Algorithms and Computability

Lecture 2: The Church-Turing Thesis

Martin Zimmermann (Aalborg University)

Last Lecture in Algorithms and Computability

We have seen

- Problems = Formal Languages
- Deterministic Turing machines (DTM) as an abstract model of computation
- The difference between computably-enumerable and computable languages:
  - L is computably-enumerable $\Leftrightarrow$ there exists TM M such that $L(M) = L$, i.e.,
    - $w \in L \Rightarrow M$ accepts w,
    - but $w \notin L \Rightarrow M$ rejects w or loops.
  - L is computable $\Leftrightarrow$ there exists halting DTM M such that $L(M) = L$, i.e.,
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- And. too. many. slides.

A Turing Machine:

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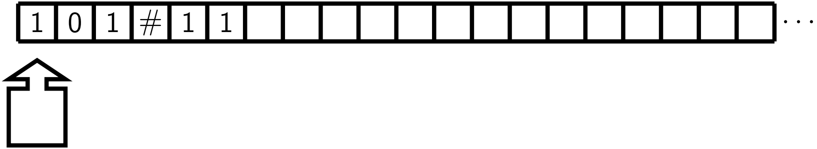

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- Symbols in some squares.
- A single square that is currently observed (with a reading/writing head).

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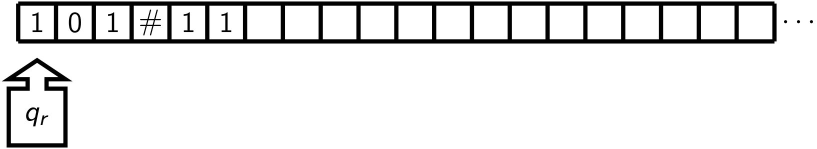

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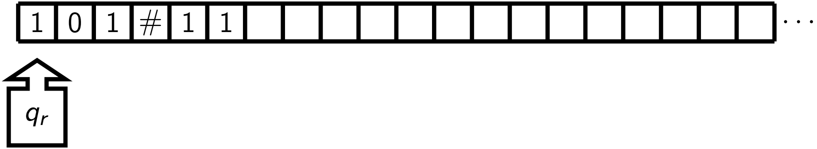
- A single square that is currently observed (with a reading/writing head).
- A “state of mind”.

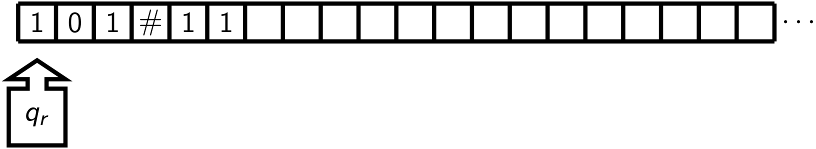
Conceptual View

A Turing Machine:



- An infinite tape of paper, divided into squares (often called cells).
- Symbols in some squares.
- A single square that is currently observed (with a reading/writing head).
- A “state of mind”.
- Rules updating the state and currently observed square.

A Turing Machine:



- If state is q_r and symbol is 0 then change to state q_r change symbol to 0 and move in direction 'right'
- If state is q_r and symbol is 1 then change to state q_r change symbol to 1 and move in direction 'right'
- If state is q_r and symbol is # then change to state q_r change symbol to # and move in direction 'right'
- If state is q_r and symbol is 'empty' then change to state q_s change symbol to 'empty' and move in direction 'left'

Conceptual View

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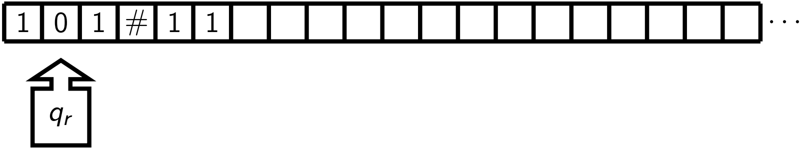

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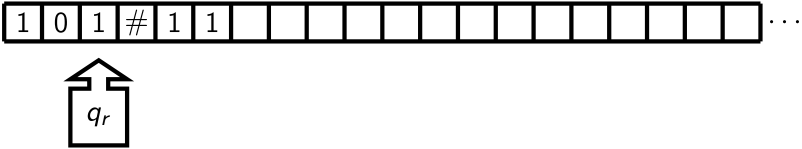

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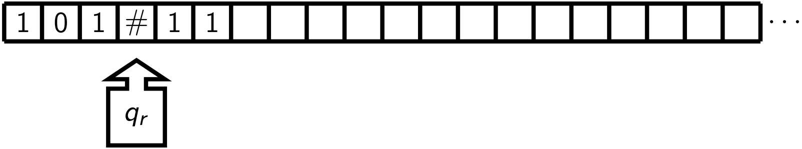

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Conceptual View

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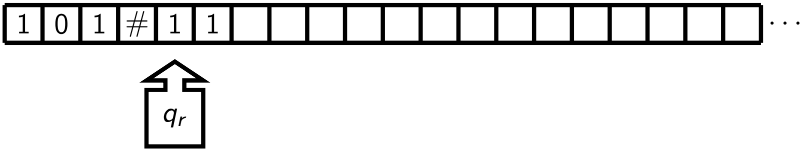

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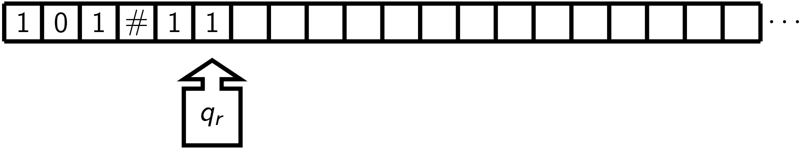

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Conceptual View

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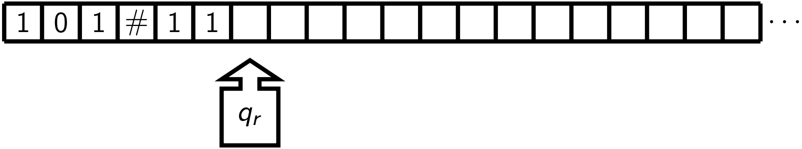

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Conceptual View

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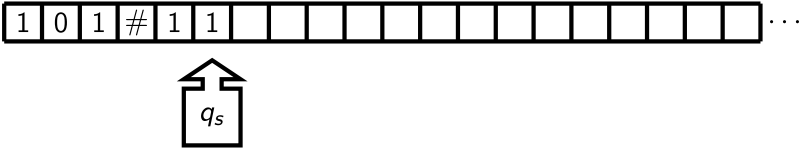

- If state is q_s and symbol is 1 then change to state q_ℓ^1 change symbol to 0 and move in direction 'left'
- If state is q_ℓ^1 and symbol is 1 then change to state q_ℓ^1 change symbol to 1 and move in direction 'left'
- If state is q_ℓ^1 and symbol is # then change to state q_a^1 change symbol to # and move in direction 'left'
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Conceptual View

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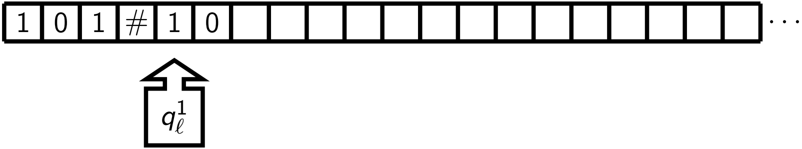

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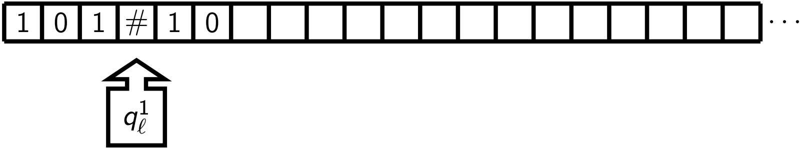

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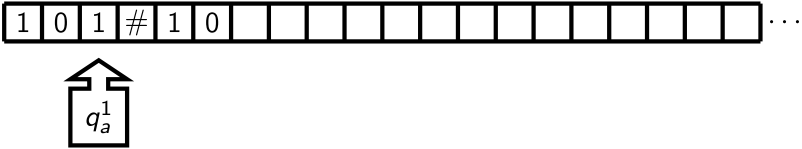

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Conceptual View

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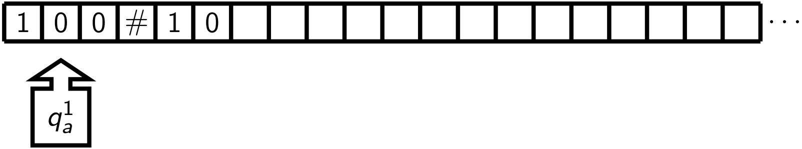
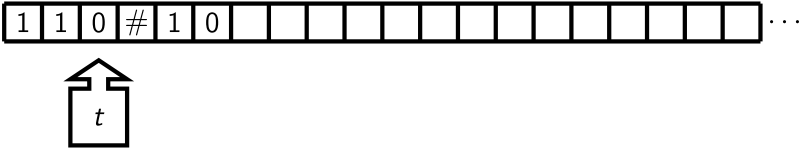


- `If state is` q_a^1 `and symbol is` 0 `then change to state` t `change symbol to` 1 `and move in direction 'right'`

Conceptual View

A Turing Machine:



- If state is q_a^1 and symbol is 0 then change to state t change symbol to 1 and move in direction 'right'

Exercise 5, Tutorial 1

Consider the language $L = \{w\#w \mid w \in \{0,1\}^*\}$.

1. Give a halting DTM for L. Explain your solution in natural language.
2. Give the accepting run on $\# \in L$.
3. Give the accepting run on $011\#011 \in L$.
4. Give the rejecting run on $01\#00 \notin L$.

Intuition

$L = \{w\#w \mid w \in \{0,1\}^*\}$

1. If current symbol is 0 or 1: remember it as b, replace it by X.
2. Go right to leftmost non-X symbol right of $\#$.
3. If it is not b, reject.
4. If it is b, replace it by X.
5. Go left to the leftmost non-X symbol left of $\#$ (if there is none go to step 7).
6. Go to step 1.
7. Check that there is no 0 or 1 left.

Intuition

$L = \{w\#w \mid w \in \{0,1\}^*\}$

1. If current symbol is 0 or 1: remember it as b, replace it by X.
   Use states s, q_0, q_1
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   Use states q_0, q_1 and $q_0^{\#}$ $q_1^{\#}$ (after $\#$)
3. If it is not b, reject.
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   Use states q_ℓ, $q_\ell^\#$
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   Use states q_ℓ, $q_\ell^\#$
6. Go to step 1.
7. Check that there is no 0 or 1 left.
   Use states q_s

Full Solution

$(Q, \Sigma, \Gamma, s, t, r, \delta)$ with

- $Q = \{s, q_0, q_1, q_0^{\#}, q_1^{\#}, q_\ell, q_\ell^{\#}, q_s, t, r\}$,
- $\Sigma = \{0, 1, \#\}$,
- $\Gamma = \{0, 1, \#, X, \llcorner\!\lrcorner\}$,
- and the following $\delta \colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

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- and the following $\delta\colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

$$
\begin{aligned}
\delta(s, 0) &= (q_0, X, +1) && \texttt{//remember 0} \\
\delta(s, 1) &= (q_1, X, +1) && \texttt{//remember 1} \\
\delta(s, \#) &= (q_s, \#, +1) && \texttt{//check that no more 0/1 on tape} \\
\delta(s, \text{␣}) &= (r, \text{␣}, +1) && \texttt{//empty word} \\
\delta(s, X) &= (r, X, +1) && \texttt{//error}
\end{aligned}
$$

Full Solution

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- and the following $\delta\colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

$\delta(q_b, 0) = (q_b, 0, +1)$ `//go right till #`

$\delta(q_b, 1) = (q_b, 1, +1)$ `//go right till #`

$\delta(q_b, \#) = (q_b^{\#}, \#, +1)$ `//reached #`

$\delta(q_b, \llcorner\!\lrcorner) = (r, \llcorner\!\lrcorner, +1)$ `//no # found`

$\delta(q_b, X) = (r, X, +1)$ `//error`

for $b \in \{0, 1\}$

$(Q, \Sigma, \Gamma, s, t, r, \delta)$ with

- $Q = \{s, q_0, q_1, q_0^{\#}, q_1^{\#}, q_\ell, q_\ell^{\#}, q_s, t, r\}$,
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- and the following $\delta\colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

$\delta(q_b^{\#}, X) = (q_b^{\#}, X, +1)$ `//go right till first 0/1`

$\delta(q_b^{\#}, b) = (q_\ell, X, -1)$ `//found b, go back left`

$\delta(q_b^{\#}, 1-b) = (r, 1-b, -1)$ `//wrong symbol found`

$\delta(q_b^{\#}, \sqcup) = (r, \sqcup, -1)$ `//no 0/1 found`

$\delta(q_b^{\#}, \#) = (r, X, +1)$ `//error`

for $b \in \{0, 1\}$

Full Solution

$(Q, \Sigma, \Gamma, s, t, r, \delta)$ with

- $Q = \{s, q_0, q_1, q_0^{\#}, q_1^{\#}, q_\ell, q_\ell^{\#}, q_s, t, r\}$,
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- and the following $\delta \colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

$$\delta(q_\ell, X) = (q_\ell, X, -1) \qquad \texttt{//go left till \#}$$

$$\delta(q_\ell, \#) = (q_\ell^{\#}, \#, -1) \qquad \texttt{//reached \#}$$

$$\delta(q_\ell, 0) = (r, 0, -1) \qquad \texttt{//error}$$

$$\delta(q_\ell, 1) = (r, 1, -1) \qquad \texttt{//error}$$

$$\delta(q_\ell, \llcorner\!\lrcorner) = (r, \llcorner\!\lrcorner, -1) \qquad \texttt{//error}$$

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- and the following $\delta\colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

$\delta(q_\ell^{\#}, 0) = (q_\ell^{\#}, 0, -1)$ `//go left till first X`

$\delta(q_\ell^{\#}, 1) = (q_\ell^{\#}, 1, -1)$ `//go left till first X`

$\delta(q_\ell^{\#}, X) = (s, X, +1)$ `//found X, check next symbol`

$\delta(q_\ell^{\#}, \text{␣}) = (r, \text{␣}, -1)$ `//error`

$\delta(q_\ell^{\#}, \#) = (r, \#, -1)$ `//error`

Full Solution

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- and the following $\delta\colon (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$:

$\delta(q_s, 0) = (r, 0, -1)$ `//reject if still a 0 on tape`

$\delta(q_s, 1) = (r, 1, -1)$ `//reject if still a 1 on tape`

$\delta(q_s, X) = (q_s, X, +1)$ `//check next cell`

$\delta(q_s, \#) = (r, \#, +1)$ `//error, found second #`

$\delta(q_s, \sqcup) = (t, \sqcup, -1)$ `//no 0/1 found`

Runs

Note: To save some space, we underline the head position instead of specifying it explicitly in a configuration and drop the $\sqcup \ldots$ in the end of the tape content!

- M accepts $\# \in L$:

Runs

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- M accepts $\# \in L$:

$$[s, \underline{\#}]$$

Runs

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- M accepts $\# \in L$:

$$[s, \underline{\#}] \ \vdash_M \ [q_s, \#\underline{\llcorner}]$$

Runs

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- M accepts $\# \in L$:

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- M accepts $\# \in L$:

$$[s, \underline{\#}] \;\vdash_M\; [q_s, \#\underline{\sqcup}] \;\vdash_M\; [t, \underline{\#}]$$

- M rejects $01\#00$:

Runs

Note: To save some space, we underline the head position instead of specifying it explicitly in a configuration and drop the ␣ . . . in the end of the tape content!

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- M rejects $01\#00$:

$[s, \underline{0}1\#00]$

Runs

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$$[s, \underline{\#}] \ \vdash_M \ [q_s, \#\underline{\sqcup}] \ \vdash_M \ [t, \underline{\#}]$$

- M rejects $01\#00$:

$[s, \underline{0}1\#00] \ \vdash_M \ [q_0, X\underline{1}\#00]$

Runs

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- M accepts $\# \in L$:

$$[s, \underline{\#}] \ \vdash_M \ [q_s, \#\underline{\sqcup}] \ \vdash_M \ [t, \underline{\#}]$$

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$$[s, \underline{0}1\#00] \ \vdash_M \ [q_0, X\underline{1}\#00] \ \vdash_M \ [q_0, X1\underline{\#}00]$$

Runs

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$$[s, \underline{\#}] \vdash_M [q_s, \#\underline{\sqcup}] \vdash_M [t, \underline{\#}]$$

- M rejects $01\#00$:

  $[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$

  $[q_0^{\#}, X1\#\underline{0}0]$

Runs

Note: To save some space, we underline the head position instead of specifying it explicitly in a configuration and drop the $\sqcup \ldots$ in the end of the tape content!

- M accepts $\# \in L$:

$$[s, \underline{\#}] \vdash_M [q_s, \#\underline{\sqcup}] \vdash_M [t, \underline{\#}]$$

- M rejects $01\#00$:

$[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$

$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0]$

Runs

Note: To save some space, we underline the head position instead of specifying it explicitly in a configuration and drop the $\textvisiblespace \ldots$ in the end of the tape content!

- M accepts $\# \in L$:

$$[s, \underline{\#}] \vdash_M [q_s, \#\underline{\textvisiblespace}] \vdash_M [t, \underline{\#}]$$

- M rejects $01\#00$:

$[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$

$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0] \vdash_M [q_\ell^{\#}, X\underline{1}\#X0]$

Runs

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- M accepts $\# \in L$:

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- M rejects $01\#00$:

$[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$
$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0] \vdash_M [q_\ell^{\#}, X\underline{1}\#X0] \vdash_M$
$[q_\ell^{\#}, \underline{X}1\#X0]$

Runs

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- M accepts $\# \in L$:

$$[s, \underline{\#}] \vdash_M [q_s, \#\underline{\llcorner\!\lrcorner}] \vdash_M [t, \underline{\#}]$$

- M rejects $01\#00$:

$[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$

$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0] \vdash_M [q_\ell^{\#}, X\underline{1}\#X0] \vdash_M$

$[q_\ell^{\#}, \underline{X}1\#X0] \vdash_M [s, X\underline{1}\#X0]$

Runs

Note: To save some space, we underline the head position instead of specifying it explicitly in a configuration and drop the $\sqcup \ldots$ in the end of the tape content!

