Week 8 Lecture Note

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Module 2 - Week 8

Seosonal ARMA models (SARMA)

Seasonal ARMA models

- For seasonal series normally dependence tend to occur more strongly at multiples of a seasonal lag ($s = 12$ for monthly data), or every quarter for quarterly data ($s = 4$).
- Economic series tend to have strong association at seasonal lags, and natural phenomena tend follow seasonal fluctuations.

ARMA models with seasonal lags: SARMA

$$\boxed{B^s X_t = X_{t-s}.}$$

Recall that $B^s X_t = X_{t-s}$. A pure seasonal ARMA model (ARMA(P,Q)) is defined as:
12

$$\boxed{\Phi_P(B^s)} X_t = \boxed{\Theta_Q}(B^s)\cancel{e_t}\ W_t.$$

where

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}$$

$$\Theta_Q(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}$$

are the seasonal autoregressive and moving average operators.

Example: First order Seasonal AR

$s = 12$

$X_t - \Phi X_{t-12} = W_t$

$$(1 - \Phi B^{12})X_t = W_t$$

causal, stationary

or

$$X_t = \Phi X_{t-12} + W_t$$

with $|\Phi| < 1$

$$Var(X_t) = Var(\Phi X_{t-12} \overset{\text{Indep.}}{+} W_t)$$

$$= \Phi^2 Var(X_{t-12}) + \sigma_w^2 \quad \text{causality}$$

$$= \Phi^2 Var(X_t) + \sigma_w^2. \quad \text{stationarity},$$

$\gamma(0) = \Phi^2 \gamma(0) + \sigma_w^2$. $\quad \gamma(0) = \dfrac{\sigma_w^2}{1-\Phi^2}$.

$k>0$ 时 $X_t = \Phi X_{t-12} + W_t$

$$= \Phi(\Phi X_{t-24} + W_{t-12}) + W_t = \Phi^2 X_{t-24} + \Phi W_{t-12} + W_t.$$

$$\vdots$$

$$= \Phi^k X_{t-12 \cdot k} + \Phi^{k-1} W_{t-12(k-1)} + \cdots + W_t.$$

$\Rightarrow Cov(X_{t-12k}, X_t)$

$= Cov(X_{t-12k}, \Phi^k X_{t-12k} + \Phi^{k-1} W_{t-12(k-1)} + \cdots + W_t)$

Indep.

$= \Phi^k Cov(X_{t-12k}, X_{t-12k})$

$= \Phi^k \gamma(0)$,

$\gamma(12k) = \gamma(-12k)$,

$$\gamma(0) = \frac{\sigma_W^2}{1-\Phi^2}$$

$$\boxed{\gamma(\pm 12k)} = \frac{\sigma_W^2 \Phi^k}{1-\Phi^2}, \quad k = 1, 2, \ldots$$

$= \Phi^k \gamma(0)$.

$$\gamma(h) = 0, \quad \text{otherwise.}$$

$h = 1, 2, 3, \cdots, 11 \to \gamma(h) = 0$.

only nonzero correlations:

$$\rho(\pm 12k) = \Phi^k, \quad k = 0, 1, 2, \ldots$$

$\frac{\gamma(\pm 12k)}{\gamma(0)} = \Phi^k$.

$$\gamma(h) = \Phi\gamma(h-12), \quad h \geq 1$$

$$\gamma(h) = Cov(X_{t-h}, X_t) = Cov(X_{t-h}, \Phi X_{t-12} + W_t).$$

$$= Cov(X_{t-h}, \Phi X_{t-12}) + \cancel{\underline{Cov(X_{t-h}, W_t)}} \quad 0.$$

