Week 7 Lecture Note

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Module 2 - Week 7

ARMA . AR, MA.

Stationary model

non-stationary models.

Stationarity through differencing

Stationarity through differencing

$$X_t - X_{t-1} = \nabla X_t$$
$$= \mu_t + Y_t - (\mu_{t-1} + Y_{t-1})$$
$$= \mu_t - \mu_{t-1} + Y_t - Y_{t-1}$$
$$= \nabla \mu_t + \nabla Y_t$$

$$X_t = \mu_t + Y_t$$

- μ_t: non-stationary trend
- Y_t: zero-mean stationary component

Y_t : stationary $\Rightarrow$ ∇Y_t is also stationary.

Examples: μ_t: non-stationary trend,

$$1.\ \nabla\mu_t = \cancel{\beta_0} + \beta_1 t - (\cancel{\beta_0} + \beta_1(t-1))$$
$$= \beta_1 \text{ : constant } \rightarrow \text{ stationary.}$$

1. $\mu_t = \beta_0 + \beta_1 t$ ✓
2. $\mu_t = \mu_{t-1} + v_t$, v_t is stationary and uncorrelated with $\cancel{Y_t}$.
3. $\boxed{\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2}$

$$2.\ \nabla\mu_t = \mu_t - \mu_{t-1} = v_t. \checkmark$$

$$3.\ \nabla\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2 - \{\beta_0 + \beta_1(t-1) + \beta_2(t-1)^2\}.$$
$$= \beta_1 + \beta_2(2t-1). \ \times \text{ stationary.}$$

$$\nabla^2\mu_t = \nabla(\nabla\mu_t)$$

$$= \nabla\left(\beta_1 + 2\beta_2 \cdot t - \beta_2\right)$$

$$= 2\beta_2 \longrightarrow \text{constant.}$$

Differencing original model ⟶ ARMA

Integrated ARMA, or ARIMA:

AR

Integrated.

MA.

Integrated ARMA, or ARIMA:

$d=1: (1-B)X_t = X_t - X_{t-1}$

$d=2: (1-B)^2 X_t = (1-2B+B^2)X_t$

$= X_t - 2X_{t-1} + X_{t-2}$

$\vdots$

A process X_t is said to be ARIMA(p,d,q) if

$$\nabla^d X_t = (1-B)^d X_t$$

is ARMA(p,q). In general, we will write the model as

$(1-\phi_1 B - \phi_2 B^2 \cdots - \phi_p B^p)$

$$\boxed{\phi(B)}(1-B)^d X_t = \boxed{\alpha} + \theta(B) W_t$$

$(1+\theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q)$

where $\alpha = \delta(1-\phi_1 - \cdots - \phi_p)$ and $\delta = \mathbb{E}[\nabla^d X_t]$.

$\nabla^d X_t \Rightarrow E[\nabla^d X_t] = \delta.$ $\quad \boxed{\nabla^d X_t - \delta} \Rightarrow$ 0-mean ARMA(p,q).

$(\nabla^d X_t - \delta) = \phi_1(\nabla^d X_{t-1} - \delta) + \phi_2(\nabla^d X_{t-2} - \delta) + \cdots + \phi_p(\nabla^d X_{t-p} - \delta)$

Example:

$+ \theta(B)W_t.$

$$\nabla^d X_t = \delta(1 - \phi_1 - \phi_2 \cdots - \phi_p) + \phi_1 \nabla^d X_{t-1} + \cdots + \phi_p \nabla^d X_{t-p} + \theta(B) W_t.$$

$$\phi(B)\nabla^d X_t = \underbrace{\delta(1 - \phi_1 - \cdots + \phi_p)}_{= \alpha.} + \theta(B) W_t.$$

$$X_t = 3 + X_{t-1} + W_t - 0.75 W_{t-1}$$

- Find p, d, q for ARIMA(p,d,q) $p=0, d=1, q=1.$ IMA(1,1)
- Find $E[\nabla X_t]$ and $Var(\nabla X_t)$ $E[\nabla X_t] = 3.$ $Var(\nabla X_t) = \sigma_w^2(1+0.75^2)$ $= \sigma_w^2 \cdot 1.5625.$

$$\nabla X_t = X_t - X_{t-1} = 3 + W_t - 0.75 W_{t-1}$$

$$\underbrace{(1-B)}_{d=1} X_t = 3 + \underbrace{(1 - 0.75B)}_{q=1.} W_t.$$

$p=0$

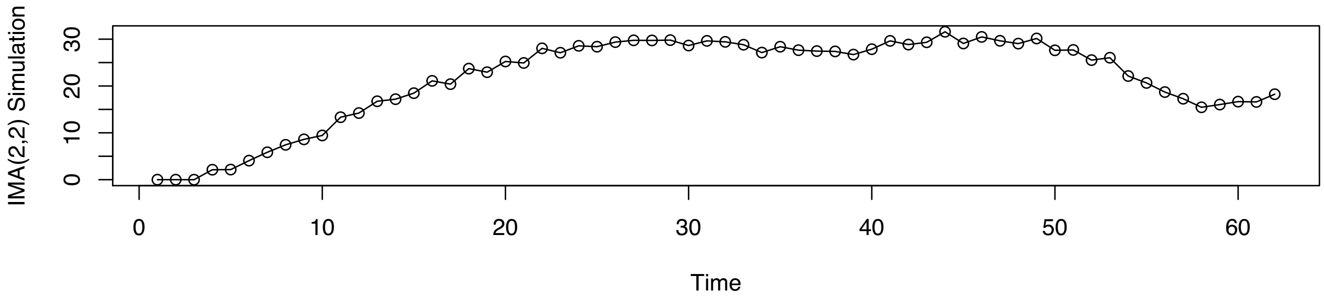
$1 - 0.75z = 0,$ $|z = \frac{1}{0.75}| > 1$

Example: simulated IMA(2,2)

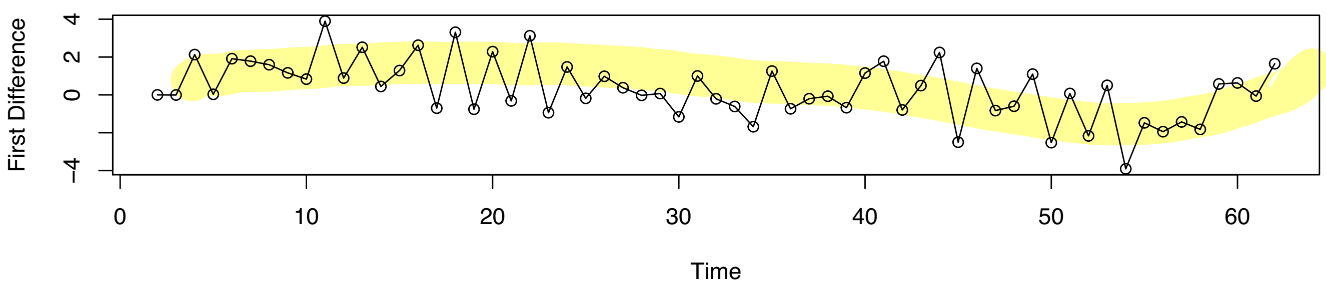
ARIMA(p=0, d=2, q=2).

library(TSA).

```
data(ima22.s)
plot(ima22.s,ylab='IMA(2,2) Simulation',type='o')
```



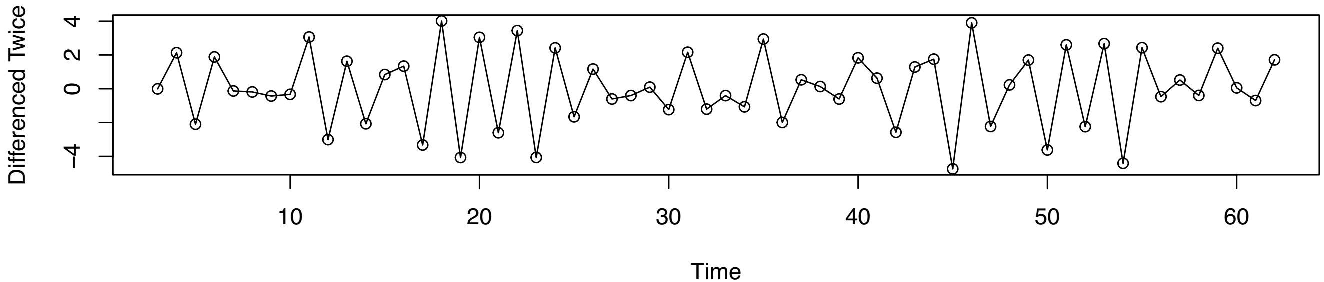
```
plot(diff(ima22.s),ylab='First Difference',type='o')
```



diff(diff(   ))   or   diff(___ , difference=2)

∇^2

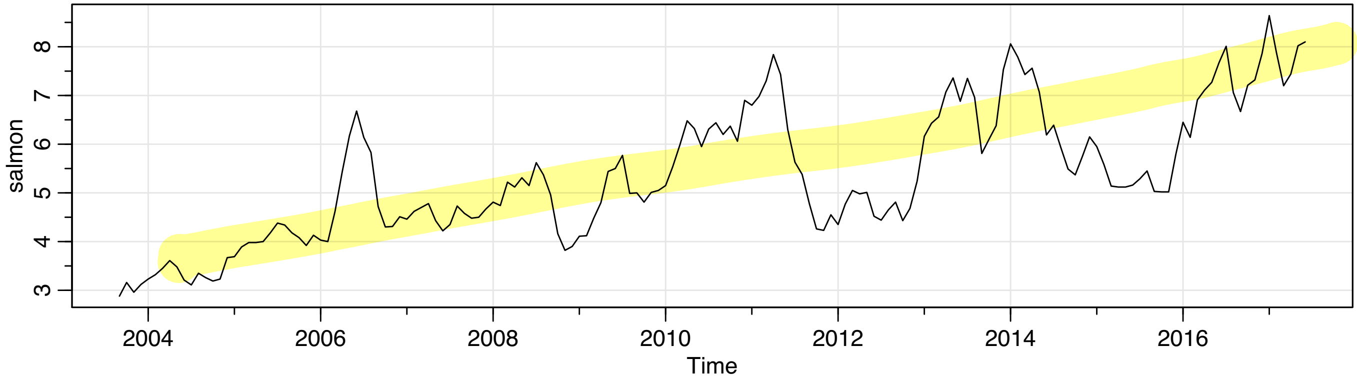
```
plot(diff(ima22.s,difference=2),ylab='Differenced Twice',ty
```