- M accepts $\# \in L$:

$$[s, \underline{\#}] \vdash_M [q_s, \#\underline{\sqcup}] \vdash_M [t, \underline{\#}]$$

- M rejects $01\#00$:

$$[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$$
$$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0] \vdash_M [q_\ell^{\#}, X\underline{1}\#X0] \vdash_M$$
$$[q_\ell^{\#}, \underline{X}1\#X0] \vdash_M [s, X\underline{1}\#X0] \vdash_M [q_1, XX\underline{\#}X0]$$

Runs

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$[s, \underline{0}1\#00] \vdash_M [q_0, X\underline{1}\#00] \vdash_M [q_0, X1\underline{\#}00] \vdash_M$

$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0] \vdash_M [q_\ell^{\#}, X\underline{1}\#X0] \vdash_M$

$[q_\ell^{\#}, \underline{X}1\#X0] \vdash_M [s, X\underline{1}\#X0] \vdash_M [q_1, XX\underline{\#}X0] \vdash_M$

$[q_1^{\#}, XX\#\underline{X}0]$

Runs

Note: To save some space, we underline the head position instead of specifying it explicitly in a configuration and drop the $\sqcup \ldots$ in the end of the tape content!

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$[s, \underline{0}1\#00] \ \vdash_M \ [q_0, X\underline{1}\#00] \ \vdash_M \ [q_0, X1\underline{\#}00] \ \vdash_M$

$[q_0^\#, X1\#\underline{0}0] \ \vdash_M \ [q_\ell, X1\underline{\#}X0] \ \vdash_M \ [q_\ell^\#, X\underline{1}\#X0] \ \vdash_M$

$[q_\ell^\#, \underline{X}1\#X0] \ \vdash_M \ [s, X\underline{1}\#X0] \ \vdash_M \ [q_1, XX\underline{\#}X0] \ \vdash_M$

$[q_1^\#, XX\#\underline{X}0] \ \vdash_M \ [q_1^\#, XX\#X\underline{0}]$

Runs

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$$[q_0^{\#}, X1\#\underline{0}0] \vdash_M [q_\ell, X1\underline{\#}X0] \vdash_M [q_\ell^{\#}, X\underline{1}\#X0] \vdash_M$$

$$[q_\ell^{\#}, \underline{X}1\#X0] \vdash_M [s, X\underline{1}\#X0] \vdash_M [q_1, XX\underline{\#}X0] \vdash_M$$

$$[q_1^{\#}, XX\#\underline{X}0] \vdash_M [q_1^{\#}, XX\#X\underline{0}] \vdash_M [r, XX\#\underline{X}0]$$

Another Run

- M accepts $011\#011 \in L$:

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011]$

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011]$

Another Run

- M accepts $011\#011 \in L$:

$$[s, \underline{0}11\#011] \ \vdash_M \ [q_0, X\underline{1}1\#011] \ \vdash_M \ [q_0, X1\underline{1}\#011]$$

Another Run

- M accepts $011\#011 \in L$:

$$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011]$$

Another Run

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$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^{\#}, X11\#\underline{0}11]$

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$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11]$

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$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11]$

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$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11]$

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$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$
$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$
$[q_\ell^{\#}, \underline{X}11\#X11]$

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$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11]$

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$$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$$

$$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11]$$

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$$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$$

$$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$$

$$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11]$$

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$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$

$[q_1^{\#}, XX1\#\underline{X}11]$

Another Run

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$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$

$[q_1^{\#}, XX1\#\underline{X}11] \vdash_M [q_1^{\#}, XX1\#X\underline{1}1]$

Another Run

- M accepts $011\#011 \in L$:

$$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$$

$$[q_0^\#, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^\#, X1\underline{1}\#X11] \vdash_M [q_\ell^\#, X\underline{1}1\#X11] \vdash_M$$

$$[q_\ell^\#, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$$

$$[q_1^\#, XX1\#\underline{X}11] \vdash_M [q_1^\#, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1]$$

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^\#, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^\#, X1\underline{1}\#X11] \vdash_M [q_\ell^\#, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^\#, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$

$[q_1^\#, XX1\#\underline{X}11] \vdash_M [q_1^\#, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1]$

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$
$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$
$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$
$[q_1^{\#}, XX1\#\underline{X}11] \vdash_M [q_1^{\#}, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1] \vdash_M$
$[q_\ell^{\#}, XX\underline{1}\#XX1]$

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^\#, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^\#, X1\underline{1}\#X11] \vdash_M [q_\ell^\#, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^\#, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$

$[q_1^\#, XX1\#\underline{X}11] \vdash_M [q_1^\#, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1] \vdash_M$

$[q_\ell^\#, XX\underline{1}\#XX1] \vdash_M [q_\ell^\#, X\underline{X}1\#XX1]$

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^\#, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^\#, X1\underline{1}\#X11] \vdash_M [q_\ell^\#, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^\#, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$

$[q_1^\#, XX1\#\underline{X}11] \vdash_M [q_1^\#, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1] \vdash_M$

$[q_\ell^\#, XX\underline{1}\#XX1] \vdash_M [q_\ell^\#, X\underline{X}1\#XX1] \vdash_M [s, XX\underline{1}\#XX1]$

Another Run

- M accepts $011\#011 \in L$:

$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$

$[q_0^\#, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^\#, X1\underline{1}\#X11] \vdash_M [q_\ell^\#, X\underline{1}1\#X11] \vdash_M$

$[q_\ell^\#, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$

$[q_1^\#, XX1\#\underline{X}11] \vdash_M [q_1^\#, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1] \vdash_M$

$[q_\ell^\#, XX\underline{1}\#XX1] \vdash_M [q_\ell^\#, X\underline{X}1\#XX1] \vdash_M [s, XX\underline{1}\#XX1] \vdash_M [q_1, XXX\underline{\#}XX1]$

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$$[s, \underline{0}11\#011] \vdash_M [q_0, X\underline{1}1\#011] \vdash_M [q_0, X1\underline{1}\#011] \vdash_M [q_0, X11\underline{\#}011] \vdash_M$$
$$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$$
$$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$$
$$[q_1^{\#}, XX1\#\underline{X}11] \vdash_M [q_1^{\#}, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1] \vdash_M$$
$$[q_\ell^{\#}, XX\underline{1}\#XX1] \vdash_M [q_\ell^{\#}, X\underline{X}1\#XX1] \vdash_M [s, XX\underline{1}\#XX1] \vdash_M [q_1, XXX\underline{\#}XX1] \vdash_M$$
$$[q_1^{\#}, XXX\#\underline{X}X1]$$

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$[q_0^{\#}, X11\#\underline{0}11] \vdash_M [q_\ell, X11\underline{\#}X11] \vdash_M [q_\ell^{\#}, X1\underline{1}\#X11] \vdash_M [q_\ell^{\#}, X\underline{1}1\#X11] \vdash_M$
$[q_\ell^{\#}, \underline{X}11\#X11] \vdash_M [s, X\underline{1}1\#X11] \vdash_M [q_1, XX\underline{1}\#X11] \vdash_M [q_1, XX1\underline{\#}X11] \vdash_M$
$[q_1^{\#}, XX1\#\underline{X}11] \vdash_M [q_1^{\#}, XX1\#X\underline{1}1] \vdash_M [q_\ell, XX1\#\underline{X}X1] \vdash_M [q_\ell, XX1\underline{\#}XX1] \vdash_M$
$[q_\ell^{\#}, XX\underline{1}\#XX1] \vdash_M [q_\ell^{\#}, X\underline{X}1\#XX1] \vdash_M [s, XX\underline{1}\#XX1] \vdash_M [q_1, XXX\underline{\#}XX1] \vdash_M$
$[q_1^{\#}, XXX\#\underline{X}X1] \vdash_M [q_1^{\#}, XXX\#X\underline{X}1]$

Another Run

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Today

Turing machines define the limits of computation:

> *Claim: Everything that can be computed can be computed by a Turing machine.*

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> *Claim: Everything that can be computed can be computed by a Turing machine.*

How can we be so sure? What about..

- parallel computing?
- quantum computing?
- neural networks?
- some technology we haven't invented yet?

Agenda

A Bit of History

The "Entscheidungsproblem" (Hilbert and Ackermann, 1928)

> *Is there an algorithm that, given a statement in some logical language (typically predicate logic), answers "Yes" or "No" according to whether the statement is universally valid.*

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Requires the formalization of "algorithms". Independently:

- **Gödel, 1933**: μ-recursive functions
- **Church, 1936**: λ-calculus
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Theorem (Church, Kleene, Turing)

All three formalizations compute the same functions (and therefore compute the same languages).

Church-Turing Thesis

This equivalence led mathematicians to believe that the intuitive notion of algorithm is precisely captured by any of these three formalizations.

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The Church-Turing Thesis

> *Claim: Everything that can be computed can be computed by a Turing machine.*

- Many other formalizations have been proposed, all equivalent to Turing machines.
- But, there are "nonphysical" models that are stronger: Zeno machines (infinite computations in finite time), time-travelling Turing machines, etc.
- It is a thesis, **not** a definition and **not** a theorem, and may be refuted in the future.

Turing-completeness

Definition

A formalization of computation is Turing-complete, if it can simulate every Turing machine.

So, a Turing-complete formalism can compute everything Turing machines can compute.

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So, a Turing-complete formalism can compute everything Turing machines can compute.