$$= \Phi\, Cov(X_{t-h}, X_{t-12})$$

$$= \Phi\, \gamma(h-12).$$

$h \neq 12k,$

$h=1.$ $\quad \gamma(1) = \Phi\gamma(-11) = \Phi\gamma(11)$

$h=11$ $\quad \gamma(11) = \Phi\gamma(11-12) = \Phi\gamma(-1) = \Phi\gamma(1).$

$$\gamma(11) = \Phi\gamma(1) = \Phi^2\gamma(11).$$

$$\rightarrow \underbrace{(1-\Phi^2)}_{\text{nonzero.}}\boxed{\gamma(11) = 0.}$$

$\gamma(11) = 0,\ \gamma(1) = 0.$

$\gamma(11+12k) = 0,\ \gamma(1+12k) = 0.$

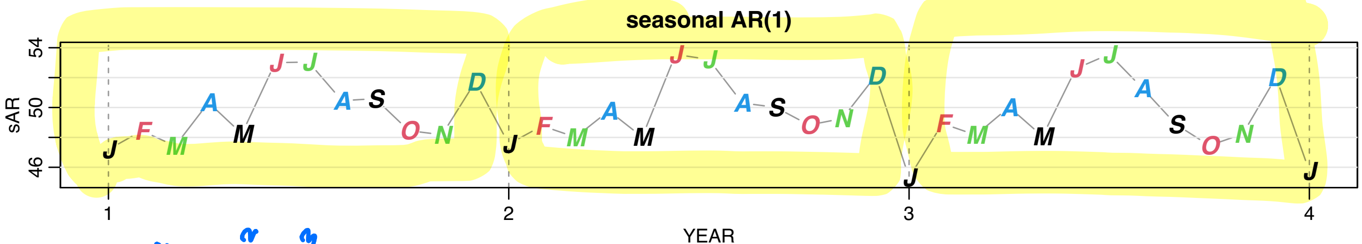
$\gamma(1) = \gamma(13) = \gamma(25) = \cdots = 0.$

$\gamma(11) = \gamma(23) = \gamma(35) = \cdots = 0.$

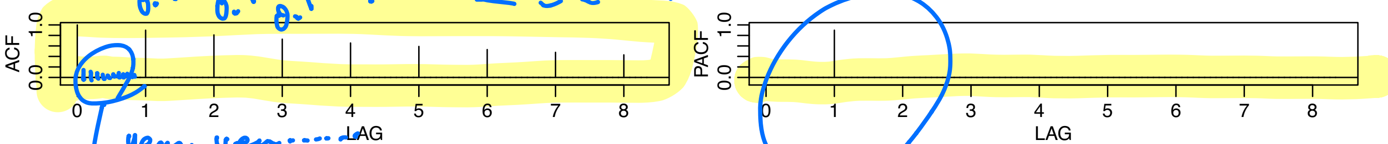
$\gamma(2) = 0$

$\gamma(10) = 0$

$\gamma(3) = \gamma(9) = \gamma(4) = \gamma(8) = \gamma(5) = \gamma(7) = \gamma(6) = 0.$



0.9 0.9^2 0.9^3 ...



year year.....

Generated $\mathrm{SAR}(1)_{12}$ model, ACF and PACF of

0.

$$X_t - 50 = \boxed{0.9}(X_{t-12} - 50) + W_t$$

$S=12$. $P=1 \to$ PACF cuts off after 1 year.

Example: Seasonal MA

$$X_t = (1 + \Theta B^s)W_t$$

or

$$X_t = W_t + \Theta W_{t-12}$$

with $|\Theta| < 1$

$\gamma(0) = Var(W_t + \Theta W_{t-12}) = (1+\Theta^2)\sigma_w^2.$

$\gamma(12) = Cov(X_t, X_{t-12})$

$= Cov(W_t + \Theta W_{t-12}, X_{t-12}),$

$= Cov(W_t + \underline{\Theta W_{t-12}}, \underline{W_{t-12}}, \Theta W_{t-24})$

$= \Theta \sigma_w^2. = \gamma(-12).$

$h \neq \pm 12$ $\gamma(h) = Cov(X_t, X_{t-h})$

$= Cov(\underbrace{\underline{W_t + \Theta W_{t-12}}, \underline{W_{t-h} + \Theta W_{t-h-12}}})$.

$= 0$.

$$\gamma(0) = (1 + \Theta^2)\sigma_W^2$$
$$\gamma(\pm 12) = (\Theta)\sigma_W^2$$
$$\gamma(h) = 0, \quad \text{otherwise.}$$

only nonzero correlations:

$$\rho(\pm 12k) = \Theta/(1 + \Theta^2)$$

$= \dfrac{\gamma(\pm 12k)}{\gamma(0)} = \dfrac{\Theta}{1+\Theta^2}$.

General behavior of the ACF and PACF for pure SARMA models

S

For stationary processes:

	$AR(P)_s$	$MA(Q)_s$	$ARMA(P,Q)_s$ $P > 0$ and $Q > 0$
ACF*	Tails off at lags ks $k = 1, 2, \ldots$	Cuts off after lag Qs	Tails off at lags ks
PACF*	Cuts off after lag Ps	Tails off at lags ks $k = 1, 2, \ldots$	Tails off at lags ks

*The values at nonseasonal lags $h \neq ks$, for$k = 1, 2, \ldots$ are **zero**

Multiplicative (Mixed) Seasonal ARMA

Multiplicative (Mixed) Seasonal ARMA

These models are denoted by:

$$ARMA\underbrace{(p,q)}_{\text{non-seasonal}} \times \underbrace{(P,Q)_s}_{\to\ \text{seasonal.}}$$

$$\Phi_P(B^s)\,\phi(B)\,X_t = \Theta_Q(B^s)\,\theta(B)\,W_t.$$

$$(1-\Phi_1 B^s \cdots - \Phi_P B^{s\cdot P})(1-\phi_1 B - \cdots - \phi_p B^p) = (1+\Theta_1 B^s \cdots + \Theta_Q B^{sQ}) \times$$

$$(1+\theta_1 B + \cdots + \theta_q B^q)\,W_t.$$

Example: $ARMA(p = 0, q = 1) \times (P = 1, Q = 0)_{12}$: $\rightarrow S=12$.

nonseasonal. seasonal

$$X_t = \Phi X_{t-12} + W_t + \theta W_{t-1}$$

For this model:

$$\gamma(0) = \frac{1 + \theta^2}{1 - \Phi^2}\sigma_W^2 \quad \checkmark$$

$|\Phi| < 1$, $|\Theta| < 1$.

$$\rho(12k) = \Phi^k, \; k = 1, 2, \ldots \quad \checkmark$$

$$\rho(12k - 1) = \rho(12k + 1) = \frac{\theta}{1 + \theta^2}\Phi^k \qquad k = 0, 1, 2, \ldots$$

$$\rho(k) = 0 \text{ otherwise}$$

$h \neq 12k$,

$\neq 12k \pm 1$,

$$X_t = \boxed{\Phi X_{t-12} + W_t + \theta W_{t-1}}.$$

$$\mathrm{Var}(X_t) = \mathrm{Var}(\underbrace{\Phi X_{t-12}} + W_t + \theta W_{t-1}).$$

$$= \mathrm{Var}(\Phi X_{t-12}) + \sigma_w^2 + \theta^2 \sigma_w^2$$

$$= \Phi^2 \mathrm{Var}(X_{t-12}) + (1+\theta^2)\sigma_w^2.$$

$$= \Phi^2 \mathrm{Var}(X_t) + (1+\theta^2)\sigma_w^2.$$

$$(1-\Phi^2)\mathrm{Var}(X_t) = (1+\theta^2)\sigma_w^2,$$

$$\mathrm{Var}(X_t) = \frac{(1+\theta^2)}{1-\Phi^2}\sigma_w^2 \; . = \gamma(0).$$

$$\gamma(12k) = Cov(X_{t-12k}, X_t)$$

$$= Cov(X_{t-12k}, \underbrace{\Phi X_{t-12} + W_t + \theta W_{t-1}})$$

$$X_t = \Phi X_{t-12} + W_t + \theta W_{t-1}$$

$$\vdots$$

$$= \Phi(\Phi X_{t-24} + W_{t-12} + \theta W_{t-13}) + W_t + \theta W_{t-1}.$$

$$\vdots$$

$$= \Phi^k X_{t-12 \cdot k} + \underbrace{\text{linear combination of } W_{t-12(k-1)}, \cdots W_t}.$$

$$= Cov(X_{t-12k}, \underbrace{\Phi^k X_{t-12 \cdot k} + \underbrace{\qquad\qquad}})$$

Indep. $Cov = 0$.

$$= \Phi^k Cov(X_{t-12k}, X_{t-12k}) = \Phi^k \gamma(0).$$

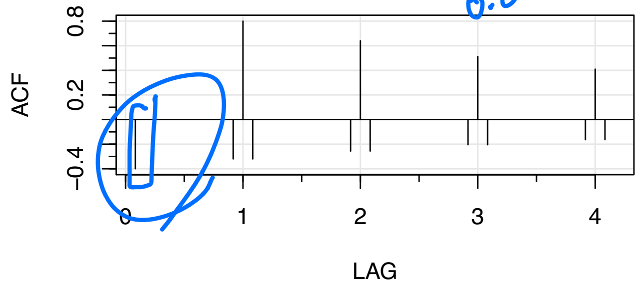
$$\gamma(12k) = \Phi^k \gamma(0), \qquad \boxed{\rho(12k) = \Phi^k.}$$

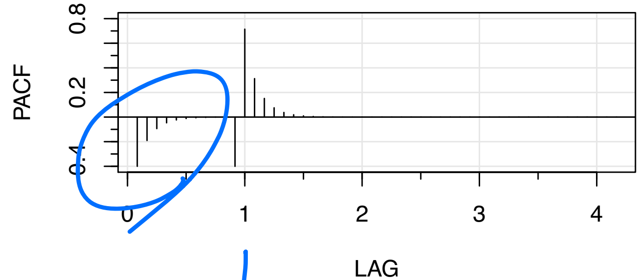
ACF and PACF of an $ARMA(0,1) \times (1,0)_{12}$

$s=12$, 1year

$$X_t = \Phi X_{t-12} + W_t + \theta W_{t-1}, \boxed{\Phi = 0.8} \text{ and } \theta = -0.5$$

0.8 0.8^2 0.8^3 ...





Seasonal: tailing off at $12 \cdot k$.

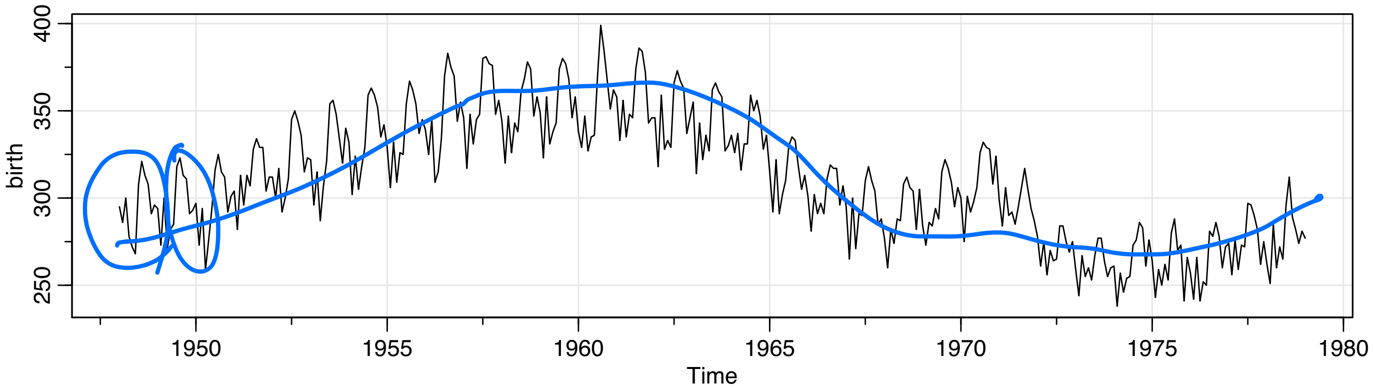
Seasonal: cutting off after $12 \cdot 1$.

$P=1$, $Q=0$.

non-seasonal: ACF: cutting off.
PACF: tailing off. → non-seasonal → $p=0$, $q=1$.

Example: Birth Series

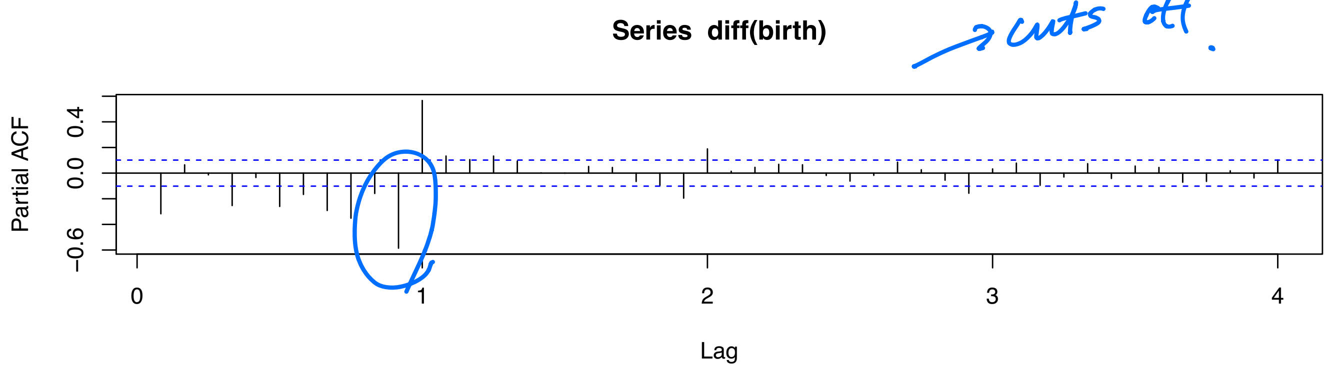
```
##-- birth series --##
tsplot(birth)  # monthly number of births in US
```



