$$X_t = \sum_{k=1}^{q} U_{k1} \cos(2\pi \omega_k t) + U_{k2} \sin(2\pi \omega_k t)$$

$$\gamma(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi \omega_k h).$$

$(\omega_k, \sigma_k^2) \rightarrow$ need to estimate .

Spectral Representation Theorem.

Any Stationary Time Series can be represented in the form

$$X_t = \sum_{k=1}^{q} U_{k1} \cos(2\pi \omega_k t) + U_{k2} \sin(2\pi \omega_k t).$$

Spectral Frequency

Suppose $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$

power of ω (frequency)

Then $$\boxed{f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h}.}$$

for $-\frac{1}{2} \leq \omega \leq \frac{1}{2}$.

(If ω is out of this range $\rightarrow$ repeats itself).

A

$f(\omega) = f(-\omega).$

$$\gamma(h) = \int_{-1/2}^{1/2} f(\omega)\, e^{2\pi i \omega h}\, d\omega, \quad \text{for } h = 0, \pm 1, \pm 2, \cdots.$$

$f(\omega)$ is called as spectral Density.

Time domain → we worked w/ $\gamma(h)$.

Frequency domain → " $f(\omega)$.

When X_t is ARMA(p,q) process,

$$f(\omega) = \sigma^2 \frac{\left|\theta\left(e^{-2\pi i \omega}\right)\right|^2}{\left|\phi\left(e^{-2\pi i \omega}\right)\right|^2}.$$

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p.$$

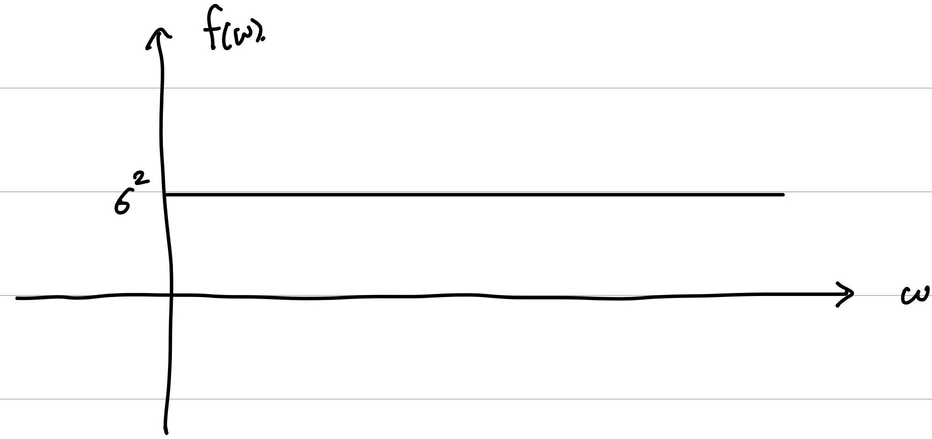
$f(\omega)$ of WN?

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)\, e^{-2\pi i \omega h}.$$

$$\gamma(h) = \begin{cases} \sigma^2 & h=0 \\ 0 & \text{otherwise } (|h| \geq 1). \end{cases}$$

$$f(\omega) = \gamma(0)\, e^{-2\pi i \omega 0}$$

$$= \sigma^2 . e^0 = \sigma^2$$



Ex) MA(1). $X_t = W_t + \theta W_{t-1}$.

$= (1+\theta B) W_t$.

$\phi(z) = 1$. $\theta(z) = 1 + \theta z$.

$$\boxed{f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h}}$$

$\gamma(0) = Var(X_t) = (1+\theta^2)\sigma^2$.

$\gamma(1) = Cov(W_t + \theta W_{t-1}, \; W_{t+1} + \theta W_t)$

$= \theta Var(W_t) = \theta \cdot \sigma^2 = \gamma(-1)$.

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2, & h=0 \\ \theta \cdot \sigma^2 & h = \pm 1. \\ 0 & \text{otherwise.} \end{cases}$$

$f(\omega) = \gamma(0) e^{-2\pi i \omega \cdot 0} + \gamma(1) e^{-2\pi i \omega \cdot 1} + \gamma(-1) e^{-2\pi i \omega (-1)}$.

$= \gamma(0) + \gamma(1) e^{-2\pi i \omega} + \gamma(-1) e^{2\pi i \omega}$.

$e^{ib} = \cos b + i \sin b$.

$= (1+\theta^2)\sigma^2 + \theta \cdot \sigma^2 \{ e^{-2\pi i \omega} + e^{2\pi i \omega} \}$.

