CMPE 185 Autonomous Mobile Robots

Coordinate Transformation

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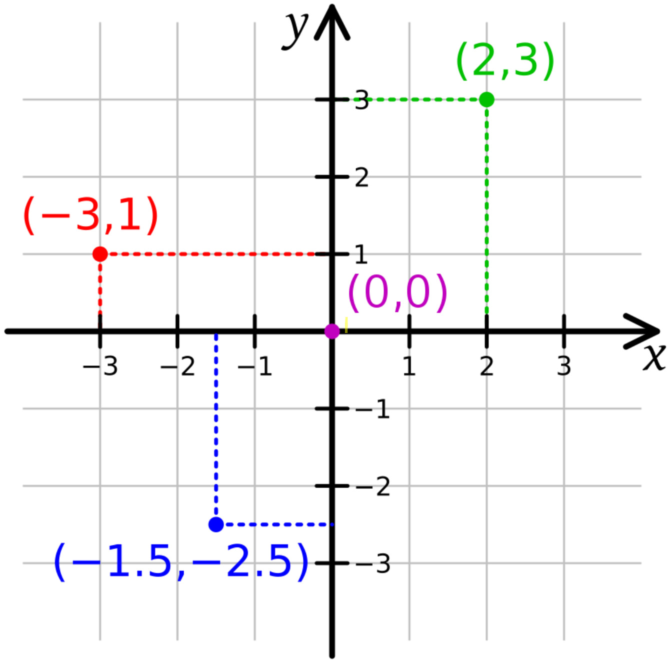
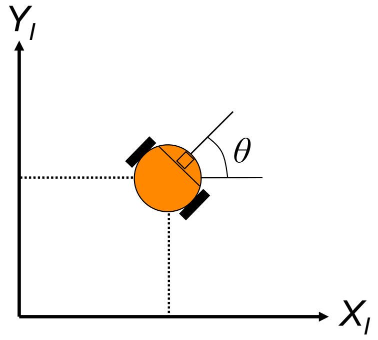
Wheeled Robot

- Wheeled locomotion
  - Highly efficient on hard surfaces
  - Generally restricted to man-made surfaces

Review: Cartesian Coordinates

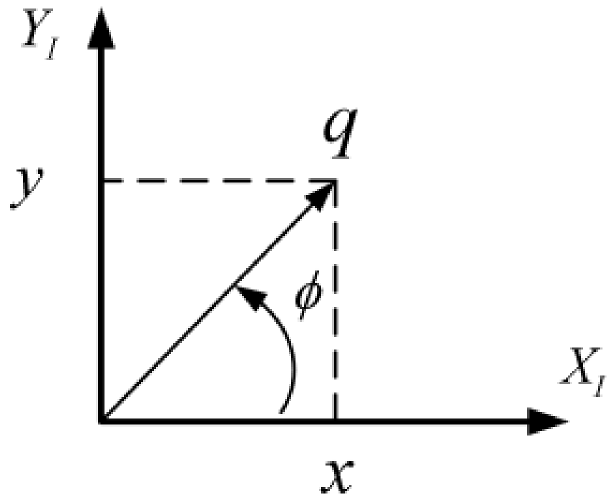
- Describes unique position of points in a plane with respect to the axis
- For each dimension there is 1 axis
- Coordinates are measured in “units” in the direction parallel to the axis
- The origin is fixed to the plane





Position

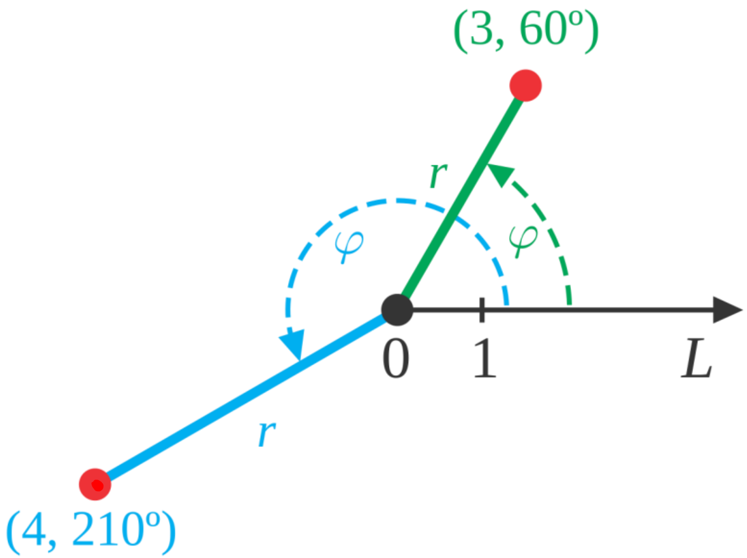
- Key Question: how can we describe the position and the orientation of a robot in 2D plane?
  - All positions must be described in a coordinate system
  - One can use Cartesian coordinates to describe a robot's position



  - Any position can be described as a vector
  - x is the projection of vector **q** onto the horizontal axis and y is the projection of q onto the vertical axis

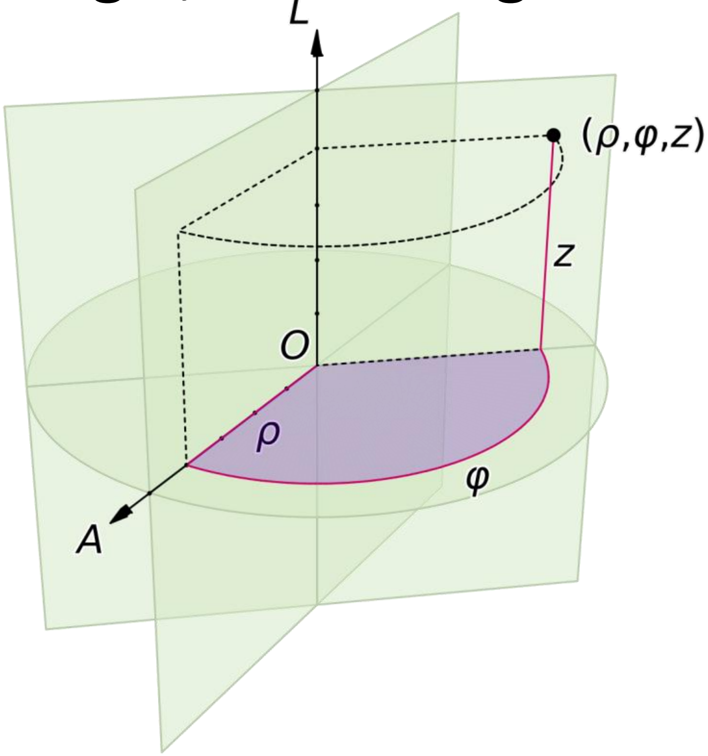
Polar Coordinates

- In polar coordinates, we specify points on a 2D plane using the length of a radius arm and an angle



