Time Series Prediction.

given data $x_1, \cdots, x_n$.

→ forecast the value of $x_{n+1}, x_{n+2}, \cdots, x_t, \cdots$.

notation

$x_t^n :=$ what we can expect x_t to be given data $x_1, \cdots, x_n$.

1-step-ahead forecast: x_{n+1}^n,

m-step-ahead forecast: x_{n+m}^n

mean-squared prediction error. (MSPE).

$$P_{n+m}^n = E\left[(x_{n+m} - x_{n+m}^n)^2 \right].$$

x_{n+m}^n → we want to minimize MSPE.

↳ $\underbrace{E[x_{n+m} \mid x_1, \cdots, x_n]}$. → minimize MSPE.

$= g(x_1, \cdots, x_n)$

Random variable X, Y.

Find $g(x)$ which minimize $E[(Y-g(x))^2]$.

$$E[(Y-g(x))^2] = E_X\Big[\underbrace{E[(Y-g(x))^2 \mid X]}\Big]$$

↳ find $g(x)$ which minimize ↓.

$$E[(Y-g(x))^2 \mid X]$$

$$= E[Y^2 - 2g(x)Y + g(x)^2 \mid X].$$

$$= \underline{E[Y^2 \mid X] - 2g(x)E[Y \mid X] + g(x)^2} \quad (*).$$

find $g(x)$ which minimize $(*)$.

$$= g(x)^2 - 2g(x)E[Y \mid X] + (E[Y \mid X])^2$$

$$+ E[Y^2 \mid X] - (E[Y \mid X])^2$$

$$= \underbrace{(g(x) - E[Y \mid X])^2} + E[Y^2 \mid X] - (E[Y \mid X])^2.$$

$g(x) = E[Y \mid X]$ will minimize $(*)$.

$$g(X) = E[Y|X]$$

minimize $E[(Y - g(X))^2]$.

$E[(X_{n+m} - X_{n+m}^{n})^2]$: MSPE

find X_{n+m}^{n} which minimize MSPE.

↳ function of $X_1, \cdots, X_n$.

↳ $g(X_1, \cdots, X_n)$.

MSPE

$$E[(X_{n+m} - g(X_1, \cdots, X_n))^2]$$

$g(X_1, \cdots, X_n) = E[X_{n+m} | X_1, \cdots, X_n]$.

minimizes MSPE.

$$X_{n+m}^{n} = E[X_{n+m} | X_1, \cdots, X_n].$$

$$X_{n+m}^{n} = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n.$$

$$= \alpha_0 + \sum_{k=1}^{n} \alpha_k X_k$$

α_k : Real numbers.

↳ depend on n, m.

Best Linear Predictor (BLP)

Linear Predictor of the form. $\boxed{X_{n+m}^{n} = \alpha_0 + \sum_{k=1}^{n} \alpha_k X_k}$

that minimize $MSPE \left(= E\left[\left(X_{n+m} - X_{n+m}^{n} \right)^2 \right] \right).$

$\boxed{\text{prop}}$ BLP for Stationary process.

given data $X_1, \cdots X_n$, the BLP. of X_{n+m} (for $m \geq 1$).

is found by solving

$$E\left[\left(X_{n+m} - X_{n+m}^{n} \right) X_k \right] = 0, \quad k = 0, \cdots, n.$$

$(X_0 = 1)$, for $\alpha_0, \cdots, \alpha_n$

pf. minimize

$$Q = E\left[\left(X_{n+m} - X_{n+m}^{n} \right)^2 \right].$$

$$X_{n+m}^{n} = \alpha_0 + \alpha_1 X_1 + \cdots + \alpha_n X_n = \sum_{k=0}^{n} \alpha_k X_k.$$

$(X_0 = 1)$.

we want to find $\boxed{\alpha_0, \cdots, \alpha_n}$

solve $\frac{\partial Q}{\partial \alpha_j} = 0$ for α_j.

