^2}$$

In order to prove $P \rightarrow ((P \rightarrow Q) \rightarrow Q)$, we have to prove $(P \rightarrow Q) \rightarrow Q$, and we're allowed to freely assume P. In order to prove $(P \rightarrow Q) \rightarrow Q$, we have to prove Q, and we're allowed to freely assume $P \rightarrow Q$. With assumptions of both P and $P \rightarrow Q$ we can use $\rightarrow$Elim to get Q. Then the two $\rightarrow$Intro steps discharge the two assumptions of $P \rightarrow Q$ and P.

Implication 5

$$\dfrac{\dfrac{\dfrac{[Q]}{P \to Q}\,{\to}\text{Intro} \qquad (P \to Q) \to (P \to R)}{P \to R}\,{\to}\text{Elim}}{Q \to (P \to R)}\,{\to}\text{Intro}$$

Our conclusion is $Q \to (P \to R)$, so we should try to prove $P \to R$ from assumptions of Q. In the first step of the proof, we apply $\to$Intro without discharging anything: Q takes us straight to $P \to Q$. After that, our big premise gets us to $P \to R$, and our final application of $\to$Intro discharges Q.

Implication 6

$$\dfrac{\dfrac{\dfrac{[Q]}{P \to Q}\,{\to}\text{Intro} \qquad (P \to Q) \to P}{P}\,{\to}\text{Elim}}{Q \to P}\,{\to}\text{Intro}$$

Our conclusion is $Q \to P$, so we can assume Q and need to prove P. Our premise $(P \to Q) \to P$ is an implication with P as its consequent, so we know we can derive P if we can provide a proof of $P \to Q$. This follows from Q by $\to$Intro (another time when we apply $\to$Intro without discharging anything), so we can then apply $\to$Elim to derive P. Finally we apply $\to$Intro to discharge Q and derive $Q \to P$.

Implication 7

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad P \to (Q \to R)}{Q \to R}\,{\to}\text{Elim} \qquad [Q]}{R}\,{\to}\text{Elim}}{P \to R}\,{\to}\text{Intro}}{Q \to (P \to R)}\,{\to}\text{Intro}$$

Here we can freely assume P and Q, and we need to get to R. Our premise gets us from the P we've assumed to $Q \to R$, and then the Q we've assumed takes us to R.

Implication 8

$$\dfrac{\dfrac{\dfrac{[P] \quad P \to (Q \to R)}{Q \to R}\,{\to}\text{Elim} \qquad \dfrac{[P] \quad P \to Q}{Q}\,{\to}\text{Elim}}{R}\,{\to}\text{Elim}}{P \to R}\,{\to}\text{Intro}$$

In this proof the conclusion is quite simple. Having to prove $P \to R$ means we can only assume P and only need to get to R. But this time we have two premises, and the P we've assumed works with both the premises: it gives us both the $Q \to R$ on the left and the Q on the right that we need to get to R.

Implication 9

$$\dfrac{\dfrac{\dfrac{\dfrac{[P]}{P \to P}\,{\scriptstyle \to\text{Intro}} \qquad (P \to P) \to Q}{Q}\,{\scriptstyle \to\text{Elim}} \qquad [Q \to R]}{R}\,{\scriptstyle \to\text{Elim}}}{(Q \to R) \to R}\,{\scriptstyle \to\text{Intro}}$$

This is a past paper question from 2009. In order to prove $(Q \to R) \to R$, we can freely assume $Q \to R$ and need to derive R. Because we can freely assume $Q \to R$, we know that we can get R if we can somehow prove Q. Because our premise is $(P \to P) \to Q$, we know that we can get Q if we can somehow prove $P \to P$. $P \to P$ is easy to prove: we assume P, and then apply $\to$Intro to discharge the assumption of P and prove $P \to P$. This then gives us Q, which then gives us R.

Implication 10

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P]^1 \quad [P \to (Q \to R)]^3}{Q \to R}\,{\scriptstyle \to\text{Elim}} \qquad \dfrac{[P]^1 \quad [P \to Q]^2}{Q}\,{\scriptstyle \to\text{Elim}}}{R}\,{\scriptstyle \to\text{Elim}}}{P \to R}\,{\scriptstyle \to\text{Intro}^1}}{(P \to Q) \to (P \to R)}\,{\scriptstyle \to\text{Intro}^2}}{(P \to (Q \to R)) \to (P \to Q) \to (P \to R)}\,{\scriptstyle \to\text{Intro}^3}$$

This proof looks nasty, but it turns out to be systematic. First of all we look at how the conclusion is composed: $P \to (Q \to R)$ is the antecendent (so we can freely assume that) and $(P \to Q) \to (P \to R)$ on the consequent, so that's what we need to prove.

$(P \to Q) \to (P \to R)$ is what we need to prove first. That means we can freely assume $P \to Q$, and we need to prove $P \to R$. To prove $P \to R$, we can freely assume P, and need to derive R.

So all this has allowed us to freely assume $P \to (Q \to R)$, $P \to Q$ and P. The P and $P \to Q$ together give us Q, which you can see on the right. The P and the $P \to (Q \to R)$ give us $Q \to R$, which you can see on the left. Together, the Q and the $Q \to R$ give us R. Then it's just a case of working backwards from there, building up the conclusion and systematically discharging all the assumptions we've made with three $\to$Intro steps.

Implication 11

$$\dfrac{\dfrac{P \wedge Q}{Q}\,{\scriptstyle \wedge\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}}$$

This is a simpler proof than it might look: $P \wedge Q$ takes us straight to Q, which then takes us to $P \to Q$ without us needing to discharge anything.

Implication 12

$$\dfrac{\dfrac{[P \wedge Q]}{P}\,{\scriptstyle \wedge\text{Elim}}}{P \wedge Q \to P}\,{\scriptstyle \to\text{Intro}}$$

Recall that, according to the bracketing conventions, $P \wedge Q \to P$ is an abbreviation of $(P \wedge Q) \to P$. This is an implication, meaning we can freely assume the antecedent $P \wedge Q$ and need to derive the consequent P. We can get from $P \wedge Q$ to P in one step using $\wedge$Elim. Then all we need to do is apply $\to$Intro discharge the $P \wedge Q$ and derive $P \wedge Q \to P$.

Implication 13

$$\dfrac{\dfrac{\dfrac{[P] \quad P \to (Q \wedge R)}{Q \wedge R}\,{\scriptstyle \to\text{Elim}}}{Q}\,{\scriptstyle \wedge\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}}$$

Here we can freely assume P and need to get to Q. Our assumption of P allows us to get at the $Q \wedge R$ in the premise, which then gives us Q.

Implication 14

$$\dfrac{\dfrac{\dfrac{[P \wedge Q]}{Q}\,{\scriptstyle \wedge\text{Elim}}}{(P \wedge Q) \to Q}\,{\scriptstyle \to\text{Intro}} \quad ((P \wedge Q) \to Q) \to (Q \to P)}{Q \to P}\,{\scriptstyle \to\text{Elim}}$$

Although our conclusion $Q \to P$ is an implication, we won't prove it by assuming Q and deriving P from it. Instead, notice that our premise $((P \wedge Q) \to Q) \to (Q \to P)$ is an implication with $Q \to P$ as its consequent. This means that if we can prove $(P \wedge Q) \to Q$ we can derive $Q \to P$ directly by $\to$Elim. $(P \wedge Q) \to Q$ can be proved by assuming $P \wedge Q$ and deriving Q by $\wedge$Elim.

Implication 15

$$\dfrac{\dfrac{\dfrac{\dfrac{[P]^2 \quad [Q]^1}{P \land Q}\,{\scriptstyle\land\text{Intro}} \qquad (P \land Q) \to R}{R}\,{\scriptstyle\to\text{Elim}}}{Q \to R}\,{\scriptstyle\to\text{Intro}^1}}{P \to (Q \to R)}\,{\scriptstyle\to\text{Intro}^2}$$

Because our conclusion $P \to (Q \to R)$ is an implication, we know that we need to prove $Q \to R$ from assumptions of P. In order to prove $Q \to R$, we need to prove R and can freely assume Q. Our assumptions of P and Q together give us $P \land Q$; that, combined with the premise, gives us R. Then we apply $\to$Intro twice to discharge our assumptions of P and Q.

Implication 16

$$\dfrac{\dfrac{\dfrac{[P] \quad \dfrac{(P \to Q) \land (P \to R)}{P \to Q}\,{\scriptstyle\land\text{Elim}}}{Q}\,{\scriptstyle\to\text{Elim}} \qquad \dfrac{[P] \quad \dfrac{(P \to Q) \land (P \to R)}{P \to R}\,{\scriptstyle\land\text{Elim}}}{R}\,{\scriptstyle\to\text{Elim}}}{Q \land R}\,{\scriptstyle\land\text{Intro}}}{P \to (Q \land R)}\,{\scriptstyle\to\text{Intro}}$$

The main connective in the conclusion $P \to (Q \land R)$ is an arrow. That means we can freely assume P and need to get to $Q \land R$. To get to $Q \land R$, we need two separate proofs: one of Q and one of R. Both of these work in a similar way. We split open the original premise $(P \to Q) \land (P \to R)$ to get conditionals $P \to Q$ and $P \to R$, and then we use our assumption of P to give us both Q and R.

Implication 17

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad P \to (Q \land R)}{Q \land R}\,{\scriptstyle\to\text{Elim}}}{Q}\,{\scriptstyle\land\text{Elim}}}{P \to Q}\,{\scriptstyle\to\text{Intro}} \qquad \dfrac{\dfrac{\dfrac{[P] \quad P \to (Q \land R)}{Q \land R}\,{\scriptstyle\to\text{Elim}}}{R}\,{\scriptstyle\land\text{Elim}}}{P \to R}\,{\scriptstyle\to\text{Intro}}}{(P \to Q) \land (P \to R)}\,{\scriptstyle\land\text{Intro}}$$

Because the conclusion $(P \to Q) \land (P \to R)$ is a conjunction, we know we need to do two proofs: one proof of what's on the left, and one proof of what's on the right. In each proof we can freely assume P (which we know we'll discharge each time we apply $\to$Intro), which works with the premise $(P \to (Q \land R))$ to give us $Q \land R$, and from there the Q and the R we need.

Bonus challenge

Using only the introduction and elimination rules for conjunction, any proof of $P_1 \wedge P_1$ from $(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5$ must consist of at least nine steps:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5}{((P_1 \wedge P_2) \wedge P_3) \wedge P_4}\,{\scriptstyle \wedge\mathrm{Elim}}}{(P_1 \wedge P_2) \wedge P_3}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1 \wedge P_2}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1}\,{\scriptstyle \wedge\mathrm{Elim}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5}{((P_1 \wedge P_2) \wedge P_3) \wedge P_4}\,{\scriptstyle \wedge\mathrm{Elim}}}{(P_1 \wedge P_2) \wedge P_3}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1 \wedge P_2}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1 \wedge P_1}\,{\scriptstyle \wedge\mathrm{Intro}}$$

The problem is that the derivation of P_1 from $(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5$ takes four steps, and we have to repeat this derivation if we want to derive $P_1 \wedge P_1$.

Using the rules for implication, we can shorten the proof so that this long derivation is only carried out once:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5}{((P_1 \wedge P_2) \wedge P_3) \wedge P_4}\,{\scriptstyle \wedge\mathrm{Elim}}}{(P_1 \wedge P_2) \wedge P_3}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1 \wedge P_2}\,{\scriptstyle \wedge\mathrm{Elim}}}{P_1}\,{\scriptstyle \wedge\mathrm{Elim}} \qquad \dfrac{\dfrac{[P_1] \quad [P_1]}{P_1 \wedge P_1}\,{\scriptstyle \wedge\mathrm{Intro}}}{P_1 \to (P_1 \wedge P_1)}\,{\scriptstyle \to\mathrm{Intro}}}{P_1 \wedge P_1}\,{\scriptstyle \to\mathrm{Elim}}$$

This means we have a proof of $P_1 \wedge P_1$ from $(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5$ taking only seven steps.

5.4 Disjunction

Disjunction 1

$$\dfrac{P \lor Q \qquad \dfrac{[P]}{Q \lor P}\,{\scriptstyle \lor\text{Intro2}} \qquad \dfrac{[Q]}{Q \lor P}\,{\scriptstyle \lor\text{Intro1}}}{Q \lor P}\,{\scriptstyle \lor\text{Elim}}$$

Our premise $P \lor Q$ splits the proof up into two cases: one where P is true and one where Q is true. Both of these are cases of $Q \lor P$.

Disjunction 2

$$\dfrac{P \lor Q \qquad \dfrac{[P]}{P \lor (Q \lor R)}\,{\scriptstyle \lor\text{Intro1}} \qquad \dfrac{\dfrac{[Q]}{Q \lor R}\,{\scriptstyle \lor\text{Intro1}}}{P \lor (Q \lor R)}\,{\scriptstyle \lor\text{Intro2}}}{P \lor (Q \lor R)}\,{\scriptstyle \lor\text{Elim}}$$

Our premise $P \lor Q$ splits the proof up into two cases: one where P is true and one where Q is true. Both of these are cases of $P \lor (Q \lor R)$.

Disjunction 3

$$\dfrac{(P \lor Q) \lor R \qquad \dfrac{[P \lor Q]^2 \qquad \dfrac{[P]^1}{P \lor (Q \lor R)}\,\lor\text{Intro1} \qquad \dfrac{\dfrac{[Q]^1}{Q \lor R}\,\lor\text{Intro1}}{P \lor (Q \lor R)}\,\lor\text{Intro2}}{P \lor (Q \lor R)}\,\lor\text{Elim}^1 \qquad \dfrac{\dfrac{[R]^2}{Q \lor R}\,\lor\text{Intro2}}{P \lor (Q \lor R)}\,\lor\text{Intro2}}{P \lor (Q \lor R)}\,\lor\text{Elim}^2$$

This proof works similarly to the previous proofs, but there are two splits: $(P \lor Q) \lor R$ splits the proof into a $P \lor Q$ case and an R case, and then the former case splits into a case of P and a case of Q. In all three cases we apply the $\lor$Intro rules to obtain $P \lor (Q \lor R)$.

Disjunction 4

$$\dfrac{(P \lor Q) \lor (R \lor P_1) \qquad \dfrac{[P \lor Q] \qquad \dfrac{\dfrac{[P]}{P \lor P_1}\,\lor\text{I1}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{I1} \qquad \dfrac{\dfrac{[Q]}{R \lor Q}\,\lor\text{I2}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{I2}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{E} \qquad \dfrac{[R \lor P_1] \qquad \dfrac{\dfrac{[R]}{R \lor Q}\,\lor\text{I1}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{I2} \qquad \dfrac{\dfrac{[P_1]}{P \lor P_1}\,\lor\text{I2}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{I1}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{E}}{(P \lor P_1) \lor (R \lor Q)}\,\lor\text{E}$$

This proof is large, but very systematic: the premise $(P \lor Q) \lor (R \lor P_1)$ splits the proof into two further disjunctions, $(P \lor Q)$ and $(R \lor P_1)$, and then these disjunctions split the cases further into cases of P, Q, R and P_1. All four of these are cases where the conclusion $(P \lor P_1) \lor (R \lor Q)$ is true.

Disjunction 5

$$\dfrac{\dfrac{P \wedge (Q \vee R)}{Q \vee R}\,\wedge\text{Elim} \qquad \dfrac{\dfrac{\dfrac{P \wedge (Q \vee R)}{P}\,\wedge\text{Elim} \quad [Q]}{P \wedge Q}\,\wedge\text{Intro}}{(P \wedge Q) \vee (P \wedge R)}\,\vee\text{Intro} \qquad \dfrac{\dfrac{\dfrac{P \wedge (Q \vee R)}{P}\,\wedge\text{Elim} \quad [P]}{P \wedge R}\,\wedge\text{Intro}}{(P \wedge Q) \vee (P \wedge R)}\,\vee\text{Intro}}{(P \wedge Q) \vee (P \wedge R)}\,\vee\text{Elim}$$

Our premise $P \wedge (Q \vee R)$ gives us two sentences to work with: a disjunction $Q \vee R$ and the sentence-letter P alone. $Q \vee R$ splits the proof into a case where Q is true (in the middle) and a case where R is true (on the right). These are, respectively, cases of $P \wedge Q$ and $P \wedge R$, because we get the P for free from our original premise. Therefore both cases are cases of $(P \wedge Q) \vee (P \wedge R)$.

Disjunction 6

$$\dfrac{\dfrac{(P \vee Q) \wedge (P \vee R)}{P \vee Q}\,\wedge\text{Elim} \qquad \dfrac{[P]^2}{P \vee (Q \wedge R)}\,\vee\text{Intro} \qquad \dfrac{\dfrac{(P \vee Q) \wedge (P \vee R)}{P \vee R}\,\wedge\text{Elim} \quad \dfrac{[P]^1}{P \vee (Q \wedge R)}\,\vee\text{Intro} \quad \dfrac{\dfrac{[Q]^2 \quad [R]^1}{Q \wedge R}\,\wedge\text{Intro}}{P \vee (Q \wedge R)}\,\vee\text{Intro}}{P \vee (Q \wedge R)}\,\vee\text{Elim}^1}{P \vee (Q \wedge R)}\,\vee\text{Elim}^2$$

Our premise $(P \vee Q) \wedge (P \vee R)$ is a conjunction, so it gives us two sentences we can use: $P \vee Q$ and $P \vee R$. Working from the bottom up, we can use $P \vee Q$ first to split the proof into a case where P is true and a case where Q is true. In the former case, we can go straight from P to $P \vee (Q \wedge R)$. In the latter case we need to split the proof again: $P \vee R$ splits the proof into a case where P is true (and therefore the conclusion $P \vee (Q \wedge R)$ is true), and a case where R is true. In this final case Q is true as well: we're still part of the case where Q is true, and our assumption of Q gets discharged in the final step of the proof. Because we have both Q and R, we can prove $Q \wedge R$ and therefore have a case where $P \vee (Q \wedge R)$ is true.

Disjunction 7

$$\dfrac{(P \wedge Q) \vee (P \wedge R) \qquad \dfrac{\dfrac{[P \wedge Q]}{P}\,{\scriptstyle \wedge\text{Elim}} \quad \dfrac{\dfrac{[P \wedge Q]}{Q}\,{\scriptstyle \wedge\text{Elim}}}{Q \vee R}\,{\scriptstyle \vee\text{Intro}}}{P \wedge (Q \vee R)}\,{\scriptstyle \wedge\text{Intro}} \qquad \dfrac{\dfrac{[P \wedge R]}{P}\,{\scriptstyle \wedge\text{Elim}} \quad \dfrac{\dfrac{[P \wedge R]}{R}\,{\scriptstyle \wedge\text{Elim}}}{Q \vee R}\,{\scriptstyle \vee\text{Intro}}}{P \wedge (Q \vee R)}\,{\scriptstyle \wedge\text{Intro}}}{P \wedge (Q \vee R)}\,{\scriptstyle \vee\text{Elim}}$$

The premise splits this proof into a case where $P \wedge Q$ is true and a case where $P \wedge R$ is true. In the former case, from Q we can derive $Q \vee R$, so it is a case where $P \wedge (Q \vee R)$ is true. Very similarly in the latter case, from R we can derive $Q \vee R$, so $P \wedge (Q \vee R)$ is also true.

Disjunction 8

$$\dfrac{P \vee (Q \wedge R) \qquad \dfrac{\dfrac{[P]}{P \vee Q}\,{\scriptstyle \vee\text{Intro}} \quad \dfrac{[P]}{P \vee R}\,{\scriptstyle \vee\text{Intro}}}{(P \vee Q) \wedge (P \vee R)}\,{\scriptstyle \wedge\text{Intro}} \qquad \dfrac{\dfrac{\dfrac{[Q \wedge R]}{Q}\,{\scriptstyle \wedge\text{Elim}}}{P \vee Q}\,{\scriptstyle \vee\text{Intro}} \quad \dfrac{\dfrac{[Q \wedge R]}{R}\,{\scriptstyle \wedge\text{Elim}}}{P \vee R}\,{\scriptstyle \vee\text{Intro}}}{(P \vee Q) \wedge (P \vee R)}\,{\scriptstyle \wedge\text{Intro}}}{(P \vee Q) \wedge (P \vee R)}\,{\scriptstyle \vee\text{Elim}}$$

The premise splits this proof into a case where P is true and a case where $Q \wedge R$ is true. In the first case P leads to both $P \vee Q$ and $P \vee R$, giving the conclusion $(P \vee Q) \wedge (P \vee R)$. In the second case, Q and R lead, respectively, to $P \vee Q$ and $P \vee R$, also giving the conclusion.

Disjunction 9

$$\dfrac{(P \to Q) \lor Q \quad [P \to Q] \quad \dfrac{[Q]}{P \to Q}\,{\scriptstyle \to\text{Intro}}}{P \to Q}\,{\scriptstyle \lor\text{Elim}}$$

Our premise $(P \to Q) \lor Q$ provides one case where $P \to Q$ is true. This is the conclusion, so we don't need to do anything else. In other case, we can freely assume Q. From this we can apply $\lor$Intro to derive $P \to Q$ in one step, so the conclusion is true in both cases.

Disjunction 10

$$\dfrac{\dfrac{P \lor Q \quad \dfrac{[P] \quad [P \to Q]}{Q}\,{\scriptstyle \to\text{Elim}} \quad [Q]}{Q}\,{\scriptstyle \lor\text{Elim}}}{(P \to Q) \to Q}\,{\scriptstyle \to\text{Intro}}$$

Our conclusion $(P \to Q) \to Q$ is an implication, so we prove it by assuming $P \to Q$ and providing a proof of Q. Because our premise $P \lor Q$ is a disjunction, we can split the proof into a case where P is true and a case where Q is true. In the case where Q is true, we don't need to do anything. In the case where P is true, we use our assumption of $P \to Q$ to derive Q by $\to$Elim, meaning we have a proof of Q in both cases.

We could also have carried out the proof in a slightly different way and applied $\lor$Elim at the very end:

$$\dfrac{P \lor Q \quad \dfrac{\dfrac{[P] \quad [P \to Q]}{Q}\,{\scriptstyle \to\text{Elim}}}{(P \to Q) \to Q}\,{\scriptstyle \to\text{Intro}} \quad \dfrac{[Q]}{(P \to Q) \to Q}\,{\scriptstyle \to\text{Intro}}}{(P \to Q) \to Q}\,{\scriptstyle \lor\text{Elim}}$$

Note that in this proof we apply $\to$Intro in both branches, so the proof is slightly longer. When using $\lor$Elim it is often possible to apply it at more than one point of the proof, but applying $\lor$Elim as early as possible usually leads to shorter proofs.

Disjunction 11

$$\dfrac{\dfrac{\dfrac{[P \lor R] \qquad \dfrac{\dfrac{\dfrac{[Q]}{P \to Q}\,{\to}\text{Intro} \qquad (P \to Q) \to (P \to R)}{P \to R}\,{\to}\text{Elim} \qquad [P]}{R}\,{\to}\text{Elim} \qquad [R]}{R}\,{\lor}\text{Elim}}{Q \to R}\,{\to}\text{Intro}}{(P \lor R) \to (Q \to R)}\,{\to}\text{Intro}$$

The conclusion $(P \lor R) \to (Q \to R)$ is an implication. This means to prove it we need to prove $Q \to R$, and we can freely assume $P \lor R$. $Q \to R$ is an implication as well; this means we can freely assume Q and need to prove R. $P \lor R$ splits the proof into two cases: in one case we have P and in the other we have R, and in both cases we need to prove R.

The right case is easy: R is just true. In the left case, proving R from P is more complex. To prove $Q \to R$, we need to get to R and we can freely assume Q. This is very helpful: Q gives us $P \to Q$, which then (thanks to our premise $(P \to Q) \to (P \to R)$) gives us $P \to R$. That, with the P we have, gives us R.

Disjunction 12

$$\dfrac{\dfrac{(P \to Q) \lor (P \to R) \qquad \dfrac{\dfrac{[P] \quad [P \to Q]}{Q}\,{\to}\text{Elim}}{Q \lor R}\,{\lor}\text{Intro} \qquad \dfrac{\dfrac{[P] \quad [P \to R]}{R}\,{\to}\text{Elim}}{Q \lor R}\,{\lor}\text{Intro}}{Q \lor R}\,{\lor}\text{Elim}}{P \to (Q \lor R)}\,{\to}\text{Intro}$$

Our conclusion is an implication, so we can freely assume P and need to work towards $Q \lor R$. Our premise $(P \to Q) \lor (P \to R)$ splits the proof into a case where $P \to Q$ is true and a case where $P \to R$ is true. These cases yield, respectively, Q and R (because P is assumed in both cases) and therefore both yield $Q \lor R$.

Disjunction 13

$$
\begin{prooftree}
\AxiomC{$[P \lor Q]$}
\AxiomC{$[P]$}
\AxiomC{$(P \to Q) \land (Q \to P)$}
\RightLabel{$\land$Elim}
\UnaryInfC{$P \to Q$}
\AxiomC{$[P]$}
\RightLabel{$\to$Elim}
\BinaryInfC{Q}
\RightLabel{$\land$Intro}
\BinaryInfC{$P \land Q$}
\AxiomC{$(P \to Q) \land (Q \to P)$}
\RightLabel{$\land$Elim}
\UnaryInfC{$Q \to P$}
\AxiomC{$[Q]$}
\RightLabel{$\to$Elim}
\BinaryInfC{P}
\AxiomC{$[Q]$}
\RightLabel{$\land$Intro}
\BinaryInfC{$P \land Q$}
\RightLabel{$\lor$Elim}
\TrinaryInfC{$P \land Q$}
\RightLabel{$\to$Intro}
\UnaryInfC{$(P \lor Q) \to (P \land Q)$}
\end{prooftree}
$$

Because our conclusion is an implication, we can freely assume $P \lor Q$ and need to prove $P \land Q$. $P \lor Q$ splits the proof into a case where P is true and a case where Q is true. Our premise tells us that P implies Q and that Q implies P. This means that in the case where P is true we can derive Q and in the case where Q is true we can derive P. Therefore $P \land Q$ is true in both cases.

Disjunction 14

$$
\begin{prooftree}
\AxiomC{$[P]$}
\RightLabel{$\lor$Intro}
\UnaryInfC{$P \lor Q$}
\AxiomC{$(P \lor Q) \to (P \land Q)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$P \land Q$}
\RightLabel{$\land$Elim}
\UnaryInfC{Q}
\RightLabel{$\to$Intro}
\UnaryInfC{$P \to Q$}
\AxiomC{$[Q]$}
\RightLabel{$\lor$Intro}
\UnaryInfC{$P \lor Q$}
\AxiomC{$(P \lor Q) \to (P \land Q)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$P \land Q$}
\RightLabel{$\land$Elim}
\UnaryInfC{P}
\RightLabel{$\to$Intro}
\UnaryInfC{$Q \to P$}
\RightLabel{$\land$Intro}
\BinaryInfC{$(P \to Q) \land (Q \to P)$}
\end{prooftree}
$$

Our conclusion is a conjunction, so there are two proofs we have to make: one proof of $P \to Q$ and another proof of $Q \to P$. In the left proof, because we are proving $P \to Q$, we can start from P and need to work towards Q. P gives us $P \lor Q$, which (thanks to our premise, $(P \lor Q) \to (P \land Q)$) gives us $P \land Q$ and therefore Q. On the other side, Q also gives us $P \lor Q$, allowing us to derive P in the same way as in the left proof.

Disjunction 15

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(Q \to R) \land (Q \lor P)}{Q \lor P}\,{\land}\text{Elim} \quad [Q] \quad \dfrac{[P] \quad [P \to Q]}{Q}\,{\to}\text{Elim}}{Q}\,{\lor}\text{Elim} \quad \dfrac{(Q \to R) \land (Q \lor P)}{Q \to R}\,{\land}\text{Elim}}{R}\,{\to}\text{Elim} \quad \dfrac{\dfrac{(Q \to R) \land (Q \lor P)}{Q \lor P}\,{\land}\text{Elim} \quad [Q] \quad \dfrac{[P] \quad [P \to Q]}{Q}\,{\to}\text{Elim}}{Q}\,{\lor}\text{Elim}}{R \land Q}\,{\land}\text{Intro}}{(P \to Q) \to (R \land Q)}\,{\to}\text{Intro}$$

Here our conclusion is an implication; we can freely assume $P \to Q$ and need to derive $R \land Q$. To derive $R \land Q$ we need to derive R (which we do on the left-hand side) and Q (which we derive on the right-hand side). Our premise is a conjunction, so we can derive two helpful sentences from it: $Q \lor P$ and $Q \to R$.

On the right-hand side (where we try to derive Q) we make use of $Q \lor P$. We have one case where we can assume Q, and another case where we can assume P. In the latter case we can use our assumption of $P \to Q$ to derive Q, so Q is true in both cases. On the left-hand side, we prove Q using the same technique we used on the left-hand side, and then use that with $Q \to R$ (which we have derived from our premise) to derive R.

5.5 Biconditional

Biconditional 1

$$\dfrac{\dfrac{[P] \quad P \leftrightarrow Q}{Q}\,{\scriptstyle \leftrightarrow\text{Elim1}} \qquad \dfrac{[Q] \quad P \leftrightarrow Q}{P}\,{\scriptstyle \leftrightarrow\text{Elim2}}}{Q \leftrightarrow P}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

The conclusion is a biconditional, so there are two proofs we need to give: one from P to Q and one from Q to P. Both of these can be done in one step, because we have $P \leftrightarrow Q$ as a premise. Our initial assumptions of P and Q are discharged in the final step.

Biconditional 2

$$\dfrac{\dfrac{\dfrac{[Q] \quad P}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{Intro}} \qquad (P \leftrightarrow Q) \leftrightarrow R}{R}\,{\scriptstyle \leftrightarrow\text{Elim1}} \qquad \dfrac{P \qquad \dfrac{[R] \quad (P \leftrightarrow Q) \leftrightarrow R}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{Elim2}}}{Q}\,{\scriptstyle \leftrightarrow\text{Elim1}}}{Q \leftrightarrow R}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

On the left we have a proof from Q to R and on the right we have a proof from R to Q. The right-hand side is fairly straightforward: we can use R with our premise $(P \leftrightarrow Q) \leftrightarrow R$ to obtain $P \leftrightarrow Q$, and we can then use that with our other premise P to obtain Q. The left-hand side has a less intuitive step: we use our assumption of Q (which will be discharged in the final step) and our premise P to go straight to $P \leftrightarrow Q$; when doing this, no assumptions are discharged. $P \leftrightarrow Q$ and our premise $(P \leftrightarrow Q) \leftrightarrow R$ then give us R.

Biconditional 3

$$\dfrac{\dfrac{\dfrac{[Q] \;\; [P \leftrightarrow Q]}{P}\,{\scriptstyle \leftrightarrow\text{E2}} \quad \dfrac{[P] \;\; [P \leftrightarrow Q]}{Q}\,{\scriptstyle \leftrightarrow\text{E1}}}{Q \leftrightarrow P}\,{\scriptstyle \leftrightarrow\text{I}} \qquad \dfrac{\dfrac{[P] \;\; [Q \leftrightarrow P]}{Q}\,{\scriptstyle \leftrightarrow\text{E2}} \quad \dfrac{[Q] \;\; [Q \leftrightarrow P]}{P}\,{\scriptstyle \leftrightarrow\text{E1}}}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{I}}}{(P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)}\,{\scriptstyle \leftrightarrow\text{I}}$$

This is a past paper question from 2014. We have to prove a biconditional: on the left we need to go from $P \leftrightarrow Q$ to $Q \leftrightarrow P$, and on the right we need to go in the other direction. Because in both cases we need to prove a biconditional, the proof splits again: we need to prove P from Q and Q from P on the left, and Q from P and P from Q on the right. All four of these can be achieved in one step because of the assumptions we have of $P \leftrightarrow Q$ and $Q \leftrightarrow P$, which are discharged in the final step.

Biconditional 4

$$\cfrac{\cfrac{\cfrac{[P]}{P \lor Q}\,{\scriptstyle \lor\text{Intro}} \qquad (P \lor Q) \leftrightarrow Q}{Q}\,{\scriptstyle \leftrightarrow\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}}$$

This proof is a one-way implication; we can freely assume P and need to work towards Q. P gives us $P \lor Q$, and this combined with our premise $(P \lor Q) \leftrightarrow Q$ gives us Q.

Biconditional 5

$$\cfrac{\cfrac{\cfrac{[P] \qquad (P \land Q) \leftrightarrow P}{P \land Q}\,{\scriptstyle \leftrightarrow\text{Elim}}}{Q}\,{\scriptstyle \land\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}}$$

Here we are proving $P \to Q$, which means we can freely assume P and have to derive Q. P and our premise $(P \land Q) \leftrightarrow P$ allow us to derive $P \land Q$, which then gives us Q.

Biconditional 6

$$\cfrac{\cfrac{[P \lor Q] \qquad \cfrac{[P] \qquad P \to Q}{Q}\,{\scriptstyle \to\text{Elim}} \qquad [Q]}{Q}\,{\scriptstyle \lor\text{Elim}} \qquad \cfrac{[Q]}{P \lor Q}\,{\scriptstyle \lor\text{Intro}}}{(P \lor Q) \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

Here we need to prove Q from $P \lor Q$ on the left and $P \lor Q$ from Q on the right. The right-hand side is easy: $P \lor Q$ can be derived from Q in a single step. On the left-hand side the proof is more complex. Our assumption of $P \lor Q$ (which is discharged in the final step) splits the proof two cases. In one, Q is true, which is what we need to derive. In the other, P is true; this, together with the premise $P \to Q$, lets us prove Q.

Biconditional 7

$$\cfrac{\cfrac{[P \land Q]}{P}\,{\scriptstyle \land\text{Elim}} \qquad \cfrac{[P] \qquad \cfrac{[P] \qquad P \to Q}{Q}\,{\scriptstyle \to\text{Elim}}}{P \land Q}\,{\scriptstyle \land\text{Intro}}}{(P \land Q) \leftrightarrow P}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

The conclusion $(P \land Q) \leftrightarrow P$ asks us to provide two proofs: a proof of P from $P \land Q$ (this is on the left and is a simple case of $\land$Elim) and a proof of $P \land Q$ from P. For this latter proof, we use our assumption of P with the premise $P \to Q$ to give us Q, and then join that with our assumption of P to obtain $P \land Q$.

Biconditional 8

$$\dfrac{\dfrac{\dfrac{(P \to Q) \land (Q \to P)}{P \to Q}\,{\scriptstyle \land\text{Elim}} \quad [P]}{Q}\,{\scriptstyle \to\text{Elim}} \qquad \dfrac{\dfrac{(P \to Q) \land (Q \to P)}{Q \to P}\,{\scriptstyle \land\text{Elim}} \quad [Q]}{P}\,{\scriptstyle \to\text{Elim}}}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

This is a past paper question from 2013. Our premise $(P \to Q) \land (Q \to P)$ gives us two sentences we can use: $P \to Q$ allows us to prove Q from P on the left-hand side and $Q \to P$ allows us to prove P from Q on the right-hand side.

Biconditional 9

$$\dfrac{\dfrac{\dfrac{[P \land Q]}{P}\,{\scriptstyle \land\text{Elim}} \quad \dfrac{\dfrac{[P \land Q]}{Q}\,{\scriptstyle \land\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}}}{(P \to Q) \leftrightarrow P}\,{\scriptstyle \leftrightarrow\text{Intro}}}{(P \land Q) \to ((P \to Q) \leftrightarrow P)}\,{\scriptstyle \to\text{Intro}}$$

The conclusion of this proof, $(P \land Q) \to ((P \to Q) \leftrightarrow P)$, is a conditional: this means we can assume $P \land Q$ and need to obtain $(P \to Q) \leftrightarrow P$. Obtaining $(P \to Q) \leftrightarrow P$ turns out to be quite easy. Although we are allowed to assume $P \to Q$ in our proof of P (on the left) and we are allowed to assume P in our proof of $P \to Q$ (on the right), we don't need either of these. Both P and $P \to Q$ follow from $P \land Q$ alone.

Biconditional 10

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad [(P \to Q) \leftrightarrow P]}{P \to Q}\,{\scriptstyle \leftrightarrow\text{Elim}} \quad [P]}{Q}\,{\scriptstyle \to\text{Elim}} \qquad \dfrac{\dfrac{[Q]}{P \to Q}\,{\scriptstyle \to\text{Intro}} \quad [(P \to Q) \leftrightarrow P]}{P}\,{\scriptstyle \leftrightarrow\text{Elim}}}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{Intro}}}{((P \to Q) \leftrightarrow P) \to (P \leftrightarrow Q)}\,{\scriptstyle \to\text{Intro}}$$

The conclusion of this proof is a conditional. This means we can assume $(P \to Q) \leftrightarrow P$ and need to derive $P \leftrightarrow Q$. To prove $P \leftrightarrow Q$ we need two proofs: a proof of Q from P (on the left) and a proof of P from Q (on the right). On the left, our assumption of P together with our assumption of $(P \to Q) \leftrightarrow Q$ gives us $P \to Q$, and we can use this with another assumption of P to get Q. On the right, from Q we can prove $P \to Q$, and this in combination with the assumption of $(P \to Q) \leftrightarrow P$ allows us to prove P.

Biconditional 11

$$\dfrac{\dfrac{\dfrac{[P]}{P \lor Q}\,{\scriptstyle \lor\mathrm{I}} \qquad \dfrac{[P] \quad ((P \lor Q) \leftrightarrow Q) \leftrightarrow P}{(P \lor Q) \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\mathrm{E}}}{Q}\,{\scriptstyle \leftrightarrow\mathrm{E}} \qquad \dfrac{\dfrac{[Q] \quad \dfrac{[Q]}{P \lor Q}\,{\scriptstyle \lor\mathrm{I}}}{(P \lor Q) \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\mathrm{I}} \quad ((P \lor Q) \leftrightarrow Q) \leftrightarrow P}{P}\,{\scriptstyle \leftrightarrow\mathrm{E}}}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\mathrm{I}}$$

For this proof we need to prove Q from assumptions of P on the left-hand side and P from Q on the right-hand side. On the left-hand side, P can be used with the premise $((P \lor Q) \leftrightarrow Q) \leftrightarrow P$ to derive $(P \lor Q) \leftrightarrow Q$. In one step we can prove $P \lor Q$ from P, and that means we can apply $\leftrightarrow$Elim to derive Q.

On the right-hand side, $(P \lor Q) \leftrightarrow Q$ is fairly easy to derive: we have Q on one side and $P \lor Q$ (which follows from Q in one step) on the other side, which means we can derive $(P \lor Q) \leftrightarrow Q$ without discharging any assumptions. This can be used with the premise $((P \lor Q) \leftrightarrow Q) \leftrightarrow P$ to derive P.

Biconditional 12

$$\dfrac{\dfrac{\dfrac{[P \land Q]}{P}\,{\scriptstyle \land\mathrm{E}} \qquad \dfrac{\dfrac{[P \land Q]}{Q}\,{\scriptstyle \land\mathrm{E}} \qquad \dfrac{\dfrac{[P \land Q]}{P}\,{\scriptstyle \land\mathrm{E}} \quad P \to (Q \leftrightarrow R)}{Q \leftrightarrow R}\,{\scriptstyle \to\mathrm{E}}}{R}\,{\scriptstyle \leftrightarrow\mathrm{E}}}{P \land R}\,{\scriptstyle \land\mathrm{I}} \qquad \dfrac{\dfrac{[P \land R]}{P}\,{\scriptstyle \land\mathrm{E}} \qquad \dfrac{\dfrac{[P \land R]}{R}\,{\scriptstyle \land\mathrm{E}} \qquad \dfrac{\dfrac{[P \land R]}{P}\,{\scriptstyle \land\mathrm{E}} \quad P \to (Q \leftrightarrow R)}{Q \leftrightarrow R}\,{\scriptstyle \to\mathrm{E}}}{Q}\,{\scriptstyle \leftrightarrow\mathrm{E}}}{P \land Q}\,{\scriptstyle \land\mathrm{I}}}{(P \land Q) \leftrightarrow (P \land R)}\,{\scriptstyle \leftrightarrow\mathrm{I}}$$

This is a past paper question from 2012. In order to prove the biconditional $(P \land Q) \leftrightarrow (P \land R)$, we need to prove $P \land R$ from assumptions of $P \land Q$ and we need to prove $P \land Q$ from assumptions of $P \land R$. On the left-hand side, our assumption of $P \land Q$ allows us to obtain P. This, combined with our premise $P \to (Q \leftrightarrow R)$, gives us $Q \leftrightarrow R$. We can use this with Q (which we also obtain from $P \land Q$) to obtain R; this and P give us $P \land R$. The right-hand side works the same way, except we work from $Q \leftrightarrow R$ and R to derive Q.

Biconditional 13

$$\dfrac{\dfrac{[P\vee(Q\wedge R)] \quad \dfrac{\dfrac{[P]}{P\vee R}\vee\mathrm{I} \quad \dfrac{[P]}{P\vee R}\vee\mathrm{I}}{(P\vee Q)\wedge(P\vee R)}\wedge\mathrm{I} \quad \dfrac{\dfrac{\dfrac{[Q\wedge R]}{Q}\wedge\mathrm{E}}{P\vee Q}\vee\mathrm{I} \quad \dfrac{\dfrac{[Q\wedge R]}{R}\wedge\mathrm{E}}{P\vee R}\vee\mathrm{I}}{(P\vee Q)\wedge(P\vee R)}\wedge\mathrm{I}}{(P\vee Q)\wedge(P\vee R)}\vee\mathrm{E} \quad \dfrac{\dfrac{[(P\vee Q)\wedge(P\vee R)]}{P\vee Q}\wedge\mathrm{E} \quad \dfrac{[P]}{P\vee(Q\wedge R)}\vee\mathrm{I} \quad \dfrac{\dfrac{[(P\vee Q)\wedge(P\vee R)]}{P\vee R}\wedge\mathrm{E} \quad \dfrac{[P]}{P\vee(Q\wedge R)}\vee\mathrm{I} \quad \dfrac{\dfrac{[Q]\quad[R]}{Q\wedge R}\wedge\mathrm{I}}{P\vee(Q\wedge R)}\vee\mathrm{I}}{P\vee(Q\wedge R)}\vee\mathrm{E}}{P\vee(Q\wedge R)}\vee\mathrm{E}}{(P\vee(Q\wedge R))\leftrightarrow((P\vee Q)\wedge(P\vee R))}\leftrightarrow\mathrm{I}$$

This is a past paper question from 2010. It breaks down into two proofs: a proof of $(P\vee Q)\wedge(P\vee R)$ from $P\vee(Q\wedge R)$ and a proof of $P\vee(Q\wedge R)$ from $(P\vee Q)\wedge(P\vee R)$.

On the left-hand side, our assumption of $P\vee(Q\wedge R)$ splits the proof further into a case where P is true and a case where $Q\wedge R$ is true. In the first case P leads to both $P\vee Q$ and $P\vee R$, giving $(P\vee Q)\wedge(P\vee R)$. In the second case, Q and R lead, respectively, to $P\vee Q$ and $P\vee R$, also giving $(P\vee Q)\wedge(P\vee R)$.

On the right-hand side, our assumption of $(P\vee Q)\wedge(P\vee R)$ is a conjunction, giving us $P\vee Q$ and $P\vee R$. $P\vee Q$ splits this part of the proof into a case where P is true and a case where Q is true. In the former case, we can derive $P\vee(Q\wedge R)$ from P. In the latter case we use $P\vee R$ to split the proof again into a case where P is true (and therefore the conclusion $P\vee(Q\wedge R)$ is true), and a case where R is true. In this final case Q is true as well: we're still part of the case where Q is true, and our assumption of Q gets discharged in the final step of the proof. Because we have both Q and R, we can prove $Q\wedge R$ and therefore have a case where $P\vee(Q\wedge R)$ is true.

5.6 Negation

Negation 1

$$\dfrac{P \qquad [\neg P]}{\neg\neg P}\,{\scriptstyle \neg\text{Intro}}$$

In order to prove $\neg\neg P$ (in other words, to prove that it is not the case that $\neg P$) we start by assuming $\neg P$ and show that it leads to a contradiction. Showing it leads to a contradiction is very easy: $\neg P$ contradicts our premise P. Applying $\neg$Intro allows us to discharge our assumption of $\neg P$ and prove $\neg\neg P$.

Negation 2

$$\dfrac{\dfrac{[P \wedge Q]}{P}\,{\scriptstyle \wedge\text{Elim}} \qquad \neg P}{\neg(P \wedge Q)}\,{\scriptstyle \neg\text{Intro}}$$

To prove $\neg(P \wedge Q)$, we start by assuming $P \wedge Q$ and try to derive a contradiction from it. $P \wedge Q$ gives us P, which contradicts our premise $\neg P$.

Negation 3

$$\dfrac{[P] \qquad \dfrac{[P] \qquad P \to \neg P}{\neg P}\,{\scriptstyle \neg\text{Elim}}}{\neg P}\,{\scriptstyle \neg\text{Intro}}$$

To prove $\neg P$, we start by assuming P and try to derive a contradiction from it. Our premise $P \to \neg P$ doesn't contradict P straight away, but because we've assumed P we can apply $\to$Elim to obtain $\neg P$; this then contradicts P and allows us to apply $\neg$Intro.

Negation 4

$$\dfrac{\dfrac{[Q]}{P \to Q}\,{\scriptstyle \to\text{Intro}} \qquad \neg(P \to Q)}{\neg Q}\,{\scriptstyle \neg\text{Intro}}$$

To prove $\neg Q$, we start by assuming Q and try to derive a contradiction. We can derive $P \to Q$ from Q in one step without discharging any assumptions; this then contradicts our premise $\neg(P \to Q)$.

Negation 5

$$\dfrac{\dfrac{\dfrac{[P] \quad [Q]}{P \wedge Q}\,{\scriptstyle \wedge\text{Intro}} \qquad \neg(P \wedge Q)}{\neg Q}\,{\scriptstyle \neg\text{Intro}}}{P \to \neg Q}\,{\scriptstyle \to\text{Intro}}$$

Our conclusion $P \to \neg Q$ is a conditional statement, so we assume P and try to derive $\neg Q$. To derive $\neg Q$, we assume Q and try to prove a contradiction. Our premise $\neg(P \wedge Q)$ is a negated statement, so we have a contradiction if we can prove $P \wedge Q$. Because we have assumptions of P and Q, we can indeed derive $P \wedge Q$. This gives us the contradiction we need to apply $\neg$Intro, derive $\neg P$ and discharge our assumption of Q. Finally we apply $\to$Intro to discharge our assumption of P and prove our conclusion.

Negation 6

$$\dfrac{\dfrac{\dfrac{[P] \quad P \to Q}{Q}\,{\scriptstyle \to\text{Elim}} \qquad [\neg Q]}{\neg P}\,{\scriptstyle \neg\text{Intro}}}{\neg Q \to \neg P}\,{\scriptstyle \to\text{Intro}}$$

Our conclusion $\neg Q \to \neg P$ is a conditional statement. This means we can assume $\neg Q$ and need to prove $\neg P$. To prove $\neg P$ we can assume P and need to derive a contradiction. Our assumption of P and the premise $P \to Q$ let us prove Q, which contradicts our assumption of $\neg Q$. P is discharged in the $\neg$Intro step, while $\neg Q$ is discharged in the $\to$Intro step.

Negation 7

$$
\dfrac{\dfrac{[(P \wedge \neg P) \vee (Q \wedge \neg Q)] \qquad \dfrac{\dfrac{[P \wedge \neg P]}{P}\,\wedge\text{Elim} \qquad \dfrac{[P \wedge \neg P]}{\neg P}\,\wedge\text{Elim}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))}\,\neg\text{Intro} \qquad \dfrac{\dfrac{[Q \wedge \neg Q]}{Q}\,\wedge\text{Elim} \qquad \dfrac{[Q \wedge \neg Q]}{\neg Q}\,\wedge\text{Elim}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))}\,\neg\text{Intro}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))}\,\vee\text{Elim} \qquad [(P \wedge \neg P) \vee (Q \wedge \neg Q)]}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))}\,\neg\text{Intro}
$$

Our conclusion $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ is a negated statement, so we can prove it by assuming $(P \wedge \neg P) \vee (Q \wedge \neg Q)$ and deriving a contradiction from it. This is a disjunction, so it splits the proof into a case where $P \wedge \neg P$ is true and a case where $Q \wedge \neg Q$ is true.

In both of these cases, it is easy to find a contradiction: in the left-hand case P contradicts $\neg P$ and in the right-hand case Q contradicts $\neg Q$. However, when we apply $\neg$Intro and derive $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ within each case we can't discharge our assumption of $(P \wedge \neg P) \vee (Q \wedge \neg Q)$, since it appears further down in the proof.

Instead what we have to do after applying $\vee$Elim (discharging $P \wedge \neg P$ and $Q \wedge \neg Q$) is assume $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ a second time. Then we can apply $\neg$Intro, derive $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ again and discharge both assumptions of $(P \wedge \neg P) \vee (Q \wedge \neg Q)$.

Negation 8

$$\dfrac{\dfrac{\dfrac{[P]}{P \lor Q}\,{\scriptstyle \lor\text{Intro}} \qquad \neg(P \lor Q)}{\neg P}\,{\scriptstyle \neg\text{Intro}} \qquad \dfrac{\dfrac{[Q]}{P \lor Q}\,{\scriptstyle \lor\text{Intro}} \qquad \neg(P \lor Q)}{\neg Q}\,{\scriptstyle \neg\text{Intro}}}{\neg P \land \neg Q}\,{\scriptstyle \land\text{Intro}}$$

Our conclusion is a conjunction, so we need to provide two proofs: a proof of $\neg P$ and a proof of $\neg Q$. On the left-hand side we assume P and try to derive a contradiction. Our premise $\neg(P \lor Q)$ is a negated statement, so we can get the contradiction we need by deriving $P \lor Q$ from P. Similarly, on the right-hand side we derive $P \lor Q$ from our assumption of Q.

Negation 9

$$\dfrac{\neg P \lor \neg Q \qquad \dfrac{\dfrac{[P \land Q]}{P}\,{\scriptstyle \land\text{Elim}} \qquad [\neg P]}{\neg(P \land Q)}\,{\scriptstyle \neg\text{Intro}} \qquad \dfrac{\dfrac{[P \land Q]}{Q}\,{\scriptstyle \land\text{Elim}} \qquad [\neg Q]}{\neg(P \land Q)}\,{\scriptstyle \neg\text{Intro}}}{\neg(P \land Q)}\,{\scriptstyle \lor\text{Elim}}$$

Our premise $\neg P \lor \neg Q$ is a disjunction, which splits the proof into a case where we can assume $\neg P$ and a case where we can assume $\neg Q$. In both cases we need to prove $\neg(P \land Q)$, which we do by assuming $P \land Q$ and showing it leads to a contradiction. From $P \land Q$ we can prove P, which contradicts $\neg P$ on the left-hand side, and Q, which contradicts $\neg Q$ on the right-hand side.

