Spectral Analysis : Estimation.

$$X_t = \mu + \sum_{k=1}^{q} U_{k1} \cos(2\pi \omega_k t) + U_{k2} \sin(2\pi \omega_k t)$$

data $x_1, \cdots, x_n$. $n = 2k+1$, $k = \frac{(n-1)}{2}$.

frequency.

$$X_t = A_0 + \sum_{j=1}^{k} A_j \cos\left(2\pi \frac{j}{n} t\right) + B_j \sin\left(2\pi \frac{j}{n} t\right)$$

$t = 1, \cdots, n$.

fundamental Freq.

frequencies $\frac{1}{n}, \frac{2}{n}, \cdots\cdots, \frac{k}{n}$. Fourier Freq.

OLS method $\Rightarrow$

$$\left.\begin{aligned} \hat{A}_0 &= \overline{X} \\ \hat{A}_j &= \frac{2}{n} \sum_{t=1}^{n} X_t \cos\left(2\pi \frac{j}{n} t\right), \\ \hat{B}_j &= \frac{2}{n} \sum_{t=1}^{n} X_t \sin\left(2\pi \frac{j}{n} t\right). \end{aligned}\right] \circledast$$

when n is even $(n = 2k)$: $\circledast$ still holds for $j = 1, \cdots, k-1$,

$\hat{A}_k = \frac{1}{n} \sum_{t=1}^{n} (-1)^t X_t$, $\hat{B}_k = 0$ $\Rightarrow$ Freq: $\frac{k}{n} = \frac{1}{2}$.

For any series of any length n.

deterministic / stochastic

with/without any true periodicities,

→ perfectly fit the model.

Periodogram.

$X_1, \dots X_n$.

$$X_t = A_0 + \sum_{j=1}^{k} A_j \cos\left(2\pi \frac{j}{n} t\right) + B_j \sin\left(2\pi \frac{j}{n} t\right)$$

$k = \frac{n-1}{2}$ If n: odd

$k = \frac{n}{2}$ If n: even.

For each frequency, $\frac{j}{n}$,

$$I\left(\frac{j}{n}\right) = \frac{n}{2}\left(\hat{A}_j^2 + \hat{B}_j^2\right). \quad : (n: \text{odd})$$

n even $\Rightarrow$ $n = 2k$, still holds for $j = 1, \dots, k-1$,

$$I\left(\frac{1}{2}\right) = n\left(\hat{A}_k\right)^2.$$

periodogram

height of periodogram shows the relative strength of cosine-sine parts at various freq.

$$\sum_{j=1}^{n} (X_j - \bar{X})^2 = \sum_{j=1}^{k} I\left(\frac{j}{n}\right). \quad (n \text{ odd}$$

n: even - similar).

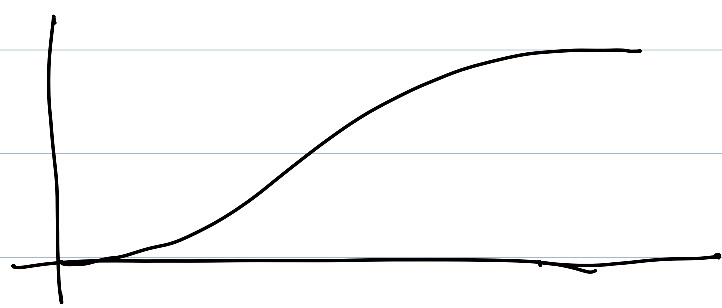
Spectral Density

all frequencies $-\frac{1}{2} < \omega \le \frac{1}{2}$,

$$f(\omega) = f(-\omega)$$

WN: $f(\omega) = \sigma^2$.

MA(1): $f(\omega) = \left(1 + \theta^2 - 2\theta \cos(2\pi\omega)\right) \sigma^2$.



$\theta > 0$

Simplest Case : WN

X_t : zero-mean WN variance σ^2.

$X_1, \ldots, X_n$.

$$X_t = A_0 + \sum_{j=1}^{k} A_j \cos(2\pi \tfrac{j}{n} t) + B_j \sin(2\pi \tfrac{j}{n} t)$$

$$\hat{A}_j = \frac{2}{n} \sum_{t=1}^{n} X_t \cos(2\pi \tfrac{j}{n} t), \quad \hat{B}_j = \frac{2}{n} \sum_{t=1}^{n} X_t \sin(2\pi \tfrac{j}{n} t).$$

$\hat{A}_j, \hat{B}_j$: normal distribution.

each have mean 0, variance $2\sigma^2/n$.

uncorrelated, independent.

$f_1 \neq f_2 \Rightarrow (\hat{A}_{f_1}, \hat{A}_{f_2}), (\hat{B}_{f_1}, \hat{B}_{f_2})$ jointly independent.

Periodogram: $I(\frac{j}{n}) = \frac{n}{2}(\hat{A}_j^2 + \hat{B}_j^2)$.

for any freq. $0 \le \omega \le \frac{1}{2}$,

$$I(\omega) = \frac{n}{2}(\hat{A}_\omega^2 + \hat{B}_\omega^2),$$

$$\hat{A}_\omega = \frac{2}{n}\sum_{t=1}^{n} X_t \cos(2\pi t\omega), \quad \hat{B}_\omega = \frac{2}{n}\sum_{t=1}^{n} X_t \sin(2\pi t\omega)$$

Sample Spectral Density

$\hat{f}(\omega) = \frac{1}{2} I(\omega)$. for all frequencies in $(-\frac{1}{2}, \frac{1}{2})$.

$\hat{S}(\frac{1}{2}) = I(\frac{1}{2})$.

estimated covariance function at lag k.

$$\hat{f}(\omega) = \hat{\gamma}_0 + 2\sum_{k=1}^{n-1} \hat{\gamma}_k \cos(2\pi\omega k)$$

$$\hat{\gamma}_k = \int_{-1/2}^{1/2} \hat{f}(\omega) \cos(2\pi \omega k) \cdot d\omega$$

$$X_t := W_t + \psi_1 W_{t-1} + \psi_2 W_{t-2} + \ldots\ldots$$

$\omega_1 \neq \omega_2$ (in btw 0 to $\frac{1}{2}$).

When $n \to \infty$.

$$\frac{2\hat{f}(\omega_1)}{f(\omega_1)} \quad \text{and} \quad \frac{2\hat{f}(\omega_2)}{f(\omega_2)}.$$

Converge in distribution to indep. χ^2 R.V.s

with df. $= 2$: