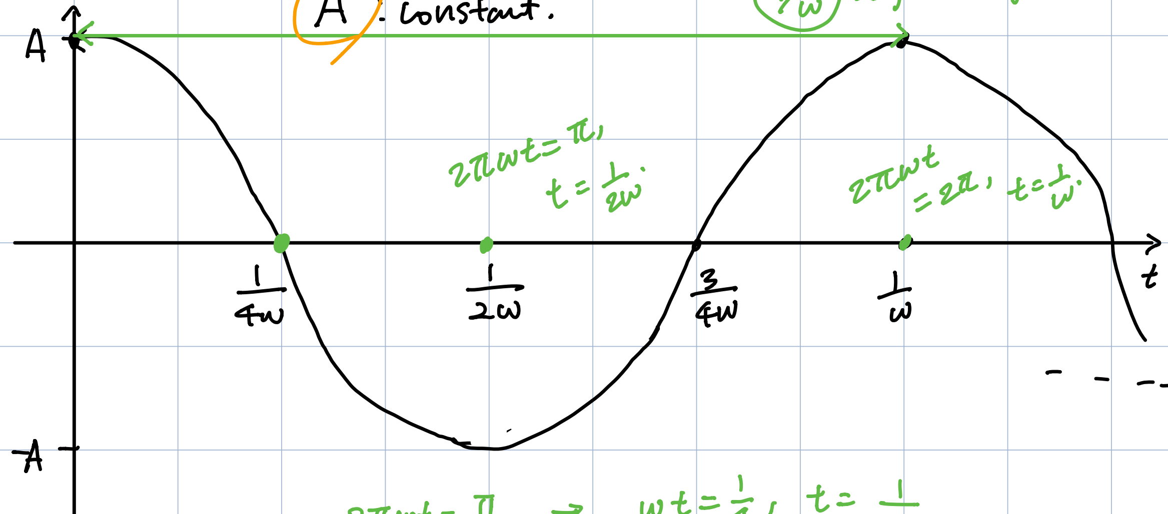
Spectral view of time series.

$f(t) = A \cos(2\pi \omega t)$, $\quad \omega > 0$.

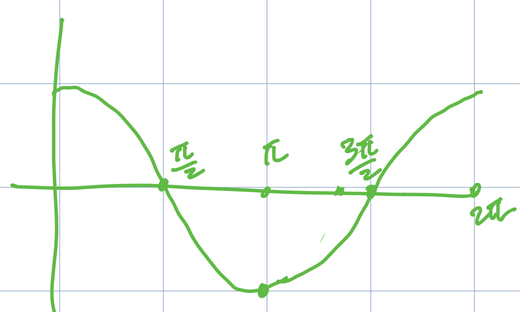
A : constant. $\qquad$ $1/\omega$: cycle length.



$2\pi\omega t = \frac{\pi}{2} \rightarrow \omega t = \frac{1}{4}, \; t = \frac{1}{4\omega}$

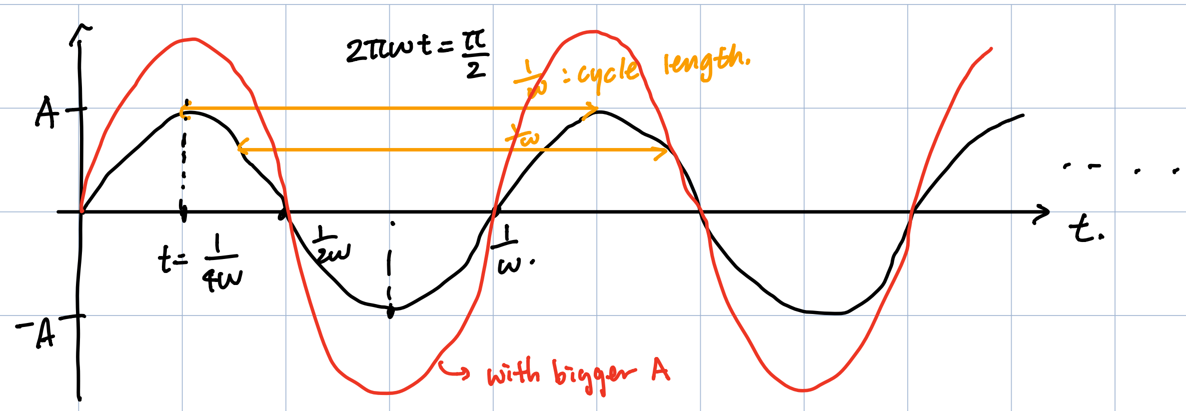
ω : frequency

cycle : one complete period of a sine or cosine function defined over a unit time interval.



$\omega \uparrow \Rightarrow$ cycle happens more quickly.

$g(t) = A \sin(2\pi\omega t)$.



A: amplitude. : height of the function.