Negation 10

$$\dfrac{\dfrac{[P] \qquad \neg P}{Q}\,{\scriptstyle \neg\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}}$$

To prove $P \to Q$ we start from an assumption of P and need to derive Q. P contradicts our premise $\neg P$, and this allows us to apply $\neg$Elim and prove Q. No assumptions are discharged until the next step, where $\to$Intro discharges P.

Negation 11

$$\dfrac{\dfrac{P \land \neg P}{P}\,{\scriptstyle \land\text{Elim}} \qquad \dfrac{P \land \neg P}{\neg P}\,{\scriptstyle \land\text{Elim}}}{Q}\,{\scriptstyle \neg\text{Elim}}$$

Here, as in the previous problem, we prove Q from a contradiction. This is provided from our premise alone: from $P \land \neg P$ we can prove both P and $\neg P$, which contradict each other.

Negation 12

$$\dfrac{\dfrac{P \lor Q \qquad \dfrac{[P] \quad [\neg P]}{Q}\,{\scriptstyle \neg\text{Elim}} \qquad [Q]}{Q}\,{\scriptstyle \lor\text{Elim}}}{\neg P \to Q}\,{\scriptstyle \to\text{Intro}}$$

This is similar to a past paper problem from 2013. In order to prove $\neg P \to Q$, we can freely assume $\neg P$ and need to prove Q. Our premise $P \lor Q$ gives us two cases to consider: one we assume Q (which is great, because that's what we need to prove) and one where we assume P. P contradicts our assumption of $\neg P$, allowing us to derive Q.

Negation 13

$$\dfrac{\dfrac{\dfrac{P \land \neg Q}{P}\,{\scriptstyle \land\text{Elim}} \qquad P \to Q}{Q}\,{\scriptstyle \to\text{Elim}} \qquad \dfrac{P \land \neg Q}{\neg Q}\,{\scriptstyle \land\text{Elim}}}{R}\,{\scriptstyle \neg\text{Elim}}$$

In order to prove R (which doesn't appear in any of our premises) we need to find a contradiction. $P \land \neg Q$ and $P \to Q$ together provide one: $P \land \neg Q$ lets us prove P, and with this and $P \to Q$ we can prove Q. This contradicts the $\neg Q$ which also follows from $P \land \neg Q$.

Negation 14

$$\dfrac{\dfrac{P \lor Q \qquad \dfrac{[P] \qquad \dfrac{[P] \quad P \leftrightarrow Q}{Q}\,{\scriptstyle \leftrightarrow\text{E}}}{P \land Q}\,{\scriptstyle \land\text{I}} \qquad \dfrac{\dfrac{[Q] \quad P \leftrightarrow Q}{P}\,{\scriptstyle \leftrightarrow\text{E}} \qquad [Q]}{P \land Q}\,{\scriptstyle \land\text{I}}}{P \land Q}\,{\scriptstyle \lor\text{E}} \qquad \neg(P \land Q)}{R}\,{\scriptstyle \neg\text{E}}$$

R doesn't appear in any of our premises, so we prove it by finding a contradiction. Of our three premises, $\neg(P \land Q)$ is a negated statement, so it makes sense to try to prove $P \land Q$ in order to find the contradiction we need. $P \lor Q$ splits the proof into two cases: one where P is true and one where Q is true. In both of these cases we apply the other premise $P \leftrightarrow Q$ to obtain the other sentence-letter and hence derive $P \land Q$. This contradicts $\neg(P \land Q)$, allowing us to derive R.

Negation 15

$$\dfrac{[\neg P] \quad \neg\neg P}{P}\,{\scriptstyle \neg\text{Elim}}$$

Here we cannot provide a proof of P directly. Instead, we prove P by assuming $\neg P$ and showing it leads to a contradiction. $\neg P$ contradicts our premise $\neg\neg P$, meaning we can apply $\neg$Elim; this discharges our assumption of $P \neg P$ and provides a proof of P.

Negation 16

$$\dfrac{\dfrac{\dfrac{\dfrac{[P]^1}{P \lor \neg P}\,{\scriptstyle \lor\text{Intro}} \quad [\neg(P \lor \neg P)]^2}{\neg P}\,{\scriptstyle \neg\text{Intro}^1}}{P \lor \neg P}\,{\scriptstyle \lor\text{Intro}} \qquad [\neg(P \lor \neg P)]^2}{P \lor \neg P}\,{\scriptstyle \neg\text{Elim}^2}$$

This proof is counter-intuitive, relying on a special technique. Our conclusion $P \lor \neg P$ is a disjunction, but we cannot show it is true by proving either disjunct: we have no proof of P and no proof of $\neg P$. Instead, we must assume that $P \lor \neg P$ is false (in other words, we assume $\neg(P \lor \neg P)$) and derive a contradiction from that.

But we still have a problem: even if we assume $\neg(P \lor \neg P)$, what do we have which it contradicts? This proof has no premises. We have to rely on a slightly counter-intuitive trick. We assume $\neg(P \lor \neg P)$ and try to derive a contradiction from it, but initially we also assume P. From P we can prove $P \lor \neg P$, which contradicts $\neg(P \lor \neg P)$.

But we can't go right ahead and apply $\neg$Elim to prove $P \lor \neg P$, because that would leave our assumption of P undischarged. So instead what we do is use the contradiction to apply $\neg$Intro, discharging P and proving $\neg P$.

From $\neg P$, we can apply $\lor$Intro to prove $P \lor \neg P$ again. Assuming $\neg(P \lor \neg P)$ leads to a contradiction again, but this time our only undischarged assumption is $\neg(P \lor \neg P)$ itself. We're free to apply $\neg$Elim, discharge our two assumptions of $\neg(P \lor \neg P)$ and prove $P \lor \neg P$.

The following is a possible alternative proof:

$$\dfrac{\dfrac{\dfrac{[P]^1}{P \lor \neg P}\,{\scriptstyle \lor\text{Intro}} \quad [\neg(P \lor \neg P)]^3}{\neg P}\,{\scriptstyle \neg\text{Intro}^1} \qquad \dfrac{\dfrac{[\neg P]^2}{P \lor \neg P}\,{\scriptstyle \lor\text{Intro}} \quad [\neg(P \lor \neg P)]^3}{P}\,{\scriptstyle \neg\text{Elim}^2}}{P \lor \neg P}\,{\scriptstyle \neg\text{Elim}^3}$$

This kind of structure is possible for many indirect proofs, but it tends to produce longer proofs than the technique above.

Negation 17

$$\dfrac{\dfrac{\dfrac{[\neg P]}{\neg P \lor \neg Q}\,{\lor}\text{Intro} \qquad \neg(\neg P \lor \neg Q)}{P}\,\neg\text{Elim} \qquad \dfrac{\dfrac{[\neg Q]}{\neg P \lor \neg Q}\,{\lor}\text{Intro} \qquad \neg(\neg P \lor \neg Q)}{Q}\,\neg\text{Elim}}{P \land Q}\,{\land}\text{Intro}$$

In order to prove $P \land Q$ we need to provide a proof of P and a proof of Q. On the left-hand side we can't prove P directly, so instead we assume $\neg P$ and show that this leads to a contradiction. From $\neg P$ we can derive $\neg P \lor \neg Q$, which contradicts our premise $\neg(\neg P \lor \neg Q)$. The contradiction allows us to apply $\neg$Elim, discharge our assumption of $\neg P$ and derive P. Similarly on the right-hand side we assume $\neg Q$ and show it leads to a contradiction in order to derive Q.

We could also prove $P \land Q$ by assuming $\neg(P \land Q)$ and deriving a contradiction from it. This proof is slightly longer: we have to assume P and Q as well as $P \land Q$, and use three contradictions to discharge these three assumptions. This alternate proof is shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad [Q]}{P \land Q}\,{\land}\text{Intro} \qquad [\neg(P \land Q)]}{\neg P}\,\neg\text{Intro}}{\neg P \lor \neg Q}\,{\lor}\text{Intro} \qquad \neg(\neg P \lor \neg Q)}{\neg Q}\,\neg\text{Intro}}{\neg P \lor \neg Q}\,{\lor}\text{Intro} \qquad \neg(\neg P \lor \neg Q)}{P \land Q}\,\neg\text{Elim}$$

Negation 18

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg P]}{\neg P \lor \neg Q}\,{\lor}\text{I} \qquad [\neg(\neg P \lor \neg Q)]}{P}\,\neg\text{E} \qquad [Q]}{P \land Q}\,{\land}\text{I} \qquad \neg(P \land Q)}{\neg Q}\,\neg\text{I}}{\neg P \lor \neg Q}\,{\lor}\text{I} \qquad [\neg(\neg P \lor \neg Q)]}{\neg P \lor \neg Q}\,\neg\text{E}$$

This is a past paper question from 2015. We want to try to prove $\neg P \lor \neg Q$, but our premise $\neg(P \land Q)$ doesn't give a direct proof of either $\neg P$ or $\neg Q$. We will need to assume $\neg(\neg P \lor \neg Q)$ and show it leads to a contradiction.

First we can assume $\neg P$. This lets us derive $\neg P \lor \neg Q$, which contradicts our assumption of $\neg(\neg P \lor \neg Q)$. Applying $\neg$Elim, we discharge our assumption of $\neg P$ and prove P.

P and $\neg(P \land Q)$ together imply $\neg Q$. Assuming Q gives us $P \land Q$, which contradicts our premise $\neg(P \land Q)$ and allows us to apply $\neg$Intro, discharge Q

and prove $\neg Q$. From this we can prove $\neg P \vee \neg Q$ again, which provides another contradiction with our assumption of $\neg(\neg P \vee \neg Q)$. Finally we discharge all of our assumptions of $\neg(\neg P \vee \neg Q)$ and derive $\neg P \vee \neg Q$.

Negation 19

$$\dfrac{\dfrac{\dfrac{[P] \quad [\neg P]}{Q}\,{\scriptstyle \neg\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}} \qquad \neg(P \to Q)}{P}\,{\scriptstyle \neg\text{Elim}}$$

The premise $\neg(P \to Q)$ doesn't allow us to prove P directly. Instead, we assume $\neg P$ and try to derive a contradiction from it. What kind of contradiction should we be looking for? Our premise, $\neg(P \to Q)$, is a negated sentence, so we will have a contradiction if we manage to prove $P \to Q$.

Proving $P \to Q$ from an assumption of P is something we've already done in problem 9. To prove $P \to Q$, we can assume P and need to derive Q. P and $\neg P$ together is a contradiction, allowing us to apply $\neg$Elim and derive Q. Applying $\to$Intro then gives us $P \to Q$, discharging P and providing a sentence which contradicts $\neg(P \to Q)$. We can then apply $\neg$Elim, discharge $\neg P$ and prove P.

Negation 20

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad [\neg P]}{Q}\,{\scriptstyle \neg\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}} \qquad (P \to Q) \to P)}{P}\,{\scriptstyle \to\text{Elim}} \qquad [\neg P]}{P}\,{\scriptstyle \neg\text{Elim}}$$

It might be surprising that we need to use a negation rule in this proof: neither the premise $(P \to Q) \to P)$ nor the conclusion P have any negation symbols in them. But we cannot prove P directly from $(P \to Q) \to P)$; again, we need to assume $\neg P$ and show that it leads to a contradiction.

Assuming $\neg P$ allows us to derive $P \to Q$: this is because assuming P (which gets discharged by $\to$Intro) allows us to apply $\neg$Elim and derive Q. Having proved $P \to Q$, our premise $(P \to Q) \to P)$ allows us to derive P. With P and our assumption of $\neg P$, we have a contradiction. This means we can discharge our assumptions of $\neg P$ and derive P.

Negation 21

$$\dfrac{\dfrac{[\neg Q] \qquad \dfrac{[P] \quad P \leftrightarrow \neg\neg Q}{\neg\neg Q}\,{\scriptstyle \leftrightarrow\text{Elim}}}{Q}\,{\scriptstyle \neg\text{Elim}} \qquad \dfrac{\dfrac{[Q] \quad [\neg Q]}{\neg\neg Q}\,{\scriptstyle \neg\text{Intro}} \qquad P \leftrightarrow \neg\neg Q}{P}\,{\scriptstyle \leftrightarrow\text{Elim}}}{P \leftrightarrow Q}\,{\scriptstyle \leftrightarrow\text{Intro}}$$

This is a past paper question from 2011. In order to prove $P \leftrightarrow Q$, we need to provide a proof of Q from assumptions of P and a proof of P from assumptions of Q.

On the left-hand side, we can use our assumption of P with the premise $P \leftrightarrow \neg\neg Q$ to derive $\neg\neg Q$. From $\neg\neg Q$ we can prove $\neg Q$ indirectly: an assumption of $\neg Q$ contradicts $\neg\neg Q$, so we can discharge $\neg Q$ and derive Q.

On the right-hand side, the premise $P \leftrightarrow \neg\neg Q$ only helps us if we have a proof of $\neg\neg Q$. Fortunately we can derive this from Q: assuming $\neg Q$ leads to a contradiction, so we can apply $\neg$Intro, discharge the $\neg Q$ and put an extra negation symbol on the front. Having $\neg\neg Q$ then allows us to prove P.

Negation 22

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad [\neg P]}{Q}\,{\scriptstyle \neg\text{Elim}}}{P \to Q}\,{\scriptstyle \to\text{Intro}} \qquad (P \to Q) \to Q}{Q}\,{\scriptstyle \to\text{Elim}} \qquad [\neg Q]}{P}\,{\scriptstyle \neg\text{Elim}}}{\neg Q \to P}\,{\scriptstyle \to\text{Intro}}$$

Our conclusion is a conditional statement, so we assume $\neg Q$ and try to prove P. Our premise $(P \to Q) \to Q$ gives us no way of proving P directly, so we have to assume $\neg P$ and try to derive a contradiction.

Because our assumption $\neg Q$ is a negated statement, it makes sense to try to prove Q in order to contradict it. Our premise is $(P \to Q) \to Q$, so we know we can prove Q if we can provide a proof of $P \to Q$.

We do this by assuming P; from this and our other assumption of $\neg P$ we can derive Q by $\neg$Elim. Applying $\to$Intro for the first time discharges our assumption of P and lets us prove $P \to Q$; we then make use of our premise to derive Q, which contradicts our assumption of $\neg Q$. Applying $\neg$Elim a second time discharges our assumption of $\neg P$ and lets us prove P. Finally, we apply $\to$Intro a second time to discharge our assumption of $\neg Q$ and prove the conclusion $\neg Q \to P$.

Negation 23

$$\dfrac{[P \lor Q] \qquad \dfrac{[P \lor Q] \qquad \dfrac{[P] \quad \dfrac{\neg P \land \neg Q}{\neg P}\,{\scriptstyle \land\text{Elim}}}{\neg(P \lor Q)}\,{\scriptstyle \neg\text{Elim}} \qquad \dfrac{[Q] \quad \dfrac{\neg P \land \neg Q}{\neg Q}\,{\scriptstyle \land\text{Elim}}}{\neg(P \lor Q)}\,{\scriptstyle \neg\text{Elim}}}{\neg(P \lor Q)}\,{\scriptstyle \lor\text{Elim}}}{\neg(P \lor Q)}\,{\scriptstyle \neg\text{Intro}}$$

This is a past paper question from 2014. To show $\neg(P \lor Q)$ we start by assuming $P \lor Q$ and try to derive a contradiction; this assumption of $P \lor Q$ splits the proof into a case where we can assume P and a case where we can assume Q. In both of these cases, contradictions are easy to get: our premise $\neg P \land \neg Q$ gives us $\neg P$ (which contradicts P on the left-hand side) and also gives us $\neg Q$ (which contradicts Q on the right-hand side).

These contradictions allow us to apply $\neg$Elim and prove $\neg(P \lor Q)$, but they don't allow us to discharge our assumption of $P \lor Q$, because it appears in a different branch of the proof. What we need to do is assume $P \lor Q$ again after applying $\lor$Elim. Then we have another contradiction which lets us discharge both assumptions of $P \lor Q$ and prove $\neg(P \lor Q)$.

Negation 24

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P]}{P \lor (P \to Q)}\,{\scriptstyle \lor\text{I}} \qquad [\neg(P \lor (P \to Q))]}{\neg P}\,{\scriptstyle \neg\text{I}} \qquad [P]}{Q}\,{\scriptstyle \neg\text{E}}}{P \to Q}\,{\scriptstyle \to\text{I}}}{P \lor (P \to Q)}\,{\scriptstyle \lor\text{I}} \qquad [\neg(P \lor (P \to Q))]}{P \lor (P \to Q)}\,{\scriptstyle \neg\text{E}}$$

This is another indirect proof of a disjunction: by assuming $\neg(P \lor (P \to Q))$, the negation of what we want to prove, we try to derive a contradiction. We start by assuming P, which lets us prove $P \lor (P \to Q)$, and therefore gives rise to a contradiction. This contradiction allows us to apply $\neg$Intro, discharge our assumption of P and derive $\neg P$.

From $\neg P$, it is possible to derive $P \to Q$: assuming P for a second time gives us a contradiction and allows us to derive Q, and we can then apply $\to$Intro to derive $P \to Q$ and discharge this second assumption of P. $P \to Q$ lets us prove $P \lor (P \to Q)$, so we have a contradiction again. This second contradiction allows us to discharge our assumptions of $\neg(P \lor (P \to Q))$ and prove $P \lor (P \to Q)$.

Negation 25

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[Q]}{P \to Q}\;{\scriptstyle \to\text{Intro}}}{(P \to Q) \lor (Q \to R)}\;{\scriptstyle \lor\text{Intro}} \qquad [\neg((P \to Q) \lor (Q \to R))]}{\neg Q}\;{\scriptstyle \neg\text{Intro}} \qquad [Q]}{R}\;{\scriptstyle \neg\text{Elim}}}{Q \to R}\;{\scriptstyle \to\text{Intro}}}{(P \to Q) \lor (Q \to R)}\;{\scriptstyle \lor\text{Intro}} \qquad [\neg((P \to Q) \lor (Q \to R))]}{(P \to Q) \lor (Q \to R)}\;{\scriptstyle \neg\text{Elim}}
$$

We prove our conclusion $(P \to Q) \lor (P \to R)$ by assuming its negation and trying to derive a contradiction, but this time the very first thing we do is assume Q.

If Q is true, which we assume initially, we can derive $P \to Q$ and then we can derive $(P \to Q) \lor (P \to R)$. We then get a contradiction, so we discharge our assumption of Q and prove $\neg Q$. From $\neg Q$ we can derive $Q \to R$ and then $(P \to Q) \lor (P \to R)$ again, so our assumption of $\neg((P \to Q) \lor (P \to R))$ still leads to a contradiction. This means we can discharge our assumptions of $\neg((P \to Q) \lor (P \to R))$ and prove $(P \to Q) \lor (P \to R)$.

The proof would still have worked if we had initially assumed $P \to Q$: this is because from $\neg(P \to Q)$ we can derive $\neg Q$, from which we can then prove $Q \to R$. Carrying out the proof this way is a tiny bit longer (requiring eight steps instead of seven), but it is reliable: to prove $\phi \lor \psi$ in general by indirect proof, it will always work to start by assuming ϕ. The alternate proof is shown below:

$$
\begin{prooftree}
\AxiomC{$[P \to Q]$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$(P \to Q) \vee (Q \to R)$}
\AxiomC{$[\neg((P \to Q) \vee (Q \to R))]$}
\RightLabel{$\neg$Intro}
\BinaryInfC{$\neg(P \to Q)$}
\AxiomC{$[Q]$}
\RightLabel{$\to$Intro}
\UnaryInfC{$P \to Q$}
\RightLabel{$\neg$Intro}
\BinaryInfC{$\neg Q$}
\AxiomC{$[Q]$}
\RightLabel{$\neg$Elim}
\BinaryInfC{R}
\RightLabel{$\to$Intro}
\UnaryInfC{$Q \to R$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$(P \to Q) \vee (Q \to R)$}
\AxiomC{$[\neg((P \to Q) \vee (Q \to R))]$}
\RightLabel{$\neg$Elim}
\BinaryInfC{$(P \to Q) \vee (Q \to R)$}
\end{prooftree}
$$

Negation 26

$$
\begin{prooftree}
\AxiomC{$R \vee \neg Q$}
\AxiomC{$[R]^1$}
\AxiomC{$\neg R \wedge \neg Q_2$}
\RightLabel{$\wedge$E}
\UnaryInfC{$\neg R$}
\RightLabel{$\neg$E}
\BinaryInfC{$\neg Q$}
\AxiomC{$[\neg Q]^1$}
\RightLabel{$\vee$E^1}
\TrinaryInfC{$\neg Q$}
\AxiomC{$[\neg P]^2$}
\AxiomC{$\neg P \to Q$}
\RightLabel{$\to$E}
\BinaryInfC{Q}
\RightLabel{$\neg$E^2}
\BinaryInfC{P}
\AxiomC{$P \to (Q_1 \vee Q_2)$}
\RightLabel{$\to$E}
\BinaryInfC{$Q_1 \vee Q_2$}
\AxiomC{$[Q_1]^3$}
\AxiomC{$[Q_2]^3$}
\AxiomC{$\neg R \wedge \neg Q_2$}
\RightLabel{$\wedge$E}
\UnaryInfC{$\neg Q_2$}
\RightLabel{$\neg$E}
\BinaryInfC{Q_1}
\RightLabel{$\vee$E^3}
\TrinaryInfC{Q_1}
\end{prooftree}
$$

This is a past paper question from 2010. Although it contains lots of different sentence letters, it is possible to work through it methodically.

First of all, we can use the premise $\neg R \wedge Q_2$to derive $\neg R$ and Q_2. $\neg R$ is helpful because one of our other premises is $R \vee \neg Q$. If either $R \vee \neg Q$ is true, and R is false (because we've proved $\neg R$), then $\neg Q$ must be true.

This then allows us to establish P by indirect proof. If we assume $\neg P$, the premise $\neg P \to Q$ allows us to derive Q, but because we've already proved $\neg Q$ we have a contradiction. Therefore we can discharge $\neg P$ and prove that P is true. Because P is true, we can use our last premise $P \to (Q_1 \vee Q_2)$ to derive $Q_1 \vee Q_2$. We have proved now that either Q_1 or Q_2 is true, and the premise $\neg R$ and Q_2 tells us that Q_2 is false. Q_1 must be true.

Negation 27

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \quad P \to (Q \lor R)}{Q \lor R}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{[Q]}{P \to Q}\,{\to}\mathrm{I}}{(P \to Q) \lor (P \to R)}\,{\lor}\mathrm{I} \qquad \dfrac{\dfrac{[R]}{P \to R}\,{\to}\mathrm{I}}{(P \to Q) \lor (P \to R)}\,{\lor}\mathrm{I}}{(P \to Q) \lor (P \to R)}\,{\lor}\mathrm{E} \qquad [\neg((P \to Q) \lor (P \to R))]}{\neg P}\,\neg\mathrm{I} \qquad [P]}{\dfrac{Q}{P \to Q}\,{\to}\mathrm{I}}\,\neg\mathrm{E}}{(P \to Q) \lor (P \to R)}\,{\lor}\mathrm{I} \qquad [\neg((P \to Q) \lor (P \to R))]}{(P \to Q) \lor (P \to R)}\,\neg\mathrm{E}$$

This proof turns out to be a lot nastier than it might look at first. The premise $P \to (Q \lor R)$ and an assumption of P give us $Q \lor R$, meaning we work with one case where we can assume Q and another case where we can assume R. Both of these assumptions bring us easily to $(P \to Q) \lor (P \to R)$, but even after applying $\lor$Elim we haven't discharged our assumption of P.

This turns out to be another proof where we prove a disjunction indirectly. If we assume $\neg((P \to Q) \lor (P \to R))$, we get a contradiction and we can derive $\neg P$ (discharging our assumption of P). From $\neg P$ we can prove $P \to Q$, which gets us back to our conclusion $(P \to Q) \lor (P \to R)$ and contradicts our assumption of $\neg((P \to Q) \lor (P \to R))$. Applying $\neg$Elim a final time discharges our assumptions of $\neg((P \to Q) \lor (P \to R))$ and lets us prove $(P \to Q) \lor (P \to R)$.

Negation 28

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg P]^1}{\neg P \vee \neg Q}\,\vee\mathrm{I} \quad [\neg(\neg P \vee \neg Q)]^3}{P}\,\neg\mathrm{E}^1 \quad [Q]^2}{P \wedge Q}\,\wedge\mathrm{I} \quad [\neg(P \wedge Q)]^7}{\neg Q}\,\neg\mathrm{I}^2}{\neg P \vee \neg Q}\,\vee\mathrm{I} \quad [\neg(\neg P \vee \neg Q)]^3}{\neg P \vee \neg Q}\,\neg\mathrm{E}^3 \quad \dfrac{[\neg P \vee \neg Q]^7 \quad \dfrac{\dfrac{[P \wedge Q]^4}{P}\,\wedge\mathrm{E} \quad [\neg P]^6}{\neg(P \wedge Q)}\,\neg\mathrm{I}^4 \quad \dfrac{\dfrac{[P \wedge Q]^5}{Q}\,\wedge\mathrm{E} \quad [\neg Q]^6}{\neg(P \wedge Q)}\,\neg\mathrm{I}^5}{\neg(P \wedge Q)}\,\vee\mathrm{E}^6}{\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)}\,\leftrightarrow\mathrm{I}^7$$

This is a past paper question from 2010. It breaks down into two proofs: a proof of $\neg P \vee \neg Q$ from $\neg(P \wedge Q)$ and a proof of $\neg(P \wedge Q)$ from $\neg P \vee \neg Q$.