Example: salmon

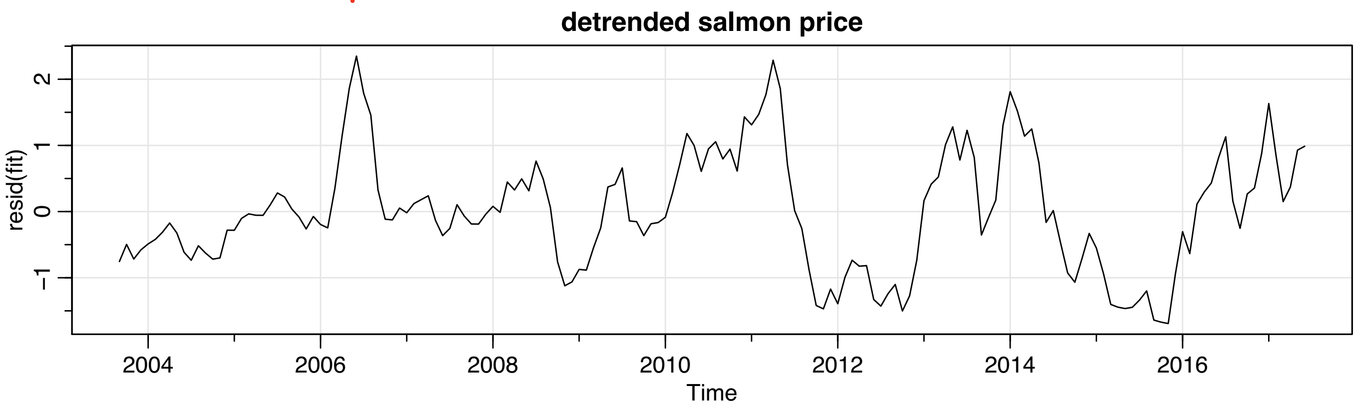
(Detrending
Differencing).

```
tsplot(salmon)
```

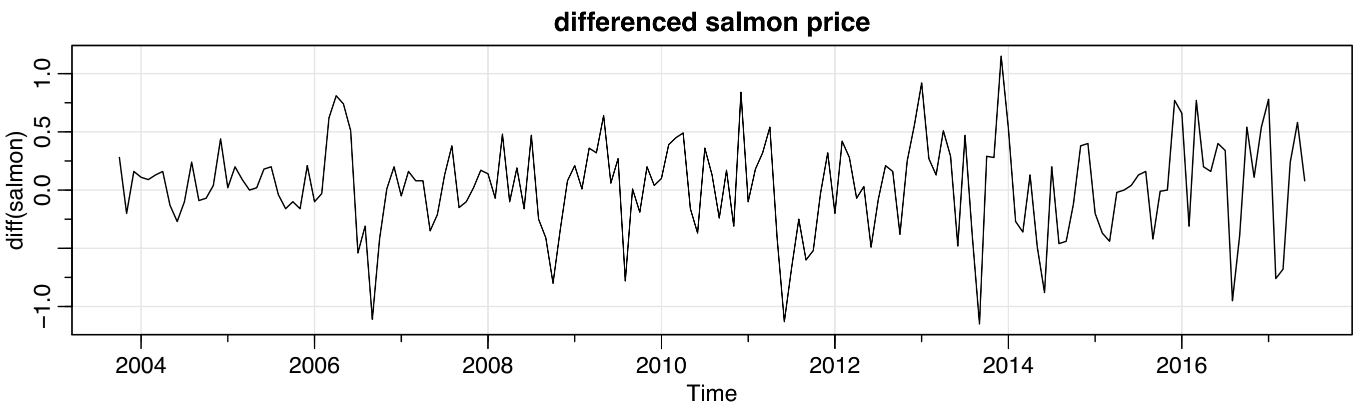



```
fit = lm(salmon~time(salmon), na.action=NULL)
# the regression
```

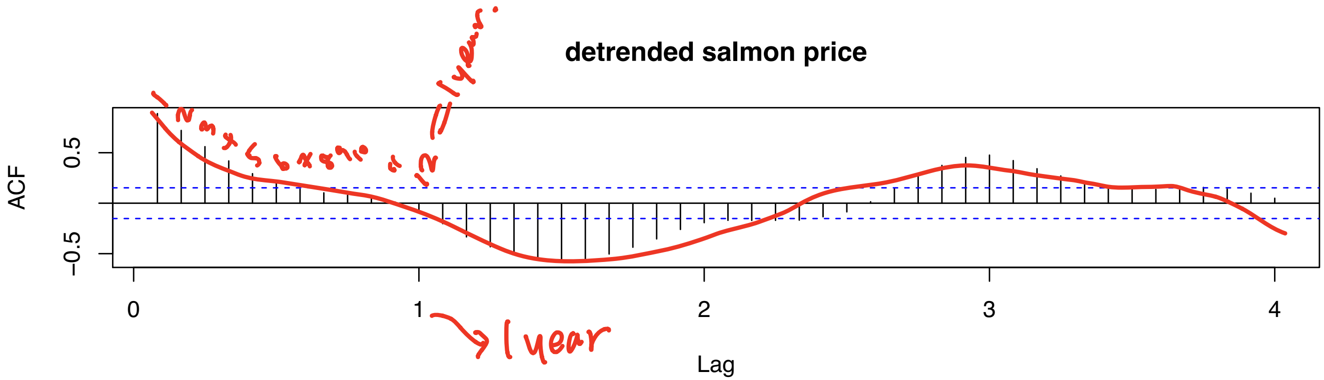
```
tsplot(resid(fit), main="detrended salmon price")
```