Cylindrical Coordinates

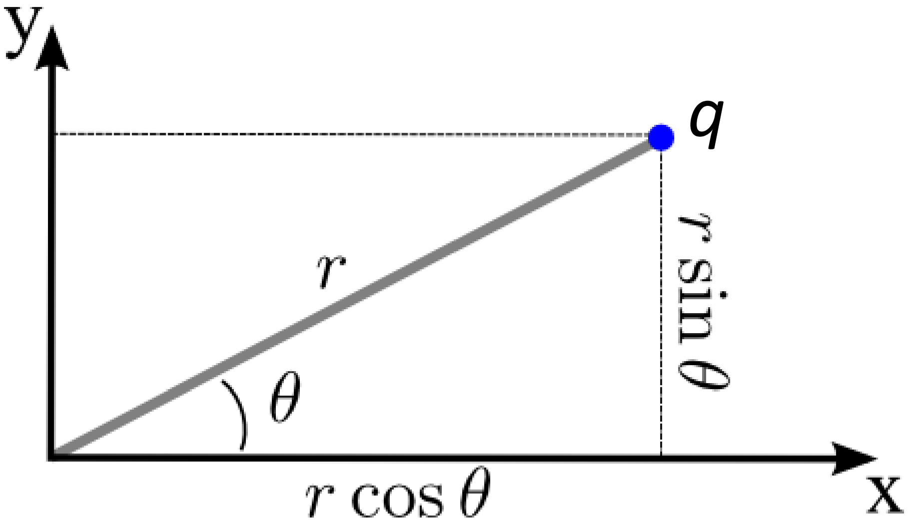
- For specifying point locations in 3D, cylindrical coordinates can be used by specifying the length of a radius arm, an angle, and a height



Polar to Cartesian

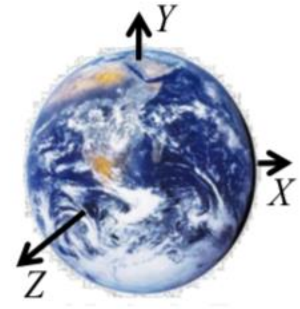
- How do we convert from polar coordinates to Cartesian coordinates?

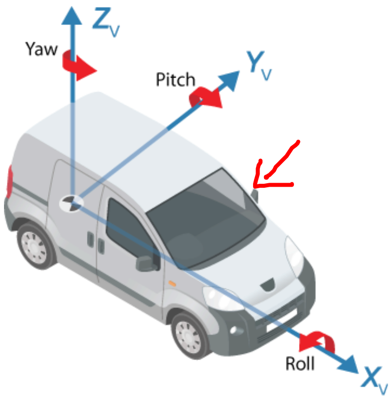
$$x = r\cos\theta$$
$$y = r\sin\theta$$

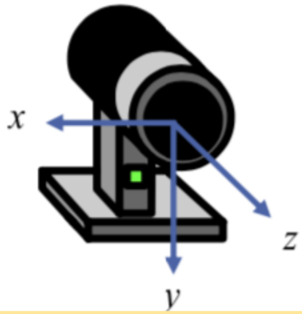


Coordinate Frames

- Right-handed by convention
- Inertial/Global frame
  - Fixed, usually relative to earth
- Body/Robot frame
  - Attached to vehicle, with the origin at the center of gravity, or center of rotation
- Sensor frame
  - Attached to sensors. Convenient for expressing sensor measurements

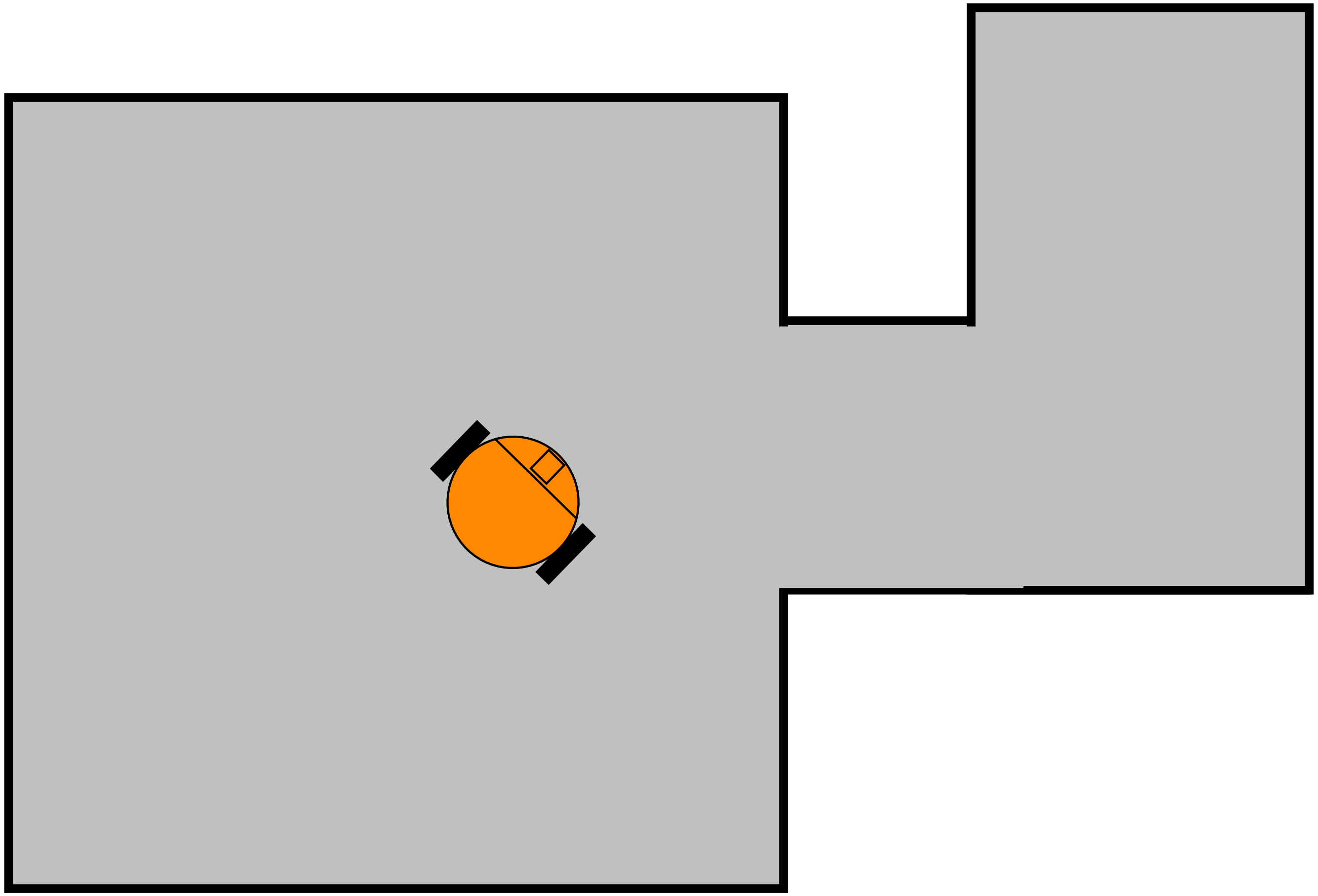






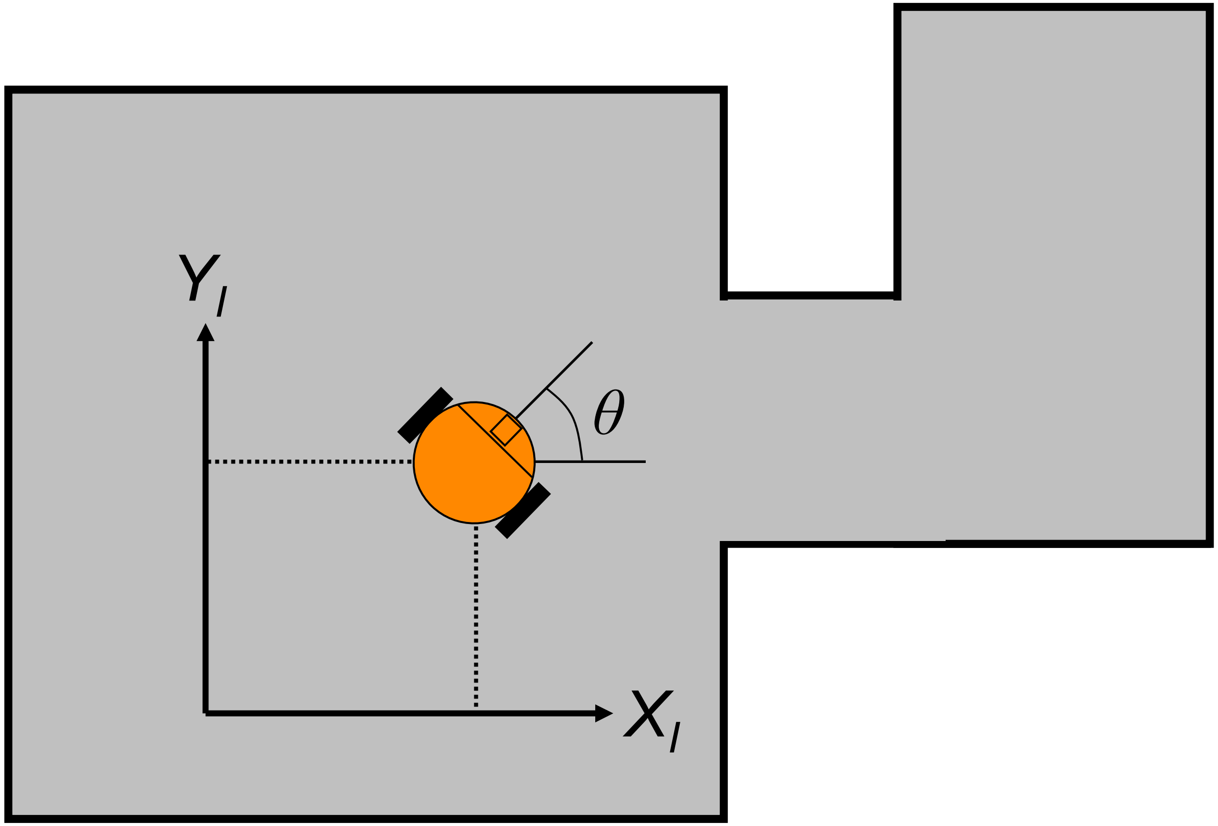
https://www.mathworks.com/help/driving/ug/coordinate-systems.html

Global (Inertial) Coordinate Frame



Global (Inertial) Coordinate Frame

- Anchor a coordinate frame to the environment
- The angle θ describes the orientation of the robot