$$Q = E\left[\left(X_{n+m} - \sum_{k=0}^{n} \alpha_k X_k \right)^2 \right].$$

$$\frac{\partial Q}{\partial \alpha_j} = E\left[2\left(X_{n+m} - \sum_{k=0}^{n} \alpha_k X_k\right)(-X_k) \right] = 0.$$

i.e. $$E\left[\left(X_{n+m} - \sum_{k=0}^{n} \alpha_k X_k\right) X_k \right] = 0.$$

i.e. $$E\left[\left(X_{n+m} - X_{n+m}^{n}\right) X_k \right] = 0$$

If $E[X_t] = \mu$.

$$E[X_{n+m}^{n}] = E[X_{n+m}] = \mu.$$

$$E\left[\left(X_{n+m} - X_{n+m}^{n}\right) \right] = 0 \iff E[X_{n+m}] = E[X_{n+m}^{n}].$$

$$\mu = E[X_{n+m}^{n}] = E\left[\alpha_0 + \sum_{k=1}^{n} \alpha_k X_k \right] = \alpha_0 + \sum_{k=1}^{n} \alpha_k E[X_k]$$

$$= \alpha_0 + \sum_{k=1}^{n} \alpha_k \mu, \qquad \alpha_0 = \mu\left(1 - \sum_{k=1}^{n} \alpha_k\right).$$

BLP =

$$X_{n+m}^{n} = \alpha_0 + \sum_{k=1}^{n} \alpha_k X_k = \mu - \mu \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \alpha_k X_k.$$

$$= \mu + \sum_{k=1}^{n} \alpha_k (X_k - \mu).$$

1-step-ahead-prediction.

given $X_1, \cdots, X_n \Rightarrow$ forecast X_{n+1}.

BLP of $\underline{X_{n+1}}$: X_{n+1}^n

$$X_{n+1}^n = \phi_{n1} X_n + \phi_{n2} X_{n-1} + \cdots + \phi_{nn} X_1.$$

$\{\phi_{n1}, \cdots \phi_{nn}\}$ satisfy

$k = n+1-l$.

$$E\left[\left(X_{n+1} - \sum_{j=1}^{n} \phi_{nj} X_{n+1-j}\right)\underline{X_{n+1-l}}\right] = 0, \quad l = 1, \cdots, n.$$

i.e. $$E\left[X_{n+1} X_{n+1-l}\right] - \sum_{j=1}^{n} \phi_{nj} E\left[X_{n+1-j} X_{n+1-l}\right] = 0.$$

i.e. $$\gamma(l) - \sum_{j=1}^{n} \phi_{nj} \gamma(l-j) = 0.$$

i.e. $$\boxed{\sum_{j=1}^{n} \phi_{nj} \gamma(l-j) = \gamma(l).}$$ for $l = 1, \cdots, n$.

$l=1$:

$$\gamma(1-1)\phi_{n1} + \gamma(1-2)\phi_{n2} + \cdots + \gamma(1-n)\phi_{nn} = \gamma(1).$$

$l=2$:

$$\gamma(2-1)\phi_{n1} + \gamma(2-2)\phi_{n2} + \cdots + \gamma(2-n)\phi_{nn} = \gamma(2).$$

$\vdots$

$l=n$:

$$\gamma(n-1)\phi_{n1} + \gamma(n-2)\phi_{n2} + \cdots + \gamma(n-n)\phi_{nn} = \gamma(n).$$

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \ddots & & \vdots \\ \vdots & & \ddots & \gamma(1) \\ \gamma(n-1) & \cdots & \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_{n1} \\ \vdots \\ \phi_{nn} \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{bmatrix}$$

Γ_n : $n \times n$ matrix : $\Gamma_n(i,j) = \gamma(i-j)$.