$b = 2\pi\omega$.

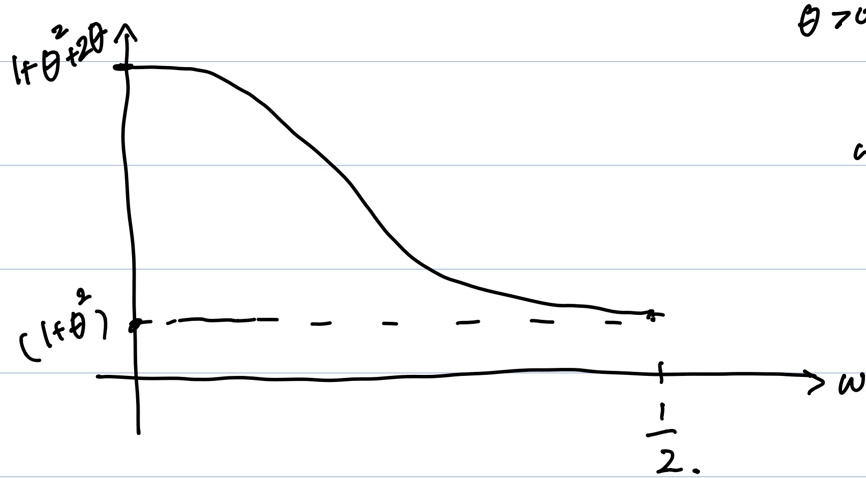
$e^{ib} + e^{-ib} = \cos b + i \sin b + \cos(-b) + i \sin(-b)$.

$= 2\cos b \cancel{+ i \sin b} \cancel{- i \sin b}$

$$= (1+\theta^2)\sigma^2 + \theta\sigma^2 \cdot 2\cos(2\pi\omega)$$

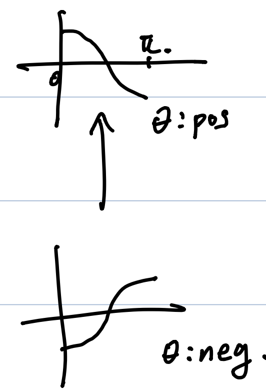
$$= \sigma^2 \{ 1+\theta^2 + 2\theta\cos(2\pi\omega) \}.$$

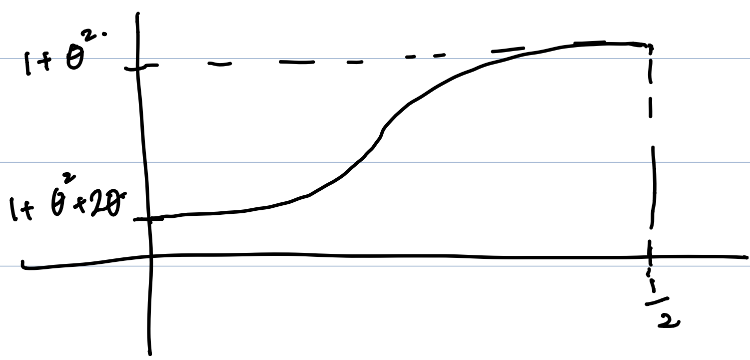
$\theta > 0.$



$\omega=0$ to $\omega=\frac{1}{2}$.

$\cos 0$ to $\cos\pi$.





$$f(\omega) = \sigma^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2} \quad \text{for ARMA.}$$

$$f(\omega) = \sigma^2 \,|1+\theta e^{-2\pi i\omega}|^2.$$

complex z : $|z|^2 = z\cdot\bar{z}$.

conjugate of $e^{ib} = e^{-ib}$.

$$e^{ib} = \cos b + i\sin b.$$

$$e^{-ib} = \cos(-b) + i\sin(-b)$$

$$= \cos b - i\sin b = \overline{e^{ib}}.$$

$$\left|1 + \theta e^{-2\pi i w}\right|^2 = (1 + \theta e^{-2\pi i w})(1 + \theta e^{2\pi i w}).$$

$$= 1 + \theta e^{-2\pi i w} + \theta e^{2\pi i w} + \theta^2 e^0.$$

$$= 1 + \theta^2 + \theta \cdot (e^{-2\pi i w} + e^{2\pi i w}).$$

$$= 1 + \theta^2 + \theta \cdot 2\cos(2\pi w).$$

$$f(w) = \sigma^2 \cdot \left|1 + \theta e^{-2\pi i w}\right|^2 = \boxed{\sigma^2 \left(1 + \theta^2 + 2\theta\cos(2\pi w)\right)}.$$