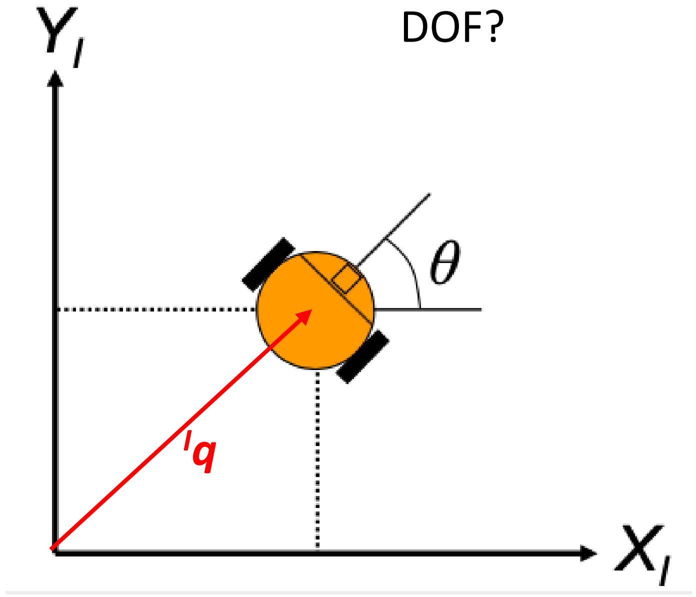
Global (Inertial) Coordinate Frame

- With this frame, we describe the robot state as position + orientation

$$\xi_I = [x, y, \theta]$$

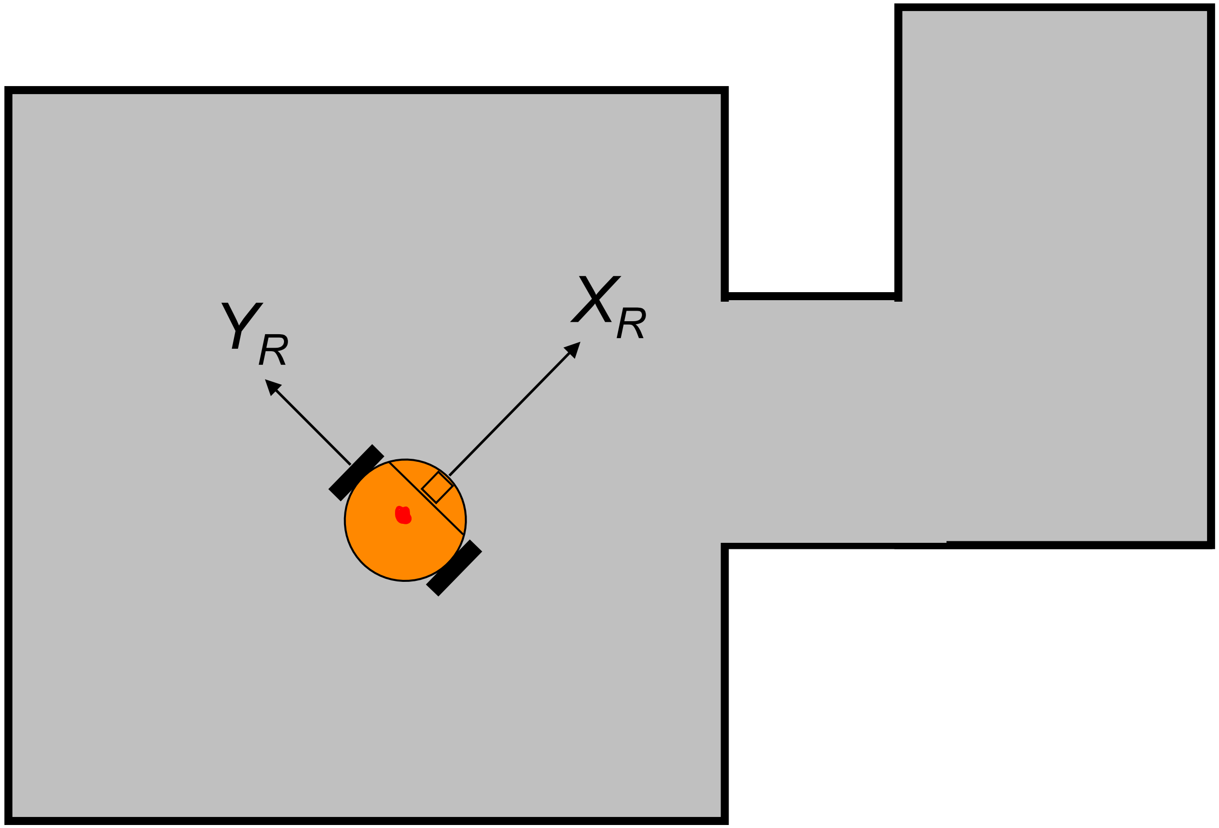
- Define the position vector as

$$^I\boldsymbol{q} = [x, y]$$



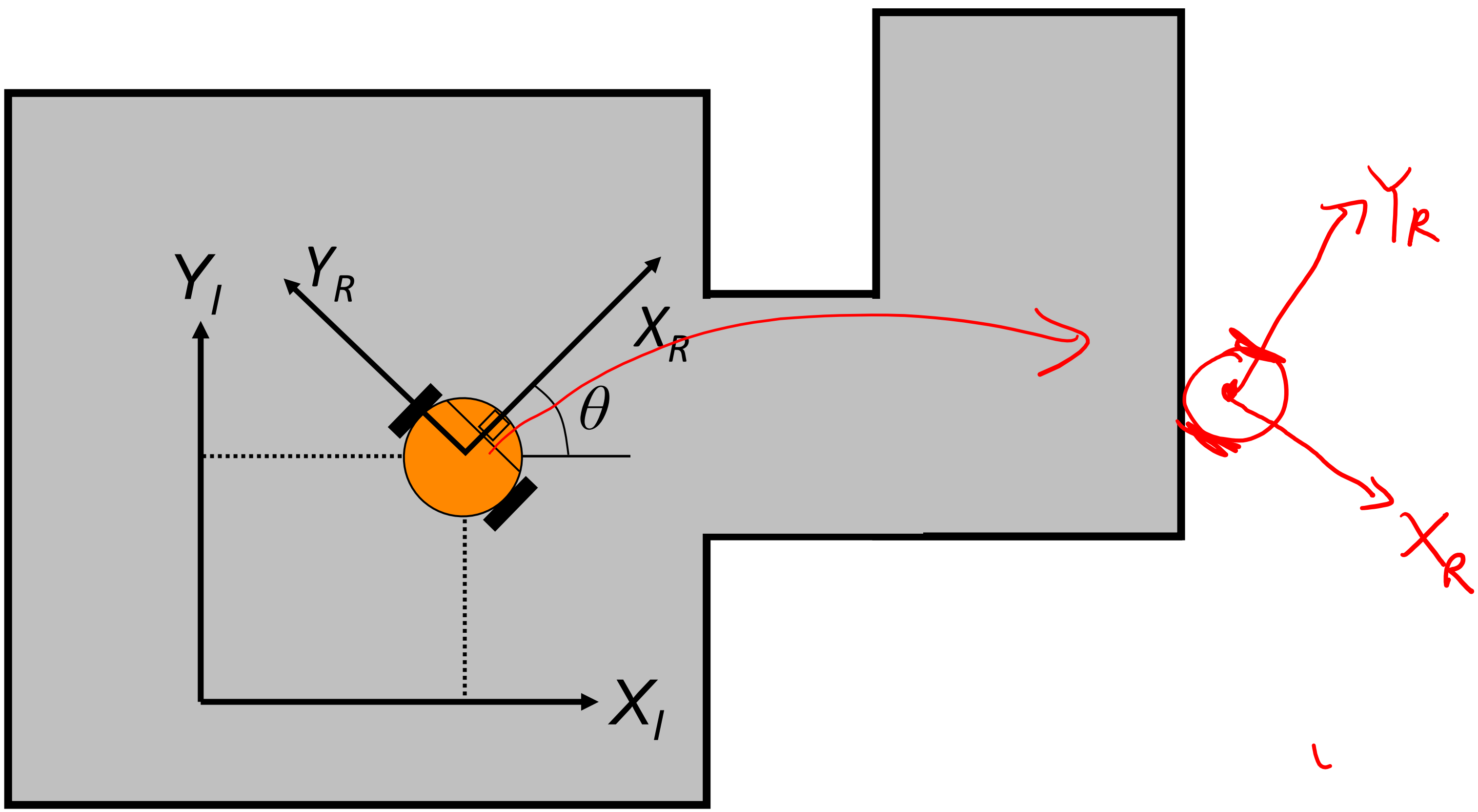
Local Coordinate Frame

- Anchor a coordinate frame to the robot



Global (Inertial) Frame & Local (Body) Frame

- Putting the global frame and local frame together…



Global (Inertial) v.s. Local Coordinate Frame

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\to X \cdot Y$

$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $= x_1 y_1 + x_2 y_2$

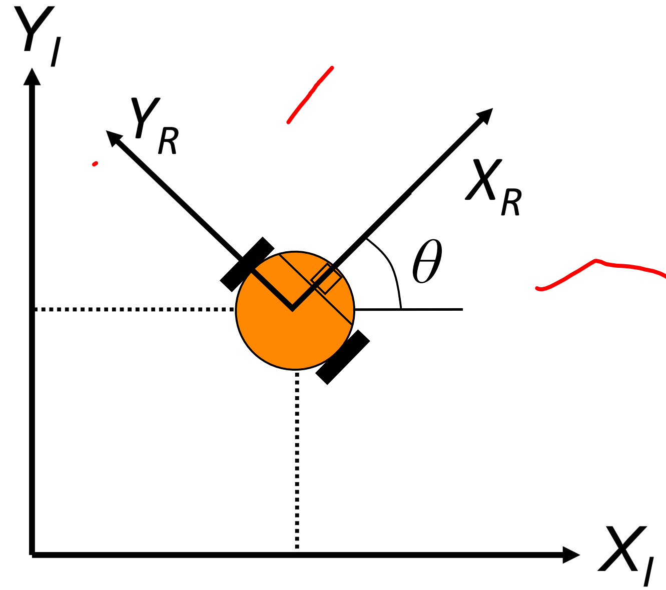
- The angle θ describes the orientation of a mobile robot

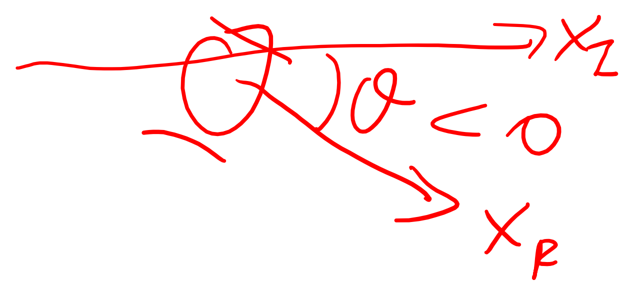
$$\cos\theta = X_R \cdot X_I$$

- It is often necessary to assign a sign to θ
- Convention:
  - We restrict the value of θ to be

$$-\pi < \theta \leq \pi$$

  - If $\theta > 0$, then X_R is in the counter-clockwise direction of X_I
  - If $\theta < 0$, then X_R is in the clockwise direction of X_I