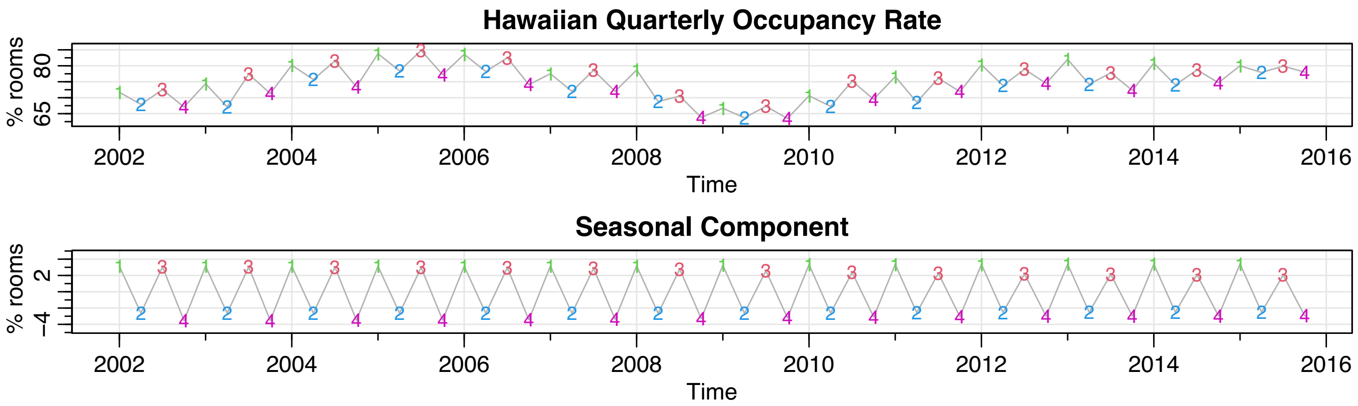
```r
acf(diff(birth), lag.max = 48)
```


```r
pacf(diff(birth), lag.max = 48)
```

→ cuts off.



Seosonal Persistence



seasonal component S_t.

$S_t \approx S_{t-4}$.

$$S_t \approx S_{t-4}$$

V_t : WN.

$$\boxed{S_t = S_{t-4} + V_t}$$

indep.

$$\boxed{X_t = S_t + W_t}$$

W_t : WN

Seasonal differencing.

$$(1 - B^s)X_t = X_t - X_{t-4} = V_t + W_t - W_{t-4}$$

$$= S_t + W_t - (S_{t-4} + W_{t-4}).$$

$$= \underbrace{S_t - S_{t-4}}_{} + W_t - W_{t-4}$$

$$= V_t + W_t - W_{t-4}.$$

Seosonal Differencing and SARIMA (seaonsal ARIMA) model

Seosonal differencing

Seosonal differencing indicated when the ACF decays slowly at multiples of some season s.

Then, a *seasonal difference of order D* is defined as

$$\nabla_s^D X_t = (1 - B^s)^D X_t$$

$D = 1, 2, \ldots$

$$\nabla_s^D X_t = (1 - B^s)^D X_t.$$

$D = 1, s = 12, \quad \nabla_{12} X_t = (1 - B^{12}) X_t.$

SARIMA model

AR differencing MA

A process X_t satisfies a multiplicative SARIMA model with non-seasonal order (p,d,q) and seasonal order (P,D,Q) if:

$$\Phi_p(B^s)\phi(B)\nabla_s^D\nabla^d X_t = \alpha + \Theta_Q(B^s)\theta(B)W_t$$

seasonal.

Example: ARIMA$(0,1,1) \times (0,1,1)_{12}$

$p=0,\ d=1,\ q=1, \qquad P=0,\ D=1,\ Q=1.$

$$\nabla_{12}\nabla X_t = \Theta(B^{12})\theta(B)W_t$$

or

seasonal MA

non-seasonal MA.

$$(1-B^{12})(1-B)X_t = (1+\Theta B^{12})(1+\theta B)W_t$$

Seasonal differencing

non-seasonal differencing.

$$(1-B^{12}-B+B^{13})X_t = (1+\theta B+\Theta B^{12}+\theta\Theta B^{13})W_t.$$

$$X_t = X_{t-1} + X_{t-12} - X_{t-13} + W_t + \theta W_{t-1} + \Theta W_{t-12} + \theta\Theta W_{t-13}$$

