Spectral analysis: Intro

Hyoeun Lee



Spectral view of Time series

Shift of thinking from time domain to frequency domain

$$f(t) = A\cos(2\pi\omega t), \quad \omega > 0$$

- ► A: amplitude
- ightharpoonup cycle length: $1/\omega$

$$g(t) = A\sin(2\pi\omega t), \quad \omega > 0$$

$$h(t) = A\cos(2\pi\omega t + \psi)$$

 ψ : phase

- ► Time Series: we play with amplitude and frequency.
- ▶ Collect (A, ω) for periodic series and superpose them.

(Orchestra music: superpose sounds of violin, flute, . . .)

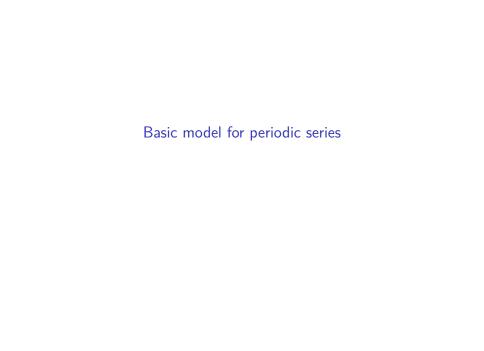


Basic concept of Periodic Time Series

$$X_t = A\cos(2\pi\omega t + \psi)$$

 $X_t = U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t)$

$$egin{aligned} X_t = &A\cos(2\pi\omega t + \psi) \ = &A\cos(\psi)\cos(2\pi\omega t) - A\sin(\psi)\sin(2\pi\omega t), \end{aligned}$$
 with $U_1 = A\cos(\psi)$, $U_2 = -A\sin(\psi)$,



Basic model for periodic series

$$X_t = U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t)$$

 U_1, U_2 : uncorrelated random variables, zero-mean, variance σ^2 .

- ▶ What is parameter here? ω, σ^2
- ► Is it stationary? Yes.

1.
$$E[X_t] = E[U_1] \cos(2\pi\omega t) + E[U_2] \sin(2\pi\omega t) = 0$$

2. Covariance:

$$Cov(X_t, X_s)$$
= $Cov(U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t), U_1 \cos(2\pi\omega s) + U_2 \sin(2\pi\omega s))$

$$=\sigma^2\cos(2\pi\omega(t-s))$$

$$\gamma(h) = \sigma^2 \cos(2\pi\omega h)$$

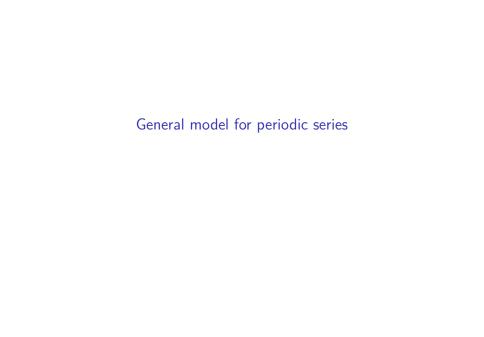
Seasonality is built in here.

Increase h does not necessarily mean correlation decrease in h.

$$ightharpoonup \gamma(0) = \sigma^2$$

•
$$\rho(h) = \cos(2\pi\omega h)$$

• Amplitude: $A = \sqrt{U_1^2 + U_2^2}$



General model for periodic series

$$X_t = \sum_{k=1}^q U_{1,k} \cos(2\pi\omega_k t) + U_{2,k} \sin(2\pi\omega_k t)$$

- estimate?
 - underlying parameters of $U_{1,k}$, $U_{2,k}$
 - how many q? $\omega_1, \ldots, \omega_q$?
- From data, we deduce a model.
- ► Fast Fourier Transform: we can figure out all parameters. (discuss later)

Assumptions for general model

For
$$k = 1, ..., q$$
,

- 1. $U_{1,k}$, $U_{2,k}$, are uncorrelated
- 2. $E[U_{1,k}] = E[U_{2,k}] = 0$
- 3. $Var(U_{1,k}) = Var(U_{2,k}) = \sigma_k^2$

Mean function:
$$E[X_t] = 0$$

Autocovaraince:

$$\gamma(t,s) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k(t-s))$$