Problem on Wallis integrals

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Mathematics

Outline

1. Statement of the problem

2. Suggested correction

Statement of the problem

Equipments

1. Statement of the problem

2. Suggested correction

Statement of the problem

Let's write : $\forall\ n \in \mathbb{N},\ I_n = \int\limits_0^{\frac{\pi}{2}} sin^n(x)dx$

1°) Calculate I_0 and I_1.
2°) When $n \in \mathbb{N} \setminus \{0, 1\}$, give a relationship between I_n and I_{n-2}. Deduce the value of $I_n I_{n-1}$ for all $n \in \mathbb{N}^*$.
3°) Calculate I_{2p} for all $p \in \mathbb{N}$. From this, deduce I_{2p+1}.
4°) Specify the monotomy of the sequence $(I_n)_{n\in\mathbb{N}}$.
5°) Provide, using the above, for $n \in \mathbb{N}^*$, a double inequality on I_n^2. Derive a simple equivalent of I_n.

Suggested correction

Equipments

Suggested correction - Question 1

1°) Calculate I_0 and I_1.

$$I_0 = \int_0^{\frac{\pi}{2}} sin^0(x)dx = \int_0^{\frac{\pi}{2}} dx = \boxed{\frac{\pi}{2}}$$

$$I_1 = \int_0^{\frac{\pi}{2}} sin^1(x)dx = \int_0^{\frac{\pi}{2}} sin(x)dx = \left[-cos\left(x\right)\right]_0^{\frac{\pi}{2}} = cos\left(0\right) - cos\left(\frac{\pi}{2}\right) = 1 - 0 = \boxed{1}$$

Suggested correction - Question 2

2°) When $n \in \mathbb{N} \setminus \{0,1\}$, give a relationship between I_n and I_{n-2}. Deduce the value of $I_n I_{n-1}$ for all $n \in \mathbb{N}^*$.

We will do an integration by parts on the quantity I_n by using the following relationship : $sin^n(x) = sin(x)sin^{n-1}(x)$. The functions $u : x \mapsto -cos(x)$ and $v : x \mapsto sin^{n-1}(x)$ are of class C^1. In addition, we have :

$$\forall\ x \in \left[0, \frac{\pi}{2}\right],\ u'(x) = sin(x) \text{ and } v'(x) = (n-1)cos(x)sin^{n-2}(x)$$

Thus :

$$\forall\ n \in \mathbb{N} \setminus \{0,1\},\ I_n = \int_0^{\frac{\pi}{2}} sin^n(x)dx = \int_0^{\frac{\pi}{2}} sin(x)sin^{n-1}(x)dx$$

$$= \int_0^{\frac{\pi}{2}} u'(x)v(x)dx = [u(x)v(x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} u(x)v'(x)dx$$

Suggested correction - Question 2

2°) When $n \in \mathbb{N} \setminus \{0, 1\}$, give a relationship between I_n and I_{n-2}. Deduce the value of $I_n I_{n-1}$ for all $n \in \mathbb{N}^*$.

$$= \left[-cos(x)sin^{n-1}(x)\right]_0^{\frac{\pi}{2}} + (n-1)\int\limits_0^{\frac{\pi}{2}} cos^2(x)sin^{n-2}(x)dx$$

$$= cos(0)sin^{n-1}(0) - cos\left(\frac{\pi}{2}\right)sin^{n-1}\left(\frac{\pi}{2}\right) + (n-1)\int\limits_0^{\frac{\pi}{2}} cos^2(x)sin^{n-2}(x)dx$$

$$= 0 - 0 + (n-1)\int\limits_0^{\frac{\pi}{2}} cos^2(x)sin^{n-2}(x)dx \quad \text{because } n \in \mathbb{N} \setminus \{0, 1\}$$

$$= (n-1)\int\limits_0^{\frac{\pi}{2}} cos^2(x)sin^{n-2}(x)dx = (n-1)\int\limits_0^{\frac{\pi}{2}} \left[1 - sin^2(x)\right] sin^{n-2}(x)dx$$

Suggested correction - Question 2

2°) When $n \in \mathbb{N} \setminus \{0,1\}$, give a relationship between I_n and I_{n-2}. Deduce the value of $I_n I_{n-1}$ for all $n \in \mathbb{N}^*$.

$$= (n-1)\int_0^{\frac{\pi}{2}} sin^{n-2}(x)dx - (n-1)\int_0^{\frac{\pi}{2}} sin^{n}(x)dx = (n-1)I_{n-2} - (n-1)I_n$$

$$\iff \forall\ n \in \mathbb{N} \setminus \{0,1\}\,,\ I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$\iff \forall\ n \in \mathbb{N} \setminus \{0,1\}\,,\ I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$\iff \forall\ n \in \mathbb{N} \setminus \{0,1\}\,,\ nI_n = (n-1)I_{n-2}$$

$$\iff \boxed{\forall\ n \in \mathbb{N} \setminus \{0,1\}\,,\ I_n = \left(\frac{n-1}{n}\right) I_{n-2}}$$

If we multiply by I_{n-1} the above equality, then we have :

$$\forall\ n \in \mathbb{N} \setminus \{0,1\}\,,\ nI_nI_{n-1} = (n-1)I_{n-1}I_{n-2}$$

Suggested correction - Question 2

2°) When $n \in \mathbb{N} \setminus \{0, 1\}$, give a relationship between I_n and I_{n-2}. Deduce the value of $I_n I_{n-1}$ for all $n \in \mathbb{N}^*$.