On the left-hand side we want to try to prove $\neg P \vee \neg Q$, but our assumption $\neg(P \wedge Q)$ doesn't give a direct proof of either $\neg P$ or $\neg Q$. We need to assume $\neg(\neg P \vee \neg Q)$ and show it leads to a contradiction. First we can assume $\neg P$. This lets us derive $\neg P \vee \neg Q$, which contradicts our assumption of $\neg(\neg P \vee \neg Q)$.

Applying $\neg$Elim, we discharge our assumption of $\neg P$ and prove P. P and $\neg(P \wedge Q)$ together imply $\neg Q$. Assuming Q gives us $P \wedge Q$, which contradicts our assumption $\neg(P \wedge Q)$ and allows us to apply $\neg$Intro, discharge Q and prove $\neg Q$. From this we can prove $\neg P \vee \neg Q$ again, which provides another contradiction with our assumption of $\neg(\neg P \vee \neg Q)$. Finally we discharge all of our assumptions of $\neg(\neg P \vee \neg Q)$ and derive $\neg P \vee \neg Q$.

On the right-hand side, the proof is slightly easier. Our assumption $\neg P \vee \neg Q$ is a disjunction, splitting the proof into a case where we can assume $\neg P$ and a case where we can assume $\neg Q$. In both cases we need to prove $\neg(P \wedge Q)$, which we do by assuming $P \wedge Q$ and showing it leads to a contradiction. From $P \wedge Q$ we can prove P, which contradicts $\neg P$ in one case, and Q, which contradicts $\neg Q$ in the other cases. In both cases we discharge $P \wedge Q$ and derive $\neg(P \wedge Q)$.

Bonus challenge 1

It is possible to derive $\neg P$ from $\neg\neg\neg P$ using only one application of $\neg$Elim, similar to the proof above of P from $\neg\neg P$:

$$\dfrac{[\neg\neg P] \quad \neg\neg\neg P}{\neg P}\ {\scriptstyle \neg\text{Elim}}$$

Without $\neg$Elim, a proof is still possible. We start by assuming P and $\neg P$. Together they give us a contradiction which lets us discharge $\neg P$ and prove $\neg\neg P$. This in turn contradicts our premise $\neg\neg\neg P$, letting us discharge P and prove $\neg P$.

$$\dfrac{\dfrac{[P]^2 \quad [\neg P]^1}{\neg\neg P}\ {\scriptstyle \neg\text{Intro}^1} \quad \neg\neg\neg P}{\neg P}\ {\scriptstyle \neg\text{Intro}^2}$$

Similarly, with $\neg$Intro we can derive $\neg\neg P$ from P in only a single step, but a longer proof is still possible using only $\neg$Elim. We start by assuming $\neg\neg P$ and $\neg\neg\neg P$, which together give us a contradiction. This lets us discharge $\neg\neg P$ and derive $\neg P$, which contradicts our premise P. This contradiction lets us discharge $\neg\neg\neg P$ and prove $\neg\neg P$.

$$\dfrac{P \quad \dfrac{[\neg\neg P]^1 \quad [\neg\neg\neg P]^2}{\neg P}\ {\scriptstyle \neg\text{Elim}^1}}{\neg\neg P}\ {\scriptstyle \neg\text{Elim}^2}$$

In fact, any proof involving $\neg$Intro can be replaced with a larger proof involving $\neg$Elim. Removing $\neg$Intro would not make the system of Natural Deduction any weaker. However, it is not always possible to do the reverse and replace $\neg$Elim by $\neg$Intro. Without $\neg$Elim we cannot (for example) prove P from $\neg\neg P$.

Bonus challenge 2

One strategy for deriving $P \wedge Q$ from $\neg\neg P \wedge \neg\neg Q$ is to provide two separate proofs of P and Q. Both of these are indirect proofs where we assume $\neg P$ or $\neg Q$ and derive a contradiction. This strategy provides a proof of only five steps, satisfying the first constraint:

$$\dfrac{\dfrac{[\neg P]^1 \quad \dfrac{\neg\neg P \wedge \neg\neg Q}{\neg\neg P}\,{\scriptstyle\wedge\text{Elim}}}{P}\,{\scriptstyle\neg\text{Elim}^1} \quad \dfrac{[\neg Q]^2 \quad \dfrac{\neg\neg P \wedge \neg\neg Q}{\neg\neg Q}\,{\scriptstyle\wedge\text{Elim}}}{Q}\,{\scriptstyle\neg\text{Elim}^2}}{P \wedge Q}\,{\scriptstyle\wedge\text{Intro}}$$

To satisfy the second constraint, we must provide a proof where $\neg$Elim only discharges assumptions in the final step of the proof. Since we probably will want to discharge assumptions using $\neg$Elim, we will have to make $\neg$Elim our last step of the proof.

Since the very last line of the proof will be $P \wedge Q$, our conclusion, we know this final $\neg$Elim step can discharge assumptions of $\neg(P \wedge Q)$. Since we can't use $\neg$Elim to discharge anything else, we can't assume $\neg P$ or $\neg Q$ and discharge them later. We must assume P and Q at the start of the proof and derive $P \wedge Q$.

We can then produce three contradictions in a row. The first contradiction with $\neg(P \wedge Q)$ lets us derive $\neg P$, discharging P. This contradicts $\neg\neg P$ and lets us derive $\neg Q$, discharging Q. This contradicts $\neg\neg Q$, which lets us discharge $\neg(P \wedge Q)$ and derive our conclusion.

This means we have at the following proof, satisfying the second constraint:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P]^1 \quad [Q]^2}{P \wedge Q}\,{\scriptstyle\wedge\text{Intro}} \quad [\neg(P \wedge Q)]^3}{\neg P}\,{\scriptstyle\neg\text{Intro}^1} \quad \dfrac{\neg\neg P \wedge \neg\neg Q}{\neg\neg P}\,{\scriptstyle\wedge\text{Elim}}}{\neg Q}\,{\scriptstyle\neg\text{Intro}^2} \quad \dfrac{\neg\neg P \wedge \neg\neg Q}{\neg\neg Q}\,{\scriptstyle\wedge\text{Elim}}}{P \wedge Q}\,{\scriptstyle\neg\text{Elim}^3}}{}$$

This is a specific instance of a general result in propositional logic. Whenever we have to provide a proof of ϕ, we can always do this by assuming $\neg\phi$, showing that it leads to a contradiction and applying $\neg$Elim at the end. If we do this, we will never need to discharge any other assumptions using $\neg$Elim anywhere else in the proof. However, this proof might not be the shortest possible proof of ϕ.

5.7 Universal quantifier

Universal 1

$$\dfrac{\dfrac{\forall x Px}{Pa}\,\forall\text{Elim}}{\forall y Py}\,\forall\text{Intro}$$

This proof involves one simple application of each of the two universal quantifier rules. The first step takes us from a universal statement (our premise $\forall x Px$) to a specific statement (Pa). In order to use the universal introduction rule, and go from a specific statement about one constant to a general statement, that constant needs to be arbitrary: it needs to be possible for the proof still to work if we replaced the constant with any other constant. This is the case here: our constant a is arbitrary. It appears in no undischarged assumptions in our proof (in fact, no assumptions at all). This means we are free to apply the $\forall$Intro rule and prove $\forall y Py$.

Universal 2

$$\dfrac{\dfrac{[Pa]}{Pa \to Pa}\,\to\text{Intro}}{\forall x(Px \to Px)}\,\forall\text{Intro}$$

Working from the bottom upwards, we know we need to prove a universal claim $\forall x(Px \to Px)$. We can derive that universal claim from a specific (but arbitrary) case $Pa \to Pa$. This is easy to prove: we can assume Pa, and then move to $Pa \to Pa$ (also discharging Pa) by applying $\to$Intro. Our move from $Pa \to Pa$ to $\forall x(Px \to Px)$ is justified because, at the time we apply the $\forall$Intro rule, a does not appear in any undischarged assumptions in the proof of $Pa \to Pa$.

Universal 3

$$\dfrac{\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall x(Pa \to Qx)}{Pa \to Qb}\,\forall\text{Elim}}{Qb}\,\to\text{Elim}}{\forall z Qz}\,\forall\text{Intro}}{Pa \to \forall z Qz}\,\to\text{Intro}$$

Our conclusion $Pa \to \forall z Qz$ is an implication. This means we can freely assume Pa and need to provide a proof of $\forall z Qz$. We can prove the universal sentence $\forall z Qz$ by showing it is true for a specific but arbitrary constant.

The premise $\forall x(Pa \to Qx)$ gives us $Pa \to Qb$, which means that (with an assumption of Pa) we have a proof of Qb. Because b hasn't appeared in any undischarged assumptions, we can generalise this to the universal $\forall z Qz$. Discharging Pa, we can then prove $Pa \to \forall z Qz$.

It is important to note that here we must use both a and b as our arbitrary constants. The following proof would not work:

$$
\dfrac{\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall x(Pa \to Qx)}{Pa \to Qa}\,\forall\text{Elim}}{Qa}\,{\to}\text{Elim}}{\forall z Qz}\,\forall\text{Intro}}{Pa \to \forall z Qz}\,{\to}\text{Intro}
$$

Here, we are not justified in applying ∀Intro. The constant a appears in the assumption Pa, which won't be discharged until the final step of the proof.

Universal 4

$$
\dfrac{\dfrac{\dfrac{\dfrac{\forall x Px \wedge \forall y Qy}{\forall x Px}\,\wedge\text{Elim}}{Pa}\,\forall\text{Elim} \qquad \dfrac{\dfrac{\forall x Px \wedge \forall y Qy}{\forall x Qx}\,\wedge\text{Elim}}{Qa}\,\forall\text{Elim}}{Pa \wedge Qa}\,\wedge\text{Intro}}{\forall z(Pz \wedge Qz)}\,\forall\text{Intro}
$$

We need to prove a universal statement $\forall z(Pz \wedge Qz)$, which we can do by proving that the statement is true for a specific but arbitrary constant. In other words, we need to prove $Pa \wedge Qa$ without a appearing in any undischarged assumptions.

To prove $Pa \wedge Qa$, we need to provide a proof of Pa and a proof of Qa. We obtain Pa from $\forall x Px$, which we derive by ∧Elim from the premise $\forall x Px \wedge \forall y Qy$. We obtain Qa from $\forall y Qy$, which we also derive from the premise $\forall x Px \wedge \forall y Qy$.

Universal 5

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\forall y_1 P y_1]}{Pa}\,\forall\text{Elim} \qquad \dfrac{\forall x(Px \to Qx)}{Pa \to Qa}\,\forall\text{Elim}}{Qa}\,{\to}\text{Elim}}{\forall y_2 Q y_2}\,\forall\text{Intro}}{\forall y_1 P y_1 \to \forall y_2 Q y_2}\,{\to}\text{Intro}
$$

Our conclusion $\forall y_1 P y_1 \to \forall y_2 Q y_2$ is an implication statement. This means we can freely assume $\forall y_1 P y_1$ and we need to try to prove $\forall y_2 Q y_2$. To prove $\forall y_2 Q y_2$ we need to show it is true for an arbitrary constant. We need to try and prove Qa. Our assumption of $\forall y_1 P y_1$ gives us Pa, and our premise $\forall x(Px \to Qx)$ gives us $Pa \to Qa$. Together, these allow us to derive Qa.

Universal 6

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall z(Pz \wedge Qz)}{Pa \wedge Qa}\,\forall\text{Elim}}{Pa}\,\wedge\text{Elim}}{\forall y Py}\,\vee\text{Intro} \qquad \dfrac{\dfrac{\dfrac{\forall z(Pz \wedge Qz)}{Pa \wedge Qa}\,\forall\text{Elim}}{Qa}\,\wedge\text{Elim}}{\forall y Qy}\,\forall\text{Intro}}{\forall y Py \wedge \forall y Qy}\,\wedge\text{Intro}$$

Our conclusion is a conjunction of two sentences, so we need to provide two proofs: a proof of $\forall y Py$ and a proof of $\forall y Qy$. On the left-hand side, we can derive $\forall y Py$ from a specific claim about an arbitrary constant, so we'll try to prove Pa. We can derive this from $Pa \wedge Qa$, which we can derive from the premise $\forall z(Pz \wedge Qz)$. The right-hand side works in a similar way: we use the premise to derive $Pa \wedge Qa$ and then Qa, which gives us $\forall y Qy$.

Universal 7

$$\dfrac{\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall x(Px \to Qx)}{Pa \to Qa}\,\forall\text{Elim}}{Qa}\,\to\text{Elim} \qquad \dfrac{\forall x \neg Qx}{\neg Qa}\,\forall\text{Elim}}{\neg Pa}\,\neg\text{Intro}}{\forall x \neg Px}\,\forall\text{Intro}$$

Our conclusion $\forall x \neg Px$ is a universal statement, so we can prove it by providing a proof of $\neg Pa$ without a appearing in any undischarged assumptions. To prove $\neg Pa$, we need to derive a contradiction from assumptions of Pa. Note that these assumptions of Pa will be discharged when we apply $\neg$Intro and prove $\neg Pa$, so they won't interfere with our application of $\forall$Intro.

Where can we find the contradiction we need? One of our premises $\forall x \neg Qx$ gives us $\neg Qa$, so we have a contradiction if we can prove Qa. We do this by using our other premise $\forall x(Px \to Qx)$ to derive $Pa \to Qa$; this and our assumption of Pa allow us to prove Qa and obtain a contradiction.

Universal 8

$$\dfrac{\dfrac{\forall x_1 Px_1 \vee \forall x_2 Qx_2 \qquad \dfrac{\dfrac{[\forall x_1 Px_1]}{Pa}\,\forall\text{Elim}}{Pa \vee Qa}\,\vee\text{Intro} \qquad \dfrac{\dfrac{[\forall x_2 Qx_2]}{Qa}\,\forall\text{Elim}}{Pa \vee Qa}\,\vee\text{Intro}}{Pa \vee Qa}\,\vee\text{Elim}}{\forall x(Px \vee Qx)}\,\forall\text{Intro}$$

In order to prove $\forall x(Px \vee Qx)$ we can prove $Pa \vee Qa$ is true without a appearing in any undischarged assumptions.

Our premise $\forall x_1 Px_1 \vee \forall x_2 Qx_2$ is a disjunction. We can apply $\vee$Elim at the bottom of the proof to split the proof into two cases: one where we can freely assume $\forall x_1 Px_1$ and one where we can freely assume $\forall x_2 Qx_2$. On both sides we need to prove $Pa \vee Qa$.

On the left-hand side, our assumption of $\forall x_1 Px_1$ gives us Pa. From this, we can derive $Pa \vee Qa$ by applying $\vee$Intro. The right-hand side works in a similar way: $\forall x_2 Qx_2$ gives us Qa, then $Pa \vee Qa$.

Universal 9

$$
\begin{prooftree}
\AxiomC{$[Pa]$}
\AxiomC{$\forall x \forall y(Px \to Qy)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\forall y(Pa \to Qy)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Pa \to Qb$}
\RightLabel{\scriptsize $\to$Elim}
\BinaryInfC{Qb}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall z Qz$}
\RightLabel{\scriptsize $\to$Intro}
\UnaryInfC{$Pa \to \forall z Qz$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x(Px \to \forall z Qz)$}
\end{prooftree}
$$

Our conclusion $\forall x(Px \to \forall z Qz)$ is a universal statement, so we can prove it by deriving $Pa \to \forall z Qz$ as long as a doesn't appear in any undischarged assumptions. This is an implication, so we can prove it by assuming Pa and deriving $\forall z Qz$. We can $\forall z Qz$ from Qb, as long as b doesn't appear in any undischarged assumptions when we apply the $\forall$Intro step; note that we cannot use Qa for this because our assumption of Pa won't be discharged when we move to $\forall z Qz$.

So how do we get from an assumption of Pa to Qb? We use our premise $\forall x \forall y(Px \to Qy)$ to derive $\forall y(Pa \to Qy)$ and then $Pa \to Qb$. This means our assumption of Pa gives us the Qb we need.

Universal 10

$$
\begin{prooftree}
\AxiomC{$[Pa]$}
\AxiomC{$\forall x(Px \to Qx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Pa \to Qa$}
\RightLabel{\scriptsize $\to$Elim}
\BinaryInfC{Qa}
\AxiomC{$\forall x(Qx \to Rx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Qa \to Ra$}
\RightLabel{\scriptsize $\to$Elim}
\BinaryInfC{Ra}
\RightLabel{\scriptsize $\to$Intro}
\UnaryInfC{$Pa \to Ra$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x(Px \to Rx)$}
\end{prooftree}
$$

Our conclusion is a universal statement, so we need to prove it is true for an arbitrary instantiation. We can do this by proving $Pa \to Ra$ without a appearing in any undischarged assumptions. $Pa \to Ra$ is a conditional statement, so we can prove it by assuming Pa and trying to derive Ra.

We get from Pa to Ra with the help of our two premises. From $\forall x(Px \to Qx)$ we can derive $Pa \to Qa$, which (with our assumption of Pa) lets us prove Qa. From $\forall x(Qx \to Rx)$ we can derive $Qa \to Ra$, which lets us prove Ra. The application of $\to$Intro discharges Pa, leaving no undischarged assumptions containing a; this means we're free to apply universal introduction and derive the conclusion.

Universal 11

$$
\begin{prooftree}
\AxiomC{$\forall x(Px \lor Qx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Pa \lor Qa$}
\AxiomC{$[Pa]$}
\AxiomC{$[Qa]$}
\AxiomC{$[\forall x \neg Qx]$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\neg Qa$}
\RightLabel{\scriptsize $\neg$Elim}
\BinaryInfC{Pa}
\RightLabel{\scriptsize $\lor$Elim}
\TrinaryInfC{Pa}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x Px$}
\AxiomC{$\neg \forall x Px$}
\RightLabel{\scriptsize $\neg$Intro}
\BinaryInfC{$\neg \forall x \neg Qx$}
\end{prooftree}
$$

Our conclusion $\neg\forall x\neg Qx$ is a negated statement, so we prove it by assuming $\forall x\neg Qx$ and deriving a contradiction. One of our premises is $\neg\forall x Px$, which is also a negated statement, so we will have the contradiction we need if we prove $\forall x Px$. A way of proving $\forall x Px$ is to prove Pa without a appearing in any undischarged assumptions.

Our other premise $\forall x(Px \lor Qx)$ is a disjunction, splitting the proof into a case where Pa is true and a case where Qa is true. In both cases we need to prove Pa; in the former case this is trivial. In the latter case, proving Pa requires making use of our assumption $\forall x\neg Qx$. This gives us $\neg Qa$, which contradicts Qa and lets us derive Pa by $\neg$Elim.

Universal 12

$$
\begin{prooftree}
\AxiomC{$\forall x(Px \land Qx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Pa \land Qa$}
\RightLabel{\scriptsize $\land$Elim}
\UnaryInfC{Pa}
\AxiomC{$\forall x(Px \land Qx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Pb \land Qb$}
\RightLabel{\scriptsize $\land$Elim}
\UnaryInfC{Qb}
\RightLabel{\scriptsize $\land$Intro}
\BinaryInfC{$Pa \land Qb$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall y(Pa \land Qy)$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x\forall y(Px \land Qy)$}
\end{prooftree}
$$

Here our conclusion $\forall x\forall y(Px\land Qy)$ features two universal quantifiers, meaning we need to apply $\forall$Intro twice. We can derive $\forall x\forall y(Px \land Qy)$ from $\forall y(Pa \land Qy)$ (provided a appears in no undischarged assumptions in the proof of $\forall y(Pa\land Qy)$) but we cannot derive $\forall y(Pa \land Qy)$ from $Pa \land Qa$. This is because the constant a appears in $\forall y(Pa \land Qy)$; this means we need to use two constants.

We will derive $\forall y(Pa \land Qy)$ from $Pa \land Qb$ and make sure that neither a nor b appear in any undischarged assumptions. Our premise $\forall x(Px \land Qx)$ gives us both $Pa \land Qa$ and $Pb \land Qb$, which allow us to obtain Pa and Qb by $\land$Elim.

Universal 13

$$\dfrac{\dfrac{\dfrac{\forall x\forall y Rxy}{\forall y Ray}\,\forall\text{Elim}}{Raa}\,\forall\text{Elim}}{\forall x Rxx}\,\forall\text{Intro}$$

Our conclusion $\forall x Rxx$ is a universal statement, so we can prove it by proving Raa without a appearing in any undischarged assumptions. We obtain Raa from our premise $\forall x\forall y Rxy$ and two applications of $\forall$Elim; in both case the variable is replaced by a.

Universal 14

$$\dfrac{\dfrac{[\forall x\forall y Rxy]}{\forall y Ray}\,\forall\text{Elim} \qquad \dfrac{\forall x\neg\forall y Rxy}{\neg\forall y Ray}\,\forall\text{Elim}}{\neg\forall x\forall y Rxy}\,\neg\text{Intro}$$

Our conclusion $\neg\forall x\forall y Rxy$ is a negated statement, so we prove it by assuming $\forall x\forall y Rxy$ and deriving a contradiction from it. Our premise $\forall x\neg\forall y Rxy$ isn't a negated statement itself, but we can apply $\forall$Elim to obtain the negated statement $\neg\forall y Ray$ from it. Since we can obtain $\forall y Ray$ from our assumption of $\forall x\forall y Rxy$, we have the contradiction we need.

Universal 15

$$\dfrac{\dfrac{\dfrac{\forall x Rxx}{Raa}\,\forall\text{Elim} \qquad \dfrac{[\forall y\neg Ray]}{\neg Raa}\,\forall\text{Elim}}{\neg\forall y\neg Ray}\,\neg\text{Intro}}{\forall x\neg\forall y\neg Ray}\,\forall\text{Intro}$$

Our conclusion $\forall x\neg\forall y\neg Ray$ is a universal statement, so we can derive it by proving $\neg\forall y\neg Ray$ without a appearing in any undischarged assumptions. This is a negated statement, so we prove it by assuming $\forall y\neg Ray$ and deriving a contradiction from it. Neither our premise $\forall x Rxx$ nor our assumption of $\forall y\neg Ray$ are negated statements, so we can't use them to obtain a contradiction immediately. However, from $\forall x Rxx$ we can derive Raa and from $\forall y\neg Ray$ w can derive $\neg Raa$, giving us the contradiction we need.

Universal 16

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\forall y(Rxy\lor Ryx)]}{\forall y(Ray\lor Rya)}\ {\scriptstyle\forall\text{Elim}}}{Raa\lor Raa}\ {\scriptstyle\forall\text{Elim}}\quad [Raa]\quad [Raa]}{Raa}\ {\scriptstyle\lor\text{Elim}}\qquad \dfrac{\forall x\neg Rxx}{\neg Raa}\ {\scriptstyle\forall\text{Elim}}}{\neg\forall x\forall y(Rxy\lor Ryx)}\ {\scriptstyle\neg\text{Intro}}$$

Our conclusion $\neg\forall x\forall y(Rxy \lor Ryx)$ is a negated statement, so we derive it by assuming $\forall x\forall y(Rxy \lor Ryx)$ and deriving a contradiction from it. Our premise $\forall x\neg Rxx$ isn't a negated statement we can use in a contradiction, but by applying $\forall$Elim we can obtain $\neg Rxx$; this means we have a contradiction if we can somehow prove Raa.

From our assumption of $\forall x\forall y(Rxy \lor Ryx)$ we can apply $\forall$Elim twice to obtain $Raa \to Raa$. This gives us Raa through a slightly bizarre application of $\lor$Elim where Raa is trivially true in both cases. With Raa and $\neg Raa$ we have the contradiction we need to apply $\neg$Intro and prove $\neg\forall x\forall y(Rxy \lor Ryx)$.

Universal 17

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall yRyb]}{Rab}\ {\scriptstyle\forall\text{Elim}}\quad \dfrac{[\forall y\neg Ray]}{\neg Rab}\ {\scriptstyle\forall\text{Elim}}}{\neg\forall yRyb}\ {\scriptstyle\neg\text{Intro}}}{\forall x\neg\forall yRyx}\ {\scriptstyle\forall\text{Intro}}\qquad \neg\forall x\neg\forall yRyx}{\neg\forall y\neg Ray}\ {\scriptstyle\neg\text{Intro}}}{\forall x\neg\forall y\neg Rxy}\ {\scriptstyle\forall\text{Intro}}$$

Our conclusion $\forall x\neg\forall y\neg Rxy$ is a universal statement, so we will try to prove $\neg\forall y\neg Ray$ (making sure that a doesn't appear in any undischarged assumptions in the proof of $\neg\forall y\neg Ray$). $\neg\forall y\neg Ray$ is a negated statement, so we can prove it by assuming $\forall y\neg Ray$ and showing that it leads to a contradiction. Since our premise $\neg\forall x\neg\forall yRyx$ is a negated statement, we will have the contradiction we need if we can prove $\forall x\neg\forall yRyx$.

