HW 02

Name: Paul Holaway, NetID: paulch2

Due: 9/8/2022 11:59pm

- Unless stated otherwise, W_t is a white noise process with variance σ_w^2.
  - (W_t are independent with zero means and variance σ_w^2.)
- Show your full work to receive full credit.

Question 1

Suppose $E[X] = 2$, $Var(X) = 16$, $E(Y) = 0$, $Var(Y) = 25$, and $Corr(X,Y) = -0.25$. Find:

(a) $Var(X+Y)$.

$$
\begin{gathered}
Var(X+Y) = Cov(X+Y, X+Y) \\
= Cov(X,X) + Cov(X,Y) + Cov(Y,X) + Cov(Y,Y) \\
= Var(X) + Var(Y) + 2Cov(X,Y) \\
Cov(X,Y) = Corr(X,Y) * \sqrt{Var(X)Var(Y)} \\
\implies Cov(X,Y) = -0.25 * \sqrt{16 * 25} = -0.25 * 20 = -5 \\
Var(X+Y) = 16 + 25 + 2(-5) = 16 + 25 - 10 \\
\implies \boxed{Var(X+Y) = 31}
\end{gathered}
$$

(b) $Cov(X, X+Y)$.

$$
\begin{gathered}
Cov(X, X+Y) = Cov(X,X) + Cov(X,Y) = Var(X) + Cov(X,Y) = 16 - 5 \\
\implies \boxed{Cov(X, X+Y) = 11}
\end{gathered}
$$

(c) $Corr(X+Y, X-Y)$.

$$
\begin{gathered}
Corr(X+Y, X-Y) = \frac{Cov(X+Y, X-Y)}{\sqrt{Var(X+Y)Var(X-Y)}} = \frac{Cov(X-Y, X+Y)}{\sqrt{Var(X+Y)Var(X-Y)}} \\
= \frac{Cov(X, X+Y) - Cov(Y, X+Y)}{\sqrt{Var(X+Y)Var(X-Y)}} \\
Cov(Y, X+Y) = Cov(Y,X) + Cov(Y,Y) = Cov(X,Y) + Var(Y) = -5 + 25 = 20 \\
Var(X-Y) = Cov(X,X) - Cov(X,Y) - Cov(Y,X) + Cov(Y,Y) \\
= Var(X) + Var(Y) - 2Cov(X,Y) = 16 + 25 - 2(-5) = 51 \\
\implies Corr(X+Y, X-Y) = \frac{11 - 20}{\sqrt{31 * 51}} \\
\implies \boxed{Corr(X+Y, X-Y) \approx -0.2263}
\end{gathered}
$$

Question 2

If X and Y are dependent but $Var(X) = Var(Y)$, find $Cov(X + Y, X - Y)$.

$$Cov(X + Y, X - Y) = Cov(X, X - Y) + Cov(Y, X - Y)$$
$$= Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y) = Cov(X, X) - Cov(X, Y) + Cov(X, Y) - Cov(Y, Y)$$
$$= Var(X) - Var(Y)$$
$$\implies Var(X) = Var(Y) \implies Var(X) - Var(Y) = 0$$
$$\implies \boxed{Cov(X + Y, X - Y) = 0}$$

Question 3

Consider the time series

$$X_t = \beta_1 + \beta_2 t + W_t,$$

where β_1 and β_2 are known constants.

(a) Find mean function, autocovariance function, and autocorrelation function of X_t.

