Appendix B: Step-by-Step Electromagnetic Induction

from Deidda et al., 2023

In [1]:
```
from IPython.display import Image, display
```

Step 1

Let us first consider two nearby coils in free space (or in free air), as in Fig. A1.
Suppose that coil `Tx` (Transmitter) is connected to an external alternating voltage source, while coil `Rx` (Receiver) is connected to a voltmeter to read voltages in it.

Let

$$I_P = I_0 e^{i\omega t} \tag{A24}$$

be the sinusoidal alternating current driven in the primary coil by the external voltage source.

According to `Ampère–Maxwell’s law`, this current produces a time-varying magnetic field intensity H_P,
or magnetic flux density $B_P = \mu_0 H_P$, around the loop, which alternates with the same frequency and phase as the current (Fig. A1).
Both magnitude and direction of this field vary with position in a complex way around the coil, but its magnitude is always proportional to the current flowing in the coil: $|B_P| \propto I_P$

In [2]:
```
display(Image("../figures/0A1.png", width=800))
```

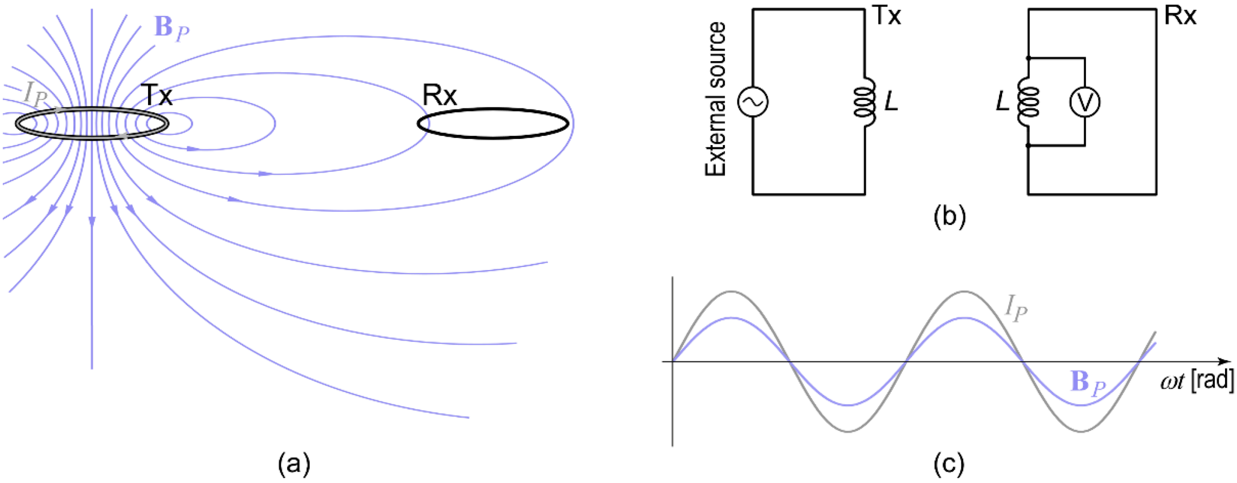

Figure A1. (a) Sketch of two magnetically coupled coils in a free space. (b) Equivalent single-loop circuits for the transmitter (on the left) and the receiver (on the right) coils. (c) Primary current and primary magnetic field as a function of time.

The time-varying magnetic field generates a changing magnetic flux through coil `Rx`, $\Phi_R(B_P)$. Therefore, the magnetic field interacts with coil `Rx` to produce an electromotive force, according to `Faraday's law`:

$$\mathcal{E}_R = -\frac{\partial \Phi_R(B_P)}{\partial t} \tag{A25}$$

Since the magnetic field is proportional to the current I_P, and the magnetic flux -by definition- is proportional to the field, the magnetic flux through coil `Rx` is proportional to the current flowing in coil `Tx`, that is:

$$\Phi_R(B_P) = M_{TR} I_P \tag{A26}$$

where M_{TR} is the `mutual inductance` between `Tx` and `Rx` coils (defined as the magnetic flux that passes through coil `Rx` due to a unit electric current circulating in coil `Tx`). Combining (A25) and (A26), the voltage sensed by coil `Rx` is

$$\mathcal{E}_{TR} = -\frac{\partial \Phi_R(B_P)}{\partial t} = -M_{TR}\frac{\partial I_P}{\partial t} = -i\omega M_{TR} I_P \tag{A27}$$

This voltage is usually employed to measure the primary magnetic field at the receiver.

Step 2

Now, let us consider again the two coils `Tx` and `Rx` in free air, but above a `half-space` containing a conductive magnetic body with electrical conductivity σ and magnetic permeability μ (Fig. A2).

In [3]:
```
display(Image("../figures/0A2.png", width=800))
```

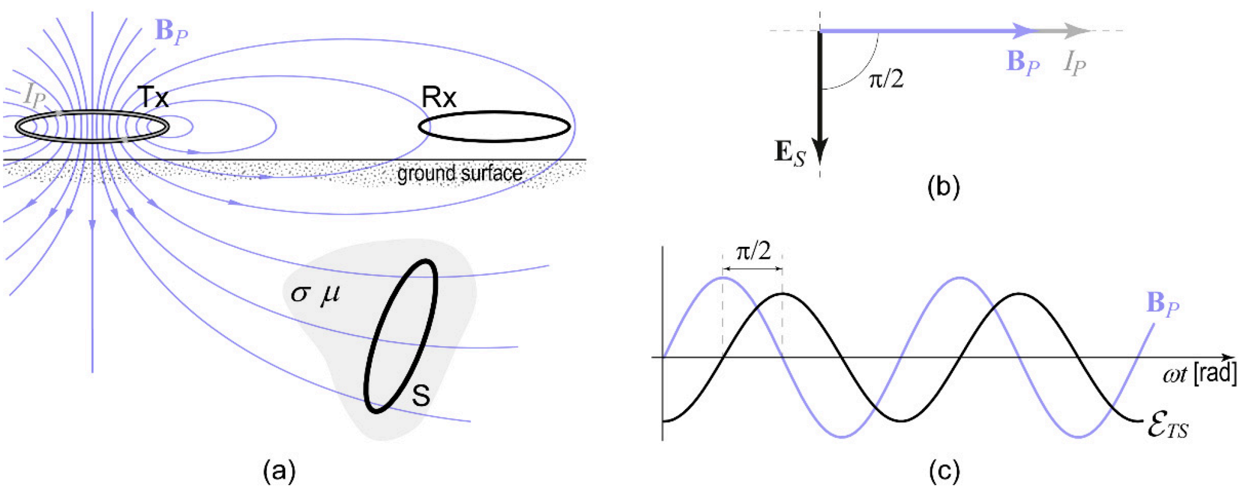

Figure A2. (a) Sketch of two magnetically coupled coils over a half-space with a conductive body. (b) Phasor diagram using as a reference the primary current. The secondary electric field lags the primary magnetic field by 90°. (c) Time dependence of $\mathcal{E}_{TS}(t)$ t and $B_p(t)$ that shows the same lagging phase as in the phasor diagram.

For a bulk material (the conductive magnetic body) there is not a loop per se, but many short-circuited loops. However, Faraday's law is general and it does not require the existence of a physical loop. Faraday's law states that when the magnetic flux through a surface changes, a time-varying electric field is induced along the boundary of that surface. This is true for any closed loop, either in empty space or in a physical material, through which the magnetic flux is changing over time. Thus, assuming S as one of these loops inside the body (Fig. A2a), the standard integral form of Faraday's law reads

$$\oint_S \mathbf{E}_S \cdot d\mathbf{l} = -\frac{\partial \Phi(B_P)}{\partial t} \tag{A28}$$

where $\mathbf{E}_S$ is the electric field at every point of such a loop and dl is an oriented displacement along the loop. The induced electromotive force $\mathcal{E}_{TS}$ is related to $\mathbf{E}_S$ by

$$\mathcal{E}_{TS} = \oint_S \mathbf{E}_S \cdot d\mathbf{l} \tag{A29}$$

Therefore, as in the case of coil `Rx` (Equation A27), by introducing the mutual induction M_{TS} the electromotive force (*emf*) induced in the loop S can be expressed in terms of the primary current I_P by

$$\mathcal{E}_{TS} = -\frac{\partial \Phi(B_P)}{\partial t} = -M_{TS}\frac{\partial I_P}{\partial t} = -i\omega M_{TS} I_P \tag{A30}$$

This *emf* alternates with the same frequency as the primary current but I_P lagging by 90° behind the current (or the primary magnetic field); see Figure A2c. The mutual inductance M_{TS} depends on the geometry of coils `Tx` and S, on their relative orientation and distance, and on the magnetic permeability μ of the core material in loop S.

Step 3

The alternating voltage induced in the conductive body by the time-varying primary magnetic field causes alternating currents to flow in the bulk material as they do through wires. These are the `eddy currents` that flow along closed loops concentrated near the boundary surface of the body (skin effect) and in planes perpendicular to the magnetic field causing them. Let S be one of these closed loops (Fig. A3a). Figure A3b shows its equivalent single-loop circuit with lumped resistance R and inductance L. Let $\mathcal{E}_{TS}(t)$ be the alternating voltage source that establishes the alternating current, I_{eddy}.