Examples

- λ-calculus, μ-recursive functions,
- Java, Python, and other programming languages (assuming your computer has infinite memory),
- Excel, LaTeX, etc.,
- Game of Life, Minecraft,
- and many other formalisms.

Robustness

We (informally) say that a formalization of computation is robust, if no reasonable extension increases its power.

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Turing machines are **very** robust! We will study three extensions (there are many more) and show that they are not more powerful than Turing machines.

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Turing machines are **very** robust! We will study three extensions (there are many more) and show that they are not more powerful than Turing machines.

To prove this, we simulate extended TM's by DTM's.

We want to add another "direction" for the reading head of a Turing machine, i.e., 0 for "stay". Thus, a transition of the form $\delta(q, a) = (q', b, 0)$ updates the state to q' and changes the symbol at the current cell to b, but does not move the head.

Show that every DTM with the "stay" direction can be simulated by a standard DTM (i.e., without the "stay" direction).

Exercise 1, Tutorial 2

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Show that every DTM with the "stay" direction can be simulated by a standard DTM (i.e., without the "stay" direction).

Note

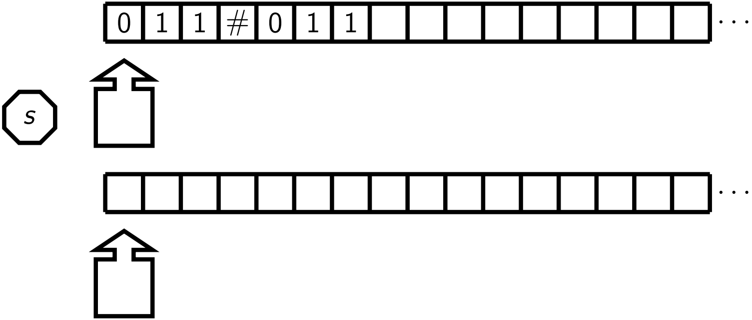
We will use the direction 0 from now on in our Turing machines (whenever convenient).

Agenda

1. The Church-Turing Thesis
2. **Multi-tape Turing Machines**
3. Nondeterministic Turing Machines

Conceptual View

A 2-tape TM for $\{w\#w \mid w \in \{0,1\}^*\}$:



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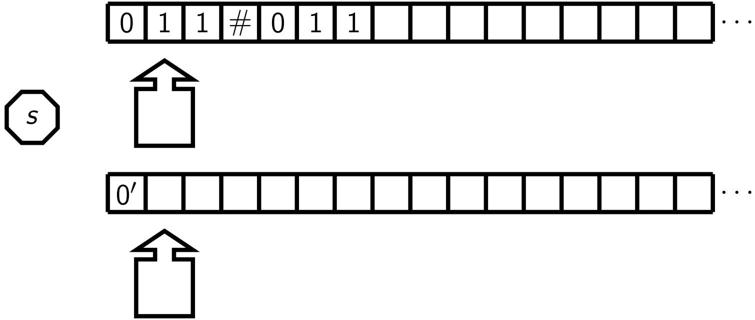

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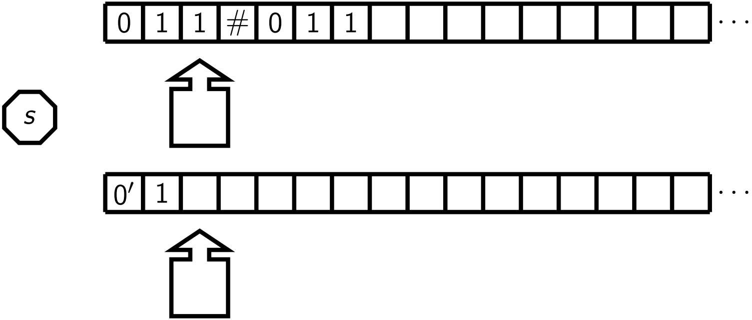

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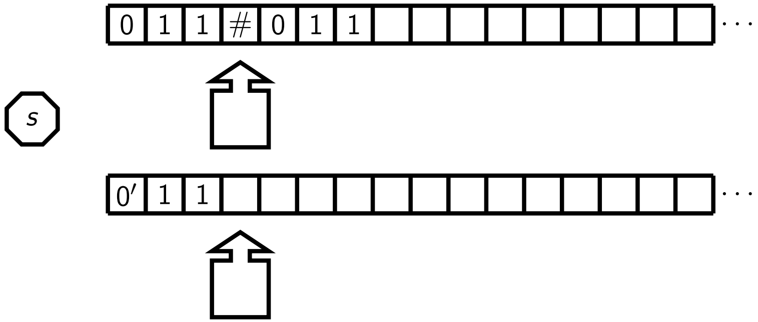

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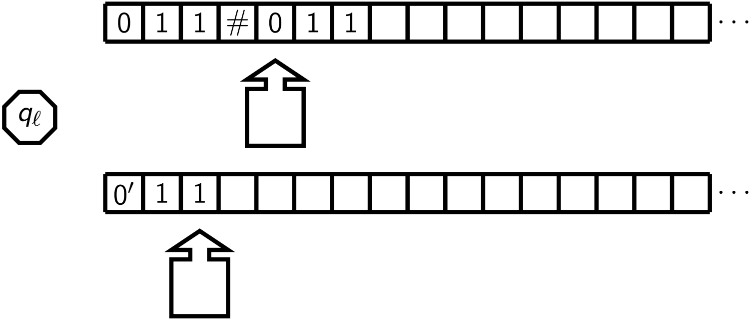

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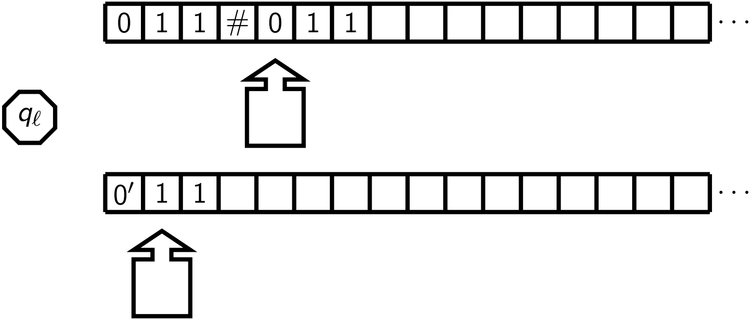

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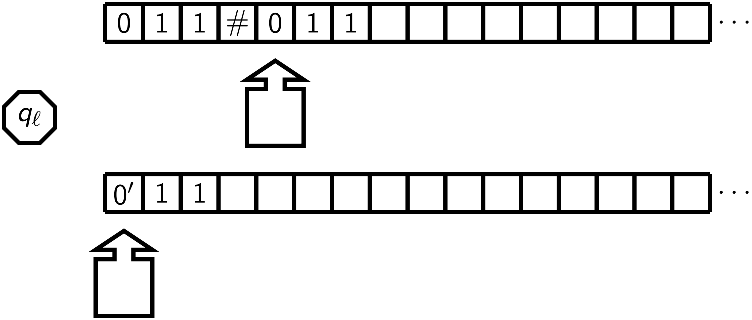

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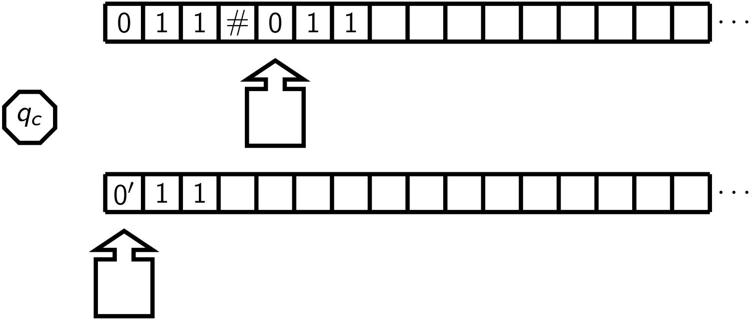

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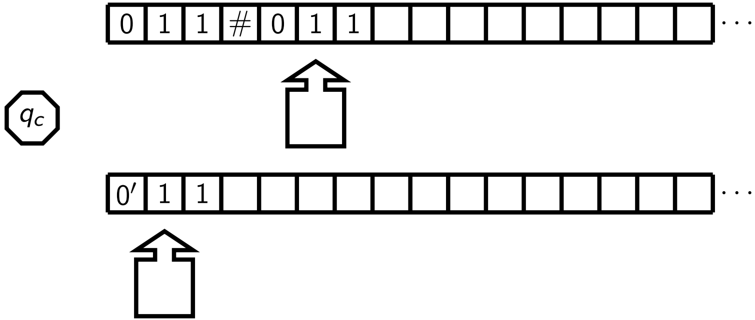

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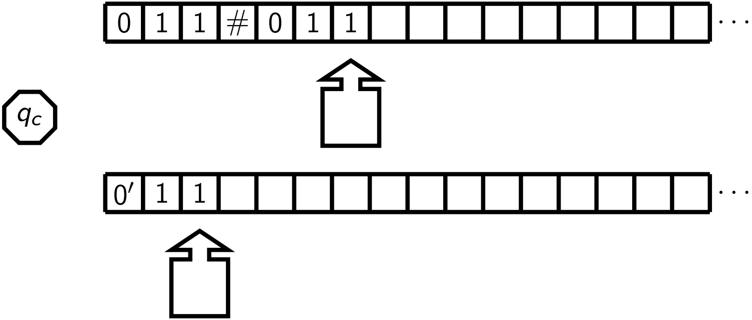

