Test functions for optimization

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In applied mathematics, **test functions**, known as **artificial landscapes**, are useful to evaluate characteristics of optimization algorithms, such as:

- Velocity of convergence.
- Precision.
- Robustness.
- General performance.

Here some test functions are presented with the aim of giving an idea about the different situations that optimization algorithms have to face when coping with these kinds of problems. In the first part, some objective functions for single-objective optimization cases are presented. In the second part, test functions with their respective Pareto fronts for multi-objective optimization problems (MOP) are given.

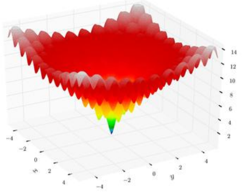
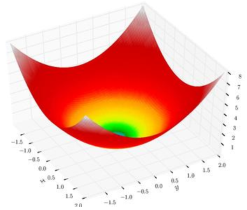
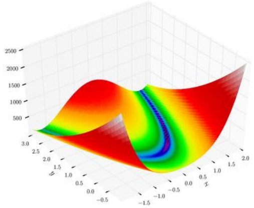
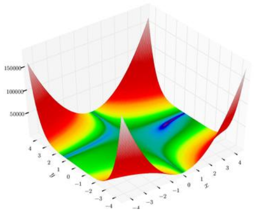
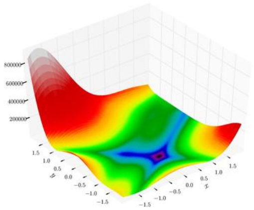
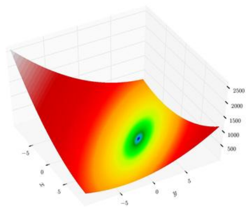
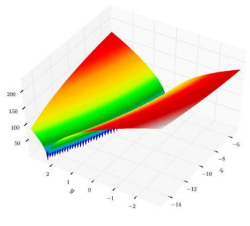
The artificial landscapes presented herein for single-objective optimization problems are taken from Bäck,[1] Haupt et. al.[2] and from Rody Oldenhuis software.[3] Given the amount of problems (55 in total), just a few are presented here. The complete list of test functions is found on the Mathworks website.[4]

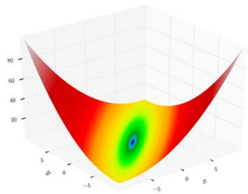
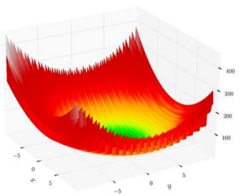
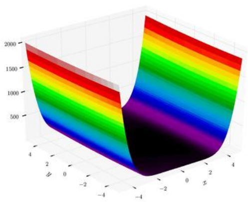
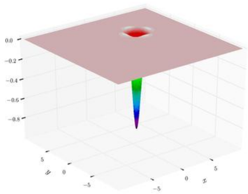
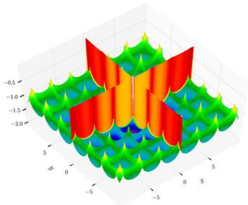
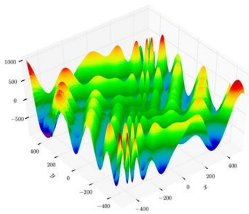
The test functions used to evaluate the algorithms for MOP were taken from Deb,[5] Binh et. al.[6] and Binh.[7] You can download the software developed by Deb,[8] which implements the NSGA-II procedure with GAs, or the program posted on Internet,[9] which implements the NSGA-II procedure with ES.

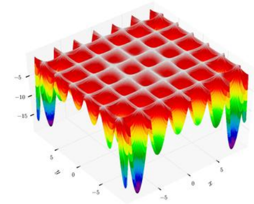
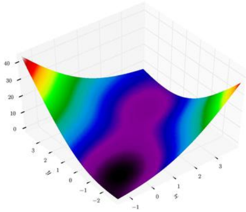
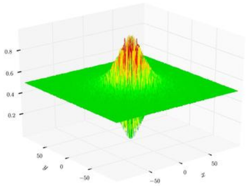
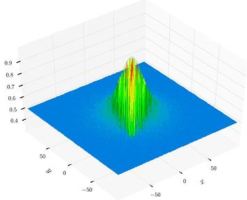
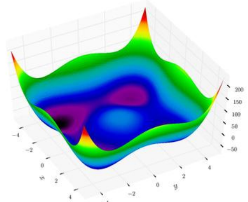
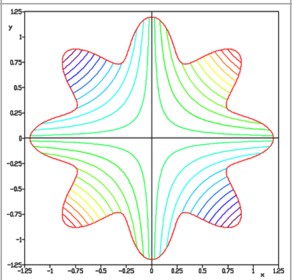
Just a general form of the equation, a plot of the objective function, boundaries of the object variables and the coordinates of global minima are given herein.