$$\mu_X(t) = \mathbb{E}(X_t) = \mathbb{E}(\beta_1 + \beta_2 t + W_t) = \mathbb{E}(\beta_1) + \mathbb{E}(\beta_2 t) + \mathbb{E}(W_t)$$
$$= \beta_1 + \beta_2 t + 0$$
$$\implies \boxed{\mu_X(t) = \beta_1 + \beta_2 t}$$
$$\gamma_X(s, t) = Cov(X_s, X_t) = Cov(\beta_1 + \beta_2 s + W_s, \beta_1 + \beta_2 t + W_t)$$
$$= Cov(\beta_2 s + W_s, \beta_2 t + W_t) = Cov(\beta_2 s, \beta_2 t + W_t) + Cov(W_s, \beta_2 t + W_t)$$
$$= Cov(\beta_2 s, \beta_2 t) + Cov(\beta_2 s, W_t + Cov(W_s, \beta_2 t)) + Cov(W_s, W_t)$$
$$= 0 + 0 + 0 + Cov(W_s, W_t) = Cov(W_s, W_t)$$
$$\implies \boxed{\gamma_X(s, t) = \begin{cases} \sigma_w^2 & ; s = t \\ 0 & ; s \neq t \end{cases}}$$
$$\rho_X(s, t) = \frac{\gamma_X(s, t)}{\sqrt{\gamma_X(s, s)\gamma_X(t, t)}} = \frac{\gamma_X(s, t)}{\sqrt{\sigma_w^2 * \sigma_w^2}} = \frac{\gamma_X(s, t)}{\sigma_w^2}$$
$$\implies \boxed{\rho_X(s, t) = \begin{cases} 1 & ; s = t \\ 0 & ; s \neq t \end{cases}}$$

(b) Find mean function, autocovariance function, and autocorrelation function of $Y_t = X_t - X_{t-1}$.

$$\mu_Y(t) = \mathbb{E}(Y_t) = \mathbb{E}(X_t - X_{t-1}) = \mathbb{E}(X_t) - \mathbb{E}(X_{t-1})$$
$$\mathbb{E}(X_{t-1}) = \mathbb{E}(\beta_1 + \beta_2(t-1) + W_{t-1}) = \mathbb{E}(\beta_1) + \mathbb{E}(\beta_2(t-1)) + \mathbb{E}(W_{t-1})$$
$$\implies \mathbb{E}(X_{t-1}) = \beta_1 + \beta_2(t-1) + 0$$
$$\mu_Y(t) = \beta_1 + \beta_2 t - (\beta_1 + \beta_2(t-1)) = \beta_1 + \beta_2 t - \beta_1 - \beta_2 t + \beta_2$$
$$\implies \boxed{\mu_Y(t) = \beta_2}$$
$$\gamma_Y(s,t) = Cov(Y_s, Y_t) = Cov(X_s - X_{s-1}, X_t - X_{t-1})$$
$$= Cov(\beta_1 + \beta_2 s + W_s - (\beta_1 + \beta_2(s-1) + W_{s-1}), \beta_1 + \beta_2 t + W_t - (\beta_1 + \beta_2(t-1) + W_{t-1}))$$
$$= Cov(\beta_2 + W_s - W_{s-1}, \beta_2 + W_t - W_{t-1}) = Cov(W_s - W_{s-1}, W_t - W_{t-1})$$
$$= Cov(W_s, W_t - W_{t-1}) - Cov(W_{s-1}, W_t - W_{t-1})$$
$$= Cov(W_s, W_t) - Cov(W_s, W_{t-1}) - Cov(W_{s-1}, W_t) + Cov(W_{s-1}, W_{t-1})$$

1. $s = t \implies \gamma_Y(s,t) = \sigma_w^2 - 0 - 0 + \sigma_w^2 = 2\sigma_w^2$
2. $s = t+1 \implies \gamma_Y(s,t) = 0 - 0 - \sigma_w^2 + 0 = -\sigma_w^2$
3. $s = t-1 \implies \gamma_Y(s,t) = 0 - \sigma_w^2 - 0 + 0 = -\sigma_w^2$
4. $s = t+2 \implies \gamma_Y(s,t) = 0 - 0 - 0 + 0 = 0$
5. $s = t-2 \implies \gamma_Y(s,t) = 0 - 0 - 0 + 0 = 0$

$$\implies \boxed{\gamma_Y(s,t) = \begin{cases} 2\sigma_w^2 & ; s = t \\ -\sigma_w^2 & ; |s-t| = 1 \\ 0 & ; |s-t| \geq 2 \end{cases}}$$
$$\rho_Y(s,t) = \frac{\gamma_Y(s,t)}{\sqrt{\gamma_Y(s,s)\gamma_Y(t,t)}} = \frac{\gamma_Y(s,t)}{\sqrt{2\sigma_w^2 * 2\sigma_w^2}} = \frac{\gamma_Y(s,t)}{2\sigma_w^2}$$
$$\implies \boxed{\rho_Y(s,t) = \begin{cases} 1 & ; s = t \\ -\frac{1}{2} & ; |s-t| = 1 \\ 0 & ; |s-t| \geq 2 \end{cases}}$$

Question 4

For a moving average process of the form

$$X_t = 0.1W_{t-2} + 0.2W_{t-1} + W_t,$$

determine mean function, autocovariance function, and autocorrelation function of X_t.

$$\begin{aligned}
\mu_X(t) = \mathbb{E}(X_t) = \mathbb{E}(0.1W_{t-2} + 0.2W_{t-1} + W_t) &= 0.1\mathbb{E}(W_{t-2}) + 0.2\mathbb{E}(W_{t-1}) + \mathbb{E}(W_t) \\
&= 0.1 * 0 + 0.2 * 0 + 0 = 0 \\
&\implies \boxed{\mu_X(t) = 0}
\end{aligned}$$

$$\begin{aligned}
\gamma_X(s,t) = Cov(X_s, X_t) &= Cov(0.1W_{s-2} + 0.2W_{s-1} + W_s, 0.1W_{t-2} + 0.2W_{t-1} + W_t) \\
&= 0.01Cov(W_{s-2}, W_{t-2}) + 0.02Cov(W_{s-2}, W_{t-1}) + 0.1Cov(W_{s-2}, W_t) \\
&+0.02Cov(W_{s-1}, W_{t-2}) + 0.04Cov(W_{s-1}, W_{t-1}) + 0.2Cov(W_{s-1}, W_t) \\
&+0.1Cov(W_s, W_{t-2}) + 0.2Cov(W_s, W_{t-1}) + Cov(W_s, W_t)
\end{aligned}$$

1. $s = t \implies \gamma_Y(s,t) = 0.01\sigma_w^2 + 0 + 0 + 0 + 0.04\sigma_w^2 + 0 + 0 + 0 + \sigma_w^2 = 1.05\sigma_w^2$
2. $s = t+1 \implies \gamma_Y(s,t) = 0 + 0.02\sigma_w^2 + 0 + 0 + 0 + 0.2\sigma_w^2 + 0 + 0 + 0 = 0.22\sigma_w^2$
3. $s = t-1 \implies \gamma_Y(s,t) = 0 + 0 + 0 + 0.02\sigma_w^2 + 0 + 0 + 0 + 0.2\sigma_w^2 + 0 = 0.22\sigma_w^2$
4. $s = t+2 \implies \gamma_Y(s,t) = 0 + 0 + 0.1\sigma_w^2 + 0 + 0 + 0 + 0 + 0 + 0 = 0.1\sigma_w^2$
5. $s = t-2 \implies \gamma_Y(s,t) = 0 + 0 + 0 + 0 + 0 + 0 + 0.1\sigma_w^2 + 0 + 0 = 0.1\sigma_w^2$
6. $s = t+3 \implies \gamma_Y(s,t) = 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$
7. $s = t-3 \implies \gamma_Y(s,t) = 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$

$$\implies \boxed{\gamma_X(s,t) = \begin{cases} 1.05\sigma_w^2 & ; s = t \\ 0.22\sigma_w^2 & ; |s-t| = 1 \\ 0.1\sigma_w^2 & ; |s-t| = 2 \\ 0 & ; |s-t| \geq 3 \end{cases}}$$

$$\rho_X(s,t) = \frac{\gamma_X(s,t)}{\sqrt{\gamma_X(s,s)\gamma_X(t,t)}} = \frac{\gamma_X(s,t)}{\sqrt{1.05\sigma_w^2 * 1.05\sigma_w^2}} = \frac{\gamma_x(s,t)}{1.05\sigma_w^2}$$

$$\implies \boxed{\rho_X(s,t) = \begin{cases} 1 & ; s = t \\ 0.2095 & ; |s-t| = 1 \\ 0.0952 & ; |s-t| = 2 \\ 0 & ; |s-t| \geq 3 \end{cases}}$$

Question 5

A time series with a periodic component can be constructed from

$$X_t = U_1 \sin(2\pi\omega_0 t) + U_2 \cos(2\pi\omega_0 t),$$

where U_1 and U_2 are independent random variables with zero means and variances σ^2. The constant ω_0 determines the period or time it takes the process to make one complete cycle. Show the mean function, autocovariance function, and autocorrelation function of X_t.

Hint: ω_0 and t are both not random.

$$\begin{aligned}
\mu_X(t) &= \mathbb{E}(U_1 sin(2\pi\omega_0 t) + U_2 cos(2\pi\omega_0 t)) = \mathbb{E}(U_1 sin(2\pi\omega_0 t)) + \mathbb{E}(U_2 cos(2\pi\omega_0 t)) \\
&= sin(2\pi\omega_0 t)\mathbb{E}(U_1) + cos(2\pi\omega_0 t)\mathbb{E}(U_2) = sin(2\pi\omega_0 t) * 0 + cos(2\pi\omega_0 t) * 0 \\
&\implies \boxed{\mu_X(t) = 0}
\end{aligned}$$

$$\begin{aligned}
\gamma_X(s,t) &= Cov(X_s, X_t) = Cov(U_1 sin(2\pi\omega_0 s) + U_2 cos(2\pi\omega_0 s), U_1 sin(2\pi\omega_0 t) + U_2 cos(2\pi\omega_0 t)) \\
&= Cov(U_1 sin(2\pi\omega_0 s), U_1 sin(2\pi\omega_0 t)) + Cov(U_1 sin(2\pi\omega_0 s), U_2 cos(2\pi\omega_0 t)) \\
&\quad + Cov(U_2 cos(2\pi\omega_0 s), U_1 sin(2\pi\omega_0 t)) + Cov(U_2 cos(2\pi\omega_0 s), U_2 cos(2\pi\omega_0 t)) \\
&= Cov(U_1 sin(2\pi\omega_0 s), U_1 sin(2\pi\omega_0 t)) + Cov(U_2 cos(2\pi\omega_0 s), U_2 cos(2\pi\omega_0 t)) \\
&= sin(2\pi\omega_0 s) sin(2\pi\omega_0 t) Cov(U_1, U_1) + cos(2\pi\omega_0 s) cos(2\pi\omega_0 t) Cov(U_2, U_2) \\
&= sin(2\pi\omega_0 s) sin(2\pi\omega_0 t) Var(U_1) + cos(2\pi\omega_0 s) cos(2\pi\omega_0 t) Var(U_2) \\
&= [sin(2\pi\omega_0 s) sin(2\pi\omega_0 t) + cos(2\pi\omega_0 s) cos(2\pi\omega_0 t)]\sigma^2 \\
&= \sigma^2 cos(2\pi\omega_0(s-t))
\end{aligned}$$

1. $s = t \implies \gamma_X(s,t) = \sigma^2$
2. $s \neq t \implies \gamma_X(s,t) = \sigma^2 cos(2\pi\omega_0(s-t))$

$$\implies \boxed{\gamma_X(s,t) = \begin{cases} \sigma^2 & ; s = t \\ \sigma^2 cos(2\pi\omega_0(s-t)) & ; s \neq t \end{cases}}$$

$$\rho_X(s,t) = \frac{\gamma_X(s,t)}{\sqrt{\gamma_X(s,s)\gamma_X(t,t)}} = \frac{\gamma_X(s,t)}{\sqrt{\sigma^2 * \sigma^2}} = \frac{\gamma_X(s,t)}{\sigma^2}$$

$$\implies \boxed{\rho_X(s,t) = \begin{cases} 1 & ; s = t \\ cos(2\pi\omega_0(s-t)) & ; s \neq t \end{cases}}$$