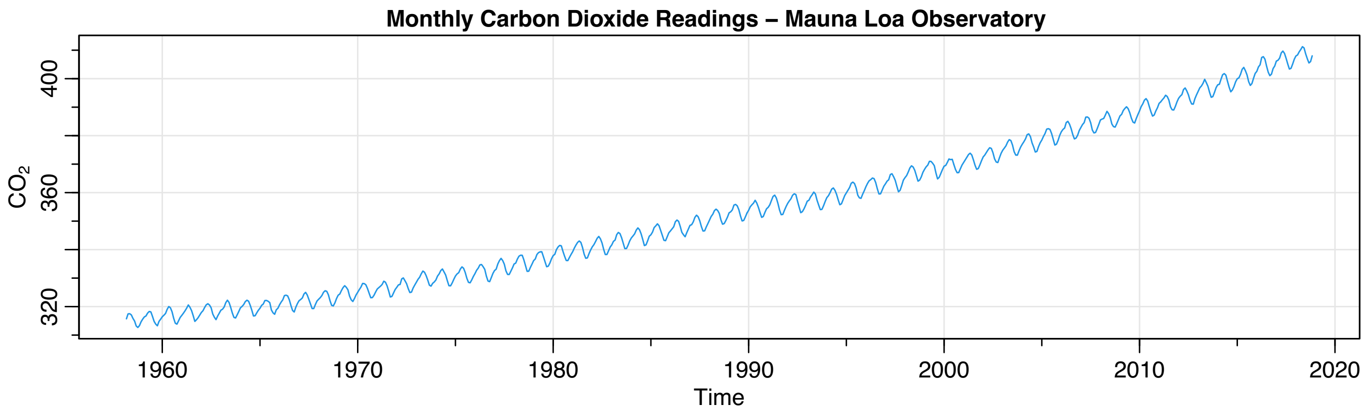
Seasonal ARIMA: model specification

Seasonal ARIMA: model specification

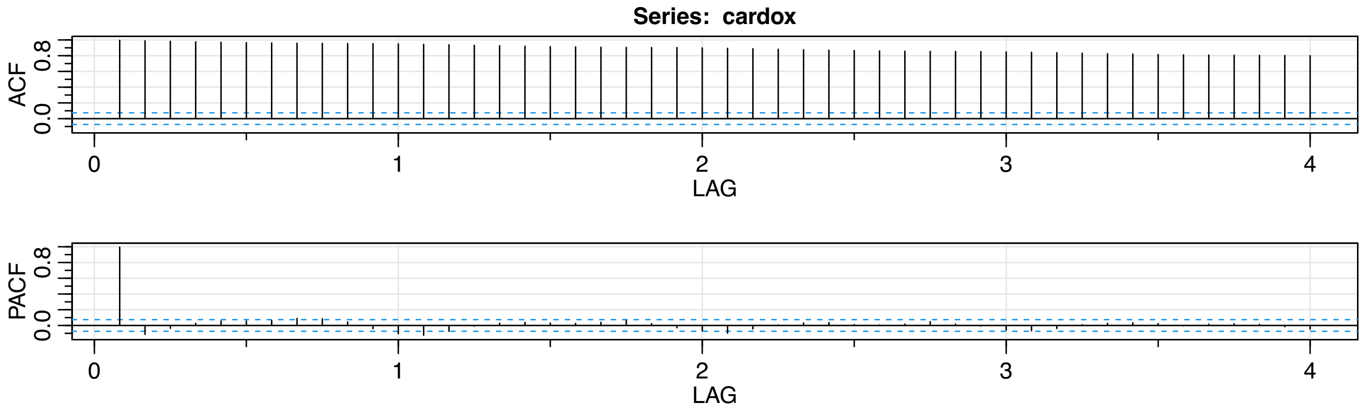
Selecting the appropriate model for a given set of data is a simple step-by-step process.