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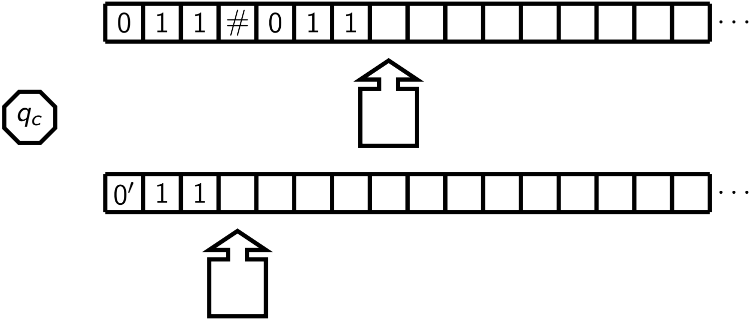

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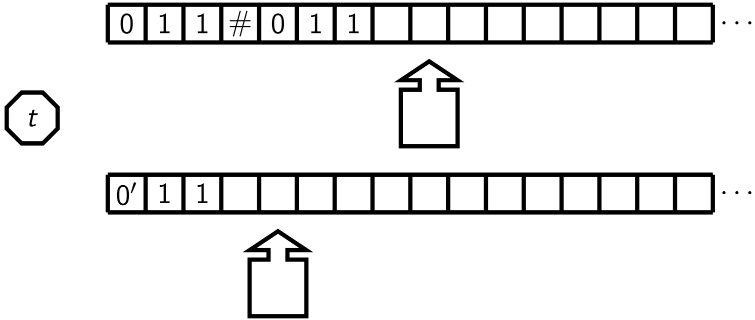

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Definition

Definition (full definition in book)

Let $k \geq 1$. A k-tape DTM has the form $(Q, \Sigma, \Gamma, s, t, r, \delta)$ where Q, Σ, Γ, s, t and r are as for DTM's and where

$$\delta \colon (Q \setminus \{t, r\}) \times \Gamma^k \to Q \times \Gamma^k \times \{-1, +1\}^k.$$

- **Configuration**: **one** state and k tapes with k (independent) heads.
- **Initial configuration**: Input word on first tape, all other tapes empty.
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Remark

1-tape DTM = DTM as defined last lecture.

Simulation

A machine M' outcome-simulates a machine M if we have the following for every input w:

- If M halts on w, then M' halts on w, and
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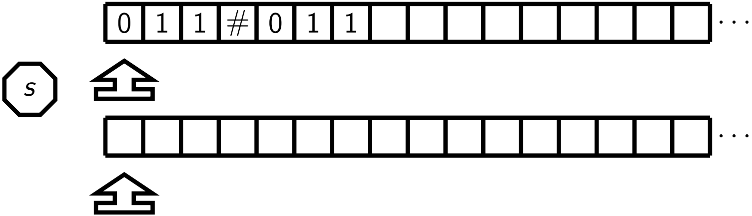
For every k-tape DTM M there is a (standard) DTM that outcome-simulates M.

The language of a multi-tape DTM and multi-tape halting DTM's are defined as expected.

Corollary

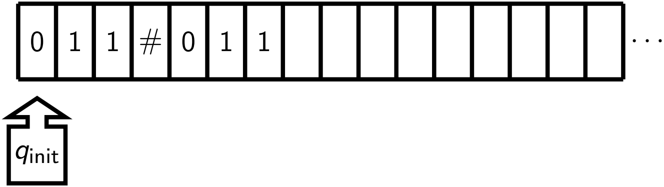
1. *A language is computably-enumerable if and only if it is the language of some multi-tape DTM.*
2. *A language is computable if and only if it is the language of some halting multi-tape DTM.*

M

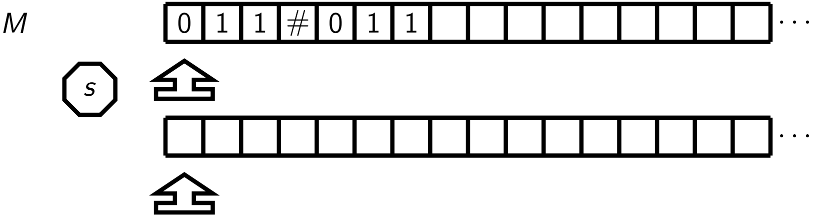


M' simulates M by storing the n-th cells of M's tapes in the n-th cell of its tape, head positions are marked with a ˆ $\Rightarrow$ requires initialization!

M'



Proof Sketch



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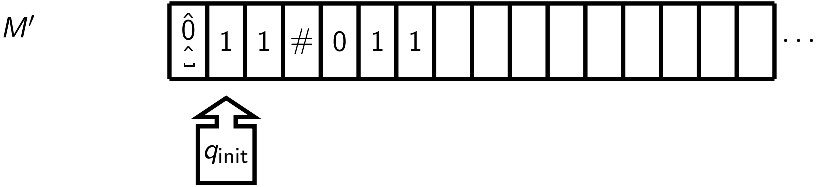

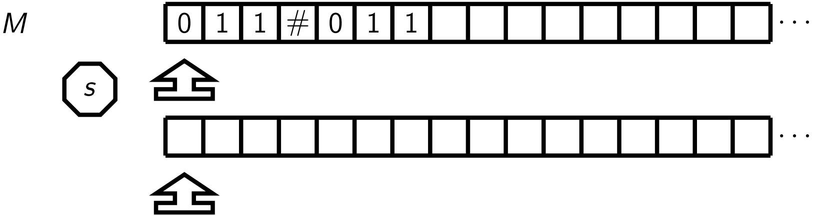

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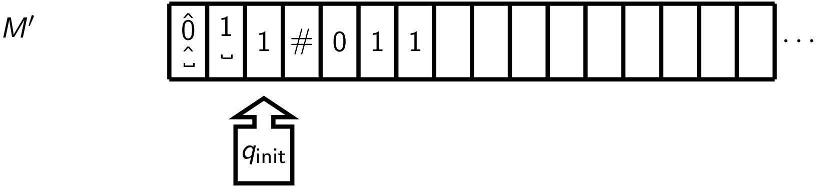

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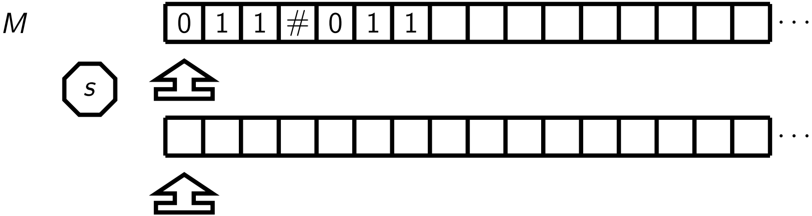

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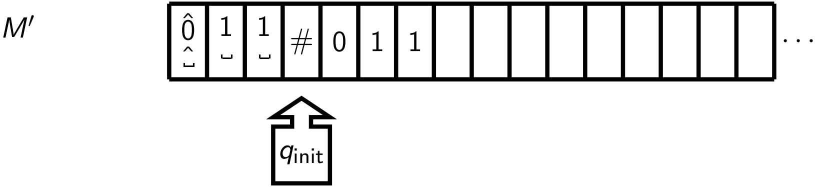

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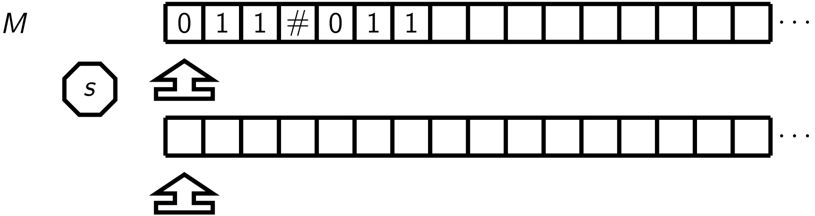

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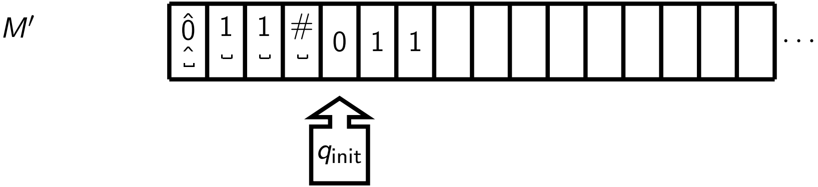

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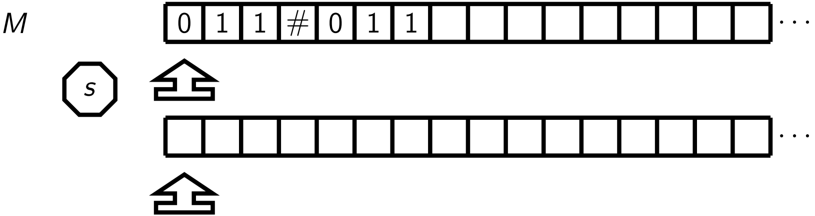

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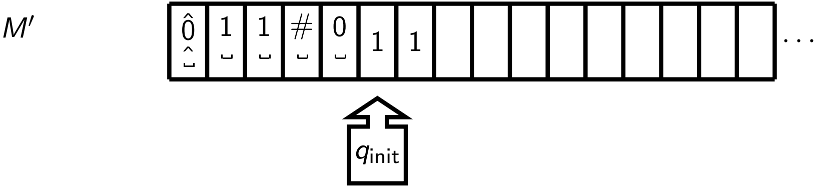