$$\gamma_n := \begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{bmatrix}. \qquad \phi_n = \begin{bmatrix} \phi_{n1} \\ \vdots \\ \phi_{nn} \end{bmatrix}$$

$$\Gamma_n \phi_n = \gamma_n. \qquad \phi_n = \Gamma_n^{-1}\gamma_n.$$

$$P_{n+1}^{n} = \gamma(0) - \gamma_n^T \Gamma_n^{-1} \gamma_n.$$

AR(2) example.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + w_t.$$

one observation X_1

$$\rightarrow \quad X_2^1 = \phi_{11} X_1.$$

$$\gamma(0)\phi_{11} = \gamma(1).$$

$$\phi_{11} = \frac{\gamma(1)}{\gamma(0)} = \rho(1).$$

$$X_2^1 = \rho(1) X_1.$$

Two Observations. $X_1, X_2. \rightarrow X_3^2.$

$$X_3^2 = \phi_{21} X_2 + \phi_{22} X_1.$$

$$\gamma(0)\phi_{21} + \gamma(1)\phi_{22} = \gamma(1)$$

$$\gamma(1)\phi_{21} + \gamma(0)\phi_{22} = \gamma(2).$$

$$\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}.$$

$$X_3^2 = \phi_1 X_2 + \phi_2 X_1$$

$$\phi_{21} = \phi_1, \quad \phi_{22} = \phi_2.$$

$$\left.\begin{array}{l} E[(X_3 - \phi_1 X_2 - \phi_2 X_1) X_1] = 0 \\ E[(X_3 - \phi_1 X_2 - \phi_2 X_1) X_2] = 0. \end{array}\right\} \text{True?}$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t.$$

$$X_3 = \phi_1 X_2 + \phi_2 X_1 + W_3.$$

$$X_3 - \phi_1 X_2 - \phi_2 X_1 = W_3.$$

$$E[W_3 X_1] = E[W_3]E[X_1] = 0.$$

$$E[W_3 X_2] = E[W_3]E[X_2] = 0.$$

$n \geq 2.$ $\Rightarrow$ have obs $X_1, \dots, X_n$.

$$\boxed{X_{n+1}^n = \phi_1 X_n + \phi_2 X_{n-1}.}$$

$E[(X_{n+1} - X_{n+1}^n) X_k] = 0$? for $k = 1, \dots, n$.

$\hookrightarrow = E[W_{n+1} X_k]$ for $k = 1, \dots n$

$= 0.$

AR(p), $n \geq p$.

$X_{n+1}^{n} = \phi_1 X_n + \phi_2 X_{n-1} + \cdots + \phi_p X_{n-p+1}$.

1-step ahead prediction.

MSPE? $\rightarrow X_{n+1} = \phi_1 X_n + \phi_2 X_{n-1} + \cdots + \phi_p X_{n-p+1} + W_{n+1}$.

$E[(X_{n+1} - X_{n+1}^{n})^2]$

$= E[(W_{n+1})^2] = \sigma_w^2$.

m-step? :

$$E[(X_{n+m} - X_{n+m}^{n}) X_k] = 0, \quad k = 0, \ldots, n.$$

Given data $X_1, \ldots X_n$.

$X_{n+m}^{n} = \phi_{n1}^{(m)} X_n + \phi_{n2}^{(m)} X_{n-1} + \cdots + \phi_{nn}^{(m)} X_1$.

$\{\phi_{n1}^{(m)}, \phi_{n2}^{(m)}, \ldots, \phi_{nn}^{(m)}\}$.

$E[(X_{n+m} - \sum_{j=1}^{n} \phi_{nj}^{(m)} X_{n+1-j}) X_{n+1-l}] = 0$. for $l = 1, \ldots, n$.

i.e.

$\Rightarrow E[X_{n+m} X_{n+1-l}] - \sum_{j=1}^{n} \phi_{nj}^{(m)} E[X_{n+1-j} X_{n+1-l}] = 0$.

i.e. $\underline{\gamma(m+l-1) = \sum_{j=1}^{n} \phi_{nj}^{(m)} \gamma(l-j)}$.

$$\begin{bmatrix} \gamma(0) & \cdots & \gamma(n-1) \\ \gamma(1) & \ddots & \vdots \\ \vdots & \ddots & \\ \gamma(n-1) & \cdots \ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_{n1}^{(m)} \\ \vdots \\ \phi_{nn}^{(m)} \end{bmatrix} = \begin{bmatrix} \gamma(m) \\ \gamma(m+1) \\ \vdots \\ \gamma(m+n-1) \end{bmatrix}$$).