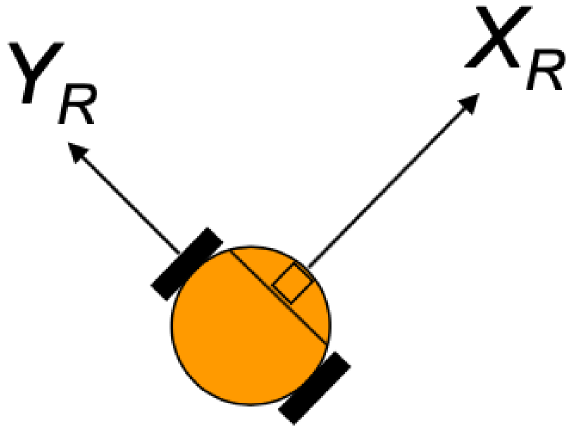




Local Coordinate Frame

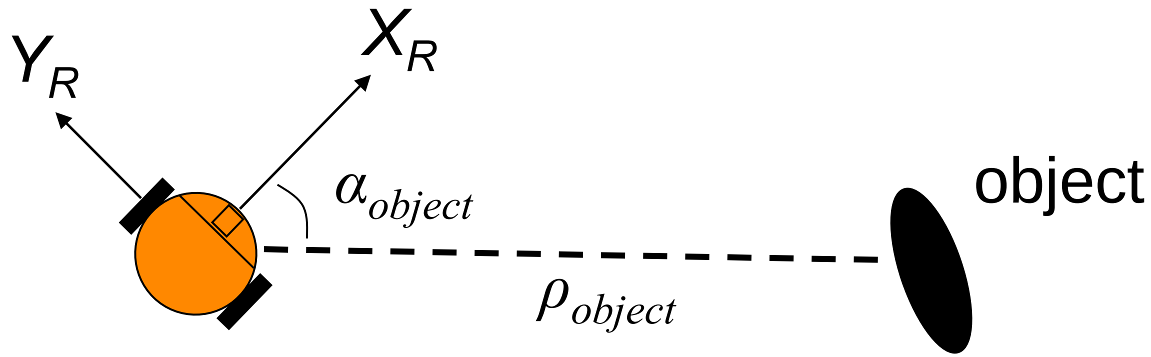
- With is coordinate frame, we describe the robot state as (position + orientation):

$$\xi_R = [x, y, \theta]_R = [0, 0, 0]$$



Local Coordinate Frame

- The local frame is useful when considering taking measurements of environment objects
- Example: consider the detection of a wall using a range finder

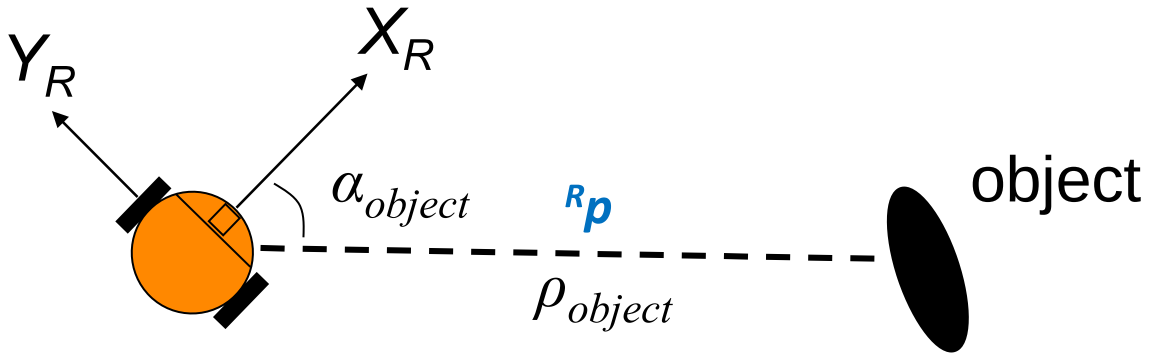


Local Coordinate Frame

- The measurement is taken relative to the robot's local coordinate frame (ρ_{object}, α_{object})
- We can calculate the position of the measurement in local coordinate frames:

$$x_{object,\,R} = \rho_{object}\cos(\alpha_{object})$$
$$y_{object,\,R} = \rho_{object}\sin(\alpha_{object})$$

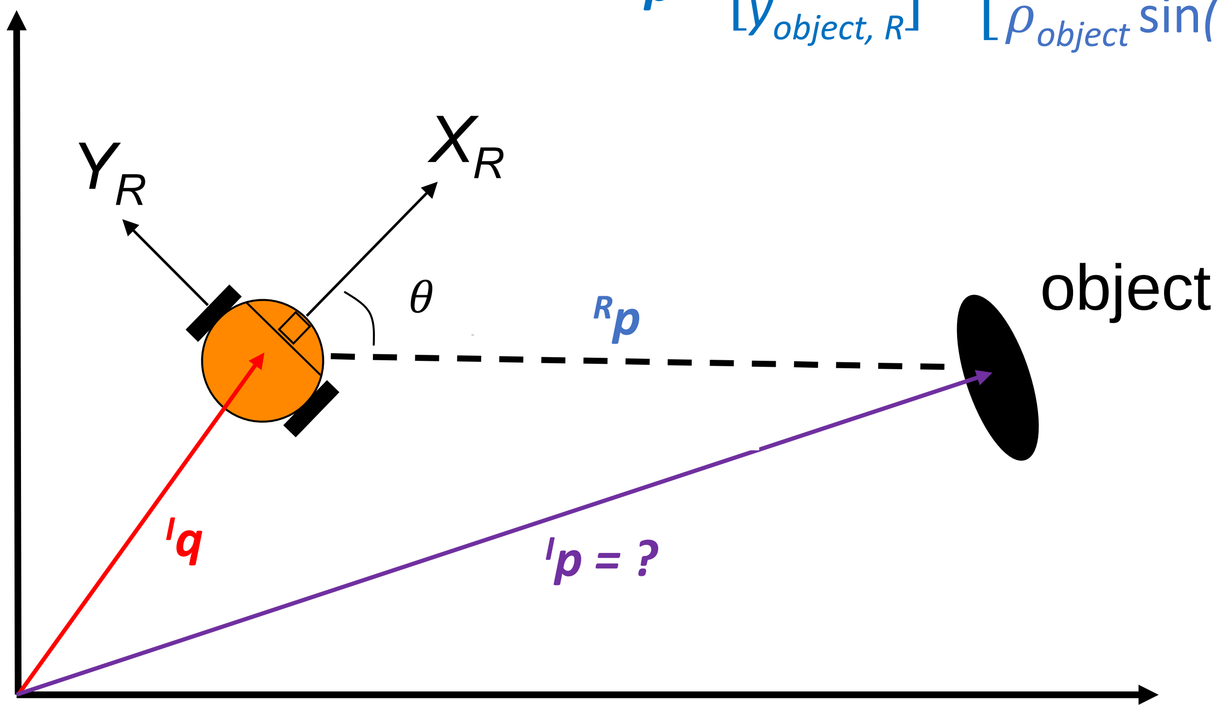
$${}^{R}\boldsymbol{p} = \begin{bmatrix} x_{object,\,R} \\ y_{object,\,R} \end{bmatrix}$$



Local to Global Coordinate Frame Transformation

- With the measurement taken in the local coordinate frame, how to convert it to the position in inertial frame?

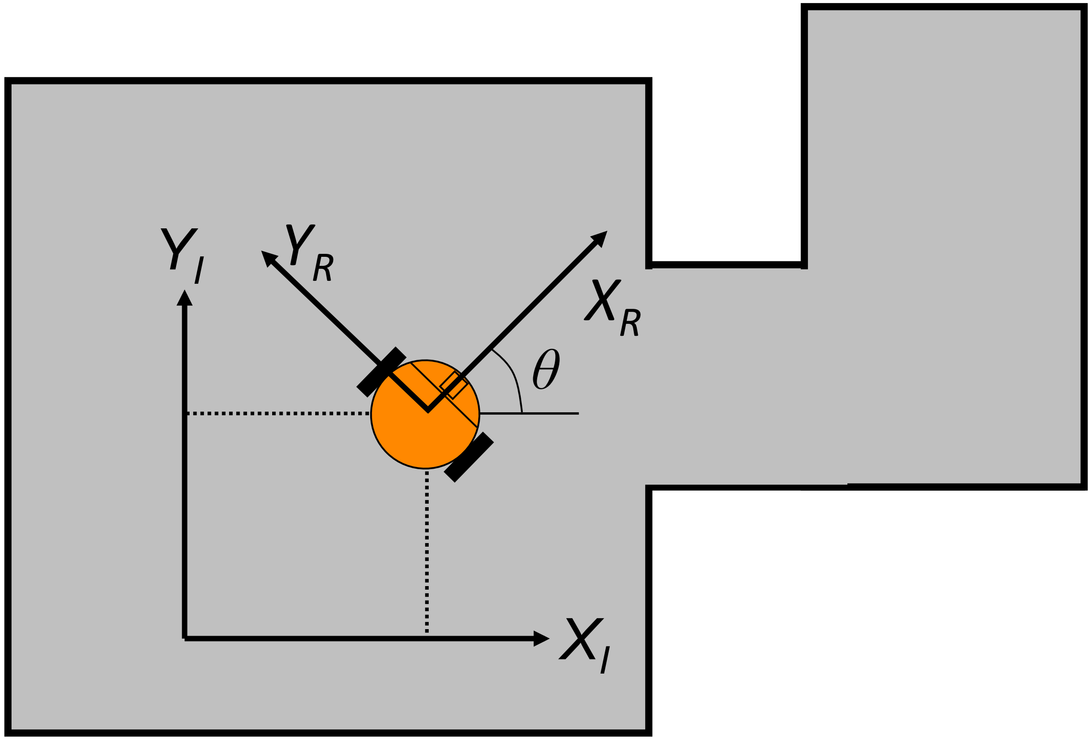
$$^{R}\mathbf{p} = \begin{bmatrix} x_{object,\ R} \\ y_{object,\ R} \end{bmatrix} = \begin{bmatrix} \rho_{object} \cos(\alpha_{object}) \\ \rho_{object} \sin(\alpha_{object}) \end{bmatrix}$$



Rotation Matrix

- The transformation between the two frames can be described by the rotation matrix ***R***

$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & con\theta \end{bmatrix} \quad R^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

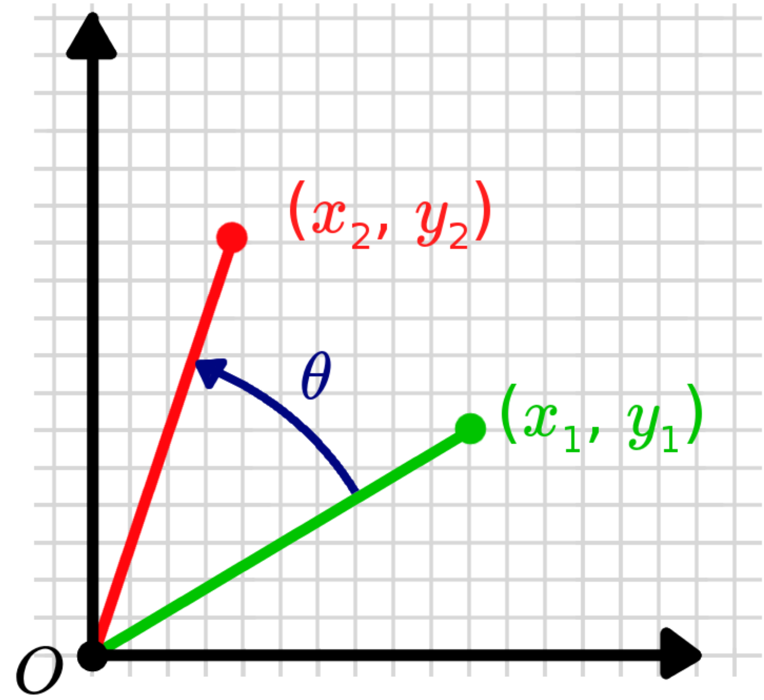


Deriving the Rotation Matrix - 1

- Say we have a point (x_1, y_1) and we want to find the 2×2 transformation matrix that will rotate it (anticlockwise) around the origin by an angle ϑ to a new point, (x_2, y_2)

- In other words, we are looking for values that satisfy the equation

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{bmatrix}$$



Deriving the Rotation Matrix - 2

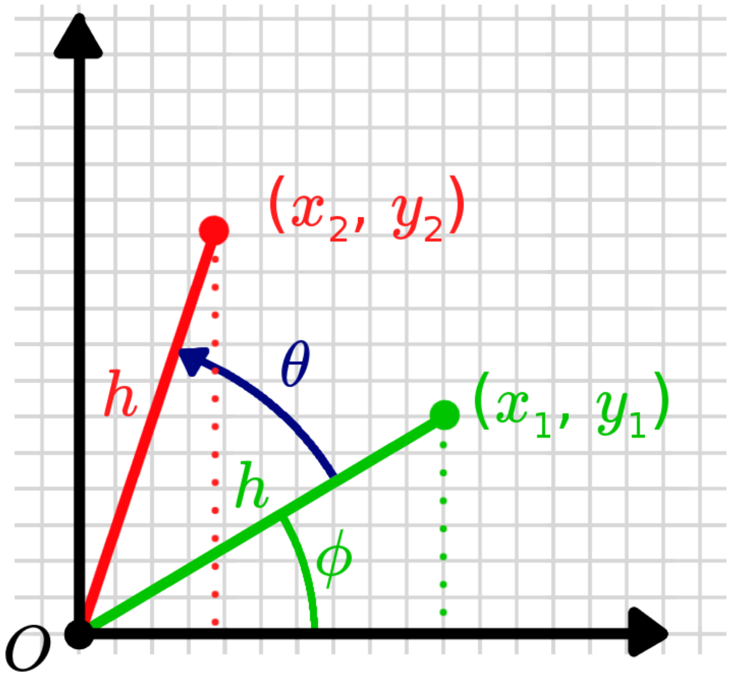
- Define h and ϕ

- Express x_1 and y_1 as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} h\cos(\phi) \\ h\sin(\phi) \end{bmatrix}$$