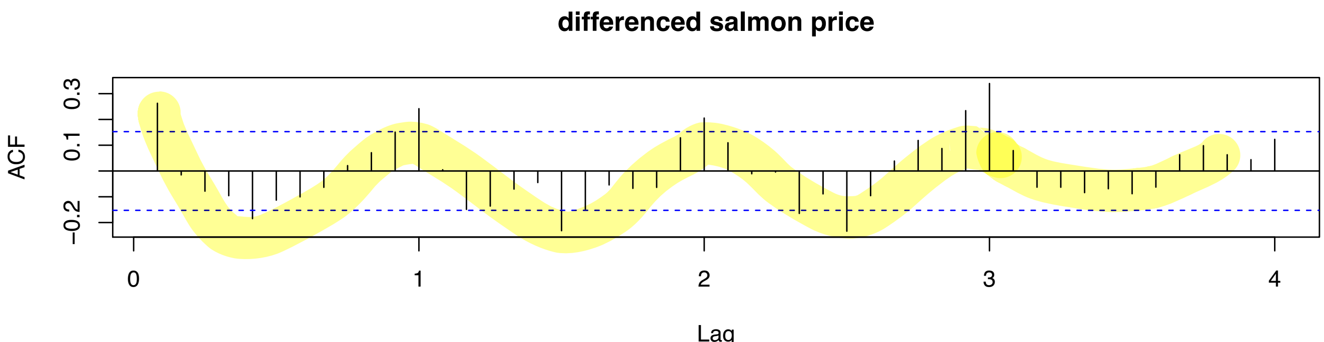

```
tsplot(diff(salmon), main="differenced salmon price")
```



```r
acf(resid(fit), 48, main="detrended salmon price")
```




```r
acf(diff(salmon), 48, main="differenced salmon price")
```



Seasonal pattern.

Building non-stationary models

There are a few basic steps to fitting ARIMA (or non-stationary) models to time series data. These steps involve

- plotting the data,
- possibly transforming the data, → coercing stationarity. d
- identifying the dependence orders of the model, — p, q. → ACF, PACF.
- parameter estimation,
- diagnostics, and
- model choice.

detrending
differencing,
log-transform.

log transformation

$$\log(1+r) = r \underbrace{- \frac{r^2}{2} + \frac{r^3}{3} - \cdots\cdots}$$

$$\log(1+r) \approx r.$$

log transformation

$X_t = \mu_t + X_t - \mu_t.$

$$\frac{X_t}{\mu_t} = 1 + \frac{X_t - \mu_t}{\mu_t}$$

$\log\left(\frac{X_t}{\mu_t}\right) = \log\left(1 + \frac{X_t-\mu_t}{\mu_t}\right)$

$\log(X_t) - \log(\mu_t) \approx \frac{X_t - \mu_t}{\mu_t}$

If higher levels of series are associated with more variation:

$$\boxed{E[X_t] = \mu_t,} \quad \sqrt{Var(X_t)} = \mu_t\sigma$$

From (Taylor) expansion,

$Var(\log(X_t)) \approx Var\left(\log(\mu_t) + \frac{X_t-\mu_t}{\mu_t}\right)$

$= Var\left(\frac{X_t-\mu_t}{\mu_t}\right)$

$= \frac{1}{\mu_t^2} Var(X_t)$

$= \frac{\cancel{\mu_t^2}}{\cancel{\mu_t^2}} \sigma^2$

$$\log(X_t) \approx \log(\mu_t) + \frac{X_t - \mu_t}{\mu_t}$$

$$E[\log(X_t)] \approx \log(\mu_t), \quad Var(\log(X_t)) \approx \sigma^2$$

ss

$E\left[\log(\mu_t) + \frac{X_t - \mu_t}{\mu_t}\right] = \log(\mu_t).$

$\frac{X_t-\mu_t}{\mu_t} \to 0$

Percentage Changes and logarithms

X_t tends to have relatively stable percentage changes from one time period to the next:

$$\boxed{\frac{X_t}{X_{t-1}} = 1+r_t}, \quad \frac{X_t}{X_{t-1}} - 1 = r_t$$

$$X_t = (1 + r_t)X_{t-1},$$

i.e. $r_t = \dfrac{X_t - X_{t-1}}{X_{t-1}}$

- $100r_t$: percentage change (possibly negative) from X_{t-1} to X_t

$$\log\left(\frac{X_t}{X_{t-1}}\right) = \boxed{\log(X_t) - \log(X_{t-1})} = \log\left(\frac{X_t}{X_{t-1}}\right) = \log(1 + r_t) \approx r_t.$$

If r_t is restricted, (approximately, $|r_t| < 0.2$), then we have approximation $\boxed{\log(1 + r_t) \approx r_t.}$