Contents

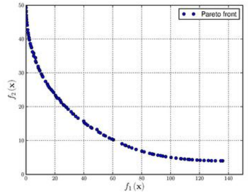
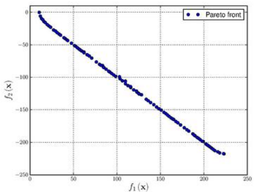
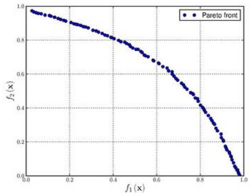
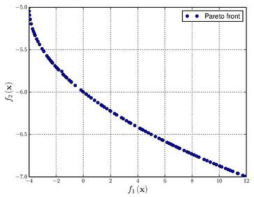
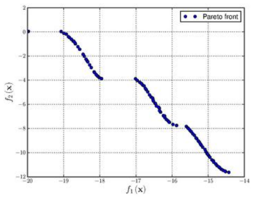
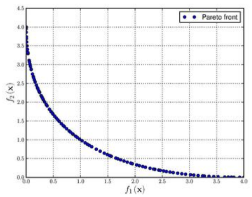
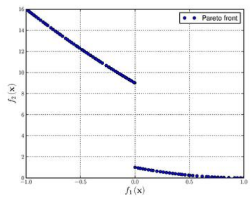
Test functions for single-objective optimization problems

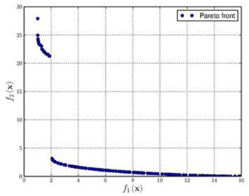
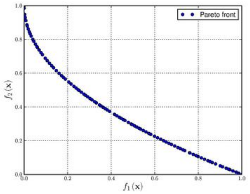
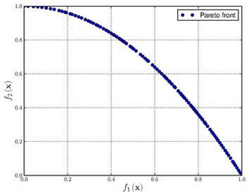
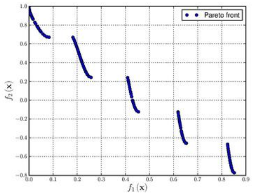
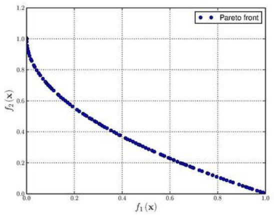
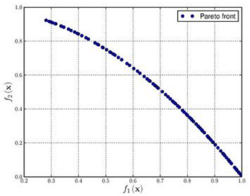
Name	Plot	Formula	Minimum	Search domain
Ackley's function:		$f(x,y) = -20\exp\left(-0.2\sqrt{0.5\left(x^2+y^2\right)}\right)$ $-\exp\left(0.5\left(\cos\left(2\pi x\right)+\cos\left(2\pi y\right)\right)\right)+e+20$	$f(0,0)=0$	$-5\le x,y\le 5$
Sphere function		$f(\boldsymbol{x}) = \sum_{i=1}^{n} x_i^2$	$f(x_1,\dots,x_n) = f(0,\dots,0) = 0$	$-\infty \le x_i \le \infty$, $1\le i\le n$
Rosenbrock function		$f(\boldsymbol{x}) = \sum_{i=1}^{n-1}\left[100\left(x_{i+1}-x_i^2\right)^2+\left(x_i-1\right)^2\right]$	$\text{Min} = \begin{cases} n=2 & \rightarrow \quad f(1,1)=0, \\ n=3 & \rightarrow \quad f(1,1,1)=0, \\ n>3 & \rightarrow \quad f\left(\underbrace{1,\dots,1}_{(n)\text{ times}}\right)=0 \end{cases}$	$-\infty \le x_i \le \infty$, $1\le i\le n$
Beale's function		$f(x,y) = \left(1.5-x+xy\right)^2+\left(2.25-x+xy^2\right)^2$ $+\left(2.625-x+xy^3\right)^2$	$f(3,0.5)=0$	$-4.5\le x,y\le 4.5$
Goldstein–Price function:		$f(x,y) = \left(1+\left(x+y+1\right)^2\left(19-14x+3x^2-14y+6xy+3y^2\right)\right)$ $\left(30+\left(2x-3y\right)^2\left(18-32x+12x^2+48y-36xy+27y^2\right)\right)$	$f(0,-1)=3$	$-2\le x,y\le 2$
Booth's function:		$f(x,y) = \left(x+2y-7\right)^2+\left(2x+y-5\right)^2$	$f(1,3)=0$	$-10\le x,y\le 10$
Bukin function N.6:		$f(x,y) = 100\sqrt{\left\lvert y-0.01x^2\right\rvert}+0.01\left\lvert x+10\right\rvert.$	$f(-10,1)=0$	$-15\le x\le -5$, $-3\le y\le 3$