- And x_2 and y_2 as

$$\begin{aligned}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} h\cos(\phi+\theta) \\ h\sin(\phi+\theta) \end{bmatrix} \\ &= \begin{bmatrix} h\cos(\phi)\cos(\theta) - h\sin(\phi)\sin(\theta) \\ h\sin(\phi)\cos(\theta) + h\cos(\phi)\sin(\theta) \end{bmatrix} \\ &= \begin{bmatrix} x_1\cos(\theta) - y_1\sin(\theta) \\ x_1\sin(\theta) + y_1\cos(\theta) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{R} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\end{aligned}$$



Rotation Matrix

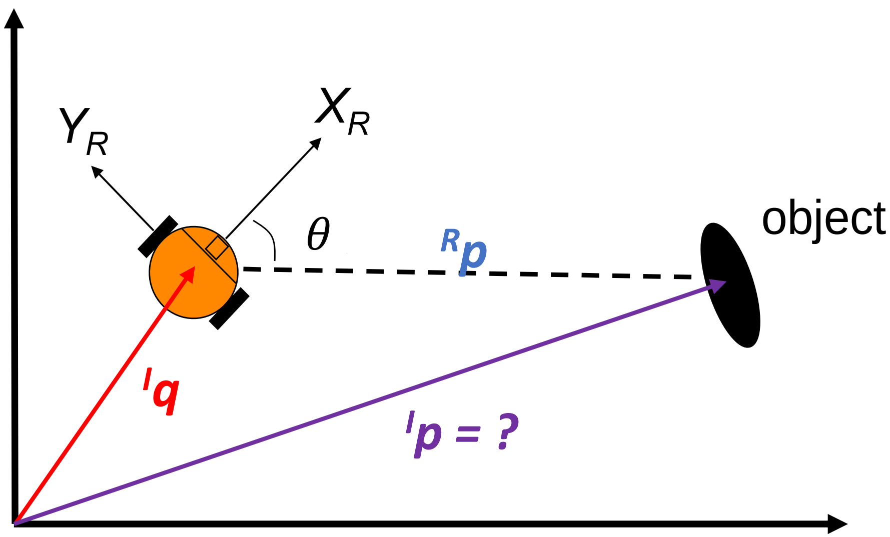
- The transformation between the two frames can be described by the rotation matrix ***R***

$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & con\theta \end{bmatrix} \quad R^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

Note: $R^{-1} = R^{T}$

$${}^{I}\mathbf{p} = R\,{}^{R}\mathbf{p} + {}^{I}\mathbf{q}$$

$${}^{R}\mathbf{p} = R^{-1}\left({}^{I}\mathbf{p} - {}^{I}\mathbf{q}\right)$$



Coordinate Change – Example

- Suppose a range sensor mounted on a robot detects an obstacle at position $[1, 3]^T$. Suppose the robot is at position $[2, 0]^T$ in the inertial frame with orientation $\pi/3$. Find the position of the obstacle in the inertial frame.
- Step 1: find the rotation matrix

$$R = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$$

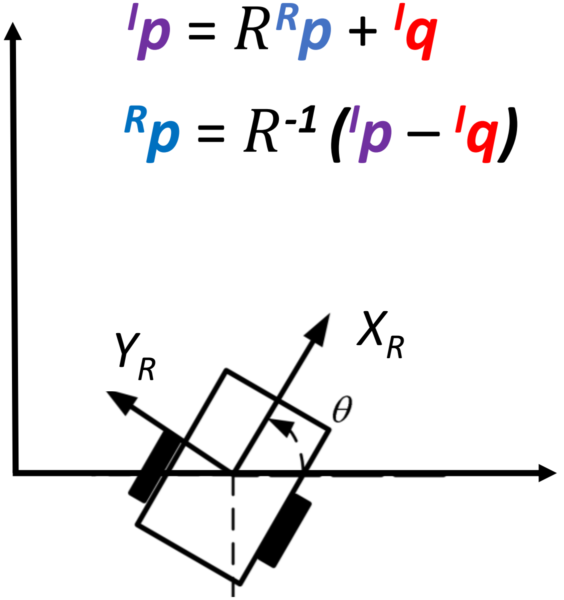
- Step 2: find the position

$$^{I}\boldsymbol{q} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ ^{R}\boldsymbol{p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$^{I}\boldsymbol{p} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \frac{3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} + \frac{3}{2} \end{bmatrix}$$

$$^{I}\boldsymbol{p} = R\,^{R}\boldsymbol{p} + {}^{I}\boldsymbol{q}$$

$$^{R}\boldsymbol{p} = R^{-1}\,(^{I}\boldsymbol{p} - {}^{I}\boldsymbol{q})$$



Recall: Global (Inertial) v.s. Local Coordinate Frame

- With inertial frame, we describe the robot state as position + orientation

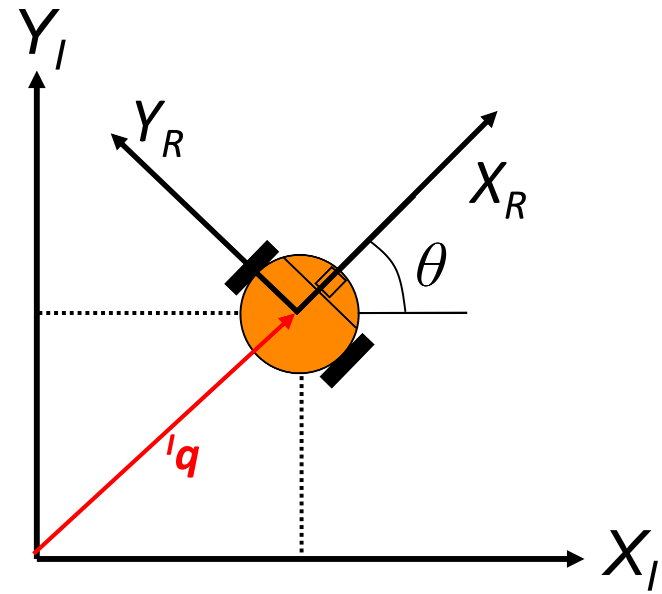
$$\xi_I = [x, y, \theta]$$

- With local frame, we describe the robot state as

$$\xi_R = [x, y, \theta]_R$$

- Orthogonal rotation matrix:

$$Rot(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Velocities in Local Coordinate Frame

- The local frame is also useful when considering velocity states:

$\frac{x_{t+1} - x_t}{\Delta t} \rightarrow V_x$

$$\frac{d\xi_R}{dt} = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{d\theta}{dt}\right]_R^T = [\dot{x}, \dot{y}, \dot{\theta}]_R^T = \dot{\xi}_R$$

- Often we know the velocities of the robot in the local coordinate frame:

$\dot{x}_R = v$ robot’s linear speed

$\dot{y}_R = 0$

$\dot{\theta}_R = w$ robot’s angular speed

Transformations

- We are also interested in the robot's velocities with respect to the global frame
- To calculate these, we need to consider the homogeneous transformation *Rot* between the two frames:

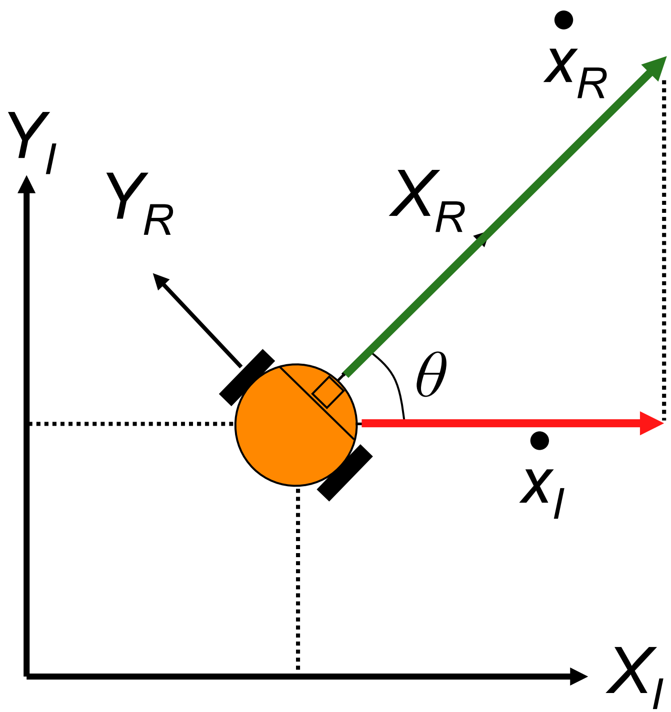
$$\dot{\xi}_R = Rot^{-1}(\theta)\,\dot{\xi}_I$$
$$\dot{\xi}_I = Rot(\theta)\dot{\xi}_R$$

- Note that *Rot* is a function of theta, the relative angle between the two frames

$$Rot(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Velocities in Global Coordinate Frame

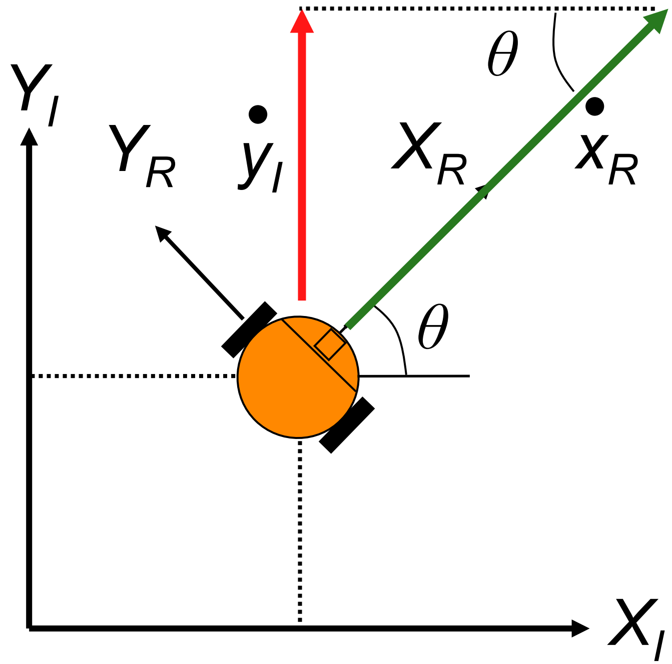
- Let's obtain the velocities in global coordinate frame
- Start with the X_I direction



$$\dot{x}_I = \dot{x}_R \cos\theta = v \cdot \cos\theta$$

Velocities in Global Coordinate Frame

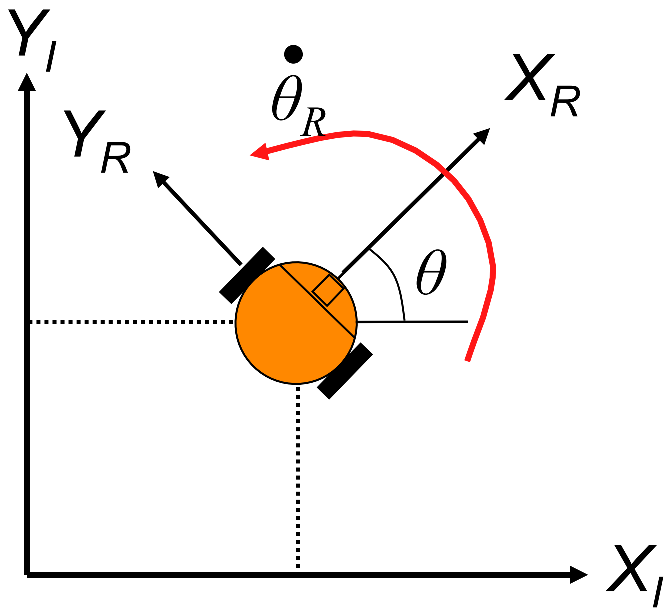
- Now the Y_I direction



$$\dot{y}_I = \dot{x}_R \sin\theta = v \cdot \sin\theta$$

Velocities in Global Coordinate Frame

- What about rotational speed?



$$\dot{\theta}_I = \dot{\theta}_R = w$$

Velocities in Global Coordinate Frame

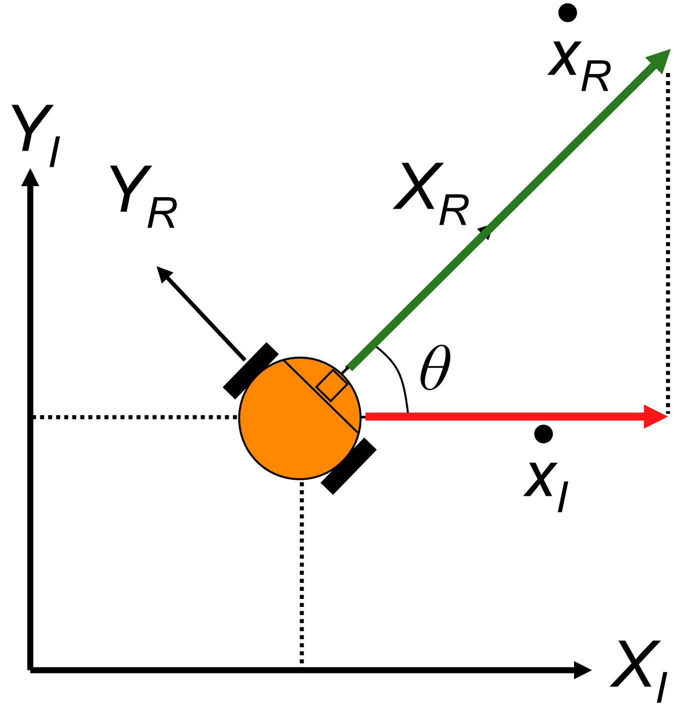
- In summary, we have

$$\dot{x}_I = v \cdot \cos\theta$$

$$\dot{y}_I = v \cdot \sin\theta$$

$$\dot{\theta}_I = \dot{\theta}_R$$

- This is the unicycle model



Transformations

- Let's put our equations in matrix form:

$$\underbrace{\begin{pmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{pmatrix}}_{\dot{\xi}_I} = \underbrace{\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Rot(\theta)} \underbrace{\begin{pmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{pmatrix}}_{\dot{\xi}_R}$$

- Or we can rewrite:

$$\dot{\xi}_I = \begin{pmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Will be used in robotics control and planning

- Thank You!