Appendix A: Brief Review of the Maxwell Equations

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Electromagnetic induction phenomena obey `Maxwell's equations`, which describe how electric and magnetic fields are generated by charges, currents, and changes of the fields. The differential form in the time domain of these equations is given by:

$$\nabla \cdot \mathbf{D} = q \quad \text{(Gauss' law)} \tag{A1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss' law for magnetic fields)} \tag{A2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)} \tag{A3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(Ampère–Maxwell's law)} \tag{A4}$$

where:

- **D** is the dielectric displacement (C/m^2)
- **B** is the magnetic flux density or magnetic induction (T)
- **E** is the electric field intensity (V/m)
- **H** is the magnetic field intensity (A/m)
- **J** is the electric current density (A/m^2)
- **q** is the electric charge density (C/m^3)

The symbols ∇· and ∇× denote the `divergence` and `curl` operators, respectively.

These equations are coupled through the following `constitutive relations`:

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{A5}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{A6}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{A7}$$

where:

- ε is the dielectric permittivity (F/m)
- μ is the magnetic permeability (H/m)
- σ is the electric conductivity (S/m)

In free space ($\sigma = 0$), the dielectric permittivity and the magnetic permeability take the values:

$$\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}, \qquad \mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}.$$

For any medium other than a vacuum, the ratio of the permeabilities of a medium to that of free space defines the dimensionless relative parameters:

$$\mu_r = \frac{\mu}{\mu_0}, \qquad \varepsilon_r = \frac{\varepsilon}{\varepsilon_0}.$$

Magnetic Susceptibility and Magnetization

For a magnetic material, Equation (A6) can be expressed using the `magnetic susceptibility` χ, which measures how much a material is susceptible to being magnetized:

$$\chi = \mu_r - 1 \tag{A8}$$

Thus, the magnetic permeability becomes:

$$\mu = \mu_0 + \mu_0\chi \tag{A9}$$

Hence, the magnetic induction field **B** can be expressed as:

$$\mathbf{B} = \mu\mathbf{H} = \mu_0(1+\chi)\mathbf{H} = \mu_0\mathbf{H} + \mu_0\chi\mathbf{H} = \mu_0\mathbf{H} + \mu_0\mathbf{M} \tag{A10}$$

where:

$$\mathbf{M} = \chi\mathbf{H} \tag{A11}$$

is the `magnetization field` that the material acquires when a magnetic field **H** acts on it.

Electromagnetic Wave Equations

Maxwell's equations together with the constitutive relations can be combined to yield the `electromagnetic wave equations` for propagation (as wave and diffusion) of electric and magnetic fields in an isotropic homogeneous lossy medium having electric conductivity σ, magnetic permeability μ, and dielectric permittivity ε.

Taking the curl of Equation (A3) and using, in order, Equations (A6), (A4), (A7), and (A5), together with the identity $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2\mathbf{E}$, where the symbol ∇^2 stands for the `Laplacian` operator, gives the `electric field equation` in time domain:

$$\nabla^2\mathbf{E} - \mu\sigma\frac{\partial\mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{A12}$$

Likewise, taking the curl of Equation (A4) and following a similar process yields the `magnetic field equation` in time domain:

$$\nabla^2\mathbf{H} - \mu\sigma\frac{\partial\mathbf{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \tag{A13}$$

Harmonic Fields and Helmholtz Equations

For harmonically varying fields at angular frequency ω, that is

$$\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}, \qquad \mathbf{H} = \mathbf{H}_0 e^{-i\omega t},$$

Equations (A12) and (A13) become the `Helmholtz equations`:

$$\nabla^2 \mathbf{E} + i\omega\mu\sigma\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = \nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{A14}$$

$$\nabla^2 \mathbf{H} + i\omega\mu\sigma\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H} = \nabla^2\mathbf{H} + k^2\mathbf{H} = 0 \tag{A15}$$

where the `complex wavenumber` is:

$$k = \sqrt{\omega^2\mu\varepsilon + i\omega\mu\sigma} = a + ib \tag{A16}$$

whose real and imaginary parts given by (Ward et al., 1987):

$$a = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right)} \tag{A17}$$

$$b = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right)} \tag{A18}$$

The imaginary part b, also called the `attenuation coefficient`, plays a key role in electromagnetism since its inverse defines the `skin depth` δ.

Appendix A.1: Quasi-Stationary Approximation

Alternating electromagnetic fields that vary slowly with time are referred to as `low-frequency alternating fields` or `quasi-stationary fields`. In the quasi-stationary case, Maxwell's equations can be simplified by neglecting the term $\frac{\partial \mathbf{D}}{\partial t}$ in Ampère–Maxwell's law (A4), but retaining the term $\frac{\partial \mathbf{B}}{\partial t}$ in Faraday's law (A3).
This means that the `displacement current is negligible` with respect to the `conduction current`, which remains the only source of the quasi-stationary magnetic field. This also means that the electromagnetic properties of the medium are such that $\sigma \gg \omega\varepsilon$.
Then, the equations (A14) and (A15) simplify to:

$$\nabla^2\mathbf{E} + i\omega\mu\sigma\mathbf{E} \approx 0 \tag{A19}$$

$$\nabla^2\mathbf{H} + i\omega\mu\sigma\mathbf{H} \approx 0 \tag{A20}$$

These are known as the `diffusion equations of electromagnetic fields`, describing the penetration of electromagnetic fields (do not consider wave propagation!) in an isotropic homogeneous lossy medium.

The complex wavenumber becomes:

$$k = \sqrt{i\omega\mu\sigma} = a + ib \tag{A21}$$

with:

$$a = b = \sqrt{|k^2|} = \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1}{\delta'} \tag{A22}$$

where the `skin depth` is defined as:

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{A23}$$

References

- Ward, Stanley H., and Gerald W. Hohmann. 1987. *Electromagnetic theory for geophysical applications.* Society of Exploration Geophysicists.
- *The Feynman lectures on physics.* California Institute of Technology (Caltech) – HTML edition. https://www.feynmanlectures.caltech.edu/II_01.html
- Deidda, G.P.; Díaz de Alba, P.; Pes, F.; Rodriguez, G. *Forward Electromagnetic Induction Modelling in a Multilayered Half-Space: An Open-Source Software Tool.* Remote Sens. 2023, 15, 1772. https://doi.org/10.3390/rs15071772

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