1-step forecast: AR(1).

$X_{n+1}^{n} = \phi X_n$

$X_{n+1} = \phi X_n + W_{n+1}$

$X_{n+1}^{n} = \phi X_n^{n} + W_{n+1}^{n}$.

we know $X_1, \cdots X_n$ $X_n^{n} = X_n$.

(X_t^{n} for $t=1,\cdots,n \Rightarrow X_t$).

(W_{n+m}^{n} : W_{n+m} is indep. of $X_1, \cdots X_n$

$\hookrightarrow = 0$. $m \geq 1$.

$X_{n+1}^{n} = \phi X_n$.

$X_{n+2}^{n} = \phi X_{n+1}^{n} + W_{n+2}^{n}$.

$= \phi^2 X_n$

$$E[(X_{n+2} - \underset{\sim}{X_{n+2}^{n}})X_k] \quad \text{for} \quad k=1,\cdots,n$$

$$= E[(\phi X_{n+1} + W_{n+2} - (\phi X_{n+1}^{n}))X_k].$$

$$= E[\phi(X_{n+1} - X_{n+1}^{n})X_k + W_{n+2}X_k].$$

$$= \phi \underbrace{E[(X_{n+1} - X_{n+1}^{n})X_k]}_{=0.} + \underbrace{E[W_{n+2}X_k]}_{=0}.$$

$$= 0.$$

$$X_{n+m}^{n} = \phi^{m} X_n.$$

MSPE.

$$P_{n+m}^{n} = E[(X_{n+m} - X_{n+m}^{n})^2]$$

$$= \sigma_w^2(1 + \phi^2 + \cdots + \phi^{2(m-1)}).$$

$$X_{n+m} = \phi X_{n+m-1} + W_{n+m}$$

$$= \phi^2 X_{n+m-2} + \phi W_{n+m-1} + W_{n+m}$$

$$\vdots$$

$$X_{n+m} - \phi^{m} X_n = \underline{\phi^{m-1} W_{n+1} \cdots + W_{n+m}}.$$

$$E[(X_{n+m} - X_{n+m}^{n})^2]$$

$$= E[(X_{n+m} - \phi^{m} X_n)^2] = E[(\phi^{m-1} W_{n+1} \cdots + W_{n+m})^2].$$

$$= \phi^{2(m-1)}\sigma_w^2 + \cdots + \phi^2\sigma_w^2$$

$$= \underbrace{\sigma_w^2\left(1+\phi^2+\cdots+\phi^{2(m-1)}\right)}.$$

$|\phi|<1$ $\quad \phi^m \to 0$ as $m\to\infty$.

$X_{n+m}^n = \phi^m X_n \to 0$ and become useless.

$$\text{MSPE} \to \sigma_w^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma_w^2}{1-\phi^2}.$$

ARMA(p,q) & MA(q) $\Rightarrow$ AR(∞) Representation

$\to$ Then proceed.

$$W_t = X_t + \sum_{j=1}^{\infty}\pi_j X_{t-j}.$$

$$X_{n+m} = -\sum_{j=1}^{\infty}\pi_j X_{n+m-j} + W_{n+m}.$$

If we have infinite history $X_1, \ldots X_n, X_0, X_{-1}, \ldots\ldots$

$\to$ predict X_{n+m} by $X_{n+m}^n = -\sum_{j=1}^{\infty}\pi_j X_{n+m-j}^n$.

$m = 1, 2, \cdots$

$X_{n+1}^{n}, \Rightarrow X_{n+2}^{n} \Rightarrow \cdots$

History $X_1, \cdots, X_n$.

$$X_{n+m}^{n} = -\sum_{j=1}^{n+m-1} \pi_j X_{n+m-j}^{n}.$$

$$= -\sum_{j=1}^{m-1} \pi_j X_{n+m-j}^{n} - \sum_{j=m}^{n+m-1} \pi_j X_{n+m-j}.$$

Calculated Recursively.