1. Consider obvious differencing transformations to remove trend (d) and to remove seasonal persistence (D) if they are present.
2. Then look at the ACF and the PACF of the possibly differenced data.
3. Consider the seasonal components (P and Q) by looking at the seasonal lags only.
4. Look at the first few lags and consider values for within seasonal components (p and q).

```r
tsplot(cardox, col = 4, ylab = expression(CO[2]))
title("Monthly Carbon Dioxide Readings - Mauna Loa Observat
```

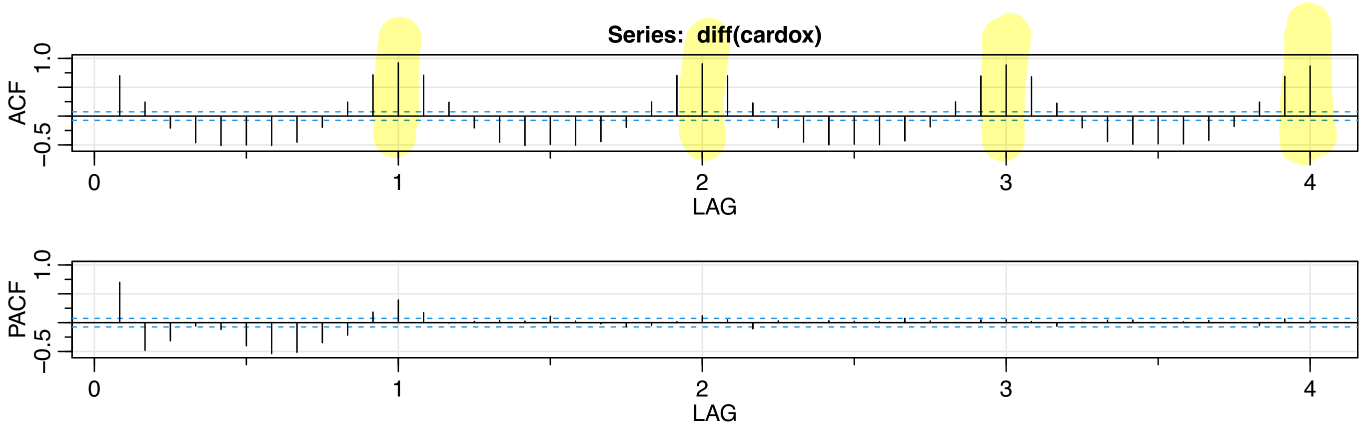


```r
acf2(cardox)
```




```
##         [,1]  [,2]  [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,1(
## ACF        1  0.99  0.98 0.98 0.97 0.97 0.96 0.96 0.96  0.9
## PACF       1 -0.11 -0.04 0.03 0.06 0.06 0.06 0.09 0.08  0.(
##        [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,2
## ACF     0.94  0.93  0.93  0.92  0.92  0.91  0.91  0.91  0.
## PACF   -0.07 -0.01  0.03  0.04  0.03  0.03  0.04  0.06  0.
##        [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,3
## ACF     0.89  0.88  0.88  0.87  0.86  0.86  0.86  0.86  0.
## PACF   -0.04  0.01  0.03  0.04  0.01  0.01  0.02  0.05  0.
##        [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,4
## ACF     0.84  0.83  0.83  0.82  0.82  0.81  0.81  0.81  0.
```

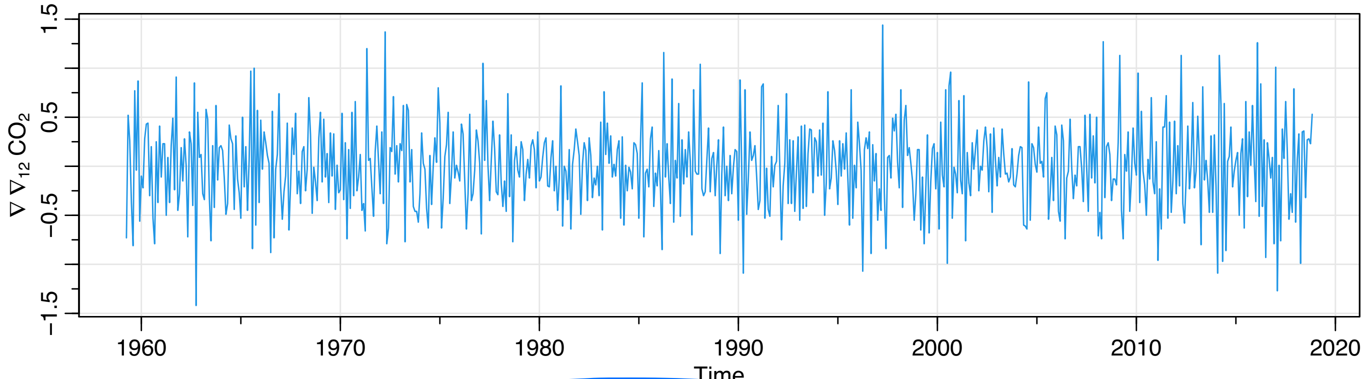
```r
acf2(diff(cardox))
```




```
##      [,1]  [,2]  [,3]  [,4]  [,5] [,6]  [,7]  [,8]  [,9]
## ACF   0.7  0.24 -0.21 -0.46 -0.51 -0.5 -0.51 -0.45 -0.19
## PACF  0.7 -0.48 -0.31 -0.06 -0.12 -0.4 -0.54 -0.51 -0.35
##      [,13] [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,2
## ACF   0.71  0.24 -0.20 -0.45 -0.51 -0.49 -0.50 -0.44 -0.
## PACF  0.17 -0.01  0.02  0.04  0.03  0.11  0.03 -0.02 -0.
##      [,25] [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,3
## ACF   0.70  0.23 -0.20 -0.45 -0.50 -0.49 -0.49 -0.43 -0.
## PACF  0.05 -0.11  0.03  0.00  0.03  0.02  0.02  0.08  0.
##      [,37] [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,4
## ACF   0.68  0.22 -0.20 -0.44 -0.49 -0.48 -0.48 -0.42 -0.
```