$$\log(X_t) - \log(X_{t-1}) = \boxed{\nabla \log(X_t)} \approx r_t$$

↪ $\log(\nabla X_t)$

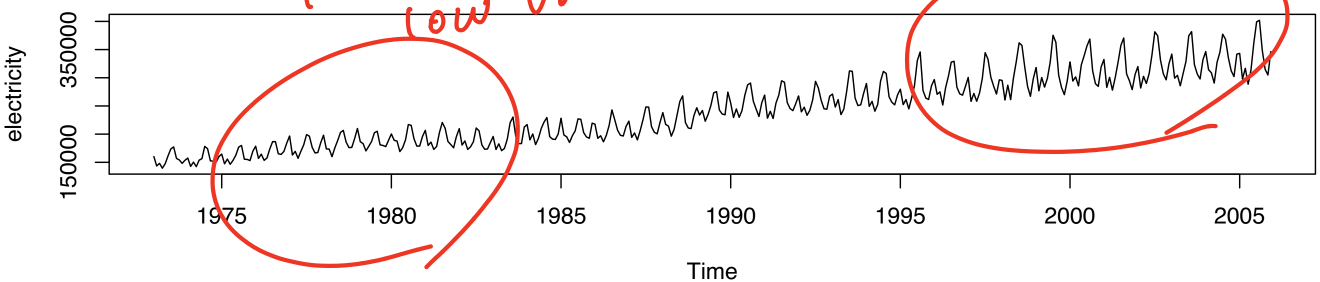
diff(log(__)) or log(diff(__))?

library(TSA)

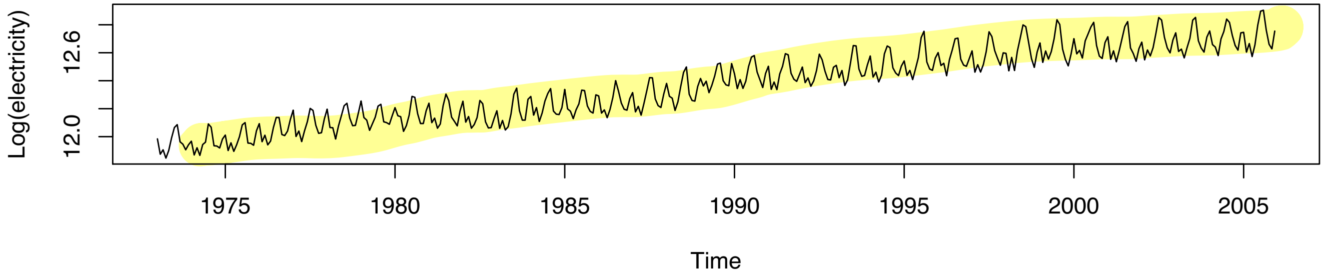
```
data(electricity)
plot(electricity)
```

higher level
var ↑

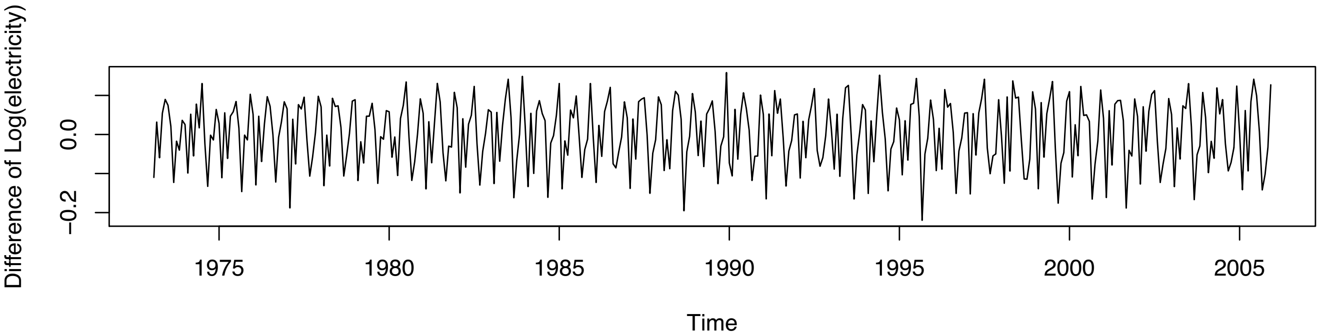
low level
low variance



```r
plot(log(electricity),ylab='Log(electricity)')
```

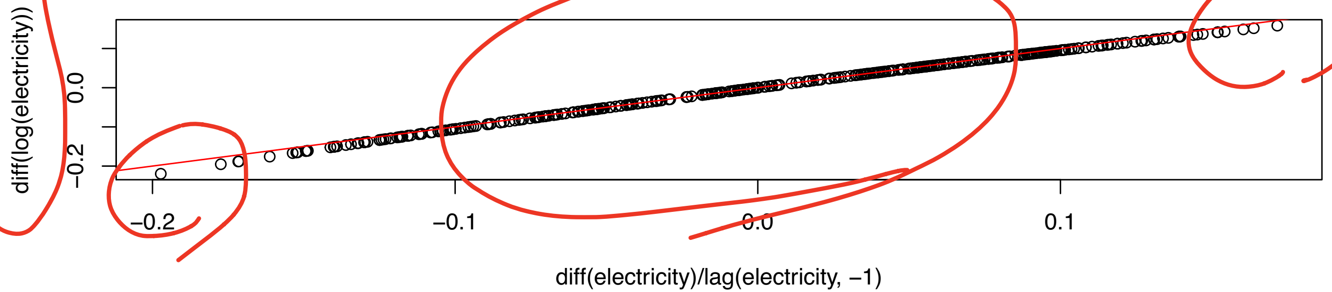


```
plot(diff(log(electricity)), ylab='Difference of Log(electr
```



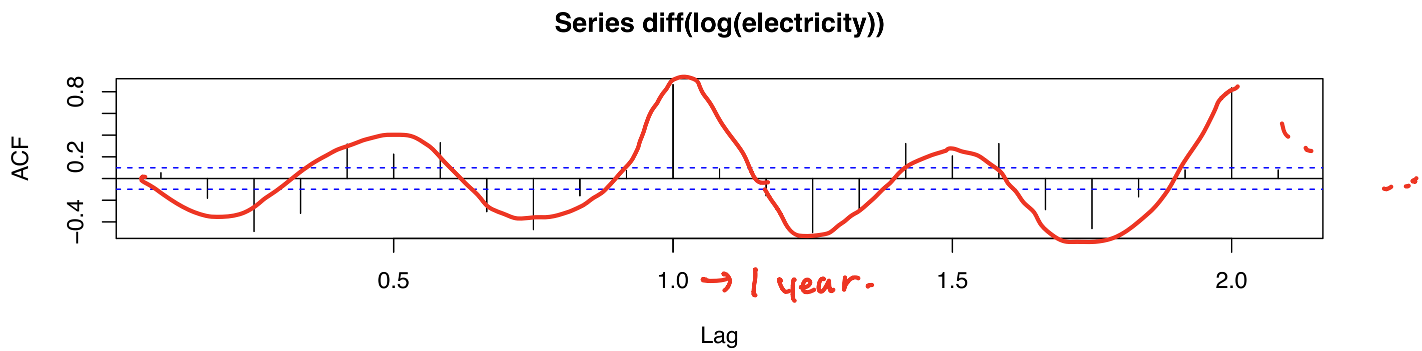
```
plot(diff(electricity)/lag(electricity,-1),diff(log(electricity)))
abline(a=0,b=1,col='red')
```

$\nabla \log(X_t) \approx r_t.$

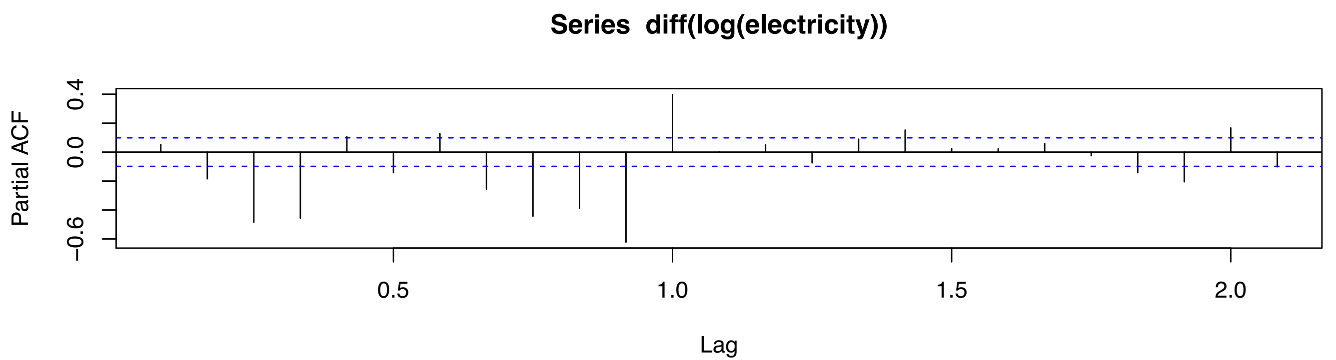


$$\frac{\nabla X_t}{X_{t-1}} = \frac{X_t - X_{t-1}}{X_{t-1}} = r_t.$$

```r
acf(diff(log(electricity)))
```

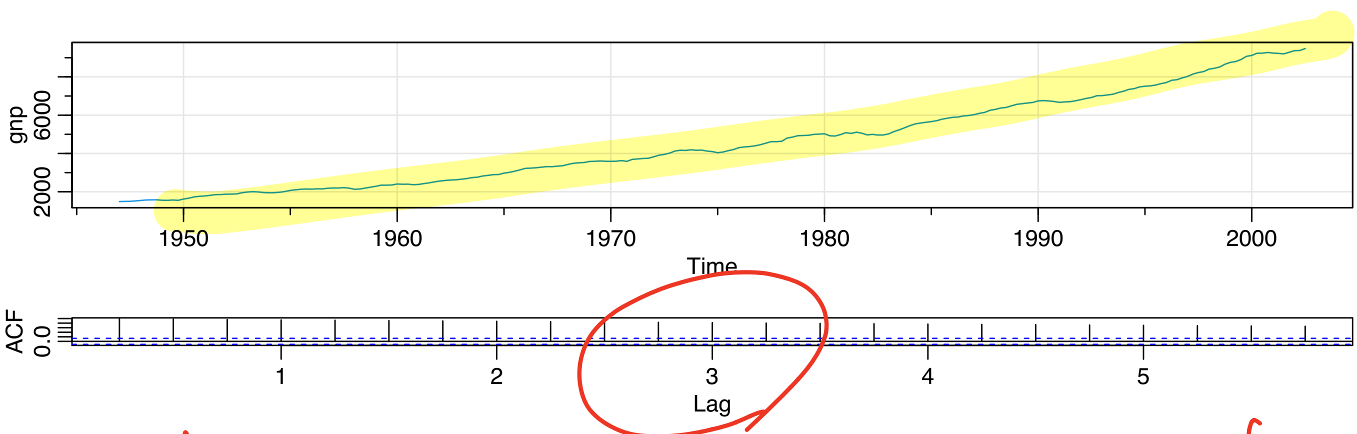



```r
pacf(diff(log(electricity)))
```