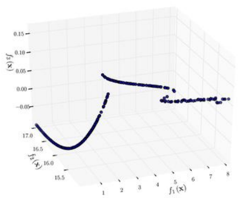
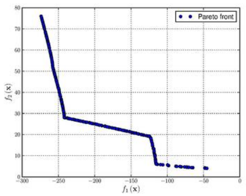
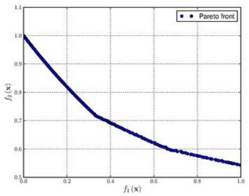
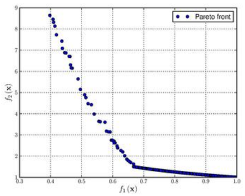
Matyas function:		$f(x,y) = 0.26\left(x^2+y^2\right) - 0.48xy$	$f(0,0)=0$	$-10 \le x,y \le 10$
Lévi function N.13:		$f(x,y) = \sin^2(3\pi x) + (x-1)^2\left(1+\sin^2(3\pi y)\right) + (y-1)^2\left(1+\sin^2(2\pi y)\right)$	$f(1,1)=0$	$-10 \le x,y \le 10$
Three-hump camel function:		$f(x,y) = 2x^2 - 1.05x^4 + \frac{x^6}{6} + xy + y^2$	$f(0,0)=0$	$-5 \le x,y \le 5$
Easom function:		$f(x,y) = -\cos(x)\cos(y)\exp\left(-\left((x-\pi)^2+(y-\pi)^2\right)\right)$	$f(\pi,\pi)=-1$	$-100 \le x,y \le 100$
Cross-in-tray function:		$f(x,y) = -0.0001\left(\left\|\sin(x)\sin(y)\exp\left(\left\|100-\frac{\sqrt{x^2+y^2}}{\pi}\right\|\right)\right\|+1\right)^{0.1}$	$\text{Min} = \begin{cases} f(1.34941,-1.34941) & = -2.06261 \\ f(1.34941,1.34941) & = -2.06261 \\ f(-1.34941,1.34941) & = -2.06261 \\ f(-1.34941,-1.34941) & = -2.06261 \end{cases}$	$-10 \le x,y \le 10$
Eggholder function:		$f(x,y) = -(y+47)\sin\left(\sqrt{\left\|\frac{x}{2}+(y+47)\right\|}\right) - x\sin\left(\sqrt{\|x-(y+47)\|}\right)$	$f(512,404.2319) = -959.6407$	$-512 \le x,y \le 512$

Hölder table function:		$f(x,y) = -\left\lvert \sin(x)\cos(y)\exp\left(\left\lvert 1 - \frac{\sqrt{x^2+y^2}}{\pi}\right\rvert\right)\right\rvert$	$\text{Min} = \begin{cases} f(8.05502, 9.66459) & = -19.2085 \\ f(-8.05502, 9.66459) & = -19.2085 \\ f(8.05502, -9.66459) & = -19.2085 \\ f(-8.05502, -9.66459) & = -19.2085 \end{cases}$	$-10 \le x, y \le 10$
McCormick function:		$f(x,y) = \sin(x+y) + (x-y)^2 - 1.5x + 2.5y + 1$	$f(-0.54719, -1.54719) = -1.9133$	$-1.5 \le x \le 4$, $-3 \le y \le 4$
Schaffer function N. 2:		$f(x,y) = 0.5 + \frac{\sin^2(x^2-y^2) - 0.5}{\left(1 + 0.001(x^2+y^2)\right)^2}$	$f(0,0) = 0$	$-100 \le x, y \le 100$
Schaffer function N. 4:		$f(x,y) = 0.5 + \frac{\cos^2(\sin(\lvert x^2-y^2\rvert)) - 0.5}{\left(1 + 0.001(x^2+y^2)\right)^2}$	$f(0, 1.25313) = 0.292579$	$-100 \le x, y \le 100$
Styblinski–Tang function:		$f(\boldsymbol{x}) = \frac{\sum_{i=1}^{n} x_i^4 - 16x_i^2 + 5x_i}{2}$	$-39.16617n < f\left(\underbrace{-2.903534, \ldots, -2.903534}_{(n)\text{ times}}\right) < -39.16616n$	$-5 \le x_i \le 5$, $1 \le i \le n$.
Simionescu function:[10]		$f(x,y) = 0.1xy$, subjected to: $x^2 + y^2 \le \left(r_T + r_S \cos\left(n \arctan\frac{x}{y}\right)\right)^2$ where: $r_T = 1, r_S = 0.2$ and $n = 8$	$f(\pm 0.85586214, \mp 0.85586214) = -0.072625$	$-1.25 \le x, y \le 1.25$

Test functions for multi-objective optimization problems

Name	Plot	Functions	Constraints	Search domain
Binh and Korn function:		$\text{Minimize} = \begin{cases} f_1(x,y) &= 4x^2 + 4y^2 \\ f_2(x,y) &= (x-5)^2 + (y-5)^2 \end{cases}$	$\text{s.t.} = \begin{cases} g_1(x,y) &= (x-5)^2 + y^2 \leq 25 \\ g_2(x,y) &= (x-8)^2 + (y+3)^2 \geq 7.7 \end{cases}$	$0 \leq x \leq 5$, $0 \leq y \leq 3$
Chakong and Haimes function:		$\text{Minimize} = \begin{cases} f_1(x,y) &= 2 + (x-2)^2 + (y-1)^2 \\ f_2(x,y) &= 9x - (y-1)^2 \end{cases}$	$\text{s.t.} = \begin{cases} g_1(x,y) &= x^2 + y^2 \leq 225 \\ g_2(x,y) &= x - 3y + 10 \leq 0 \end{cases}$	$-20 \leq x, y \leq 20$
Fonseca and Fleming function:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) &= 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2\right) \\ f_2(\boldsymbol{x}) &= 1 - \exp\left(-\sum_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2\right) \end{cases}$		$-4 \leq x_i \leq 4$, $1 \leq i \leq n$
Test function 4:[7]		$\text{Minimize} = \begin{cases} f_1(x,y) &= x^2 - y \\ f_2(x,y) &= -0.5x - y - 1 \end{cases}$	$\text{s.t.} = \begin{cases} g_1(x,y) &= 6.5 - \frac{x}{6} - y \geq 0 \\ g_2(x,y) &= 7.5 - 0.5x - y \geq 0 \\ g_3(x,y) &= 30 - 5x - y \geq 0 \end{cases}$	$-7 \leq x, y \leq 4$
Kursawe function:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) &= \sum_{i=1}^{2}\left[-10\exp\left(-0.2\sqrt{x_i^2 + x_{i+1}^2}\right)\right] \\ f_2(\boldsymbol{x}) &= \sum_{i=1}^{3}\left[\lvert x_i\rvert^{0.8} + 5\sin\left(x_i^3\right)\right] \end{cases}$		$-5 \leq x_i \leq 5$, $1 \leq i \leq 3$.
Schaffer function N. 1:		$\text{Minimize} = \begin{cases} f_1(x) &= x^2 \\ f_2(x) &= (x-2)^2 \end{cases}$		$-A \leq x \leq A$. Values of A form 10 to 10^5 have been used successfully. Higher values of A increase the difficulty of the problem.
Schaffer function N. 2:		$\text{Minimize} = \begin{cases} f_1(x) &= \begin{cases} -x, & \text{if } x \leq 1 \\ x-2, & \text{if } 1 < x \leq 3 \\ 4-x, & \text{if } 3 < x \leq 4 \\ x-4, & \text{if } x > 4 \end{cases} \\ f_2(x) &= (x-5)^2 \end{cases}$		$-5 \leq x \leq 10$.