$\nabla_{12} \nabla X_t.$

```
tsplot(diff(diff(cardox, 12)), col = 4, ylab = expression(r
```



$\boxed{ARIMA(0,1,1) \times (0,1,1)_{12}}$

Seasonal : ACF cutting off after 1 year

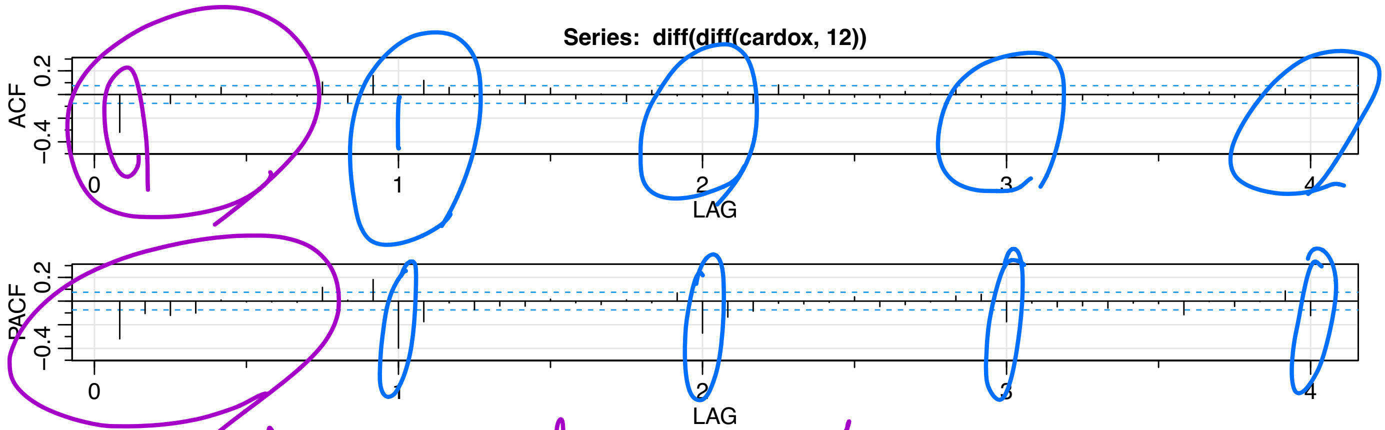
$s=12$, $\boxed{12 \cdot 1 \text{ month}}$

PACF : tailing off.

$P=0$

$Q=1$

```r
acf2(diff(diff(cardox, 12)))
```



→ non-seasonal : p=0, q=1.

```
##        [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8] [,9
## ACF   -0.32  0.01 -0.08 -0.02  0.06 -0.02  0.00 -0.01 0.1
## PACF  -0.32 -0.11 -0.12 -0.10  0.01 -0.01 -0.01 -0.01 0.1
##       [,13] [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,2
## ACF    0.12  0.06 -0.02  0.04 -0.08  0.05 -0.04  0.00 -0.
## PACF  -0.17 -0.02 -0.07 -0.04 -0.04  0.02 -0.01 -0.03  0.
##       [,25] [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,3
## ACF    0.02 -0.07  0.08 -0.03  0.04 -0.02 -0.03  0.03  0.
## PACF  -0.14 -0.09 -0.02 -0.02 -0.01  0.04 -0.05 -0.03 -0.
##       [,37] [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,4
## ACF    0.03  0.00 -0.05 -0.01  0.02 -0.02  0.01  0.02 -0.
```