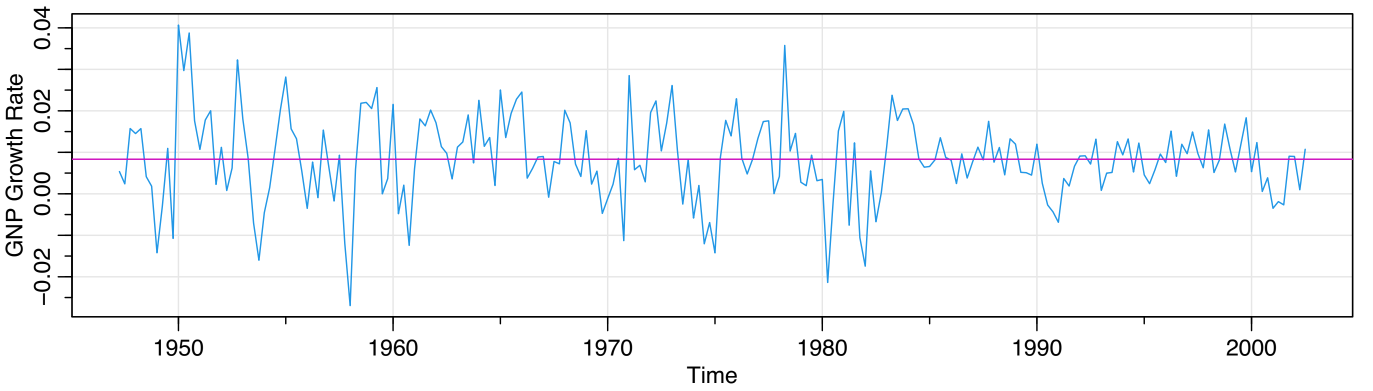
gnp example

```
layout(1:2, heights=2:1)
tsplot(gnp, col=4)
acf(gnp, main="")
```

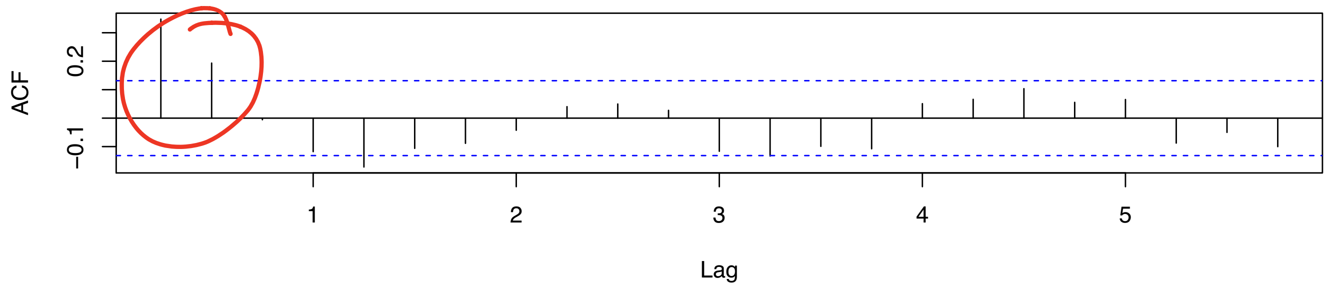


ACF Decreasing very slow → Indication of Not stationary.

```r
tsplot(diff(log(gnp)), ylab="GNP Growth Rate", col=4)
abline(h=mean(diff(log(gnp))), col=6)
```

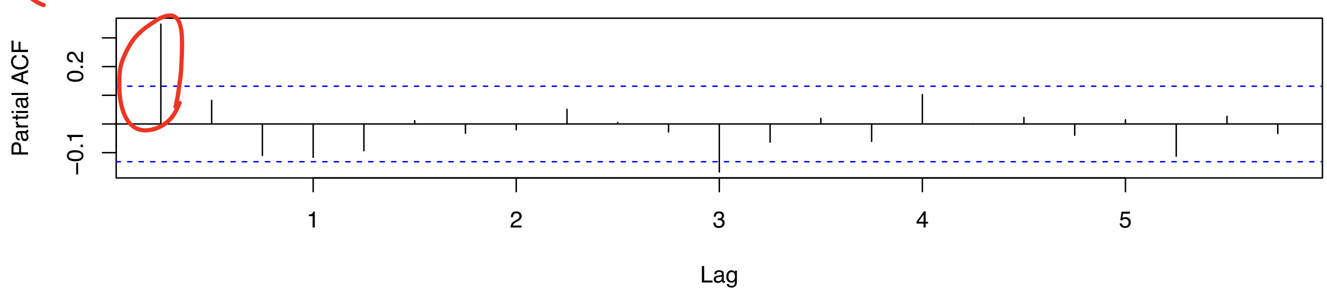


```r
acf(diff(log(gnp)), main="")
```




```r
pacf(diff(log(gnp)), main="")
```

(PACF cut off after lag 1, ACF tails off.) → AR(1).



