$$dS = S\mu\, dt + S\sigma\, dW \quad , \quad ¿S?$$

$\boxed{\text{Regla de Ito}}$ ~ Proceso de Ito :

$$dX = a(x,t)\, dt + b(x,t)\, dW \qquad \& \qquad g = f(x,t)$$

$$dg = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dx + \frac{1}{2}\frac{\partial g^2}{\partial x^2} dx^2$$

Sustituyendo valores ...

$$dg = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x}\left[a(x,t)\, dt + b(x,t)\, dW\right] + \frac{1}{2}\frac{\partial g^2}{\partial x^2}\left[\cancel{a^2(x,t)\, dt^2} + b^2(x,t)\, \overset{dt}{\cancel{dW^2}} + \cancel{2a(x,t)\, b(x,t)\, dt\, dW}\right]$$

$$dg = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x}\left[a(x,t)\, dt + b(x,t)\, dW\right] + \frac{1}{2}\frac{\partial g^2}{\partial x^2}\left[b^2(x,t)\, dt\right]$$

Factorizamos ...

$$\boxed{dg = \left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} a(x,t) + \frac{1}{2}\frac{\partial g^2}{\partial x^2} b^2(x,t)\right] dt + \frac{\partial g}{\partial x} b(x,t)\, dW}$$

Rendimientos :

$$R_{t+1} = \ln\left(\frac{S_{t+1}}{S_t}\right) = \underbrace{\ln(S_{t+1}) - \ln(S_t)}_{\text{"El precio tiene un comportamiento log-normal"}} \Rightarrow \text{Normal}$$

Entonces ...

$$d\tilde{G} = \ln(S_{t+1}) - \ln(S_t) = \text{algo}$$

$$\boxed{G = \ln S_t}$$

$$dG \longrightarrow S \qquad \boxed{dS = S\mu\, dt + S\sigma\, dW}$$

Aplicando el lema de Ito ...

$$dG = \left[\emptyset + \frac{1}{S}(S\mu) + \frac{1}{2}\left(-\frac{1}{S^2}\right)\left(S^2\sigma^2\right)\right] dt + \frac{1}{S}(S\sigma)\, dW$$

$$\boxed{dG = \left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW}$$

$$\ln(S_{t+1}) - \ln(S_t) = \left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW$$

$$\ln(S_{t+1}) = \ln(S_t) + \left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW$$

$$\underline{E[\ln(S_{t+1})] = \ln(S_t) + \left[\mu - \frac{1}{2}\sigma^2 \right] dt}$$

$$\underline{V[\ln(S_{t+1})] = \sigma^2 dt}$$

$$\boxed{S_{t+1} = S_t \, e^{\left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW}} \Rightarrow \text{Comportamiento log - normal}$$

¿$E[S_{t+1}]$? Nota: Propiedades de una distribución log - normal

Si $y = \ln(x) \longrightarrow x = e^y$, entonces $E[x] = E[e^y] = e^{a + \frac{1}{2}b^2}$

donde $E[y] = a$, $V[y] = b^2$

$$V[x] = V[e^y] = e^{2a + 2b^2} - e^{2a + b^2}$$

$$E[S_{t+1}] = E\left[S_t \, e^{\left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW} \right] = S_t \, E\left[e^{\left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW} \right]$$

Si $R = \left[\mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW$, $E[R] = \underbrace{\left[\mu - \frac{1}{2}\sigma^2 \right] dt}_{a}$ & $\underbrace{V[R] = \sigma^2 dt}_{b}$

entonces: $E[S_{t+1}] = S_t \, e^{\left[\mu - \frac{1}{2}\sigma^2 \right] dt + \frac{1}{2}\sigma^2 dt} = \boxed{S_t \, e^{\mu dt}}$

Ejemplos de aplicación:

1. Considera que el rendimiento anual de un activo es de 15 % y su vol. anual es de 65 %. El día de hoy el activo tiene un precio de 45.

¿Probabilidad precio sea mayor a 47 en 3 meses?

Solo usar $P(ds > z)$ con intervalos de tiempo dt pequeños (max. 1 mes)

$P(S > 47) = P(\ln S > \ln 47)$:

$E[\ln S] = \ln S_t + \left[\mu - \frac{1}{2}\sigma^2\right] dt = \ln(45) + \left[0.15 - \frac{1}{2}(0.65)^2\right]\frac{1}{4}$

$= \underline{3.7913}$

$V[\ln S] = \sigma^2 dt = (0.65)^2 \frac{1}{4} = \underline{0.1056}$

$z = (\ln 45 - E[\ln S]) / \sqrt{V[\ln S]} = (\ln 45 - 3.7913) / \sqrt{0.1056} = \underline{0.05}$

$P(S > 47) = 1 - 0.5714 = \underline{42.86\%}$

II. Precio esperado en 3 meses:

$e^{3.7913} \approx \underline{\$\,44.31}$