In [4]:
```
display(Image("../figures/0A3.png", width=800))
```

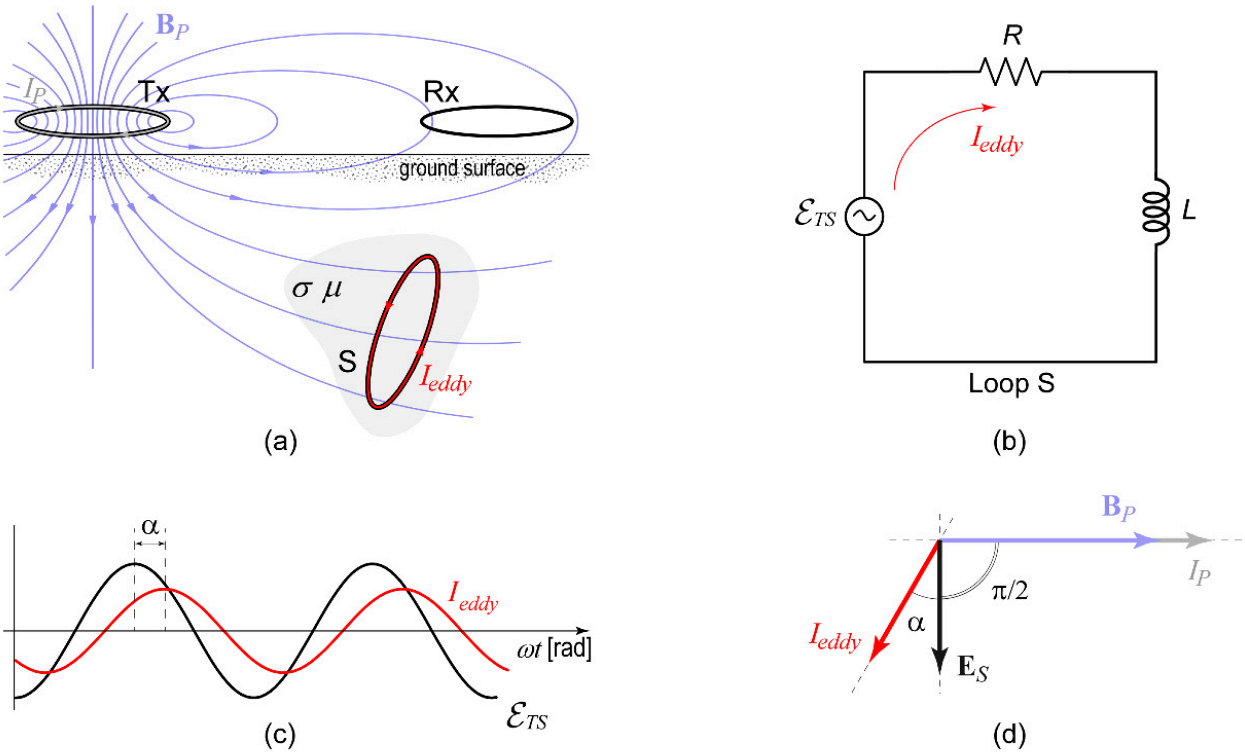

Figure A3. (a) Sketch showing one eddy current loop. (b) Equivalent circuit with lumped resistance R and inductance L, connected across an alternating voltage source. (c) Time dependence of $\mathcal{E}_{TS}(t)$ and I_{eddy} across loop S; the current lags the voltage. (d) Phasor diagram using as a reference the primary current. The secondary electric field lags by 90° the primary magnetic field, while eddy currents show an additional phase lag.

According to `Kirchhoff's voltage rule`, the circuit equation read

$$\mathcal{E}_{TS} - RI_{\text{eddy}} - L\frac{dI_{\text{eddy}}}{dt} = 0$$

which, for the present time-harmonic case, yields

$$\mathcal{E}_{TS} = (R + i\omega L)I_{\text{eddy}} \tag{A32}$$

The complex quantity in the brackets is the impedance of the RL circuit, whose amplitude is given by

$$|Z| = \sqrt{R^2 + \omega^2 L^2} \tag{A33}$$

which, for the present time-harmonic case, yields the phase

$$\alpha = \arctan\left(\frac{\omega L}{R}\right) \tag{A34}$$

Therefore, letting $E_{TS}(t) = E_0 e^{i(\omega t - \pi/2)}$, the sinusoidal alternating current circulating in the circuit is

$$I_{\text{eddy}}(t) = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} e^{i(\omega t - \pi/2 - \alpha)} \tag{A35}$$

which lags the voltage by α radians (Fig. A3c) and the primary magnetic field (or primary current) by α+π/2 radians (Fig. A3d). The phase shift α depends only on the response parameter β=$\omega L/R$, also known as the dimensionless `induction number`.

- When $\beta \to 0$ or equivalently $R \to \infty$ (for a given value of ωL), the circuit becomes purely resistive as the amplitude and phase of the impedance becomes $|Z|$=R and α=0, respectively. In this case, the current circulating in the circuit is in-phase with the induced voltage $\mathcal{E}_{TS}$ and is given by

$$I_{\text{eddy}}(t) = \frac{\mathcal{E}_0}{R} e^{i(\omega t - \pi/2)} \tag{A36}$$

- When $\beta \to \infty$ or equivalently $R \to 0$ (for a given value of ωL), the circuit becomes purely inductive as the amplitude impedance takes the value $|Z|$=ωL and the phase approaches $\pi/2$ radians:

$$\alpha = \lim_{R \to 0} [\arctan(\omega L/R)] = \frac{\pi}{2} \tag{A37}$$

In this case, thus, the current circulating in the circuit is in quadrature with the induced voltage $\mathcal{E}_{TS}(t)$, lags the primary current by π radians, and is given by:

$$I_{\text{eddy}}(t) = \frac{\mathcal{E}_0}{\omega L} e^{i(\omega t - \pi)} \tag{A38}$$

Step 4

Eddy currents induced in the body generate a time-varying magnetic field around the body (Fig. A4a), according to Ampère-Maxwell's law and (A32). This field, called `secondary magnetic field`, generates in turn a secondary voltage in coil `Rx`, according to Faraday's law:

$$E_{SR} = -i\omega M_{SR} I_{\text{eddy}} = -\frac{\omega^2 M_{TS} M_{SR}}{R + i\omega L} I_P \tag{A39}$$

where M_{SR} denotes the mutual inductance between the coils S and `Rx`.

In [5]:
```
display(Image("../figures/0A4.png", width=800))
```

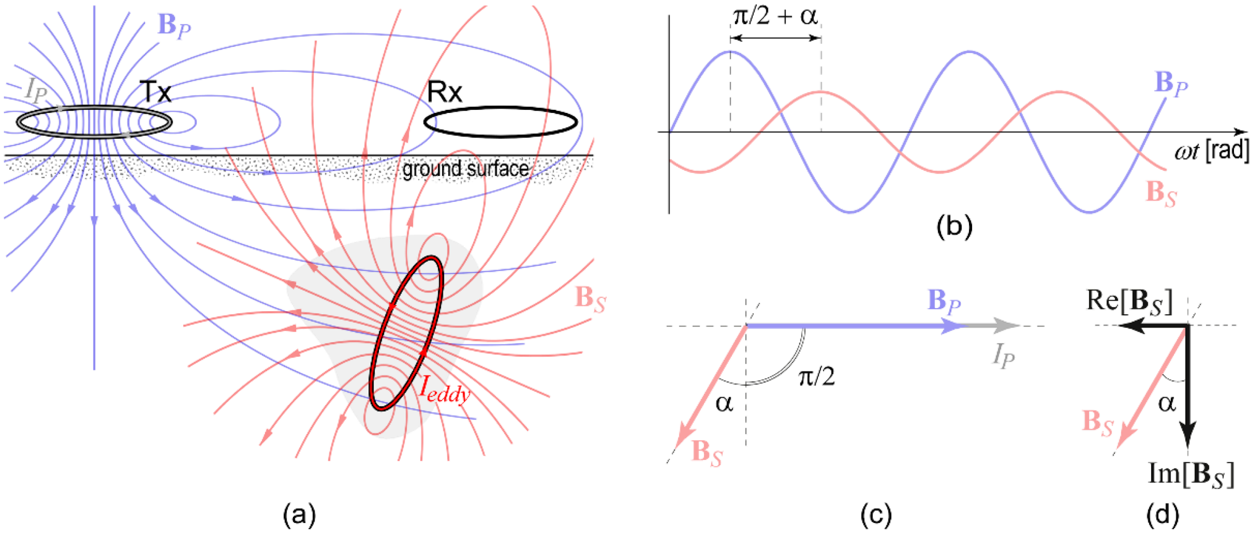

Figure A4. (a) Sketch showing the secondary magnetic field, H_S, generated by the eddy current loop. (b) Time dependence of H_P and H_S; the secondary magnetic field lags the primary field. (c) Phasor diagram using as a reference the primary current: the secondary magnetic field lags the primary magnetic field by $\alpha+\pi/2$ radians. (d) Decomposition of the secondary magnetic field in its real and imaginary parts, which are the in-phase and in quadrature components with respect to the primary field, respectively.