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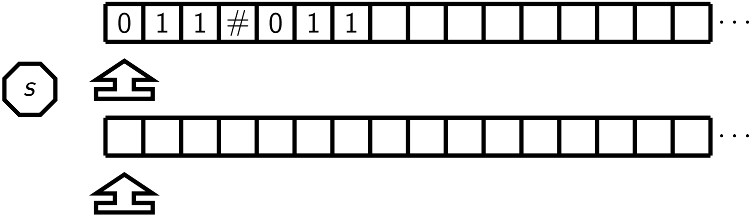

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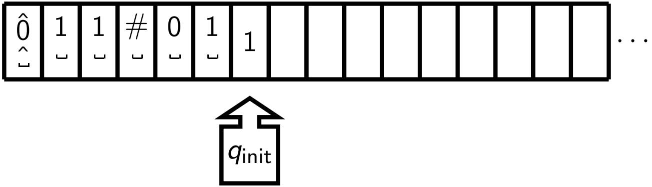

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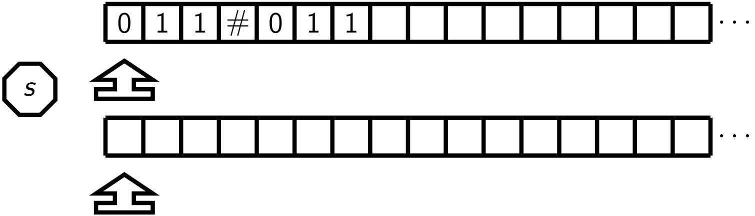

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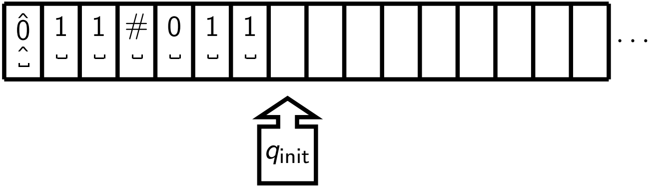

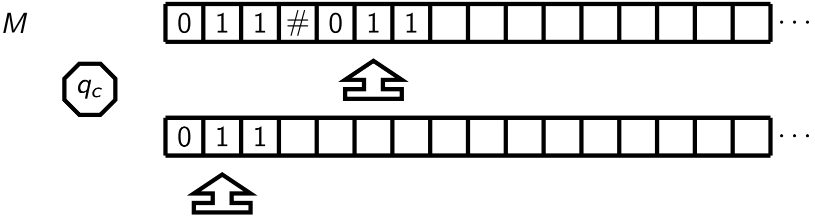

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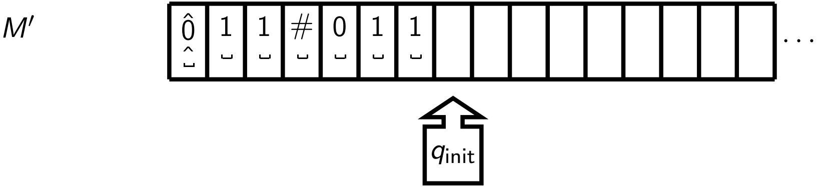

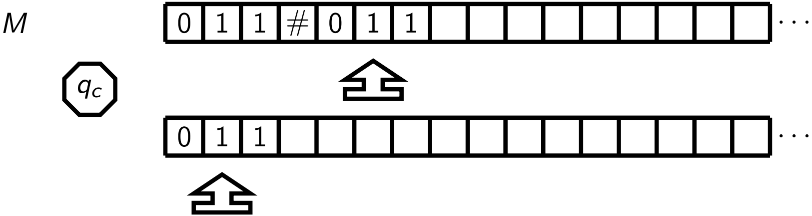

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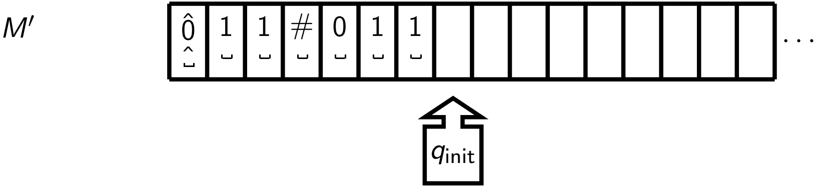

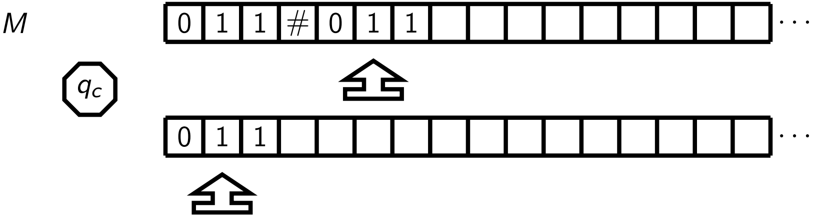

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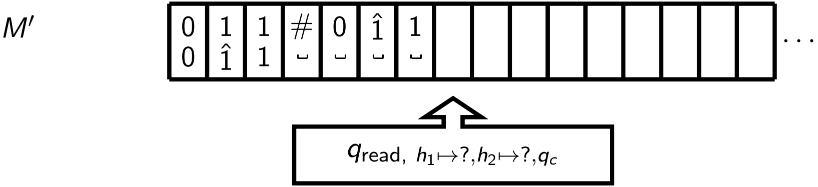

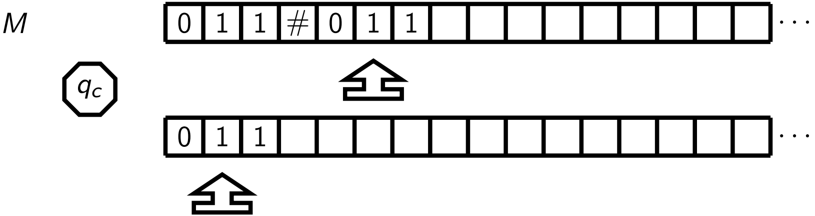

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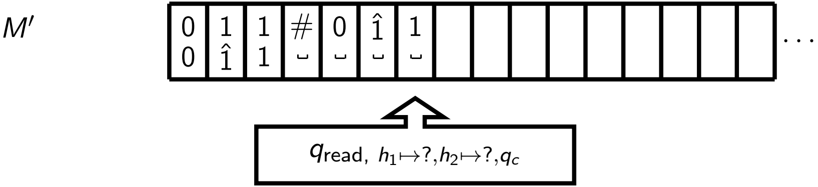

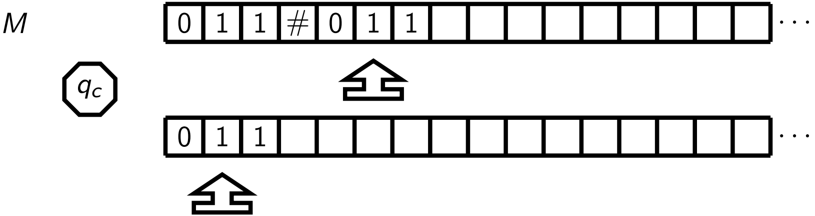

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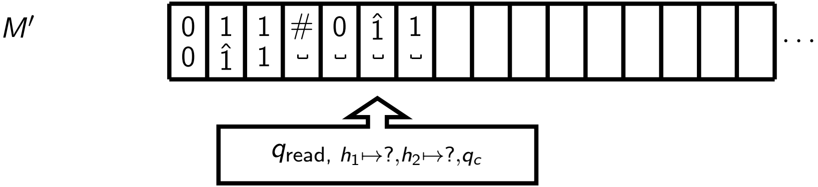

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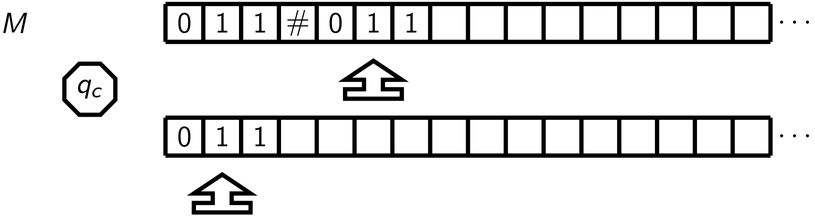

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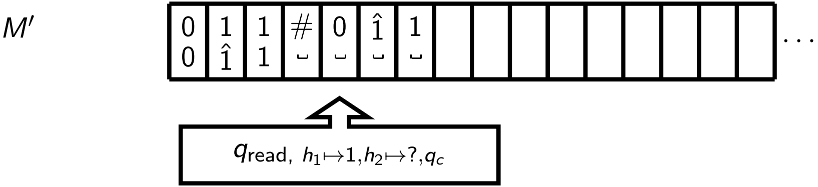

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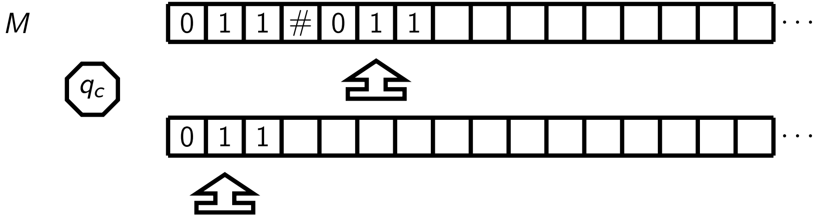