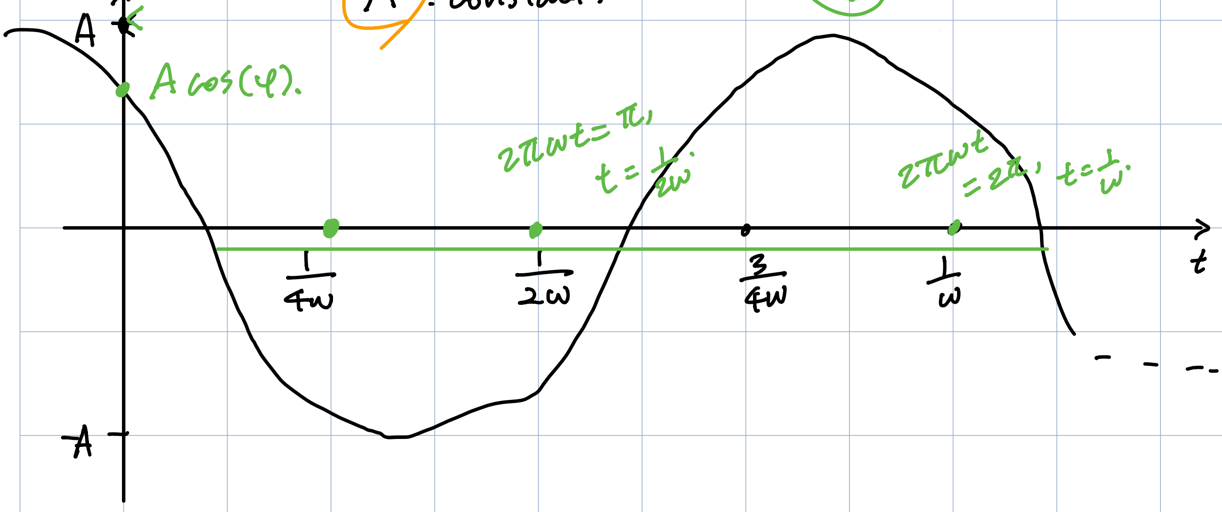
$$h(t) = A \cos(2\pi w t + \varphi)$$

φ: phase → start point.

→ $t = 0 \rightarrow h(0) = A\cos(\varphi)$.

A . constant.

$1/w$: cycle length.



cycle length, amplitude do not change.

(A, w). $\quad A$: How big oscillate.

w : How quickly oscillate

series 1 $\quad (A_1, w_1)$

2 $\quad (A_2, w_2)$

$\vdots$

add up $\rightarrow$ TS.

$$X_t = A\cos(2\pi w t + \varphi).$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$X_t = A\cos(2\pi w t)\cos(\varphi) - A\sin(2\pi w t)\sin(\varphi).$$

$$= \underbrace{A\cos(\varphi)}_{=U_1}\cos(2\pi w t) + \underbrace{(-A\sin(\varphi))}_{=U_2}\sin(2\pi w t).$$

with $U_1 = A\cos(\varphi)$, $U_2 = -A\sin(\varphi)$,

$$= U_1\cos(2\pi w t) + U_2\sin(2\pi w t).$$

$\boxed{\text{Basic Model}}$

amplitude : $A = \sqrt{U_1^2 + U_2^2} = \sqrt{A^2\cos^2(\varphi) + A^2\sin^2(\varphi)}$

$$= \sqrt{A^2(\cos^2(\varphi) + \sin^2(\varphi))} = \sqrt{A^2} = A.$$

$X_t = U_1 \cos(2\pi w t) + U_2 \sin(2\pi w t).$

assumption: U_1, U_2 uncorrelated Random Variables.

$[\ \mathrm{cov}(U_1, U_2) = 0.\ ].$

& U_1, U_2 : mean 0, variance σ^2,

Is (X_t) Stationary?

①. $E[X_t]$

$$= E[\ U_1 \cos(2\pi w t) + U_2 \sin(2\pi w t)\ ].$$

$$= \cos(2\pi w t)\, E[U_1] + \sin(2\pi w t)\, E[U_2].$$

$$= \cos(2\pi w t) \cdot 0 + \sin(2\pi w t) \cdot 0 = 0.$$

②. $\mathrm{Cov}(X_t, X_s)$?