The receiver, then, simultaneously senses both `primary` and `secondary` magnetic fields, measuring both primary and secondary electromotive forces. In particular, the receiver records the whole electromagnetic response of the buried loop as the ratio of the secondary to the primary magnetic fields, which is equal to the ratio of the secondary to the primary voltages:

$$\frac{\mathcal{E}_S}{\mathcal{E}_P} = -\frac{M_{TS}M_{SR}}{M_{PR}L} \cdot \frac{i\beta}{1+i\beta} = \kappa \cdot \frac{i\beta}{1+i\beta} = \kappa \left(\frac{\beta^2 + i\beta}{1+\beta^2} \right). \tag{A40}$$

The first factor

$$\kappa = -\frac{M_{TS}M_{SR}}{M_{PR}L} \tag{A41}$$

is the `coupling coefficient`. It depends only on relative size, shape, position, and orientation of the coils. The other factor, called the `response function`, is a complex-valued function of β, which depends on the frequency ω and on the target's electromagnetic properties:

$$G(\beta) = \frac{i\beta}{1+i\beta} = \frac{\beta^2}{1+\beta^2} + i\frac{\beta}{1+\beta^2} \tag{A42}$$

Thus, the `electromagnetic response` of the measuring device to the buried body is given by

$$M = \frac{\mathcal{E}_S}{\mathcal{E}_P} = \kappa G(\beta) \tag{A43}$$

with real and imaginary parts:

$$\mathrm{Re}M = \kappa \frac{\beta^2}{1+\beta^2}, \quad \mathrm{Im}M = \kappa \frac{\beta}{1+\beta^2} \tag{A44–A45}$$

The real part of the response, having the same phase as the primary magnetic field, is usually designated the `In-phase component` $\rightarrow \mathrm{Re}(M)$, while the imaginary part, or `Quadrature component` $\rightarrow \mathrm{Im}(M)$, is out-of-phase with the primary by 90° (Fig. A4c). The response function becomes purely real when $\beta \rightarrow \infty$ (inductive limit), and when the instrument works at high frequency, or the target is highly conductive (low R) or highly inductive. Otherwise, the response function is purely imaginary when $\beta \rightarrow 0$ (resistive limit), which means using a low frequency, or being in the presence of a poorly conductive target (high R). Figure A5 shows the graph of the real and imaginary parts of the response function $G(\beta)$.

In [7]:
```
display(Image("../figures/0A5.png", width=400))
```

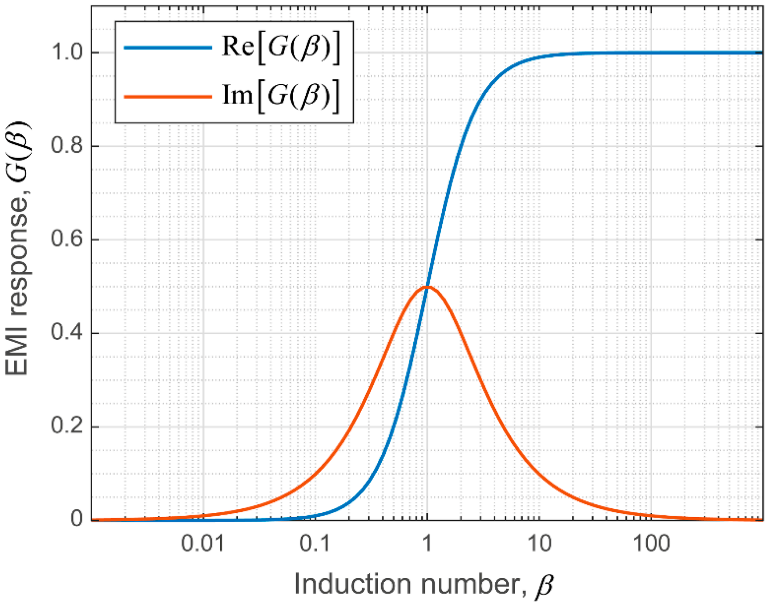

Figure A5. Real and Imaginary parts of the electromagnetic response function.

References

- Deidda, G.P.; Díaz de Alba, P.; Pes, F.; Rodriguez, G. *Forward Electromagnetic Induction Modelling in a Multilayered Half-Space: An Open-Source Software Tool.* Remote Sens. 2023, 15, 1772. https://doi.org/10.3390/rs15071772