Poloni's two objective function:		$\text{Minimize} = \begin{cases} f_1(x,y) & = \left[1 + (A_1 - B_1(x,y))^2 + (A_2 - B_2(x,y))^2\right] \\ f_2(x,y) & = (x+3)^2 + (y+1)^2 \end{cases}$ $\text{where} = \begin{cases} A_1 & = 0.5\sin(1) - 2\cos(1) + \sin(2) - 1.5\cos(2) \\ A_2 & = 1.5\sin(1) - \cos(1) + 2\sin(2) - 0.5\cos(2) \\ B_1(x,y) & = 0.5\sin(x) - 2\cos(x) + \sin(y) - 1.5\cos(y) \\ B_2(x,y) & = 1.5\sin(x) - \cos(x) + 2\sin(y) - 0.5\cos(y) \end{cases}$		$-\pi \le x, y \le \pi$
Zitzler–Deb–Thiele's function N. 1:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) & = x_1 \\ f_2(\boldsymbol{x}) & = g(\boldsymbol{x})\, h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) \\ g(\boldsymbol{x}) & = 1 + \frac{9}{29}\sum_{i=2}^{30} x_i \\ h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) & = 1 - \sqrt{\frac{f_1(\boldsymbol{x})}{g(\boldsymbol{x})}} \end{cases}$		$0 \le x_i \le 1$, $1 \le i \le 30$.
Zitzler–Deb–Thiele's function N. 2:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) & = x_1 \\ f_2(\boldsymbol{x}) & = g(\boldsymbol{x})\, h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) \\ g(\boldsymbol{x}) & = 1 + \frac{9}{29}\sum_{i=2}^{30} x_i \\ h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) & = 1 - \left(\frac{f_1(\boldsymbol{x})}{g(\boldsymbol{x})}\right)^2 \end{cases}$		$0 \le x_i \le 1$, $1 \le i \le 30$.
Zitzler–Deb–Thiele's function N. 3:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) & = x_1 \\ f_2(\boldsymbol{x}) & = g(\boldsymbol{x})\, h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) \\ g(\boldsymbol{x}) & = 1 + \frac{9}{29}\sum_{i=2}^{30} x_i \\ h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) & = 1 - \sqrt{\frac{f_1(\boldsymbol{x})}{g(\boldsymbol{x})}} - \left(\frac{f_1(\boldsymbol{x})}{g(\boldsymbol{x})}\right)\sin(10\pi f_1(\boldsymbol{x})) \end{cases}$		$0 \le x_i \le 1$, $1 \le i \le 30$.
Zitzler–Deb–Thiele's function N. 4:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) & = x_1 \\ f_2(\boldsymbol{x}) & = g(\boldsymbol{x})\, h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) \\ g(\boldsymbol{x}) & = 91 + \sum_{i=2}^{10}\left(x_i^2 - 10\cos(4\pi x_i)\right) \\ h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) & = 1 - \sqrt{\frac{f_1(\boldsymbol{x})}{g(\boldsymbol{x})}} \end{cases}$		$0 \le x_1 \le 1$, $-5 \le x_i \le 5$, $2 \le i \le 10$
Zitzler–Deb–Thiele's function N. 6:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) & = 1 - \exp(-4x_1)\sin^6(6\pi x_1) \\ f_2(\boldsymbol{x}) & = g(\boldsymbol{x})\, h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) \\ g(\boldsymbol{x}) & = 1 + 9\left[\frac{\sum_{i=2}^{10} x_i}{9}\right]^{0.25} \\ h(f_1(\boldsymbol{x}), g(\boldsymbol{x})) & = 1 - \left(\frac{f_1(\boldsymbol{x})}{g(\boldsymbol{x})}\right)^2 \end{cases}$		$0 \le x_i \le 1$, $1 \le i \le 10$.

Viennet function:		$\text{Minimize} = \begin{cases} f_1(x,y) &= 0.5\left(x^2+y^2\right)+\sin\left(x^2+y^2\right) \\ f_2(x,y) &= \frac{(3x-2y+4)^2}{8}+\frac{(x-y+1)^2}{27}+15 \\ f_3(x,y) &= \frac{1}{x^2+y^2+1}-1.1\exp\left(-\left(x^2+y^2\right)\right) \end{cases}$		$-3 \le x,y \le 3$.
Osyczka and Kundu function:		$\text{Minimize} = \begin{cases} f_1(\boldsymbol{x}) &= -25\left(x_1-2\right)^2-\left(x_2-2\right)^2-\left(x_3-1\right)^2-\left(x_4-4\right)^2-\left(x_5-1\right)^2 \\ f_2(\boldsymbol{x}) &= \sum_{i=1}^{6} x_i^2 \end{cases}$	$\text{s.t.} = \begin{cases} g_1(\boldsymbol{x}) &= x_1+x_2-2 \ge 0 \\ g_2(\boldsymbol{x}) &= 6-x_1-x_2 \ge 0 \\ g_3(\boldsymbol{x}) &= 2-x_2+x_1 \ge 0 \\ g_4(\boldsymbol{x}) &= 2-x_1+3x_2 \ge 0 \\ g_5(\boldsymbol{x}) &= 4-\left(x_3-3\right)^2-x_4 \ge 0 \\ g_6(\boldsymbol{x}) &= \left(x_5-3\right)^2+x_6-4 \ge 0 \end{cases}$	$0 \le x_1, x_2, x_6 \le 10$, $1 \le x_3, x_5 \le 5$, $0 \le x_4 \le 6$.
CTP1 function (2 variables):[5]		$\text{Minimize} = \begin{cases} f_1(x,y) &= x \\ f_2(x,y) &= (1+y)\exp\left(-\frac{x}{1+y}\right) \end{cases}$	$\text{s.t.} = \begin{cases} g_1(x,y) &= \frac{f_2(x,y)}{0.858\exp\left(-0.541 f_1(x,y)\right)} \ge 1 \\ g_1(x,y) &= \frac{f_2(x,y)}{0.728\exp\left(-0.295 f_1(x,y)\right)} \ge 1 \end{cases}$	$0 \le x,y \le 1$.
Constr-Ex problem:[5]		$\text{Minimize} = \begin{cases} f_1(x,y) &= x \\ f_2(x,y) &= \frac{1+y}{x} \end{cases}$	$\text{s.t.} = \begin{cases} g_1(x,y) &= y+9x \ge 6 \\ g_1(x,y) &= -y+9x \ge 1 \end{cases}$	$0.1 \le x \le 1$, $0 \le y \le 5$

See also

- Himmelblau's function
- Rosenbrock function
- Rastrigin function
- Shekel function

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References

1. Bäck, Thomas (1995). *Evolutionary algorithms in theory and practice : evolution strategies, evolutionary programming, genetic algorithms*. Oxford: Oxford University Press. p. 328. ISBN 0-19-509971-0.
2. Haupt, Randy L. Haupt, Sue Ellen (2004). *Practical genetic algorithms with CD-Rom* (2nd ed.). New York: J. Wiley. ISBN 0-471-45565-2.
3. Oldenhuis, Rody. "Many test functions for global optimizers". Mathworks. Retrieved 1 November 2012.
4. Ortiz, Gilberto A. "Evolution Strategies (ES)". Mathworks. Retrieved 1 November 2012.
5. Deb, Kalyanmoy (2002) Multiobjective optimization using evolutionary algorithms (Repr. ed.). Chichester [u.a.]: Wiley. ISBN 0-471-87339-X.
6. Binh T. and Korn U. (1997) MOBES: A Multiobjective Evolution Strategy for Constrained Optimization Problems. In: Proceedings of the Third International Conference on Genetic Algorithms. Czech Republic. pp. 176-182
7. Binh T. (1999) A multiobjective evolutionary algorithm. The study cases. Technical report. Institute for Automation and Communication. Barleben, Germany
8. Deb K. (2011) Software for multi-objective NSGA-II code in C. Available at URL:http://www.iitk.ac.in/kangal/codes.shtml. Revision 1.1.6
9. Ortiz, Gilberto A. "Multi-objective optimization using ES as Evolutionary Algorithm.". Mathworks. Retrieved 1 November 2012.
10. Simionescu, P.A. (2014). *Computer Aided Graphing and Simulation Tools for AutoCAD Users* (1st ed.). Boca Raton, FL: CRC Press. ISBN 9-781-48225290-3.