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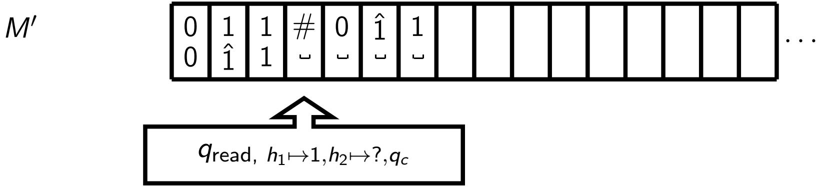

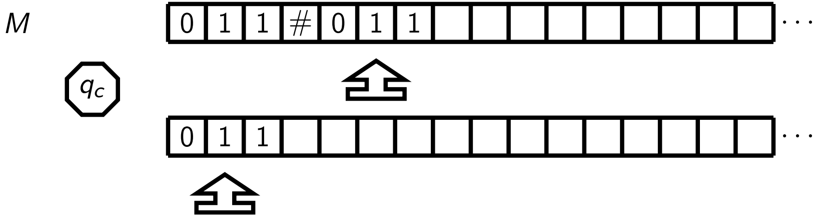

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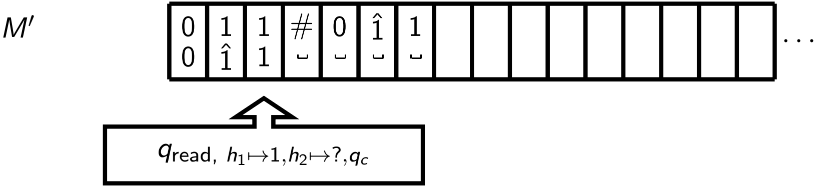

Proof Sketch

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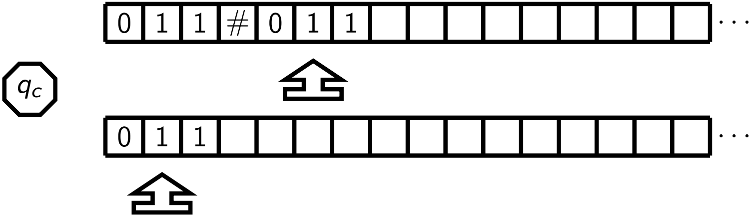

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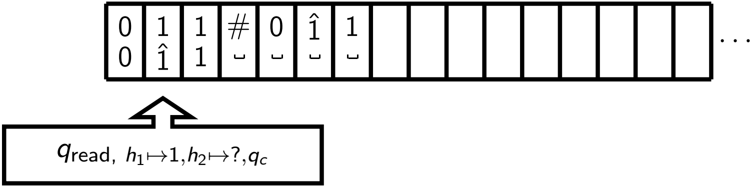

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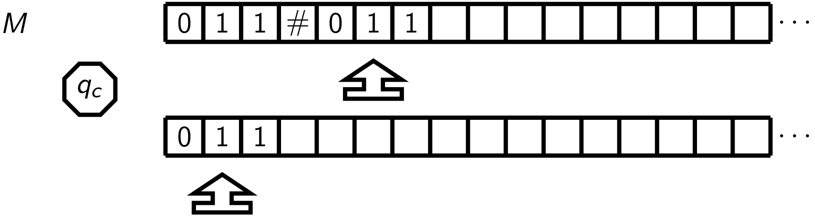

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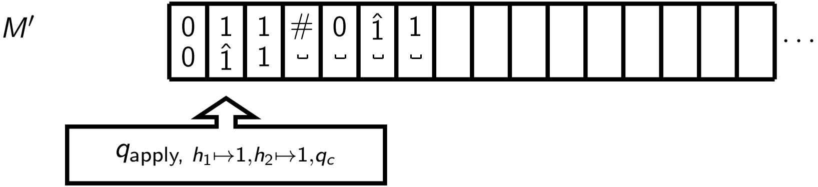

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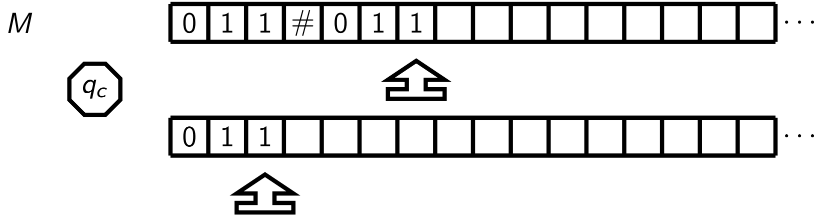
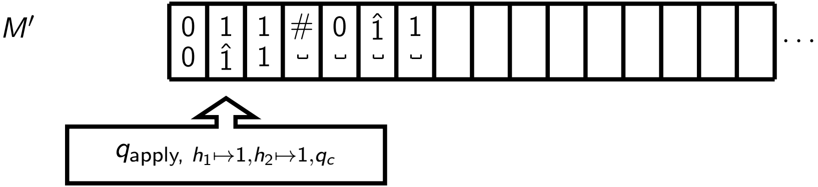
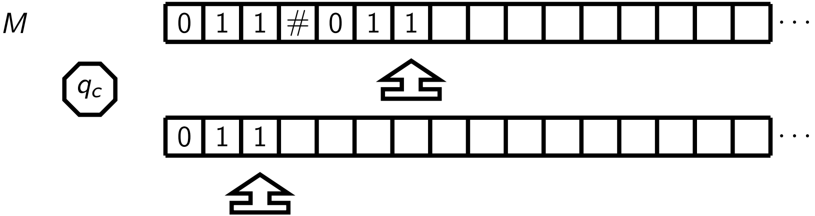


Let us simulate a transition of M. Then, M' can simulate the transition by updating cells and head positions.





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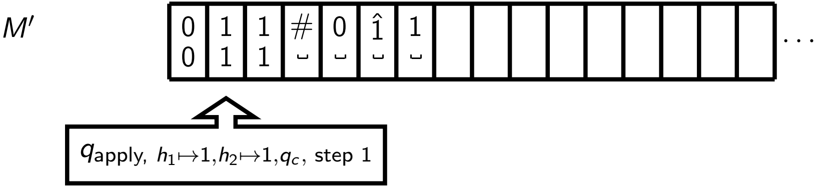

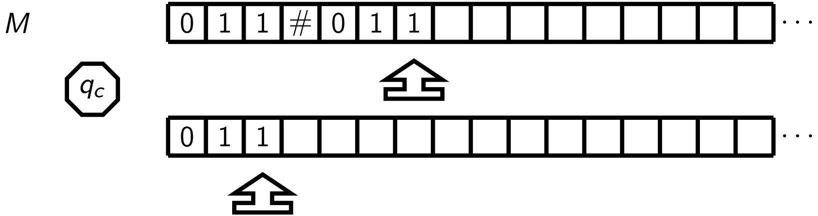

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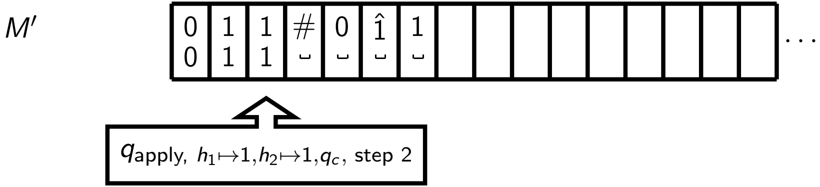

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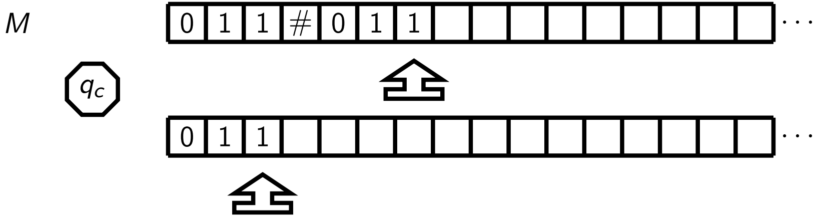

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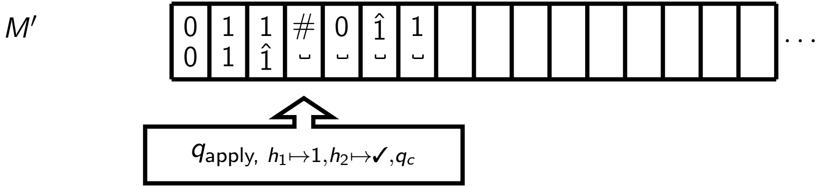
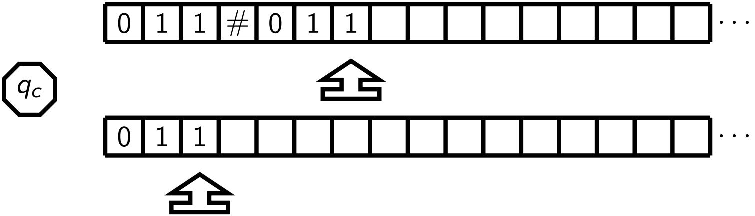