Thus, the sequence $(nI_nI_{n-1})_{n\in\mathbb{N}^*}$ is constant. Let's take $n = 1$, then :

$$1I_1I_{1-1} = I_1I_0 = 1 \times \frac{\pi}{2} = \frac{\pi}{2}$$

Finally :

$$\forall\ n \in \mathbb{N}^*,\ nI_nI_{n-1} = \frac{\pi}{2}$$

$$\Longleftrightarrow \boxed{\forall\ n \in \mathbb{N}^*,\ I_nI_{n-1} = \frac{\pi}{2n}}$$

Suggested correction - Question 3

3°) Calculate I_{2p} for all $p \in \mathbb{N}$. From this, deduce I_{2p+1}.

According to the question **2°)**, we have :

$$\forall\ p \in \mathbb{N}^*,\ I_{2p} = \left(\frac{2p-1}{2p}\right) I_{2p-2} = \left(\frac{2p-1}{2p}\right)\left[\frac{(2p-2)-1}{2p-2}\right] I_{(2p-2)-2}$$

$$= \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right) I_{2p-4} = \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right)\left[\frac{(2p-4)-1}{2p-4}\right] I_{(2p-4)-2}$$

$$= \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right)\left(\frac{2p-5}{2p-4}\right) I_{2p-6}$$

$$= \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right)\left(\frac{2p-5}{2p-4}\right)\ldots\left(\frac{1}{2}\right) I_0 = \frac{\pi}{2}\left[\frac{\prod_{k=1}^{p}(2k-1)}{\prod_{l=1}^{p}(2l)}\right]$$

Suggested correction - Question 3

3°) Calculate I_{2p} for all $p \in \mathbb{N}$. From this, deduce I_{2p+1}.

$$= \frac{\pi}{2}\left[\frac{\prod_{k=1}^{p}(2k-1)}{\prod_{l=1}^{p}(2l)}\right]\left[\frac{\prod_{n=1}^{p}(2n)}{\prod_{m=1}^{p}(2m)}\right] = \frac{\pi\,(2p)!}{2\left[\prod_{k=1}^{p}(2k)\right]^2} = \frac{\pi\,(2p)!}{2^{2p+1}\,(p!)^2}$$

It can be seen that for $p = 0$, we have the formula that is always right. Thus :

$$\boxed{\forall\ p \in \mathbb{N},\ \ I_{2p} = \frac{\pi(2p)!}{2^{2p+1}(p!)^2}}$$

Using the answer to the question **2°)**, we have :

$$\forall\ p \in \mathbb{N},\ \ I_{2p+1} = \frac{\pi}{2I_{2p}\,(2p+1)} \Longleftrightarrow \boxed{\forall\ p \in \mathbb{N},\ \ I_{2p+1} = \frac{(p!)^2}{(2p+1)!}}$$

Suggested correction - Question 4

4°) Specify the monotomy of the sequence $(I_n)_{n\in\mathbb{N}}$.

We have :

$$\forall\ x \in \left[0, \frac{\pi}{2}\right],\ 0 \leq sin(x) \leq 1$$

$$\Longrightarrow \forall\ x \in \left[0, \frac{\pi}{2}\right],\ \forall\ n \in \mathbb{N},\ 0 \leq sin^{n+1}(x) \leq sin^n(x)$$

$$\Longrightarrow \forall\ n \in \mathbb{N},\ 0 \leq \int_0^{\frac{\pi}{2}} sin^{n+1}(x)dx \leq \int_0^{\frac{\pi}{2}} sin^n(x)dx$$

$$\Longleftrightarrow \forall\ n \in \mathbb{N},\ 0 \leq I_{n+1} \leq I_n$$

Thus :

$\boxed{\text{The sequence } (I_n)_{n\in\mathbb{N}} \text{ is decreasing.}}$

Suggested correction - Question 5

5°) Provide, using the above, for $n \in \mathbb{N}^*$, a double inequality on I_n^2. Derive a simple equivalent of I_n.

According to the question **4°)**, we have :

$$\forall\ n \in \mathbb{N},\ 0 \leq I_{n+1} \leq I_n \Longrightarrow \forall\ n \in \mathbb{N},\ 0 \leq I_{n+1} I_n \leq I_n^2$$

$$\forall\ n \in \mathbb{N}^*,\ 0 \leq I_n \leq I_{n-1} \Longrightarrow \forall\ n \in \mathbb{N}^*,\ 0 \leq I_n^2 \leq I_n I_{n-1}$$

Thus :

$$\boxed{\forall\ n \in \mathbb{N}^*,\ 0 \leq I_{n+1} I_n \leq I_n^2 \leq I_n I_{n-1}}$$

According to the question **2°)**, we have :

$$\forall\ n \in \mathbb{N}^*,\ 0 \leq \frac{\pi}{2\left(n+1\right)} \leq I_n^2 \leq \frac{\pi}{2n}$$

We can deduce that :

$$\lim_{n \to +\infty} \frac{2n I_n^2}{\pi} = 1$$

Suggested correction - Question 5

5°) Provide, using the above, for $n \in \mathbb{N}^*$, a double inequality on I_n^2. Derive a simple equivalent of I_n.

In addition, we know that :

$$\forall\, n \in \mathbb{N},\ I_n \geq 0$$

Thus :

$$\boxed{\mathrm{I}_n = \int_0^{\frac{\pi}{2}} sin^n(x)dx \sim \sqrt{\frac{\pi}{2n}} \quad \text{when } n \to +\infty}$$