$\forall x\neg\forall yRyx$ is a universal statement, so we can prove it by proving $\neg\forall yRyb$ (as long as b doesn't appear in any undischarged assumptions in the proof of $\neg\forall yRyb$). This is a negated statement, so we can prove it by assuming $\forall yRyb$ and deriving a contradiction from it. $\forall yRyb$ gives us Rab, and our other assumption of $\forall y\neg Ray$ gives us $\neg Rab$, so we have the contradiction we need.

Note that $\forall yRyb$ is discharged before we apply the first $\forall$Intro step, and $\forall y\neg Ray$ is discharged before we apply the second $\forall$Intro step, so both applications are permitted.

Universal 18

$$
\begin{prooftree}
\AxiomC{$\forall xRxx$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{Raa}
\AxiomC{$[Rab]$}
\RightLabel{\scriptsize $\wedge$Intro}
\BinaryInfC{$Raa \wedge Rab$}
\AxiomC{$[\forall z\neg(Raz \wedge Rzb)]$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\neg(Raa \wedge Rab)$}
\RightLabel{\scriptsize $\neg$Intro}
\BinaryInfC{$\neg\forall z\neg(Raz \wedge Rzb)$}
\RightLabel{\scriptsize $\to$Intro}
\UnaryInfC{$Rab \to \neg\forall z\neg(Raz \wedge Rzb)$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall y(Ray \to \neg\forall z\neg(Raz \wedge Rzy))$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x\forall y(Rxy \to \neg\forall z\neg(Rxz \wedge Rzy))$}
\end{prooftree}
$$

Our conclusion is a universal statement with two quantifiers, so we can prove it by proving $Rab \to \neg\forall z\neg(Raz \wedge Rzb)$, as long as a and b don't appear in any undischarged assumptions. This is a conditional statement, so we prove it by assuming Rab and deriving $\neg\forall z\neg(Raz \wedge Rzb)$. This in turn is a negated statement, so we can assume $\forall z\neg(Raz \wedge Rzb)$ and need to show it leads to a contradiction.

From $\forall z\neg(Raz \wedge Rzb)$ we can derive the negated statement $\neg(Raa \wedge Rab)$, so we have a contradiction if we can derive $Raa \wedge Rab$. We can obtain $Raa \wedge Rab$ from our premise (which gives us Raa) and our assumption of Rab. $\forall z\neg(Raz \wedge Rzb)$ and Rab are both discharged before the end of the proof, so we are free to apply the two $\forall$Intro steps.

We could also have carried out the proof in a slightly different way, using $\forall xRxx$ to derive Rbb and using the assumption of $\forall z\neg(Raz \wedge Rzb)$ to derive $\neg(Rab \wedge Rbb)$. This would still have given us the contradiction we needed.

Universal 19

$$
\begin{prooftree}
\AxiomC{$\forall x\forall yRxy$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\forall yRay$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{Raa}
\AxiomC{$\forall x\forall yRxy$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\forall yRby$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{Rba}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall yRya$}
\RightLabel{\scriptsize $\wedge$Intro}
\BinaryInfC{$Raa \wedge \forall yRya$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x(Rxx \wedge \forall yRyx)$}
\end{prooftree}
$$

Our conclusion $\forall x(Rxx \wedge \forall yRyx)$ is a negated statement, so we can prove it by proving $Raa \wedge \forall yRya$ without a appearing in any undischarged assumptions. $Raa \wedge \forall yRya$ is a conjunction, so we need to provide a proof of Raa and a proof of $\forall yRya$. On the left-hand side, we can derive Raa from our premise $\forall x\forall yRxy$ by applying $\forall$Elim twice.

On the right-hand side, we need to prove the universal statement $\forall yRya$, which we can prove by providing a proof of Rba without b appearing in any undischarged assumptions. Note that we cannot derive $\forall yRya$ from Raa, even if a appears in no undischarged assumptions in the proof of Raa; this is because the constant a occurs in $\forall yRya$. We can prove Rba from the premise $\forall x\forall yRxy$ by applying $\forall$Elim twice.

Universal 20

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y Rxy}{\forall y Ray}\;\forall\text{Elim}}{Rab}\;\forall\text{Elim} \qquad \dfrac{\dfrac{\forall x \forall y Rxy}{\forall y Rby}\;\forall\text{Elim}}{Rba}\;\forall\text{Elim}}{Rab \wedge Rba}\;\wedge\text{Intro}}{\forall y(Ray \wedge Rya)}\;\forall\text{Intro}}{\forall x \forall y(Rxy \wedge Ryx)}\;\forall\text{Intro}$$

Our conclusion $\forall x \forall y(Rxy \wedge Ryx)$ is a universal statement, which we can derive from $\forall y(Ray \wedge Rya)$ (as long as a appears in no undischarged assumptions). This in turn we can derive from $Rab \wedge Rba$ (as long as b appears in no undischarged assumptions). Note that we would not be able to move from $Raa \wedge Raa$ to $\forall y(Ray \wedge Rya)$, because a still appears in $\forall y(Ray \wedge Rya)$; we have to use two separate constants.

To prove $Rab \wedge Rba$, we need to provide a proof of Rab and a proof of Rba. Both of these can be derived from our premise $\forall x \forall y Rxy$ through two applications of $\forall$Elim.

Universal 21

$$
\begin{prooftree}
\AxiomC{$[Rab]$}
\AxiomC{$[Rab]$}
\AxiomC{$\forall x \forall y (Rxy \to Ryx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\forall y (Ray \to Rya)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$Rab \to Rba$}
\RightLabel{\scriptsize $\to$Elim}
\BinaryInfC{Rba}
\RightLabel{\scriptsize $\wedge$Intro}
\BinaryInfC{$Rab \wedge Rba$}
\AxiomC{$\forall x \forall y \neg (Rxy \wedge Ryx)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\forall y \neg (Ray \wedge Rya)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\neg (Rab \wedge Rba)$}
\RightLabel{\scriptsize $\neg$Intro}
\BinaryInfC{$\neg Rab$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall y \neg Ray$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x \forall y \neg Rxy$}
\end{prooftree}
$$

This is adapted from a past paper question from 2009. Our conclusion $\forall x \forall y \neg Rxy$ has two universal quantifiers, so we can prove it by proving $\neg Rab$, provided a and b appear in no undischarged assumptions. $\neg Rab$ is a negation, so we prove it by assuming Rab and showing it leads to a negation. The premise $\forall x \forall y \neg (Rxy \wedge Ryx)$ gives us $\neg (Rab \wedge Rba)$, so we have the contradiction we need if we can provide a proof of $Rab \wedge Rba$.

To do this we need to provide proofs of Rab and Rba. We have a proof of Rab because we've assumed it (it will be discharged when we apply $\neg$Elim). We use our other premise $\forall x \forall y (Rxy \to Ryx)$ to derive $Rab \to Rba$, meaning we can obtain Rba with our assumption of Rab and $\to$Elim.

Universal 22

$$
\begin{prooftree}
\AxiomC{$\forall x \forall y(\neg Qxy \vee \neg Qyx)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall y(\neg Qay \vee \neg Qya)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\neg Qab \vee \neg Qba$}
\AxiomC{$[\neg Qab]$}
\AxiomC{$[Qab]$}
\AxiomC{$\forall x \forall y(Qxy \to Qyx)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall y(Qay \to Qya)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$Qab \to Qba$}
\RightLabel{$\to$Elim}
\BinaryInfC{Qba}
\AxiomC{$[\neg Qba]$}
\RightLabel{$\neg$Intro}
\BinaryInfC{$\neg Qab$}
\RightLabel{$\vee$Elim}
\TrinaryInfC{$\neg Qab$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall y \neg Qay$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall x \forall y \neg Qxy$}
\end{prooftree}
$$

Our conclusion $\forall x \forall y \neg Qxy$ features two universal quantifiers, so we can prove it by proving $\neg Qab$ and applying $\forall$Intro twice. One of our premises $\forall x \forall y(\neg Qxy \vee \neg Qyx)$ gives us the disjunction $\neg Qab \vee \neg Qba$, splitting the proof into a case where we can assume $\neg Qab$ and a case where we can assume $\neg Qba$.

In the first case, $\neg Qab$ is exactly what we want to prove. In the other case, we can prove $\neg Qab$ by assuming Qab and showing it leads to a contradiction. Since the disjunction gives us an assumption of $\neg Qba$, it makes sense to try and prove Qba in order to give us the contradiction we need. We do this using our other premise $\forall x \forall y(Qxy \to Qyx)$, which gives us $Qab \to Qba$.

Universal 23

$$
\begin{prooftree}
\AxiomC{$[Rab]^2$}
\AxiomC{$[Rbc]^1$}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$Rab \wedge Rbc$}
\AxiomC{$[\forall y \forall z \neg (Ray \wedge Ryz)]^3$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall z \neg (Rab \wedge Rbz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\neg (Rab \wedge Rbc)$}
\RightLabel{$\neg$Intro^1}
\BinaryInfC{$\neg Rbc$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall y \neg Rby$}
\AxiomC{$\forall x \neg \forall y \neg Rxy$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\neg \forall y \neg Rby$}
\RightLabel{$\neg$Intro^2}
\BinaryInfC{$\neg Rab$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall y \neg Ray$}
\AxiomC{$\forall x \neg \forall y \neg Rxy$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\neg \forall y \neg Ray$}
\RightLabel{$\neg$Intro^3}
\BinaryInfC{$\neg \forall y \forall z \neg (Ray \wedge Ryz)$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall x \neg \forall y \forall z \neg (Rxy \wedge Ryz)$}
\end{prooftree}
$$

Our conclusion $\forall x \neg \forall y \forall z \neg (Rxy \wedge Ryz)$ is a universal statement, so we prove it by proving $\neg \forall y \forall z \neg (Ray \wedge Ryz)$ without a appearing in any undischarged assumptions. This is a negated statement, so we need to derive a contradiction from an assumption of $\forall y \forall z \neg (Ray \wedge Ryz)$. Since our premise $\forall x \neg \forall y \neg Rxy$ gives us $\neg \forall y \neg Ray$, we have the contradiction we need if we can prove $\forall y \neg Ray$.

We can derive $\forall y \neg Ray$ from $\neg Rab$ as long as b doesn't appear in any undischarged assumptions. Proving $\neg Rab$ requires assuming Rab and deriving another contradiction. To get this second contradiction, we need to use our premise again, but this time we derive $\neg \forall y \neg Rby$. We have a contradiction if we can prove $\forall y \neg Rby$.

This in turn can be derived from $\neg Rbc$, as long as c doesn't appear in any undischarged assumptions. $\neg Rbc$ can be proved by assuming Rbc and showing that it leads to a contradiction.

We've now made three assumptions: Rab, Rac and $\forall y \forall z \neg (Ray \wedge Ryz)$. From the latter we can derive $\neg (Rab \wedge Rbc)$, which gives us the first contradiction we need. Because this contradiction discharges Rbc, c is left in no undischarged assumptions and we can derive $\forall y \neg Rby$. Because the second contradiction discharges Rab, b is left in no undischarged assumptions and we are free to derive $\forall y \neg Ray$. Because the final contradiction discharges $\forall y \forall z \neg (Ray \wedge Ryz)$, a is left in no undischarged assumptions and we can derive our conclusion.

Universal 24

$$
\begin{prooftree}
\AxiomC{$[Rab]$}
\AxiomC{$[Rab]$}
\AxiomC{$\forall x\forall y(Rxy \to Ryx)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall y(Ray \to Rya)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$Rab \to Rba$}
\RightLabel{$\to$Elim}
\BinaryInfC{Rba}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$Rab \wedge Rba$}
\AxiomC{$\forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall y\forall z((Ray \wedge Ryz) \to Raz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall z((Rab \wedge Rbz) \to Raz)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$(Rab \wedge Rba) \to Raa$}
\RightLabel{$\to$Elim}
\BinaryInfC{Raa}
\AxiomC{$[\neg Raa]$}
\RightLabel{$\neg$Intro}
\BinaryInfC{$\neg Rab$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall y\neg Ray$}
\AxiomC{$\forall x\neg\forall y\neg Rxy$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\neg\forall y\neg Ray$}
\RightLabel{$\neg$Elim}
\BinaryInfC{Raa}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall x Rxx$}
\end{prooftree}
$$

This problem, adapted from a 2010 past paper question, is quite nasty. We want to prove $\forall x Rxx$, so we know that we will need to derive Raa (and make sure that a doesn't appear in any undischarged assumptions in the proof of Raa). However, we don't have any way of proving Raa directly. Instead, we need to assume $\neg Raa$ and show that this leads to a contradiction.

What kind of contradiction are we looking for? From $\forall x\neg\forall y\neg Rxy$, one of our premises, we can derive $\neg\forall y\neg Ray$. This means that if we can prove $\forall y\neg Ray$ we will have the contradiction we need. We can prove this by proving $\neg Rab$ (as always, making sure b doesn't appear in any undischarged assumptions in its proof). This, in turn, we prove by assuming Rab and showing it leads to a contradiction.

What can we do with an assumption of Rab? It turns out we can do quite a lot. From the premise $\forall x\forall y(Rxy \to Ryx)$ we can derive $Rab \to Rba$, which lets us derive Rba. From the premise $\forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$ we can derive $(Rab \wedge Rba) \to Raa$ (by replacing x and z with the same constant), which gives us Raa. This contradicts our assumption of $\neg Raa$, letting us discharge Rab and derive $\neg Rab$. Now that all assumptions involving b have been discharged, we are free to derive $\forall y\neg Ray$, contradicting $\neg\forall y\neg Ray$. This contradiction lets us discharge $\neg Raa$ and derive Raa; finally we apply $\forall$Intro to derive the conclusion $\forall x Rxx$.

Universal 25

$$
\begin{prooftree}
\AxiomC{$[(Rab \wedge Rbc) \wedge Rca]$}
\RightLabel{$\wedge$E}
\UnaryInfC{$Rab \wedge Rbc$}
\AxiomC{$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y \forall z((Ray \wedge Ryz) \to Raz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall z((Rab \wedge Rbz) \to Raz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$(Rab \wedge Rbc) \to Rac$}
\RightLabel{$\to$E}
\BinaryInfC{Rac}
\AxiomC{$[(Rab \wedge Rbc) \wedge Rca]$}
\RightLabel{$\wedge$E}
\UnaryInfC{Rca}
\RightLabel{$\wedge$I}
\BinaryInfC{$Rac \wedge Rca$}
\AxiomC{$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y \forall z((Ray \wedge Ryz) \to Raz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall z((Rac \wedge Rcz) \to Raz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$(Rac \wedge Rca) \to Raa$}
\RightLabel{$\to$E}
\BinaryInfC{Raa}
\AxiomC{$\forall x \neg Rxx$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Raa$}
\RightLabel{$\neg$I}
\BinaryInfC{$\neg((Rab \wedge Rbc) \wedge Rca)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall z \neg((Rab \wedge Rbz) \wedge Rza)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall y \forall z \neg((Ray \wedge Ryz) \wedge Rza)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y \forall z \neg((Rxy \wedge Ryz) \wedge Rzx)$}
\end{prooftree}
$$

This is adapted from a past paper question from 2014. In order to prove $\forall x \forall y \forall z \neg((Rxy \wedge Ryz) \wedge Rzx)$ we can prove $\neg((Rab \wedge Rbc) \wedge Rca)$ without a, b or c appearing in any undischarged assumptions; this is a negated statement, so we need to derive a contradiction from assumptions of $(Rab \wedge Rbc) \wedge Rca$. Our premise $\forall x \neg Rxx$ could give us $\neg Raa$, $\neg Rbb$ and $\neg Rcc$, so we have a contradiction if we can prove Raa, Rbb or Rcc.

Although it is possible to prove any of them, Raa turns out to be the easiest to prove. From our assumption $(Rab \wedge Rbc) \wedge Rca$ we can derive $Rab \wedge Rbc$; we can use this with our premise to derive Rac. Our assumption provides Rca, so with both of them we can derive $Rac \wedge Rca$. In order to go from $Rac \wedge Rca$ to Raa, we need to use our premise to derive $(Rac \wedge Rca) \to Raa$. Raa gives us the contradiction we need to discharge $(Rab \wedge Rbc) \wedge Rca$ and derive $\neg((Rab \wedge Rbc) \wedge Rca)$.

Universal 26

$$
\begin{prooftree}
\AxiomC{$[Rab \wedge Rbc]$}
\RightLabel{$\wedge$E}
\UnaryInfC{Rab}
\AxiomC{$\forall x Rxx$}
\RightLabel{$\forall$E}
\UnaryInfC{Raa}
\RightLabel{$\wedge$I}
\BinaryInfC{$Rab \wedge Raa$}
\AxiomC{$\forall x \forall y \forall z((Rxy \wedge Rxz) \to Ryz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y \forall z((Ray \wedge Raz) \to Ryz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall z((Rab \wedge Raz) \to Rbz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$(Rab \wedge Raa) \to Rba$}
\RightLabel{$\to$E}
\BinaryInfC{Rba}
\AxiomC{$[Rab \wedge Rbc]$}
\RightLabel{$\wedge$E}
\UnaryInfC{Rbc}
\RightLabel{$\wedge$I}
\BinaryInfC{$Rba \wedge Rbc$}
\AxiomC{$\forall x \forall y \forall z((Rxy \wedge Rxz) \to Ryz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y \forall z((Rby \wedge Rbz) \to Ryz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall z((Rba \wedge Rbz) \to Raz)$}
\RightLabel{$\forall$E}
\UnaryInfC{$(Rba \wedge Rbc) \to Rac$}
\RightLabel{$\to$E}
\BinaryInfC{Rac}
\RightLabel{$\to$I}
\UnaryInfC{$(Rab \wedge Rbc) \to Rac$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall z((Rab \wedge Rbz) \to Raz)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall y \forall z((Ray \wedge Ryz) \to Raz)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$}
\end{prooftree}
$$

This is adapted from a past paper question from 2011. In order to prove $\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$ we can prove $(Rab \wedge Rbc) \to Rac$, as long as a, b and c don't appear in any undischarged assumptions. This is an implication, so we can assume $Rab \wedge Rbc$ and need to derive Rac. This much is straightforward; more tricky is working out how to use our premises to get from $Rab \wedge Rbc$ to Rac. $\forall x Rxx$ allows us to derive Raa, Rbb or Rcc if we want them; $\forall x \forall y \forall z((Rxy \wedge Rxz) \to Ryz)$ gives us many different sentences involving a, b and c. We need to work out which sentences we need.

Rac is the sentence we're aiming for, so ideally we want to use our premise $\forall x \forall y \forall z((Rxy \wedge Rxz) \to Ryz)$ to derive a conditional with Rac at the end. We could derive $(Raa \wedge Rac) \to Rac$, but this wouldn't help us very much: in order to use it we would need a proof of Rac, and a proof of Rac is what we're looking for in the first place. We could also try $(Rca \wedge Rcc) \to Rac$, and this would work, but would lead to a proof which is much longer than necessary. So instead we will use all three of our constants and derive $(Rba \wedge Rba) \to Rac$.

This means we need a proof of $Rba \wedge Rbc$, which requires a proof of Rba and a proof of Rbc. Rbc is easy; it comes directly from our assumption. Proving Rba requires using our big premise to derive another conditional, this time with Rba as the consequent. $(Rab \wedge Raa) \to Rba$ is the ideal conditional to derive, because we have both Rab (from our assumption) and Raa (from our premise). This means we have a complete proof.

5.8 Existential quantifier

Existential 1

$$\dfrac{\exists x Px \qquad \dfrac{[Pa]}{\exists y Py}\,{\scriptstyle \exists\text{Intro}}}{\exists y Py}\,{\scriptstyle \exists\text{Elim}}$$

This proof involves one use of each rule for the existential quantifier. Although there are only two steps, this proof exemplifies the unusual structure which proofs involving the existential quantifier take: we eliminate any existential quantifiers at the bottom, and introduce them at the top. This means there are two ways we can look at this proof: one from top to bottom and one from bottom to top.

From top to bottom, we start by assuming Pa, and then apply $\exists$Intro to derive $\exists y Py$. Following this we make use of our premise $\exists x Px$ and apply $\exists$Elim. This allows us to discharge our assumptions of Pa, provided that a doesn't appear in any other undischarged assumptions, and doesn't appear in either $\exists y Py$ or $\exists x Px$. All of these conditions are satisfied, so we are allowed to apply $\exists$Elim and discharge Pa.

From bottom to top, we can see our premise $\exists x Px$ as giving us a 'free' assumption of Pa, provided that a is an arbitrary constant. a can't appear in $\exists x Px$ or in what we ultimately plan to prove. It also can't appear in any other undischarged assumptions we use. It is this assumption of Pa which we use to apply $\exists$Intro and derive our conclusion, $\exists y Py$.

Existential 2

$$\dfrac{\dfrac{\dfrac{[Pa]}{\exists x Px}\,{\scriptstyle \exists\text{Intro}} \qquad \neg\exists x Px}{\neg Pa}\,{\scriptstyle \neg\text{Intro}}}{\exists x \neg Px}\,{\scriptstyle \exists\text{Intro}}$$

Our conclusion is an existential statement, so we can prove it from $\neg Pa$ by applying $\exists$Intro. $\neg Pa$ is a negation, so we prove it by assuming Pa and showing that it leads to a contradiction. Our premise $\neg\exists x Px$ is a negated statement, so we have a contradiction if we can prove $\exists x Px$. We can derive this from our assumption of Pa by applying $\exists$Intro, giving us the contradiction we need.

We could have used any constant in place of a in this proof, because the two $\exists$Intro steps would still have worked. Although our choice of a here was arbitrary, in other proofs we might not have so much flexibility.

Existential 3

$$\dfrac{\exists x_1(Pa \to Qx_1) \qquad \dfrac{\dfrac{\dfrac{[Pa] \quad [Pa \to Qb]}{Qb}\,{\to}\text{Elim}}{\exists x_2 Qx_2}\,\exists\text{Intro}}{Pa \to \exists x_2 Qx_2}\,{\to}\text{Intro}}{Pa \to \exists x_2 Qx_2}\,\exists\text{Elim}$$

Our premise $\exists x_1(Pa \to Qx_1)$ gives us a free assumption of $Pa \to Qb$ which we can use to derive our conclusion $Pa \to \exists x_2 Qx_2$. Note that I've chosen to use b as our arbitrary constant because we can't use a: a can't be our arbitrary constant because it appears in $\exists x_1(Pa \to Qx_1)$.

Our conclusion is a conditional, so we can assume Pa and need to try to derive $\exists x_2 Qx_2$. With our assumptions of Pa and $Pa \to Qb$ we can derive Qb, which means we can derive $\exists x_2 Qx_2$ by $\exists$Intro.

In the proof above we apply $\exists$Elim at the end of the proof to discharge $Pa \to Qb$; we could also have applied it before applying $\to$Intro, as shown in the proof below. Usually it is most straightforward to wait until the very end of the proof before applying $\exists$Elim, but in later examples we'll see cases where this isn't possible.

$$\dfrac{\dfrac{\exists x_1(Pa \to Qx_1) \qquad \dfrac{\dfrac{[Pa] \quad [Pa \to Qb]}{Qb}\,{\to}\text{Elim}}{\exists x_2 Qx_2}\,\exists\text{Intro}}{\exists x_2 Qx_2}\,\exists\text{Elim}}{Pa \to \exists x_2 Qx_2}\,{\to}\text{Intro}$$

Existential 4

$$\dfrac{\exists x(Px \wedge Qx) \qquad \dfrac{\dfrac{\dfrac{[Pa \wedge Qa]}{Pa}\,\wedge\text{Elim}}{\exists y Py}\,\exists\text{Intro} \quad \dfrac{\dfrac{[Pa \wedge Qa]}{Qa}\,\wedge\text{Elim}}{\exists z Qz}\,\exists\text{Intro}}{\exists y Py \wedge \exists z Qz}\,\wedge\text{Intro}}{\exists y Py \wedge \exists z Qz}\,\exists\text{Elim}$$

Our premise $\exists x(Px \wedge Qx)$ is an existential statement, which gives us a free assumption of $Pa \wedge Qa$. We need to derive $\exists y Py \wedge \exists z Qz$, so we need to provide a proof of $\exists y Py$ and a proof of $\exists z Qz$. On the left, our assumption of $Pa \wedge Qa$ gives us Pa, which gives us $\exists y Py$; on the right, our assumption gives us Qa, which gives us $\exists z Qz$. When we apply $\exists$Elim, both assumptions of $Pa \wedge Qa$ are discharged at once, which the rules governing $\exists$Elim do allow.

Again it's possible for us to apply $\exists$Elim sooner than the end of the proof. This results in a proof which is slightly longer, as shown below:

$$\dfrac{\dfrac{\exists x(Px \wedge Qx) \qquad \dfrac{\dfrac{[Pa \wedge Qa]}{Pa}\,{\scriptstyle\wedge\text{Elim}}}{\exists y Py}\,{\scriptstyle\exists\text{Intro}}}{\exists y Py}\,{\scriptstyle\exists\text{Elim}} \qquad \dfrac{\exists x(Px \wedge Qx) \qquad \dfrac{\dfrac{[Pa \wedge Qa]}{Qa}\,{\scriptstyle\wedge\text{Elim}}}{\exists z Qz}\,{\scriptstyle\exists\text{Intro}}}{\exists y Py}\,{\scriptstyle\exists\text{Elim}}}{\exists y Py \wedge \exists z Qz}\,{\scriptstyle\wedge\text{Intro}}$$

In this proof we apply $\exists$Elim twice, and on both occasions we use a as our arbitrary constant. This is allowed, but isn't necessary: we could use a different constant in each branch. In certain proofs we apply $\exists$Elim twice in the same branch, so that one application follows another; in these proofs we are obliged to use different constants each time.

Existential 5

$$\dfrac{\exists x(Px \vee Qx) \qquad \dfrac{[Pa \vee Qa] \qquad \dfrac{\dfrac{[Pa]}{\exists y Py}\,{\scriptstyle\exists\text{Intro}}}{\exists y Py \vee \exists z Qz}\,{\scriptstyle\vee\text{Intro}} \qquad \dfrac{\dfrac{[Qa]}{\exists z Qz}\,{\scriptstyle\exists\text{Intro}}}{\exists y Py \vee \exists z Qz}\,{\scriptstyle\vee\text{Intro}}}{\exists y Py \vee \exists z Qz}\,{\scriptstyle\vee\text{Elim}}}{\exists y Py \vee \exists z Qz}\,{\scriptstyle\exists\text{Elim}}$$

Our premise is an existential statement, giving us a free assumption of $Pa \vee Qa$. This is a disjunction, splitting the proof into a case where Pa is true and a case where Qa is true. In the case where Pa is true we can derive $\exists y Py$ and hence our conclusion $\exists y Py \vee \exists z Qz$; in the other case we can derive $\exists y Py$ and can also derive the conclusion $\exists y Py \vee \exists z Qz$. Our assumptions of Pa and Qa are discharged when we apply $\vee$Elim, so they do not interfere with our application of $\exists$Elim.

Existential 6

$$\dfrac{\exists x Px \vee \exists y Qy \qquad \dfrac{[\exists x Px] \qquad \dfrac{\dfrac{[Pa]}{Pa \vee Qa}\,{\scriptstyle\vee\text{Intro}}}{\exists z(Pz \vee Qz)}\,{\scriptstyle\exists\text{Intro}}}{\exists z(Pz \vee Qz)}\,{\scriptstyle\exists\text{Elim}} \qquad \dfrac{[\exists y Qy] \qquad \dfrac{\dfrac{[Qa]}{Pa \vee Qa}\,{\scriptstyle\vee\text{Intro}}}{\exists z(Pz \vee Qz)}\,{\scriptstyle\exists\text{Intro}}}{\exists z(Pz \vee Qz)}\,{\scriptstyle\exists\text{Elim}}}{\exists z(Pz \vee Qz)}\,{\scriptstyle\vee\text{Elim}}$$

Our premise is a disjunction, splitting the proof into a case where $\exists x Px$ is true (on the left) and a case where $\exists y Qy$ is true (on the right). On the left-hand side we use our assumption of $\exists x Px$ to discharge another assumption of Pa; this gives us $Pa \vee Qa$ and hence our conclusion $\exists z(Pz \vee Qz)$. The right-hand side works similarly: $\exists y Qy$ gives us a free assumption of Qa, from which we can derive $Pa \vee Qa$ and $\exists z(Pz \vee Qz)$.

Existential 7

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Pa]^2 \quad Pa \to \exists x Qx}{\exists x Qx}{\scriptstyle\to\mathrm{Elim}} \quad \dfrac{\dfrac{[Qb]^1}{Pa \to Qb}{\scriptstyle\to\mathrm{Intro}}}{\exists x(Pa \to Qx)}{\scriptstyle\exists\mathrm{Intro}}}{\exists x(Pa \to Qx)}{\scriptstyle\exists\mathrm{Elim}^1} \quad [\neg\exists x(Pa \to Qx)]^3}{\neg Pa}{\scriptstyle\neg\mathrm{Intro}^2} \quad [Pa]}{Qc}{\scriptstyle\neg\mathrm{Elim}}}{Pa \to Qc}{\scriptstyle\to\mathrm{Intro}}}{\exists x(Pa \to Qx)}{\scriptstyle\exists\mathrm{Intro}} \quad [\neg\exists x(Pa \to Qx)]^3}{\exists x(Pa \to Qx)}{\scriptstyle\neg\mathrm{Elim}^3}$$

This proof is nastier than it might look at first. Our premise $Pa \to \exists x Qx$ isn't much help to us on its own, but with an assumption of Pa it allows us to derive $\exists x Qx$. This gives us a free assumption of Qb which we can use to derive $\exists x(Pa \to Qx)$; we can't assume Qa because it appears in $\exists x(Pa \to Qx)$ and Pa (which won't be discharged by the time we apply $\exists$Elim).

From Qb we can straightforwardly derive $Pa \to Qb$ and then $\exists x(Pa \to Qx)$, and we can then apply $\exists$Elim to discharge Qb. But there is a problem: our assumption of Pa still hasn't been discharged, so we don't have a complete proof of $\exists x(Pa \to Qx)$.

It turns out that this is an indirect proof: we need to assume $\neg\exists x(Pa \to Qx)$ and show that this leads to a contradiction. Our first assumption of $\neg\exists x(Pa \to Qx)$ lets us discharge Pa and derive $\neg Pa$; from $\neg Pa$ we can derive $\exists x(Pa \to Qx)$ again.

One way of deriving $\exists x(Pa \to Qx)$ is to assume Pa, derive Qc in one step by $\neg$Elim, and apply $\to$Intro to discharge Pa and derive $Pa \to Qc$. $\exists x(Pa \to Qx)$ again contradicts $\neg\exists x(Pa \to Qx)$, so we apply $\neg$Elim, discharge our assumptions of $\neg\exists x(Pa \to Qx)$ and derive $\exists x(Pa \to Qx)$.

Existential 8

$$\dfrac{\dfrac{\dfrac{\dfrac{[Raa]}{Raa \to Raa}\,{\scriptstyle \to\mathrm{Intro}}}{\exists y(Ray \to Rya)}\,{\scriptstyle \exists\mathrm{Intro}}}{\exists x\exists y(Rxy \to Ryx)}\,{\scriptstyle \exists\mathrm{Intro}}}{}$$

The statement we want to prove is $\exists x\exists y(Rxy \to Ryx)$. We can apply $\exists$Intro twice to derive this from a number of different statements, but not all of them will be useful. For example, we could derive $\exists x\exists y(Rxy \to Ryx)$ from $Rab \to Rba$, but we have no way of providing a proof of $Rab \to Rba$ from no premises.

Instead, we will derive $\exists x\exists y(Rxy \to Ryx)$ from $Raa \to Raa$. We can prove $Raa \to Raa$ from no premises by assuming Raa and applying implication-introduction, discharging the assumption.

Note that the rules of Natural Deduction do allow us to make the first $\exists$Intro step and only replace some of the as with ys. This is a difference between $\exists$Intro and $\forall$Intro: when applying the latter rule, all occurrences of the constant need to be replaced with the variable.

Existential 9

$$\dfrac{\exists x\exists yRxy \qquad \dfrac{[\exists yRay]^2 \qquad \dfrac{\dfrac{[Rab]^1}{\exists yRyb}\,{\scriptstyle \exists\mathrm{Intro}}}{\exists x\exists yRyx}\,{\scriptstyle \exists\mathrm{Intro}}}{\exists x\exists yRyx}\,{\scriptstyle \exists\mathrm{Elim}^1}}{\exists x\exists yRyx}\,{\scriptstyle \exists\mathrm{Elim}^2}$$

In this proof our premise $\exists x\exists yRxy$ contains two existential quantifiers. This means we need to apply $\exists$Elim twice. The way we do this is quite mechanical, but produces an odd-looking proof structure.

Applying $\exists$Elim with $\exists x\exists yRxy$ at the bottom of the proof allows us to discharge an assumption of $\exists yRay$. Applying $\exists$Elim with $\exists yRay$ allows us to discharge an assumption of Rab. Using this assumption, we apply $\exists$Intro twice to derive our conclusion $\exists x\exists yRyx$.

It's worth verifying that our $\exists$Elim steps are allowed. The first application of $\exists$Elim (higher up in the proof) replaces y with b; it is allowed because b doesn't appear in $\exists yRay$, $\exists x\exists yRyx$ or in any undischarged assumptions in the proof of $\exists x\exists yRyx$ other than Rab, which is then discharged. Our second application (at the bottom of the proof) replaces x with a; it is allowed because a doesn't appear in $\exists x\exists yRxy$, $\exists x\exists yRyx$ or in any undischarged assumptions other than $\exists yRay$, which is then discharged.

If we had used only one constant throughout the proof, the proof would not have worked. An example of an incorrect proof is shown below:

$$\dfrac{\exists x\exists y Rxy \qquad \dfrac{[\exists y Ray]^2 \qquad \dfrac{\dfrac{[Raa]^1}{\exists y Rya}\,\exists\text{Intro}}{\exists x\exists y Ryx}\,\exists\text{Intro}}{\exists x\exists y Ryx}\,\exists\text{Elim}^1}{\exists x\exists y Ryx}\,\exists\text{Elim}^2$$

Here, the first application of ∃Elim (attempting to discharge Raa) is not allowed, because the constant a appears in $\exists y Ray$.

Existential 10

$$\dfrac{\exists x Rxx \qquad \dfrac{\dfrac{\dfrac{[Raa] \quad [Raa]}{Raa \wedge Raa}\,\wedge\text{Intro}}{\exists y(Ray \wedge Rya)}\,\exists\text{Intro}}{\exists x\exists y(Rxy \wedge Ryx)}\,\exists\text{Intro}}{\exists x\exists y(Rxy \wedge Ryx)}\,\exists\text{Elim}$$

Our premise $\exists x Rxx$ allows us to discharge an assumption of Raa. This is useful because we can use it twice to derive $Raa \wedge Raa$, which allows us to derive $\exists x\exists y(Rxy \wedge Ryx)$ by ∃Intro.

Existential 11

$$\dfrac{\dfrac{[\exists y Ryy] \qquad \dfrac{\dfrac{[Raa]}{\exists y Ray}\,\exists\text{Intro}}{\exists x\exists y Rxy}\,\exists\text{Intro}}{\exists x\exists y Rxy}\,\exists\text{Elim} \qquad \neg\exists x\exists y Rxy}{\neg\exists y Ryy}\,\neg\text{Intro}$$

Here our conclusion is a negated statement, meaning we need to derive a contradiction from an assumption of $\exists y Ryy$. Our premise $\neg\exists x\exists y Rxy$ is a negated statement, so we have a contradiction if we can derive $\exists x\exists y Rxy$. Applying ∃Elim to this assumption of $\exists y Ryy$ gives us an assumption of Raa. From this assumption we can derive $\exists x\exists y Rxy$, giving us the contradiction we need.

Unusually, we don't apply ∃Elim at the very end of the proof. When we apply ¬Intro we need to discharge our assumption of $\exists y Ryy$, so we need to apply ∃Elim before we apply ¬Intro.

Existential 12

$$\dfrac{\dfrac{[\exists x\exists y(Rxy \wedge \neg Rxy)]^3 \qquad \dfrac{[\exists y(Ray \wedge \neg Ray)]^2 \qquad \dfrac{\dfrac{[Rab \wedge \neg Rab]^1}{Rab}\,{\scriptstyle \wedge\text{Elim}} \qquad \dfrac{[Rab \wedge \neg Rab]^1}{\neg Rab}\,{\scriptstyle \wedge\text{Elim}}}{\neg\exists x\exists y(Rxy \wedge \neg Rxy)}\,{\scriptstyle \neg\text{Intro}}}{\neg\exists x\exists y(Rxy \wedge \neg Rxy)}\,{\scriptstyle \exists\text{Elim}^1}}{\neg\exists x\exists y(Rxy \wedge \neg Rxy)}\,{\scriptstyle \exists\text{Elim}^2} \qquad [\exists x\exists y(Rxy \wedge \neg Rxy)]^3}{\neg\exists x\exists y(Rxy \wedge \neg Rxy)}\,{\scriptstyle \neg\text{Intro}^3}$$

Here our conclusion $\neg\exists x\exists y(Rxy \wedge \neg Rxy)$ is a negated statement, so we prove it by assuming $\exists x\exists y(Rxy \wedge \neg Rxy)$ and deriving a contradiction from it. With this assumption we can apply $\exists$Elim to discharge an assumption of $\exists y(Ray \wedge \neg Ray)$, provided a doesn't appear in any other undischarged assumptions by the time we apply $\exists$Elim. $\exists y(Ray \wedge \neg Ray)$ in turn lets us discharge an assumption of $Rab \wedge \neg Rab$, provided b doesn't appear in any other undischarged assumptions when we make this $\exists$Elim step.

With this assumption of $Rab \wedge \neg Rab$ we have what we need to derive a contradiction: by $\wedge$Elim we can derive both Rab and $\neg Rab$, which contradict each other. However, if we apply $\neg$Intro at this stage to derive $\neg\exists x\exists y(Rxy \wedge \neg Rxy)$ we won't be able to discharge $\exists x\exists y(Rxy \wedge \neg Rxy)$, because that assumption appears much further down in the proof.

This means we need to apply $\neg$Intro twice. First we apply $\neg$Intro at the top of the proof to derive $\neg\exists x\exists y(Rxy \wedge \neg Rxy)$. Then we apply $\exists$Elim twice, using $\exists y(Ray \wedge \neg Ray)$ to discharge $Rab \wedge \neg Rab$ and $\exists x\exists y(Rxy \wedge \neg Rxy)$ to discharge $\exists y(Ray \wedge \neg Ray)$. Finally we assume $\exists x\exists y(Rxy \wedge \neg Rxy)$ again, and apply $\neg$Intro a second time. This discharges both assumptions of $\exists x\exists y(Rxy \wedge \neg Rxy)$ and provides a proof of the conclusion.

Existential 13

$$
\begin{prooftree}
\AxiomC{$[\exists x Rxx]^1$}
\RightLabel{$\vee$I}
\UnaryInfC{$\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$}
\AxiomC{$[\neg(\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx))]^3$}
\RightLabel{$\neg$I^1}
\BinaryInfC{$\neg\exists x Rxx$}
\AxiomC{$[Raa]^2$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists x Rxx$}
\RightLabel{$\neg$E}
\BinaryInfC{$\neg\exists y Rya$}
\RightLabel{$\to$I^2}
\UnaryInfC{$Raa \to \neg\exists y Rya$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists x(Rxx \to \neg\exists y Ryx)$}
\RightLabel{$\vee$I}
\UnaryInfC{$\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$}
\AxiomC{$[\neg(\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx))]^3$}
\RightLabel{$\neg$E^3}
\BinaryInfC{$\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$}
\end{prooftree}
$$

What we want to prove is a disjunction, but we can't provide a direct proof of either disjunct. This turns out to be an indirect proof, of a very similar form to the proof of $P \vee (P \to Q)$ given in the negation section. We have to assume $\neg\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$, the negation of our conclusion, and show that it leads to a contradiction.

We start by assuming $\exists x Rxx$, from which we can derive the conclusion $\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$. Assuming $\neg(\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx))$ gives us a contradiction, which lets us discharge $\exists x Rxx$ and derive $\neg\exists x Rxx$.

We do the actual legwork of the proof when we derive $\exists x(Rxx \to \neg\exists y Ryx)$ from $\neg\exists x Rxx$. We can derive this from $Raa \to \neg\exists y Rya$, which is an implication: so we assume Raa and derive $\neg\exists y Rya$.

This assumption of Raa allows us to derive $\exists x Rxx$, which contradicts $\neg\exists x Rxx$ and lets us derive $\neg\exists y Rya$ by $\neg$Elim. This means we can apply implication-introduction (discharging Raa), $\exists$Intro and $\vee$Intro to prove $\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$.

Finally we assume $\neg(\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx))$ a second time, letting us discharge both occurrences of $\neg(\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx))$ and derive $\exists x Rxx \vee \exists x(Rxx \to \neg\exists y Ryx)$ by $\neg$Elim.

The above is not the only way in which this proof could have been carried out. We could have begun, for example, by assuming Raa. It would also be possible to carry out the proof by first assuming $\exists x(Rxx \to \neg\exists y Ryx)$, or by first assuming $\neg Raa$.

Existential 14

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg Qa \quad [Qb]}{\neg Qa \wedge Qb}\,{\scriptstyle \wedge\mathrm{I}} \quad \dfrac{Rab \wedge Rbc}{Rab}\,{\scriptstyle \wedge\mathrm{E}}}{(\neg Qa \wedge Qb) \wedge Rab}\,{\scriptstyle \wedge\mathrm{I}}}{\exists y((\neg Qa \wedge Qy) \wedge Ray)}\,{\scriptstyle \exists\mathrm{I}}}{\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)}\,{\scriptstyle \exists\mathrm{I}} \quad [\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)]}{\neg Qb}\,{\scriptstyle \neg\mathrm{I}} \quad Qc}{\neg Qb \wedge Qc}\,{\scriptstyle \wedge\mathrm{I}} \quad \dfrac{Rab \wedge Rbc}{Rbc}\,{\scriptstyle \wedge\mathrm{E}}}{(\neg Qb \wedge Qc) \wedge Rbc}\,{\scriptstyle \wedge\mathrm{I}}}{\exists y((\neg Qb \wedge Qy) \wedge Rby)}\,{\scriptstyle \exists\mathrm{I}}}{\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)}\,{\scriptstyle \exists\mathrm{I}} \quad [\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)]}{\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)}\,{\scriptstyle \neg\mathrm{E}}$$

This is adapted from a past paper question from 2016. The good news is our premises have no quantifiers, but the bad news is that we have no way of proving our conclusion $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ directly from these premises.

What we will do first is assume Qb and derive the conclusion $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ from this. Then we will assume $\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ and apply $\neg$Intro to discharge Qb and derive $\neg Qb$. We will then show that even with $\neg Qb$ we can derive $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$. At this point we will assume $\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ again and apply $\neg$Elim, discharging both assumptions of $\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ and providing a proof of $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$.

The proofs of $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ from Qb and from $\neg Qb$ are fairly straightforward, and similar to each other. In the first proof we aim to prove $(\neg Qa \wedge Qb) \wedge Rab$, which we can do easily with the conjunction rules, our premises and our assumption of Qb. In the second proof we aim to prove $(\neg Qb \wedge Qc) \wedge Rbc$, which we can do with our premises and proof of $\neg Qb$. Then both times we apply $\exists$Intro twice to arrive at $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$.

Existential 15

$$
\dfrac{\exists x \exists y \neg \exists z \neg Pxyz \qquad \dfrac{[\exists y \neg \exists z \neg Payz]^5 \qquad \dfrac{\dfrac{[\exists x \neg \exists y \exists z \neg Pyzx]^3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Pabc]^1}{\exists z Pazc}\,{\scriptstyle \exists\mathrm{Intro}}}{\exists y \exists z Pyzc}\,{\scriptstyle \exists\mathrm{Intro}} \qquad [\neg \exists y \exists z Pyzc]^2}{\neg Pabc}\,{\scriptstyle \neg\mathrm{Intro}^1}}{\exists z \neg Pbz}\,{\scriptstyle \exists\mathrm{Intro}}}{\exists z \neg Pbz}\,{\scriptstyle \exists\mathrm{Elim}^2} \qquad [\neg \exists z \neg Pabz]^4}{\neg \exists x \neg \exists y \exists z Pyzx}\,{\scriptstyle \neg\mathrm{Intro}^3}}{\neg \exists x \neg \exists y \exists z Pyzx}\,{\scriptstyle \exists\mathrm{Elim}^4}}{\neg \exists x \neg \exists y \exists z Pyzx}\,{\scriptstyle \exists\mathrm{Elim}^5}
$$

This proof involves ternary predicates, lots of discharging of assumptions and three applications of ∃Elim. The good news is that the proof is quite systematic.

With our premise $\exists x \exists y \neg \exists z \neg Pxyz$ we can apply ∃Elim at the end of the proof to discharge assumptions of $\exists y \neg \exists z \neg Payz$, as long as a doesn't appear in any other undischarged assumptions by that stage. $\exists y \neg \exists z \neg Payz$ is itself an existential statement, so we can use it to discharge assumptions of $\neg \exists z \neg Pabz$, as long as b doesn't appear in any undischarged assumptions by the time we make this application of ∃Elim.

What we want to prove is $\neg \exists x \neg \exists y \exists z Pyzx$, a negation, so we need to assume $\exists x \neg \exists y \exists z Pyzx$ and show that this leads to a contradiction. The assumption which our premise gives us is $\neg \exists z \neg Pabz$, so we can obtain a contradiction by deriving $\exists z \neg Pabz$.

Assuming $\exists x \neg \exists y \exists z Pyzx$ (which we have assumed for our first contradiction), we can apply ∃Elim to discharge assumptions of $\neg \exists y \exists z Pyzc$, as long as c doesn't appear in any undischarged assumptions at this point. From this we need to derive $\exists z \neg Pabz$, which can be obtained from $\neg Pabc$. $\neg Pabc$ is itself a negated statement, so we obtain it by assuming $Pabc$ and deriving a second contradiction. To do this derive $\exists y \exists z Pyzc$ from $Pabc$, contradicting our assumption of $\neg \exists y \exists z Pyzc$.

Existential 16

$$\dfrac{\dfrac{\dfrac{[Pa]}{\exists x Px}\,{\scriptstyle \exists\text{Intro}} \qquad \neg\exists x Px}{\neg Pa}\,{\scriptstyle \neg\text{Intro}}}{\forall x \neg Px}\,{\scriptstyle \vee\text{Intro}}$$

This is the first of four proofs illustrating the duality between the universal quantifier and the existential quantifier. The proofs aren't very intuitive, but the techniques appear in lots of harder proofs.

We can prove $\forall x \neg Px$ by proving $\neg Pa$ as long as Pa doesn't appear in any undischarged assumptions. This is a negation, so we can derive it by assuming Pa and deriving a contradiction. Our premise $\neg\exists x Px$ is also a negated statement, so we have a contradiction if we can prove $\exists x Px$. This follows from our assumption of Pa by $\exists$Intro, so we have the contradiction we need.

Because our application of $\neg$Intro discharges our assumption of Pa, we are free to apply $\forall$Intro in the last step and derive the conclusion $\forall x \neg Px$.

Existential 17

$$\dfrac{\exists x \neg Px \qquad \dfrac{\dfrac{[\forall x Px]}{Pa}\,{\scriptstyle \forall\text{Elim}} \qquad [\neg Pa]}{\neg\forall x Px}\,{\scriptstyle \neg\text{Intro}}}{\neg\forall x Px}\,{\scriptstyle \exists\text{Elim}}$$

Our premise $\exists x \neg Px$ is an existential statement, so by applying $\exists$Elim at the end of the proof we can discharge an assumption of $\neg Pa$ (as long as a doesn't appear in any other undischarged assumptions).

Our conclusion $\neg\forall x Px$ is a negated statement, so we prove it by assuming $\forall x Px$ and deriving a contradiction. Since $\exists x \neg Px$ gives us an assumption of $\neg P$, we have a contradiction if we can derive Pa. This follows from $\forall x Px$ by $\forall$Elim.

Existential 18

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa]}{\exists x \neg Px}\,\exists\text{Intro} \qquad [\neg\exists x\neg Px]}{Pa}\,\neg\text{Elim}}{\forall x Px}\,\forall\text{Intro} \qquad \neg\forall x Px}{\exists x\neg Px}\,\neg\text{Elim}$$

Our conclusion $\exists x\neg Px$ is an existential statement, but it turns out we have no way of proving it directly. We have to assume $\neg\exists x\neg Px$ and show that it leads to a contradiction. Our premise $\neg\forall x Px$ is a negated statement, so we have the contradiction we need if we can derive $\forall x Px$.

This is a universal statement, so we can prove it if we can prove Pa without a appearing in any undischarged assumptions. Unfortunately, we have no way of proving Pa directly either: we have to assume $\neg Pa$ and show that this leads to a contradiction.

Because $\neg\exists x\neg Px$ is a negated statement, we have a contradiction if we can derive $\exists x\neg Px$. This follows from $\neg Pa$ by ∃Intro. Because the first application of ¬Elim discharges $\neg Pa$ we are free to apply ∀Intro and derive $\forall x Px$. This then gives us the contradiction which allows us to derive $\exists x\neg Px$.

Existential 19

$$\dfrac{\dfrac{[\exists x Px] \qquad \dfrac{[Pa] \qquad \dfrac{\forall x\neg Px}{\neg Pa}\,\forall\text{Elim}}{\neg\exists x Px}\,\neg\text{Intro}}{\neg\exists x Px}\,\exists\text{Elim} \qquad [\exists x Px]}{\neg\exists x Px}\,\neg\text{Intro}$$

Our conclusion $\neg\exists x Px$ is a negated statement, so we assume $\exists x Px$ and try to derive a contradiction. $\exists x Px$ is an existential statement, so we can use it to discharge a proof of Pa (as long as a doesn't appear in any other undischarged assumptions when we apply ∃Elim).

Because we can derive $\neg Pa$ from our premise $\forall x\neg Px$ we have the contradiction we need, but applying ¬Intro at this stage won't let us discharge $\exists x Px$. This means after applying ∃Elim we assume $\exists x Px$ again and apply ¬Intro a second time; this second application lets us derive the conclusion $\neg\exists x Px$ and discharge both assumptions of $\exists x Px$.