$$= \mathrm{Cov}(U_1 \cos(2\pi w t) + U_2 \sin(2\pi w t),\ U_1 \cos(2\pi w s) + U_2 \sin(2\pi w s)).$$

$$= \mathrm{Cov}(U_1 \cos(2\pi w t),\ U_1 \cos(2\pi w s))$$

$$+ \mathrm{Cov}(U_1 \cos(2\pi w t),\ U_2 \sin(2\pi w s))$$

$$+ \mathrm{Cov}(U_2 \sin(2\pi w t),\ U_1 \cos(2\pi w s))$$

$$+ \mathrm{Cov}(U_2 \sin(2\pi w t),\ U_2 \sin(2\pi w s)).$$

$$
\begin{aligned}
= \; & \cos(2\pi\omega t)\cos(2\pi\omega s)\;\mathrm{Cov}(U_1,U_1) = \sigma^2 \\
+ \; & \cancel{\cos(2\pi\omega t)\sin(2\pi\omega s)\;\mathrm{Cov}(U_1,U_2)} \to 0 \\
+ \; & \cancel{\sin(2\pi\omega t)\cos(2\pi\omega s)\;\mathrm{Cov}(U_2,U_1)} \to 0 \\
+ \; & \sin(2\pi\omega t)\sin(2\pi\omega s)\;\mathrm{Cov}(U_2,U_2). = \sigma^2
\end{aligned}
$$

$$= \sigma^2 \left\{ \cos(2\pi\omega t)\cos(2\pi\omega s) + \sin(2\pi\omega t)\sin(2\pi\omega s) \right\}.$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b.$$

$$= \sigma^2 \cos(2\pi\omega t - 2\pi\omega s)$$

$$= \sigma^2 \cos(2\pi\omega(t-s)). \to \text{Stationary!}$$

$$\gamma(h) = \sigma^2 \cos(2\pi\omega h).$$

Basic Model

$$X_t = U_1 \cos 2\pi \omega t + U_2 \sin 2\pi \omega t.$$

Assumption : U_1, U_2 $\begin{cases} \text{uncorrelated} \\ \text{mean } 0 \\ \text{var} : \sigma^2. \end{cases}$

X_t : Stationary. $\gamma(h) = \sigma^2 \cos(2\pi\omega h)$.

$\gamma(0) = \sigma^2$

$\rho(h) = \cos(2\pi\omega h)$.

$-1 \le \rho(h) \le ?$ ω : controls frequency

↳ control correlation.

General Model

$$X_t = \sum_{k=1}^{q} \left[U_{k1} \cos(2\pi \omega_k t) + U_{k2} \sin(2\pi \omega_k t) \right].$$

mixture (superposition) of periodic series.

$\omega_1, \cdots \omega_q$.

Assumptions:

(1) U_{k1}, U_{k2} , $(k=1, \cdots, q)$ uncorrelated.

(2) $E[U_{k1}] = E[U_{k2}] = 0$, $k=1,2,\cdots q$.

(3) $Var(U_{k1}) = Var(U_{k2}) = \sigma_k^2$ $k=1,\cdots,q$.

$$E[X_t] = E\left[\sum_{k=1}^{q} \left[U_{k1} \cos(2\pi \omega_k t) + U_{k2} \sin(2\pi \omega_k t)\right]\right].$$

$$= \sum_{k=1}^{q} \underbrace{E[U_{k1}]}_{0} \cos(2\pi \omega_k t) + \underbrace{E[U_{k2}]}_{0} \sin(2\pi \omega_k t).$$

$$= 0.$$

$$\gamma(t,s) = Cov(X_t, X_s)$$

$$= Cov\Big(\sum_{k=1}^{q} \left[U_{k1} \cos(2\pi \omega_k t) + U_{k2} \sin(2\pi \omega_k t) \right],$$

$$\sum_{j=1}^{q} \left[U_{j1} \cos(2\pi \omega_j s) + U_{j2} \sin(2\pi \omega_j s) \right] \Big),$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} \Big\{ \underbrace{Cov(U_{k1}, U_{j1})}_{=\sigma^2 \text{ if } j=k,\ 0 \text{ otherwise.}} \cos(2\pi \omega_k t) \cos(2\pi \omega_j s)$$

$$+ \cancel{Cov(U_{k1}, U_{j2})}^{\,0} \cos(2\pi \omega_k t) \sin(2\pi \omega_j s)$$

$$+ \cancel{Cov(U_{k2}, U_{j1})}^{\,0} \sin(2\pi \omega_k t) \cos(2\pi \omega_j s)$$

$$+ \underbrace{Cov(U_{k2}, U_{j2})}_{=\sigma^2 \text{ if } j=k,\ 0 \text{ otherwise.}} \sin(2\pi \omega_k t) \sin(2\pi \omega_j s) \Big\}.$$

$$= \sum_{k=1}^{q} \sigma_k^2 \left\{ \cos(2\pi \omega_k t) \cos(2\pi \omega_k s) + \sin(2\pi \omega_k t) \sin(2\pi \omega_k s) \right\}.$$

$$= \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi \omega_k (t-s)). \rightarrow \text{Stationary!}$$

$$\gamma(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi \omega_k h).$$

$$\gamma(0) = \sum_{k=1}^{q} \sigma_k^2.$$ ← variance of superposed series = addition of each series.