Proof Sketch

M



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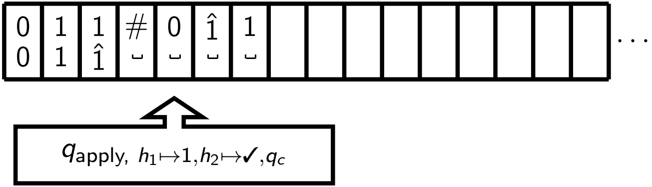

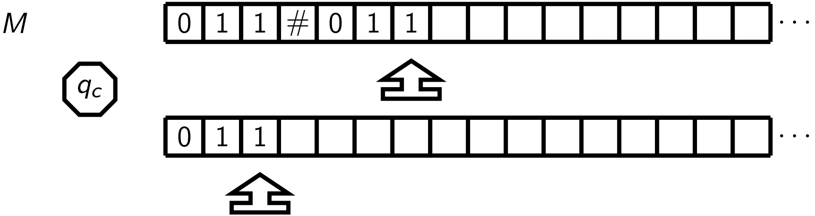

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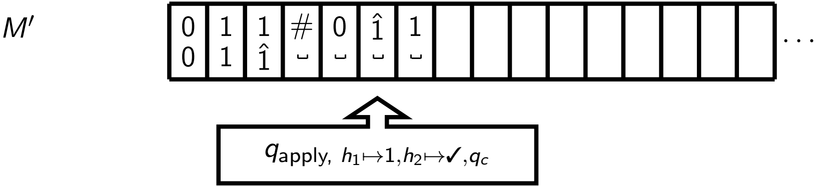

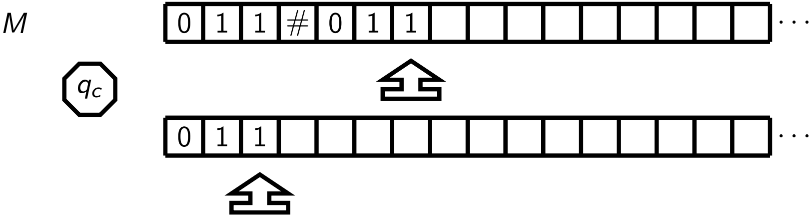

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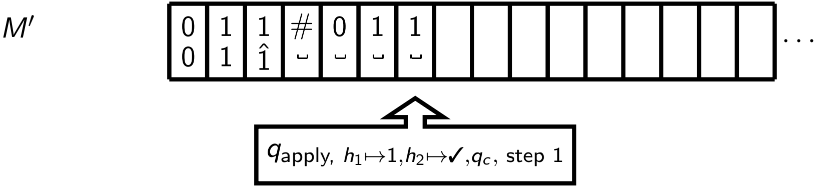

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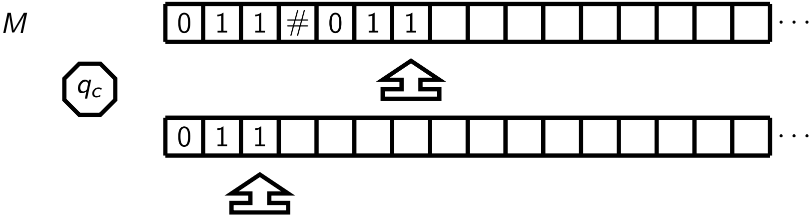

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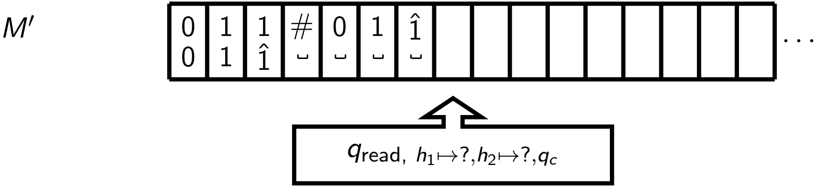
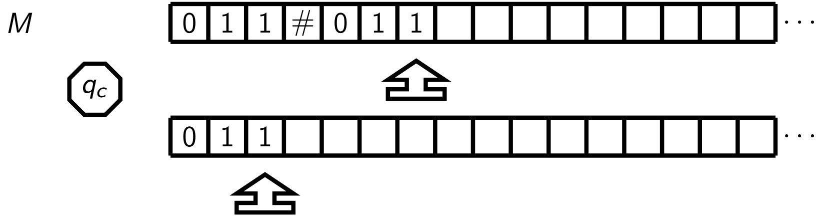
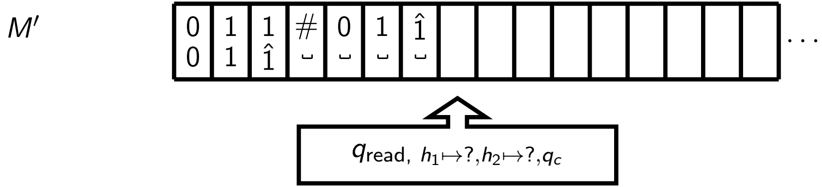


Proof Sketch



This process is repeated until M halts. M' accepts/rejects if and only if M does so.



Agenda

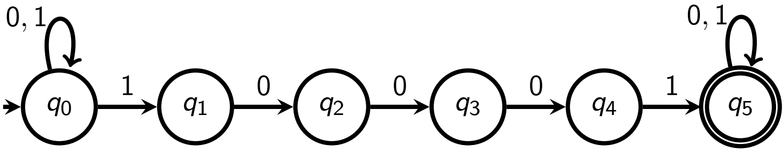
1. The Church-Turing Thesis
2. Multi-tape Turing Machines
3. Nondeterministic Turing Machines

A nondeterministic finite automaton (NFA) has the form $(Q, \Sigma, q_I, \delta, F)$ where

- Q is a finite set of states,
- Σ is an alphabet,
- $q_I \in Q$ is the initial state,
- $\delta\colon Q \times \Sigma \to 2^Q$ is the transition function, and
- $F \subseteq Q$ is a set of accepting states.

Example

An NFA for the language $\{\{0,1\}^*10001\{0,1\}^* \mid n \geq 0\}$:



Nondeterministic Turing Machines

Definition (full definition in book)

A nondeterministic Turing machine (NTM) has the form $(Q, \Sigma, \Gamma, s, t, r, \delta)$ where Q, Σ, Γ, s, t and r are as for DTM's and where

$$\delta \colon (Q \setminus \{t, r\}) \times \Gamma \to 2^{Q \times \Gamma \times \{-1,+1\}}.$$

Intuition:

- Configurations and initial configuration as for DTM.
- But: A configuration can have multiple successor configurations.
- Acceptance: An accepting configuration is reachable from the initial one.

Intuition

$\delta(q, b) = \{(q_1, c, 0), (q_2, d, +1), (q_3, e, -1)\}$:

$$[q, abcw' \llcorner\!\lrcorner \ldots, 2]$$

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$$[q_1, accw' \sqcup \ldots, 2] \quad [q_2, adcw' \sqcup \ldots, 3] \quad [q_3, aecw' \sqcup \ldots, 1]$$

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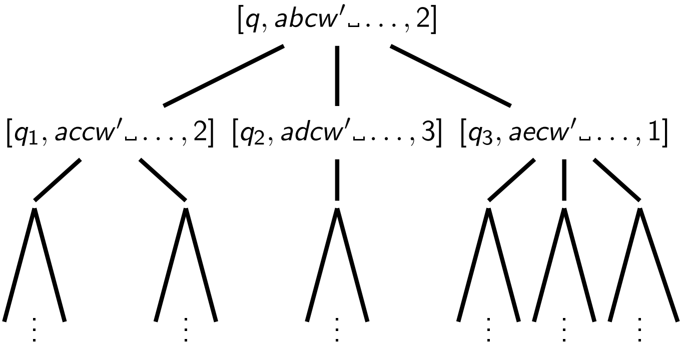

Computation tree of a NTM on an input w:

- Root: initial configuration on w.
- Children of a configuration: All its successor configurations.
- May be infinite (if and only if it has an infinite branch).

More Definitions

Definition

A NTM M accepts an input w if the computation tree of M on w contains an accepting configuration.
As before:

$$L(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}.$$

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Definition

A NTM M is a halting NTM, if for every input, the computation tree of M on w is finite.

So, every branch ends in an accepting or rejecting configuration.

Simulation

Theorem

For every NTM M there is a (standard) DTM that outcome-simulates M.

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Let M' be a DTM that does the following when given an input w:

- $\Gamma_0 := \{\alpha_w\}$, where α_w is the initial configuration of M on w.
- $i := 0$.
- Iterate:
  - If Γ_i contains an accepting configuration, accept.
  - If Γ_i is empty, then reject.
  - $\Gamma_{i+1} := \{\gamma' \mid \gamma \vdash_M \gamma' \text{ for some } \gamma \in \Gamma_i\}$ (the set of successor configurations of the configurations in Γ_i).
  - $i := i + 1$.

Theorem

For every NTM M there is a (standard) DTM that outcome-simulates M.

Corollary

1. *A language is computably-enumerable if and only if it is the language of some NTM.*
2. *A language is computable if and only if it is the language of some halting NTM.*

Conclusion

We have seen

- the Church-Turing Thesis,
- multi-tape DTM's and
- NTM's, and
- their equivalence.
- Not covered: multi-tape NTM's (can also be simulated by DTM's).

Reading:

- Sections 2.6 and 2.7.1 of "Computability and Complexity" (pages 98 to 115)

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Watching:

- Tom Wildenhain: "On The Turing Completeness of PowerPoint"