Existential 20

$$\cfrac{\cfrac{\cfrac{\cfrac{[Pb]}{\exists y Py}\,\exists\text{Intro} \qquad \cfrac{\forall x(\exists y Py \to Qx)}{\exists y Py \to Qa}\,\forall\text{Elim}}{Qa}\,\to\text{Elim}}{Pb \to Qa}\,\to\text{Intro}}{\cfrac{\exists y(Py \to Qa)}{\forall x \exists y(Py \to Qx)}\,\forall\text{Intro}}\,\exists\text{Intro}$$

This is a past paper question from 2011. The statement we want to prove is $\forall x \exists y(Py \to Qx)$, a universal statement. This means we can prove it from $\exists y(Py \to Qa)$, as long as a doesn't appear in any undischarged assumptions. This is an existential statement, so there are lots of statements we can prove it from. We'll try $Pb \to Qa$.

$Pb \to Qa$ is an implication, so we prove it by assuming Pb and trying to derive Qa. We can get to Qa with the help of our premise, $\forall x(\exists y Py \to Qx)$. From this we can derive $\exists y Py \to Qa$, which is an implication with Qa as its consequent. All we need to do is prove $\exists y Py$; fortunately this follows from our assumption of Pb.

The proof would also have worked if we had assumed Pa at the very top of the proof, and then derived $\exists y(Py \to Qa)$ from $Pa \to Qa$. Our $\forall$Intro step would still have been allowed because the assumption of Pa would have been discharged before applying $\forall$Intro. This alternate proof is shown below:

$$\cfrac{\cfrac{\cfrac{\cfrac{[Pa]}{\exists y Py}\,\exists\text{Intro} \qquad \cfrac{\forall x(\exists y Py \to Qx)}{\exists y Py \to Qa}\,\forall\text{Elim}}{Qa}\,\to\text{Elim}}{Pa \to Qa}\,\to\text{Intro}}{\cfrac{\exists y(Py \to Qa)}{\forall x \exists y(Py \to Qx)}\,\forall\text{Intro}}\,\exists\text{Intro}$$

Existential 21

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{[Pab]}{\exists y Pay}\,{\scriptstyle \exists\text{Intro}} \qquad [\neg\exists y Pay]
}{Qab}\,{\scriptstyle \neg\text{Elim}}
}{Pab \to Qab}\,{\scriptstyle \to\text{Intro}}
}{\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Intro}}
\qquad
\dfrac{\forall x \neg \forall y(Pxy \to Qxy)}{\neg\forall y(Pay \to Qay)}\,{\scriptstyle \forall\text{Elim}}
}{\exists y Pay}\,{\scriptstyle \neg\text{Elim}}
}{\forall x \exists y Pxy}\,{\scriptstyle \forall\text{Intro}}
$$

This is a past paper question from 2013. Our conclusion $\forall x \exists y Pxy$ is a universal statement, so we can prove it from $\exists y Pay$ as long as a doesn't appear in any undischarged assumptions. This is an existential statement, but we have no way of proving it directly. Instead we have to assume $\neg\exists y Pay$ and show that it leads to a contradiction. Since our premise $\forall x \neg\forall y(Pxy \to Qxy)$ gives us the negated statement $\neg\forall y(Pay \to Qay)$, we have the contradiction we need if we can provide a proof of $\forall y(Pay \to Qay)$.

$\forall y(Pay \to Qay)$ is a universal statement, so we can prove it from $Pab \to Qab$, as long as b doesn't appear in any undischarged assumptions. Remember that the conditions for $\forall$Intro prevent us from using a twice. $Pab \to Qab$ is an implication, so we can assume Pab and need to prove Qab. We can't obtain Qab from anything directly, but from our assumption of Pab we can obtain $\exists y Pay$, which contradicts our assumption of $\neg\exists y Pay$. This lets us obtain Qab by $\neg$Elim.

Existential 22

$$
\dfrac{
\dfrac{
\dfrac{\forall x(Pxx \lor \forall y Qxy)}{Paa \lor \forall y Qay}\,{\scriptstyle \forall\text{Elim}}
\quad
\dfrac{\dfrac{[Paa]}{\exists y Pay}\,{\scriptstyle \exists\text{Intro}}}{\exists y Pay \lor Qaa}\,{\scriptstyle \lor\text{Intro}}
\quad
\dfrac{\dfrac{[\forall y Qay]}{Qaa}\,{\scriptstyle \forall\text{Elim}}}{\exists y Pay \lor Qaa}\,{\scriptstyle \lor\text{Intro}}
}{\exists y Pay \lor Qaa}\,{\scriptstyle \lor\text{Elim}}
}{\forall x(\exists y Pxy \lor Qxx)}\,{\scriptstyle \forall\text{Intro}}
$$

This is a past paper question from 2009. Our conclusion $\forall x(\exists y Pxy \lor Qxx)$ is a universal statement, so we can prove it from $\exists y Pay \lor Qaa$ as long as a doesn't appear in any undischarged assumptions. Our premise $\forall x(Pxx \lor \forall y Qxy)$ gives us $Paa \lor \forall y Qay$, a disjunction splitting the proof into a case where Paa is true and a case where $\forall y Qay$ is true. In the left-hand case we can obtain $\exists y Pay$ by $\exists$Intro, and in the right-hand case we can obtain Qaa by $\forall$Elim. This means that in both cases we have proofs of $\exists y Pay \lor Qaa$.

Existential 23

$$
\dfrac{\exists x(Pxx \land \forall y Qxy) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Paa \land \forall y Qay]}{Paa}\ \land\text{Elim}}{\exists y Pay}\ \exists\text{Intro} \qquad \dfrac{\dfrac{[Paa \land \forall y Qay]}{\forall y Qay}\ \land\text{Elim}}{Qaa}\ \forall\text{Elim}}{\exists y Pay \land Qaa}\ \land\text{Intro}}{\exists x(\exists y Pay \land Qaa)}\ \exists\text{Intro}}{\exists x(\exists y Pay \land Qaa)}\ \exists\text{Elim}
$$

This is a past paper question from 2009. Our premise $\exists x(Pxx \land \forall y Qxy)$ is an existential statement, so we can apply $\exists$Elim at the end of the proof to discharge assumptions of $Paa \land \forall y Qay$, provided a doesn't appear in any other undischarged assumptions. Our conclusion $\exists x(\exists y Pay \land Qaa)$ is an existential statement; since our premise is going to discharge assumptions involving a, it makes sense to try to derive $\exists x(\exists y Pay \land Qaa)$ from $\exists y Pay \land Qaa$.

This is a conjunction, so we need to provide a proof of $\exists y Pay$ and a proof of Qaa. Both of these can be derived from our assumption of $Paa \land \forall y Qay$. From Paa we can derive $\exists y Pay$ by $\exists$Intro, and from $\forall y Qay$ we can derive Qaa by $\forall$Elim.

Existential 24

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\forall xRxx]}{\forall x\exists yRxy \lor \neg\forall xRxx}\,{\lor}\text{Intro} \qquad [\neg(\forall x\exists yRxy \lor \neg\forall xRxx)]}{\forall xRxx}\,\neg\text{Elim}}{Raa}\,\forall\text{Elim}}{\exists yRay}\,\exists\text{Intro}}{\forall x\exists yRxy}\,\forall\text{Intro}}{\forall x\exists yRxy \lor \neg\forall xRxx}\,{\lor}\text{Intro} \qquad [\neg(\forall x\exists yRxy \lor \neg\forall xRxx)]}{\forall x\exists yRxy \lor \neg\forall xRxx}\,\neg\text{Elim}
$$

This is a past paper question from 2013. Our conclusion $\forall x\exists yRxy \lor \neg\forall xRxx$ is a disjunction which we are asked to prove from no premises. This makes it pretty likely that we'll need to carry out an indirect proof, assuming $\neg(\forall x\exists yRxy \lor \neg\forall xRxx)$ and showing that it leads to a contradiction.

We follow the usual strategy for indirectly proving disjunctions. First we assume one disjunct, $\neg\forall xRxx$, and derive the conclusion $\forall x\exists yRxy \lor \neg\forall xRxx$ from it. Using our assumption of $\neg(\forall x\exists yRxy \lor \neg\forall xRxx)$ we apply ¬Elim and derive $\forall xRxx$. With this proof of $\forall xRxx$ we want to derive the other disjunct, $\forall x\exists yRxy$.

This is a universal statement, so we can derive it from $\exists yRay$ (as long as a doesn't appear in any undischarged assumptions). This in turn can be derived from Raa, which can be derived by ∀Elim from $\forall xRxx$.

From $\forall x\exists yRxy$ we derive $\forall x\exists yRxy \lor \neg\forall xRxx$ a second time and assume $\neg(\forall x\exists yRxy \lor \neg\forall xRxx)$ a second time. Finally we apply ¬Elim, discharging both assumptions of $\neg(\forall x\exists yRxy \lor \neg\forall xRxx)$ and deriving $\forall x\exists yRxy \lor \neg\forall xRxx$.

An alternate proof, starting from $\forall x\exists yRxy$, is shown below:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\exists yRxy]}{\forall x\exists yRxy \lor \neg\forall xRxx}\,{\lor}\text{Intro} \qquad [\neg(\forall x\exists yRxy \lor \neg\forall xRxx)]}{\neg\forall x\exists yRxy}\,\neg\text{Intro} \qquad \dfrac{\dfrac{\dfrac{[\forall xRxx]}{Raa}\,\forall\text{Elim}}{\exists yRay}\,\exists\text{Intro}}{\forall x\exists yRxy}\,\forall\text{Intro}}{\neg\forall xRxx}\,\neg\text{Intro}}{\forall x\exists yRxy \lor \neg\forall xRxx}\,{\lor}\text{Intro} \qquad [\neg(\forall x\exists yRxy \lor \neg\forall xRxx)]}{\forall x\exists yRxy \lor \neg\forall xRxx}\,\neg\text{Elim}
$$

Existential 25

$$
\begin{prooftree}
\AxiomC{$\exists x\forall y Ryx$}
\AxiomC{$[Rbb]$}
\RightLabel{\scriptsize $\exists$Intro}
\UnaryInfC{$\exists x Rxx$}
\AxiomC{$[\forall y Rya]$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{Rca}
\RightLabel{\scriptsize $\exists$Intro}
\UnaryInfC{$\exists y Rcy$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x\exists y Rxy$}
\AxiomC{$\forall x\exists y Rxy \to \neg\exists x Rxx$}
\RightLabel{\scriptsize $\to$Elim}
\BinaryInfC{$\neg\exists x Rxx$}
\RightLabel{\scriptsize $\neg$Intro}
\BinaryInfC{$\neg Rbb$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x\neg Rxx$}
\RightLabel{\scriptsize $\exists$Elim}
\BinaryInfC{$\forall x\neg Rxx$}
\end{prooftree}
$$

This is adapted from a past paper question from 2010. Our premise $\exists x\forall y Ryx$ is an existential statement, letting us discharge assumptions of $\forall y Rya$, as long as a doesn't appear in any other undischarged assumptions by the end of the proof.

We want to prove $\forall x\neg Rxx$, which we can derive from $\neg Rbb$ (as long as b doesn't appear in any undischarged assumptions when we apply $\forall$Intro). This is a negated statement, so we prove it by assuming Rbb and showing that it leads to a contradiction.

We don't have any negated statements readily available, but our other premise $\forall x\exists y Rxy \to \neg\exists x Rxx$ is a conditional with a negated consequent. This means that if we can prove $\forall x\exists y Rxy$ we will be able to derive $\neg\exists x Rxx$ by $\to$Elim.

$\forall x\exists y Rxy$ follows from our assumption of $\forall y Rya$, so we have a proof of $\neg\exists x Rxx$. Since $\exists x Rxx$ follows from Rbb, we have the contradiction we need to discharge Rbb and apply $\neg$Intro. This then lets us derive our conclusion $\forall x\neg Rxx$; finally an $\exists$Elim step discharges our assumption of $\forall y Rya$.

Existential 26

$$
\begin{prooftree}
\AxiomC{$[\neg \forall x Px]$}
\RightLabel{$\vee$I}
\UnaryInfC{$\forall x \exists z Rzx \vee \neg \forall x Px$}
\AxiomC{$[\neg(\forall x \exists z Rzx \vee \neg \forall x Px)]$}
\RightLabel{$\neg$E}
\BinaryInfC{$\forall x Px$}
\RightLabel{$\forall$E}
\UnaryInfC{Pa}
\AxiomC{$\forall x(Px \to \exists y Ryx)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Pa \to \exists y Rya$}
\RightLabel{$\to$E}
\BinaryInfC{$\exists y Rya$}
\AxiomC{$[Rba]$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists z Rza$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists z Rza$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \exists z Rzx$}
\RightLabel{$\vee$I}
\UnaryInfC{$\forall x \exists z Rzx \vee \neg \forall x Px$}
\AxiomC{$[\neg(\forall x \exists z Rzx \vee \neg \forall x Px)]$}
\RightLabel{$\neg$E}
\BinaryInfC{$\forall x \exists z Rzx \vee \neg \forall x Px$}
\end{prooftree}
$$

This is adapted from a past paper question from 2013. This is another indirect proof of a disjunction, so we follow the usual strategy of starting by assuming $\neg \forall x Px$, one of the disjuncts, deriving the conclusion by $\vee$Intro and then assuming the negation of the conclusion to create a contradiction. With this contradiction we apply $\neg$Elim to derive $\forall x Px$.

The legwork of the proof lies in using our premise $\forall x(Px \to \exists y Ryx)$ to derive $\forall x \exists z Rzx$ from $\forall x Px$. $\forall x \exists z Rzx$ is a universal statement, which we can derive from $\exists z Rza$ (as long as a doesn't appear in any undischarged assumptions when we apply $\forall$Intro). From $\forall x Px$ we can derive Pa and from $\forall x(Px \to \exists y Ryx)$ we can derive $Pa \to \exists y Rya$, meaning we can apply $\to$Elim to derive $\exists y Rya$. This is almost what we need, but we need to apply $\exists$Intro and $\exists$Elim to convert it to $\exists z Rza$.

With $\forall x \exists z Rzx$ we can apply $\vee$Intro again to derive the conclusion. We assume the negation of the conclusion a second time and apply $\neg$Elim to discharge both assumptions of $\neg(\forall x \exists z Rzx \vee \neg \forall x Px)$ and derive $\forall x \exists z Rzx \vee \neg \forall x Px$.

Bonus challenge

The following proof has the two features specified in the challenge:

$$\dfrac{\exists x P \qquad [P]}{P}\ \exists\text{Elim}$$

This requires us to apply ¬Elim in a bizarre way, but it is indeed allowed. Recall the formulation of the ∃Elim rule:

$$\dfrac{\exists v\phi \qquad \begin{array}{c}[\phi[t/v]] \\ \vdots \\ \psi\end{array}}{\psi}\ \exists\text{Elim}$$

The ∃Elim rule lets us discharge all assumptions of $\phi[t/v]$, where $\phi[t/v]$ is the result of replacing all occurrences of v (x in this case) in ϕ (P in this case) with the constant t. But here there are no occurrences of x in P, P is what we discharge. ψ happens to be P as well in this case, so P is what we conclude.

Furthermore, all additional conditions for ∃Elim are satisfied: there are no constants in the proof at all, so none appear in ψ or ϕ or in any undischarged assumptions in the proof of ψ.

This is an absurd proof, but it highlights an unusual way in which the quantifier rules can be applied. Similar proofs exist for ∃Intro and the two rules for the universal quantifier:

$$\dfrac{P}{\exists x P}\ \exists\text{Intro} \qquad \dfrac{\forall x P}{P}\ \exists\text{Elim} \qquad \dfrac{P}{\forall x P}\ \exists\text{Intro}$$

5.9 Identity

Identity 1

$$\dfrac{\dfrac{[a=a]^{=}}{\exists y a = y}\ \exists\text{Intro}}{\exists x \exists y x = y}\ \exists\text{Intro}$$

We aren't given any premises, but we can apply =Intro to make and immediately discharge the assumption $a = a$. From this, we can derive $\exists x \exists y x = y$ by applying $\exists$Intro twice.

Identity 2

$$\dfrac{\dfrac{[\exists x \forall y x = y]^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\forall y a = y]^1}{a = b}\ \forall\text{Elim} \qquad \dfrac{[\forall y a = y]^1}{a = c}\ \forall\text{Elim}}{b = c}\ =\text{Elim}}{\forall y b = y}\ \forall\text{Intro}}{\forall x \forall y x = y}\ \forall\text{Intro}}{\forall x \forall y x = y}\ \exists\text{Elim}^1 \qquad \dfrac{\dfrac{[\forall x \forall y x = y]^2}{\forall y a = y}\ \forall\text{Elim}}{\exists x \forall y x = y}\ \exists\text{Intro}}{\exists x \forall y x = y \leftrightarrow \forall x \forall y x = y}\ \leftrightarrow\text{Intro}^2$$

This is a past paper question from 2009. Our conclusion is a biconditional, so we need to provide a proof of $\forall x \forall y x = y$ from $\exists x \forall y x = y$ and a proof of $\exists x \forall y x = y$ from $\forall x \forall y x = y$.

The right-hand side is easy: we use $\forall$Elim to replace x with any constant (a is used in the proof above) and then we use $\exists$Intro to replace that constant with x again.

On the left-hand side, we have an existential assumption $\exists x \forall y x = y$ which lets us discharge an assumption of $\forall y a = y$. We want to prove $\forall x \forall y x = y$, which we can derive from $b = c$. We can't derive it from anything involving a because our assumption of $\forall y a = y$ won't be discharged when we apply $\forall$Intro. We need to use our assumption of $\forall y a = y$ twice, deriving $a = b$ and $a = c$. Then we can apply =Elim to derive $b = c$.

Identity 3

$$\cfrac{\cfrac{Pa \quad [a=b]}{Pb}\,{\scriptstyle =\text{Elim}} \qquad \neg Pb}{\neg a = b}\,{\scriptstyle \neg\text{Intro}}$$

Our conclusion $\neg a = b$ is a negation, so we derive it by assuming $a = b$ and trying to derive a contradiction. One of our premises $\neg Pb$ is a negation, so we have a contradiction if we can derive Pb.

To obtain Pb we need to make use of our assumption of $a = b$ and apply =Elim, replacing the a in Pa (our other premise) with b.

We could also have carried out the proof in a different way, applying =Elim to obtain $\neg Pa$ which contradicts Pa:

$$\cfrac{Pa \qquad \cfrac{\neg Pb \quad [a=b]}{\neg Pa}\,{\scriptstyle =\text{Elim}}}{\neg a = b}\,{\scriptstyle \neg\text{Intro}}$$

Identity 4

$$\cfrac{\cfrac{Pb \wedge Qb}{Qb}\,{\scriptstyle \wedge\text{Elim}} \qquad \cfrac{\cfrac{Pb \wedge Qb}{Pb}\,{\scriptstyle \wedge\text{Elim}} \qquad \cfrac{\forall x(Px \to x = a)}{Pb \to b = a}\,{\scriptstyle \forall\text{Elim}}}{b = a}\,{\scriptstyle \to\text{Elim}}}{Qa}\,{\scriptstyle =\text{Elim}}$$

Our conclusion is Qa. This doesn't have any connectives or quantifiers in it, so we know the last line of our proof won't be an introduction rule. Qa doesn't appear explicitly in any of our premises, and we can't derive it using $\neg$Elim (assuming $\neg Qa$ is no help).

However, we do have the premise $Pb \wedge Qb$, which gives us Qb; we can derive Qa by =Elim if we can prove $b = a$. We can show $b = a$ by $\to$Elim, since our first premise gives us Pb and our other premise gives us $\forall x(Px \to x = a)$ gives us $Pb \to b = a$.

Identity 5

$$\dfrac{\exists x\forall y(Py\leftrightarrow x=y) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Pb] \quad \dfrac{[\forall y(Py\leftrightarrow a=y)]}{Pb\leftrightarrow a=b}{\scriptstyle\forall\text{Elim}}}{a=b}{\scriptstyle\leftrightarrow\text{Elim}}}{Pb\to a=b}{\scriptstyle\to\text{Intro}}}{\forall y(Py\to a=y)}{\scriptstyle\forall\text{Intro}}}{\exists x\forall y(Py\to x=y)}{\scriptstyle\exists\text{Intro}}}{\exists x\forall y(Py\to x=y)}{\scriptstyle\exists\text{Elim}}$$

This is a past paper question from 2011. It turns out we don't need to use any of the rules for identity, but we do need all four quantifier rules.

Because we have an existential premise $\exists x\forall y(Py\leftrightarrow x=y)$, we should look at this first and apply $\exists$Elim at the end of the proof to discharge $\forall y(Py\leftrightarrow a=y)$ (making sure that a doesn't appear in any other undischarged assumptions by the time we apply $\exists$Elim).

Our conclusion $\exists x\forall y(Py\to x=y)$ is also an existential statement, and it's reasonable to suspect that we'll derive it from $\forall y(Py\to a=y)$ (that both the premise and the conclusion refer to the same object). This is a universal statement, so we'll derive it from $Pb\to a=b$. This is an implication, so we assume Pb and try to derive $a=b$. We do this by $\leftrightarrow$Elim using $Pb\leftrightarrow a=b$, which can be derived from our assumption of $\forall y(Py\leftrightarrow a=y)$.

Identity 6

$$\dfrac{\exists xPx \qquad \dfrac{\dfrac{\dfrac{\forall x(x=a\lor x=b)}{c=a\lor c=b}{\scriptstyle\forall\text{E}} \quad \dfrac{\dfrac{[Pc]^3 \quad [c=a]^1}{Pa}{\scriptstyle=\text{E}} \quad [\neg Pa]^2}{Pb}{\scriptstyle\neg\text{E}} \quad \dfrac{[Pc]^3 \quad [c=b]^1}{Pb}{\scriptstyle=\text{E}}}{Pb}{\scriptstyle\lor\text{E}^1}}{\neg Pa\to Pb}{\scriptstyle\to\text{I}^2}}{\neg Pa\to Pb}{\scriptstyle\exists\text{E}^3}$$

One of our premises $\exists xPx$ is an existential statement, so we should think about that first. If we apply $\exists$Elim at the end of the proof we can discharge assumptions of Pc, as long as c doesn't appear in any undischarged assumptions. We can't use a or b because they both appear in our conclusion $\neg Pa\to Pb$.

$\neg Pa\to Pb$ is an implication, so we derive it by assuming $\neg Pa$ and deriving Pb from it. Our premise $\forall x(x=a\lor x=b)$ is a universal statement. We can derive lots of disjunctions from it, but not all of them will be useful. $a=a\lor a=b$, for example, is something we could derive without any premises if we wanted to: when we can use =Intro to assume and discharge $a=a$ and then use $\lor$Intro to derive $a=a\lor a=b$.

$c=a\lor c=b$ is useful: it splits the proof into a case where $c=a$ is true and a case where $c=b$ is true. On the right-hand side, our assumptions of Pc and $c=b$ let us derive Pb by =Elim. On the left-hand side, we can use =Elim to derive Pa, which contradicts our other assumption of $\neg Pa$ and lets us derive Pb by $\neg$Elim.

Identity 7

$$
\dfrac{\exists x(Px \land Qx \land Rax) \qquad \dfrac{\dfrac{[Pc \land Qc \land Rac]}{Rac}\,{\scriptstyle \land\mathrm{E}} \qquad \dfrac{\dfrac{Pb \land Qb \qquad \dfrac{[Pc \land Qc \land Rac]}{Pc \land Qc}\,{\scriptstyle \land\mathrm{E}}}{(Pb \land Qb) \land (Pc \land Qc)}\,{\scriptstyle \land\mathrm{I}} \qquad \dfrac{\dfrac{\forall x \forall y((Px \land Qx) \land (Py \land Qy) \to x = y)}{\forall y((Pb \land Qb) \land (Py \land Qy) \to b = y)}\,{\scriptstyle \forall\mathrm{E}}}{(Pb \land Qb) \land (Pc \land Qc) \to b = c}\,{\scriptstyle \forall\mathrm{E}}}{b = c}\,{\scriptstyle \to\mathrm{E}}}{Rab}\,{\scriptstyle =\mathrm{E}}}{Rab}\,{\scriptstyle \exists\mathrm{E}}
$$

This is a past paper question from 2010. Our premise $\exists x(Px \land Qx \land Rax)$ is an existential statement, so we will apply $\exists$Elim at the end of the proof to discharge assumptions of $Pc \land Qc \land Rac$ and make sure that c doesn't appear in any other undischarged assumptions. Not that we're using c as our constant here: we can't use a or b because they appear in our conclusion Rab, the existential premise $\exists x(Px \land Qx \land Rax)$ and our other premise $Pb \land Qb$.

We want to prove Rab; because our assumption $Pc \land Qc \land Rac$ gives us Rac, we can obtain Rab using =Elim. To do this, we need to show $b = c$. Our big premise $\forall x \forall y((Px \land Qx) \land (Py \land Qy) \to x = y)$ gives us $(Pb \land Qb) \land (Pc \land Qc) \to b = c$, so we can prove $b = c$ by $\to$Elim if we can prove $(Pb \land Qb) \land (Pc \land Qc)$. We can obtain this by $\land$Intro and $\land$Elim from our premise $Pb \land Qb$ and our assumption $Pc \land Qc \land Rac$.

Identity 8

$$
\begin{prooftree}
\AxiomC{$[Pa \wedge Qa]$}
\RightLabel{$\wedge$Elim}
\UnaryInfC{Qa}
\AxiomC{$[Pa \wedge Qa]$}
\RightLabel{$\wedge$Elim}
\UnaryInfC{Pa}
\AxiomC{$[a = a]^{=}$}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$Pa \wedge a = a$}
\AxiomC{$\forall x \forall y(Px \wedge x = y \to \neg Qy)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\forall y(Pa \wedge a = y \to \neg Qy)$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$Pa \wedge a = a \to \neg Qa$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\neg Qa$}
\RightLabel{$\neg$Intro}
\BinaryInfC{$\neg(Pa \wedge Qa)$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall z \neg(Pz \wedge Qz)$}
\end{prooftree}
$$

This is a past paper question from 2012. Our conclusion is a universal statement, so we can derive it from $\neg(Pa \wedge Qa)$ (as long as a doesn't appear in any undischarged assumptions by the end of the proof). This is a negated statement, so we assume $Pa \wedge Qa$ and show that it leads to a contradiction.

We don't have any negated statements immediately available which can give us the contradiction we need, but our premise $\forall y(Pa \wedge a = y \to \neg Qy)$ can give us one. If we apply $\forall$Elim twice, we can obtain $Pa \wedge a = a \to \neg Qa$, which is an abbreviation of $(Pa \wedge a = a) \to \neg Qa$. This means that if we can prove $Pa \wedge a = a$ we can prove $\neg Qa$ by $\to$Elim.

Pa comes from our assumption of $Pa \wedge Qa$ and $a = a$ can be assumed and discharged by $=$Intro, so a proof of $Pa \wedge a = a$ is easy to provide. With a proof of $\neg Qa$ and a proof of Qa (which also follows from our assumption of $Pa \wedge Qa$) we have the contradiction we need.

Identity 9

$$\dfrac{\dfrac{[a=a]^{=} \qquad \dfrac{\dfrac{\forall x\forall y(Rxy \leftrightarrow x=y)}{\forall y(Ray\leftrightarrow a=y)}\,{\scriptstyle\forall\text{Elim}}}{Raa\leftrightarrow a=a}\,{\scriptstyle\forall\text{Elim}}}{Raa}\,{\scriptstyle\leftrightarrow\text{Elim}}}{\forall x Rxx}\,{\scriptstyle\forall\text{Intro}}$$

Our conclusion $\forall xRxx$ is a universal statement, so we can derive it from Raa (as long as a appears in no undischarged assumptions). By applying $\forall$Elim twice on our premise we can derive $Raa \leftrightarrow a = a$; this allows us to apply $\leftrightarrow$Elim and derive Raa if we can provide a proof of $a = a$. =Intro gives us the proof of $a = a$ we need by allowing us to make and immediately discharge an assumption of $a = a$.

Identity 10

$$\dfrac{\dfrac{\dfrac{\dfrac{Rab \quad [a=b]}{Raa}\,{\scriptstyle =\text{Elim}} \qquad \dfrac{\forall x\neg Rxx}{\neg Raa}\,{\scriptstyle\forall\text{Elim}}}{\neg a=b}\,{\scriptstyle\neg\text{Intro}}}{\exists y\neg a=y}\,{\scriptstyle\exists\text{Intro}}}{\exists x\exists y\neg x=y}\,{\scriptstyle\exists\text{Intro}}$$

This is a past paper question from 2012. The conclusion $\exists x\exists y\neg x = y$ has two existential quantifiers, so it's likely we will derive it by applying $\exists$Intro, but we also need to determine which statement we should try to derive it from. The constants a and b appear in the premise Rab, so $\neg a = b$ would be a good sentence to try and prove.

This is a negation, so we prove it by assuming $a = b$ and showing that it leads to a contradiction. There are actually lots of ways we can derive a contradiction from this assumption and our premises Rab and $\forall x\neg Rxx$. In the proof above, we use =Elim and $a = b$ to replace the b in Rab with a, giving Raa. Because we can derive $\neg Raa$ from $\forall x\neg Rxx$, we have the contradiction we need.

Identity 11

$$
\begin{prooftree}
\AxiomC{$\exists x \forall y x = y$}
\AxiomC{$\exists x \exists y Rxy$}
\AxiomC{$[\exists y Rby]^2$}
\AxiomC{$[Rbc]^1$}
\AxiomC{$[\forall y a = y]^3$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$a = b$}
\RightLabel{$=$Elim}
\BinaryInfC{Rac}
\AxiomC{$[\forall y a = y]^3$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$a = c$}
\RightLabel{$=$Elim}
\BinaryInfC{Raa}
\AxiomC{$[\forall y a = y]^3$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$a = b_1$}
\RightLabel{$=$Elim}
\BinaryInfC{$Rb_1 a$}
\AxiomC{$[\forall y a = y]^3$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$a = c_1$}
\RightLabel{$=$Elim}
\BinaryInfC{$Rb_1 c_1$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall y R b_1 y$}
\RightLabel{$\forall$Intro}
\UnaryInfC{$\forall x \forall y Rxy$}
\RightLabel{$\exists\text{Elim}^1$}
\BinaryInfC{$\forall x \forall y Rxy$}
\RightLabel{$\exists\text{Elim}^2$}
\BinaryInfC{$\forall x \forall y Rxy$}
\RightLabel{$\exists\text{Elim}^3$}
\BinaryInfC{$\forall x \forall y Rxy$}
\end{prooftree}
$$

We have two existential premises, so we should think about those first. From $\exists x \forall y x = y$ we can apply $\exists$Elim to discharge an assumption of $\forall y a = y$, and from $\exists x \exists y Rxy$ we can apply $\exists$Elim twice to discharge an assumption of Rbc.

Our conclusion is $\forall x \forall y Rxy$, a universal statement, so we derive it from Rb_1c_1. We can't derive it from any statements involving a, b or c because they appear in our assumptions of Rbc and $\forall y a = y$, which won't be discharged until the very end of the proof.

Now all we need to do is move from our assumptions of Rbc and $\forall y a = y$ to Rb_1c_1. This is easy to do using =Elim, because our assumption of $\forall y a = y$ tells us that everything is identical to a. First we replace the b and c with Rbc with as, and then we replace those as with b_1 and c_1.

Identity 12

$$
\begin{prooftree}
\AxiomC{$[\forall x Pxx]^4$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{Paa}
\AxiomC{$[\neg Pab]^2$}
\AxiomC{$[a = b]^1$}
\RightLabel{\scriptsize $=$Elim}
\BinaryInfC{$\neg Paa$}
\RightLabel{\scriptsize $\neg$Intro^1}
\BinaryInfC{$\neg a = b$}
\RightLabel{\scriptsize $\rightarrow$Intro^2}
\UnaryInfC{$\neg Pab \rightarrow \neg a = b$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall y(\neg Pay \rightarrow \neg a = y)$}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x \forall y(\neg Pxy \rightarrow \neg x = y)$}
\AxiomC{$[a = a]^=$}
\AxiomC{$[\neg Paa]^3$}
\AxiomC{$[\forall x \forall y(\neg Pxy \rightarrow \neg x = y)]^4$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\forall y(\neg Pay \rightarrow \neg a = y)$}
\RightLabel{\scriptsize $\forall$Elim}
\UnaryInfC{$\neg Paa \rightarrow \neg a = a$}
\RightLabel{\scriptsize $\rightarrow$Elim}
\BinaryInfC{$\neg a = a$}
\RightLabel{\scriptsize $\neg$Elim^3}
\BinaryInfC{Paa}
\RightLabel{\scriptsize $\forall$Intro}
\UnaryInfC{$\forall x Pxx$}
\RightLabel{\scriptsize $\leftrightarrow$Intro^4}
\BinaryInfC{$\forall x Pxx \leftrightarrow \forall x \forall y(\neg Pxy \rightarrow \neg x = y)$}
\end{prooftree}
$$

This is a past paper question from 2009. Our conclusion is a biconditional, so we need to provide a proof of $\forall x \forall y(\neg Pxy \rightarrow \neg x = y)$ from $\forall x Pxx$ and a proof of $\forall x Pxx$ $\forall x \forall y(\neg Pxy \rightarrow \neg x = y)$.

On the left-hand side, we can derive $\forall x Pxx$ $\forall x \forall y(\neg Pxy \rightarrow \neg x = y)$ by deriving $\neg Pab \rightarrow \neg a = b$ and applying $\forall$Intro twice, as long as neither a or b appear in any undischarged assumptions when we apply $\forall$Intro. $\neg Pab \rightarrow \neg a = b$ is an implication, so we assume $\neg Pab$ and try to derive $\neg a = b$. Because $\neg a = b$ is a negated statement, we assume $a = b$ and try to derive a contradiction.

There are many ways we can derive a contradiction from $\forall x Pxx$, $\neg Pab$ and $a = b$. In the proof above, we use $a = b$ to replace the b in $\neg Pab$ with a, giving us $\neg Paa$. This contradicts Paa, which can be derived from $\forall x Pxx$.

On the right-hand side, we need to prove the universal statement $\forall x Pxx$, which can be derived from Paa (as long as a doesn't appear in any undischarged assumptions when we apply $\forall$Intro). Sadly we don't have any way of deriving Paa directly; instead we prove it by deriving a contradiction from assumptions of $\neg Paa$.

From our assumption of $\forall x Pxx$ $\forall x \forall y(\neg Pxy \rightarrow \neg x = y)$ we can derive $\neg Paa \rightarrow \neg a = a$, so we can derive $\neg a = a$ by $\rightarrow$Elim. This gives us the contradiction we need, because $=$Intro lets us make and immediately discharge an assumption of $a = a$.

Notice that this proof involves a particular symmetry: on the left-hand side we apply $\forall$Intro (twice), $\rightarrow$Intro, $\neg$Intro, $=$Elim and $\forall$Elim; on the right-hand side we apply $\forall$Elim (twice), $\rightarrow$Elim, $\neg$Elim, $=$Intro and $\forall$Intro. Each time an introduction rule appears on one side, the corresponding elimination rule appears on the other side.

Identity 13

$$
\begin{prooftree}
\AxiomC{$\forall x \exists y(Rxy \wedge Py)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\exists y(Ray \wedge Py)$}
\AxiomC{$\forall x \exists y(Rxy \wedge Py)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\exists y(Rby \wedge Py)$}
\AxiomC{$[Rbc \wedge Pc]^3$}
\RightLabel{$\wedge$E}
\UnaryInfC{Rbc}
\AxiomC{$[Rbc \wedge Pc]^3$}
\RightLabel{$\wedge$E}
\UnaryInfC{Pc}
\AxiomC{$[Rab \wedge Pb]^2$}
\RightLabel{$\wedge$E}
\UnaryInfC{Pb}
\AxiomC{$[\forall x \forall y(Px \to (Py \to x = y))]^1$}
\RightLabel{$\forall$E}
\UnaryInfC{$\forall y(Pb \to (Py \to b = y))$}
\RightLabel{$\forall$E}
\UnaryInfC{$Pb \to (Pc \to b = c)$}
\RightLabel{$\to$E}
\BinaryInfC{$Pc \to b = c$}
\RightLabel{$\to$E}
\BinaryInfC{$b = c$}
\RightLabel{$=$E}
\BinaryInfC{Rbb}
\AxiomC{$\forall x \neg Rxx$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Rbb$}
\RightLabel{$\neg$I^1}
\BinaryInfC{$\neg \forall x \forall y(Px \to (Py \to x = y))$}
\RightLabel{$\exists$E^2}
\BinaryInfC{$\neg \forall x \forall y(Px \to (Py \to x = y))$}
\RightLabel{$\exists$E^3}
\BinaryInfC{$\neg \forall x \forall y(Px \to (Py \to x = y))$}
\end{prooftree}
$$

This is a past paper question from 2012. Because our conclusion is a negated statement, we assume $\forall x \forall y(Px \to (Py \to x = y))$ and show that it leads to a contradiction. However, our two premises are difficult to use. $\forall x \neg Rxx$ could give us $\neg Raa$, $\neg Rbb$ or any of infinitely many other negated statements, so we can't be certain which one will give us the contradiction we need. Similarly $\forall x \exists y(Rxy \wedge Py)$ could give us any number of different existential statements, and we can't be certain how many times we'll need to use it.

For a hint we can look at our assumption of $\forall x \forall y(Px \to (Py \to x = y))$. This is an implication involving two occurrences of the predicate P. In order to derive the consequent by $\to$Elim, we need two different things for which P holds. Each time we apply $\forall$Elim on $\forall x \exists y(Rxy \wedge Py)$ we obtain an existential statement asserting the existence of one thing for which P holds. This suggests we want to use $\forall$Elim on $\forall x \exists y(Rxy \wedge Py)$ twice.

So we derive $\exists y(Ray \wedge Py)$ and apply $\exists$Elim at the very end of the proof to discharge assumptions of $Rab \wedge Pb$. Then before the end of the proof we derive a second existential statement, $\exists y(Ray \wedge Py)$, which we use to discharge assumptions of $Rbc \wedge Pc$. We're justified in doing this: note that by the very end of the proof, b appears in no undischarged assumptions other than $Rab \wedge Pb$ because $Rbc \wedge Pc$ has already been discharged.

From these assumptions of $Rab \wedge Pb$ and $Rbc \wedge Pc$ we can derive Pb and Pc, which (using our assumption of $\forall x \forall y(Px \to (Py \to x = y)))$ gives us $b = c$. We could use $b = c$ with Rab to obtain Rac, but this isn't very useful. Instead, we use $b = c$ with Rbc to obtain Rbb. Because we can derive $\neg Rbb$ from $\forall x \neg Rxx$, we have the contradiction we need.

5.10 Additional challenges

Admissible rules 1

The rule $\star 1$ is admissible. We can show that any proof making use of $\star 1$ can be rewritten using $\wedge$Elim.

Suppose we have a proof involving one or more applications of $\star 1$. Each application of $\star 1$ corresponds to a subproof of the following form:

$$\dfrac{\begin{array}{c}\vdots \\ \phi \wedge \psi\end{array} \qquad \begin{array}{c}[\phi] \\ \vdots \\ \chi\end{array}}{\chi}\ \star 1$$

We can rewrite this subproof by moving the proof of $\phi \wedge \psi$ to the top of the proof of χ, and applying $\wedge$Elim to derive ϕ:

$$\begin{array}{c}\vdots \\ \dfrac{\phi \wedge \psi}{\phi}\ \wedge\text{Elim} \\ \vdots \\ \chi\end{array}$$

We can repeat this process for each application of $\star 1$ (starting with the smallest subproof) until we are left with a proof only using the original Natural Deduction rules.

Admissible rules 2

The rule $\star 2$ is not admissible. $P \to Q, \neg P \vdash \neg Q$ is not possible in unaugmented Natural Deduction, but is possible with $\star 2$:

$$\dfrac{\neg P \quad P \to Q}{\neg Q}\ \star 2$$

Admissible rules 3

The rule $\star 3$ is not admissible. $\vdash P$ is not possible in unaugmented Natural Deduction, but is possible with $\star 3$:

$$\dfrac{\dfrac{\dfrac{[Q]}{Q \to Q}\ \to\text{Intro} \qquad [(Q \to Q) \to P]}{P}\ \to\text{Elim}}{P}\ \star 3$$

Admissible rules 4

The rule $\star 4$ is admissible. We can rewrite any proof using $\star 4$ as a proof using $\neg$Intro, $\neg$Elim and $\lor$Intro.

Suppose we have a proof involving one or more applications of $\star 4$. Each application corresponds to a subproof of the following form:

$$\begin{array}{c} [\neg\phi] \\ \vdots \\ \dfrac{\psi}{\phi \lor \psi}\ \star 4 \end{array}$$

We replace the $\star 4$ step itself with an application of $\lor$Intro. At the top of the subproof we assume ϕ and $\neg(\phi \lor \psi)$ and apply $\lor$Intro and $\neg$Intro to derive $\neg\phi$. At the bottom of the subproof we assume $\neg(\phi \lor \psi)$ again and apply $\neg$Elim to discharge assumptions of $\neg(\phi \lor \psi)$ and derive $\phi \lor \psi$.

The resultant subproof will look like this:

$$\begin{array}{c} \dfrac{\dfrac{[\phi]^1}{\phi \lor \psi}\ {\scriptstyle \lor\text{Intro}} \qquad [\neg(\phi \lor \psi)]^2}{\neg\phi}\ {\scriptstyle \neg\text{Intro}^1} \\ \vdots \\ \dfrac{\dfrac{\psi}{\phi \lor \psi}\ {\scriptstyle \lor\text{Intro}} \qquad\qquad [\neg(\phi \lor \psi)]^2}{\neg\phi}\ {\scriptstyle \neg\text{Elim}^2} \end{array}$$

This is a familiar proof structure: it is how we proved disjunctions such as $P \lor \neg P$ by indirect proof.

We can repeat this process for each application of $\star 4$ (starting with the smallest subproof) until we are left with a proof only using the original Natural Deduction rules.

Admissible rules 5

The rule $\star 5$ is admissible. We can rewrite any proof using $\star 5$ as a proof using $\to$Intro and $\neg$Elim.

Suppose we have a proof involving one or more applications of $\star 5$. Each application corresponds to a subproof of the following form:

$$\begin{array}{c} [\phi \to \psi] \\ \vdots \\ \dfrac{\phi}{\phi}\ \star 5 \end{array}$$

At the top of the subproof we assume $\neg\phi$ and ϕ and apply $\neg$Elim and $\to$Intro to derive $\phi \to \psi$. At the bottom of the subproof we assume $\neg\phi$ again and apply $\neg$Elim to discharge assumptions of $\neg\phi$ and derive ϕ.

The resultant subproof will look like this:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{[\phi]^1 \quad [\neg\phi]^2}{\psi}\,{\scriptstyle \neg\text{Elim}}}{\phi \to \psi}\,{\scriptstyle \to\text{Intro}^1} \\ \vdots \\ \phi\end{array} \qquad\qquad [\neg\phi]^2}{\phi}\,{\scriptstyle \neg\text{Elim}^2}$$

We can repeat this process for each application of $\star 5$ (starting with the smallest subproof) until we are left with a proof only using the original Natural Deduction rules.

Contraposition 1

$$\dfrac{\dfrac{\neg P}{\neg Q \to \neg P}\,{\scriptstyle \to\text{Intro}}}{P \to Q}\,{\scriptstyle \text{C}}$$

Our conclusion $P \to Q$ is an implication, but now we have two different ways of proving an implication. We could assume P and prove Q from it, or we could prove $\neg Q \to \neg P$. It turns out that we need to do the latter. $\neg Q \to \neg P$ follows from $\neg P$ by ordinary $\to$Intro, and then we can apply contraposition to derive $P \to Q$.

Contraposition 2

$$\dfrac{\dfrac{P \qquad \dfrac{\dfrac{[\neg P]}{\neg Q \to \neg P}\,{\scriptstyle \to\text{Intro}}}{P \to Q}\,{\scriptstyle \text{C}}}{Q}\,{\scriptstyle \to\text{Elim}}}{\neg P \to Q}\,{\scriptstyle \to\text{Intro}}$$

We prove $\neg P \to Q$ by providing a proof of Q from assumptions of $\neg P$. To obtain Q we derive $P \to Q$ from $\neg P$ (as in the first contraposition problem) and then use our premise P to obtain Q by $\to$Elim.

Contraposition 3

$$\dfrac{\neg\neg P \qquad \dfrac{\dfrac{\dfrac{\neg\neg P}{\neg\neg\neg\neg P \to \neg\neg P}\,{\scriptstyle \to\text{Intro}}}{\neg P \to \neg\neg\neg P}\,{\scriptstyle \text{C}}}{\neg\neg P \to P}\,{\scriptstyle \text{C}}}{P}\,{\scriptstyle \to\text{Elim}}$$

We want to get from $\neg\neg P$ to P, and we know we can't do this using $\neg$Elim. Because P doesn't have any connectives in it, we obviously can't arrive at P by applying any introduction rules. Instead it seems reasonable to try and arrive at P by applying $\to$Elim. If we were able to provide a proof of $\neg\neg P \to P$, then we could go from $\neg\neg P$ to P by $\to$Elim.

How can we provide a proof of $\neg\neg P \to P$? This is an implication, and here it seems reasonable to apply contraposition. By applying contraposition, we can derive $\neg\neg P \to P$ from $\neg P \to \neg\neg\neg P$. $\neg P \to \neg\neg\neg P$ doesn't look very easy to prove, but this in turn can be derived from $\neg\neg\neg\neg P \to \neg\neg P$ by contraposition. $\neg\neg\neg\neg P \to \neg\neg P$ follows straight from our premise $\neg\neg P$ by $\to$Intro.

Another way of proving P from $\neg\neg P$ would have been to apply the method shown below. The proof is slightly longer, but we only need to assume $\neg\neg P$ once:

$$\dfrac{\dfrac{[Q]}{Q \to Q}\,{\scriptstyle \to\text{Intro}} \qquad \dfrac{\dfrac{\dfrac{\neg\neg P}{\neg\neg(Q \to Q) \to \neg\neg P}\,{\scriptstyle \to\text{Intro}}}{\neg P \to \neg(Q \to Q)}\,{\scriptstyle \text{C}}}{(Q \to Q) \to P}\,{\scriptstyle \text{C}}}{P}\,{\scriptstyle \to\text{Elim}}$$

There is still an intuition behind this. $P \to P$ is a simple tautology (one we can easily derive), so $\neg(P \to P)$ is a contradiction and $\neg\neg(P \to P)$ is also a tautology. After taking $\neg\neg P$ as our premise, we infer that a tautology implies $\neg\neg P$. From this, we infer (by contraposition) that $\neg P$ implies a contradiction. Then we infer (by contraposition again) that a tautology we can easily derive implies P. We then derive this tautology and derive P.

We will use a similar method to this when we tackle question five, and show that any proof in ordinary Natural Deduction can be converted to a proof using contraposition.

Contraposition 4

Suppose that $\Gamma \vdash_C \phi$. This means that there is a proof of ϕ from Γ which potentially uses contraposition (but doesn't use $\neg$Intro or $\neg$Elim).

If this proof involves no applications of contraposition, it is trivially true that $\Gamma \vdash \phi$.

If the proof does involve at least one application of contraposition, this application corresponds to a subproof of the following shape:

$$\dfrac{\begin{matrix}\vdots\\ \neg\psi \to \neg\phi\end{matrix}}{\phi \to \psi}\,\mathrm{C}$$

We replace this subproof with a subproof of the following shape:

$$\dfrac{\dfrac{[\phi]^2 \qquad \dfrac{[\neg\psi]^1 \qquad \begin{matrix}\vdots\\ \neg\psi \to \neg\phi\end{matrix}}{\neg\phi}\,{\to}\mathrm{Elim}}{\psi}\,\neg\mathrm{Elim}^1}{\phi \to \psi}\,{\to}\mathrm{Intro}^2$$

We repeat this process for each application of contraposition, starting with the smallest subproof. The resultant proof has no applications of contraposition, but does have applications of ¬Elim. Hence $\Gamma \vdash \phi$.

Contraposition 5

Suppose that $\Gamma \vdash \phi$. This means that there is a proof of ϕ from Γ which potentially uses ¬Intro and ¬Elim.

If this proof involves no applications of ¬Intro or ¬Elim, it is trivially true that $\Gamma \vdash_C \phi$. More difficult is the case where this proof involves applications ¬Intro or ¬Elim.

What we need to do is replace all applications of ¬Intro and ¬Elim with subproofs using contraposition.

First, we note that an application of ¬Intro corresponds to a subproof of the following shape:

$$\dfrac{\begin{matrix}[\phi]\\ \vdots\\ \psi\end{matrix} \qquad \begin{matrix}[\phi]\\ \vdots\\ \neg\psi\end{matrix}}{\neg\phi}\,\neg\mathrm{Intro}$$

It's entirely possible to replace these subproofs with different subproofs involving contraposition, but this is quite cumbersome. Instead what we'll do is replace all ¬Intro steps with ¬Elim steps, and then replace all the ¬Elim steps with subproofs involving contraposition.

To do this we replace all applications of ¬Intro (if there are any) with subproofs of the following shape:

$$\dfrac{\begin{matrix}\dfrac{[\neg\phi]^1 \quad [\neg\neg\phi]^3}{\phi}\,\neg\mathrm{Elim}^1\\ \vdots\\ \psi\end{matrix} \qquad \begin{matrix}\dfrac{[\neg\phi]^2 \quad [\neg\neg\phi]^3}{\phi}\,\neg\mathrm{Elim}^2\\ \vdots\\ \neg\psi\end{matrix}}{\neg\phi}\,\neg\mathrm{Elim}^3$$

Now our proof has no ¬Intro steps, but still has ¬Elim steps. We note that each ¬Elim step corresponds to a subproof of the following shape:

$$\dfrac{\begin{matrix}[\neg\phi]\\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\neg\phi]\\ \vdots \\ \neg\psi\end{matrix}}{\phi}\ \neg\text{Elim}$$

We replace all applications of ¬Elim with subproofs of the following shape:

$$\dfrac{\dfrac{[P]^1}{P\to P}\,{\to}\text{Intro}^1 \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[\neg\phi]^2\\ \vdots\\ \psi\end{matrix}\quad \dfrac{\dfrac{\begin{matrix}[\neg\phi]^2\\ \vdots\\ \neg\psi\end{matrix}}{\neg\neg(P\to P)\to\neg\psi}\,{\to}\text{Intro}}{\psi\to\neg(P\to P)}\,\text{C}}{\neg(P\to P)}\,{\to}\text{Elim}}{\neg\phi\to\neg(P\to P)}\,{\to}\text{Intro}^2}{(P\to P)\to\phi}\,\text{C}}{\phi}\,{\to}\text{Elim}$$

The resultant proof now contains no applications of ¬Intro or ¬Elim, but may contain applications of contraposition.

Hence $\Gamma \vdash_C \phi$.