(ACF cut off after lag 2, PACF tails off) → MA(2)

AR(1) $\quad X_t = 0.35 X_{t-1} + W_t$

$\downarrow$

MA(∞)?

$$(1 - 0.35B) X_t = W_t.$$

$$X_t = \frac{1}{1 - 0.35B} W_t$$

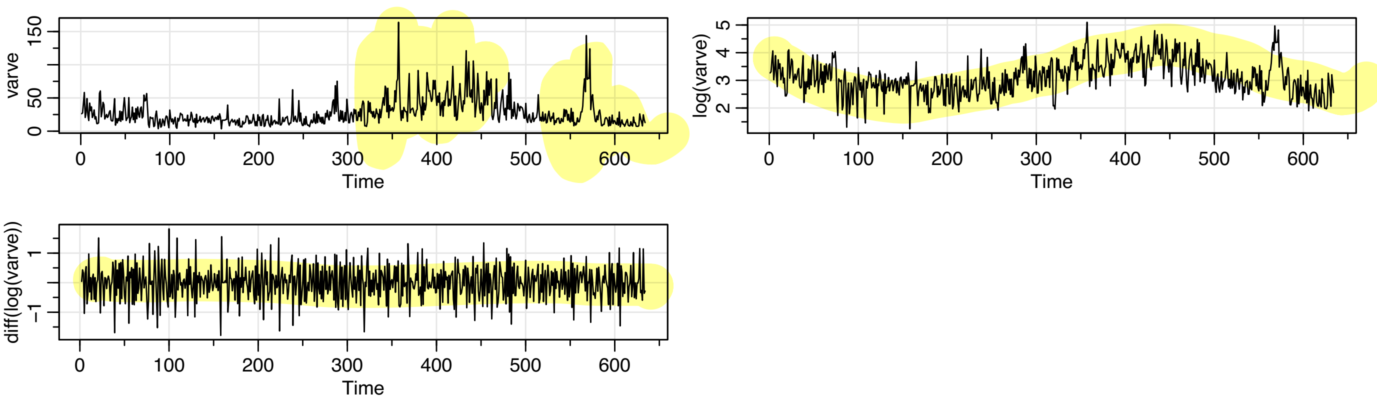
$$X_t = (1 + 0.35B + 0.35^2 B^2 + 0.35^3 B^3 + \cdots) W_t.$$

$$= W_t + 0.35 W_{t-1} + 0.1225 W_{t-2} + \underline{0.043 W_{t-3} + \cdots}$$

$$\approx W_t + 0.35 W_{t-1} + 0.1225 W_{t-2}$$

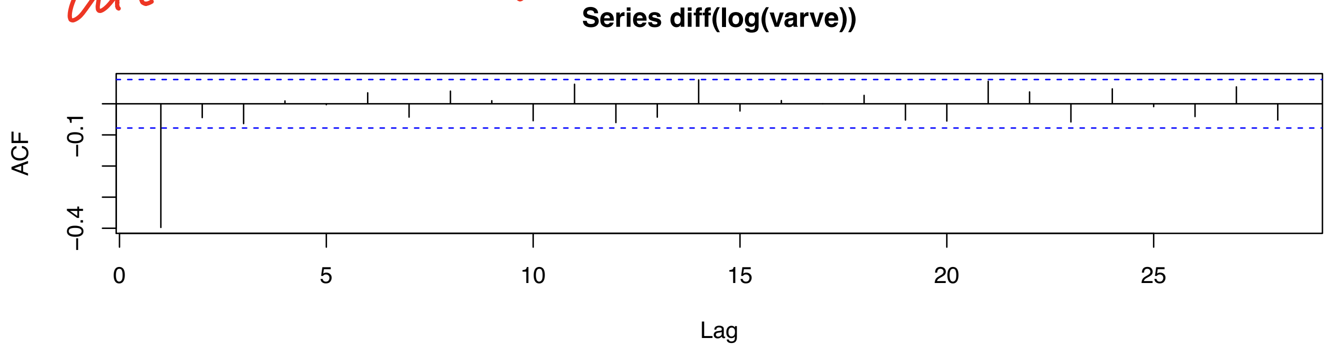
varve example

```
par(mfrow=c(2,2))
tsplot(varve)
tsplot(log(varve))
tsplot(diff(log(varve)))
```



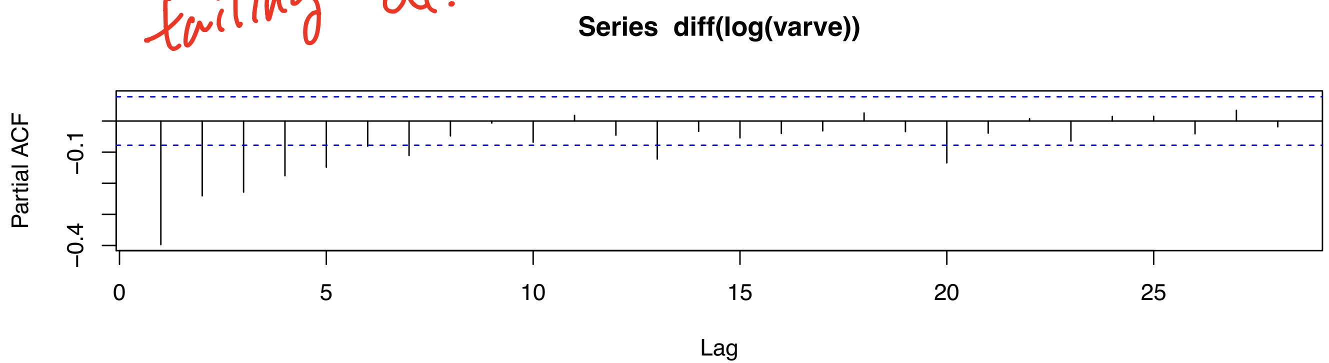
```r
acf(diff(log(varve)))
```

cut off after lag 1




```r
pacf(diff(log(varve)))
```

tailing off.



MA(1) for diff(log(varve)) data.

$\log(\text{varve}) \Rightarrow \text{IMA}(1,1).$