Spectral analysis: Intro

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Spectral view of Time series

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Shift of thinking from time domain to frequency domain

$$f(t) = A\cos(2\pi\omega t), \quad \omega > 0$$

- A: amplitude
- cycle length: $1/\omega$

$$g(t) = A\sin(2\pi\omega t), \quad \omega > 0$$

$$h(t) = A\cos(2\pi\omega t + \psi)$$

ψ: phase

- Time Series: we play with amplitude and frequency.
- Collect (A, ω) for periodic series and superpose them.

(Orchestra music: superpose sounds of violin, flute, $\ldots$)

Basic concept of Periodic Time Series

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$$X_t = A\cos(2\pi\omega t + \psi)$$

$$\begin{aligned} X_t =& A\cos(2\pi\omega t + \psi) \\ =& A\cos(\psi)\cos(2\pi\omega t) - A\sin(\psi)\sin(2\pi\omega t), \end{aligned}$$

with $U_1 = A\cos(\psi)$, $U_2 = -A\sin(\psi)$,

$$X_t = U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t)$$

Basic model for periodic series

Basic model for periodic series

$$X_t = U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t)$$

U_1, U_2 : uncorrelated random variables, zero-mean, variance σ^2.

- What is parameter here? ω, σ^2
- Is it stationary? Yes.

1. $E[X_t] = E[U_1]\cos(2\pi\omega t) + E[U_2]\sin(2\pi\omega t) = 0$
2. Covariance:

$$
\begin{aligned}
&Cov(X_t, X_s) \\
&= Cov(U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t), U_1\cos(2\pi\omega s) + U_2\sin(2\pi\omega s)) \\
&= \sigma^2\cos(2\pi\omega(t-s))
\end{aligned}
$$

$$\gamma(h) = \sigma^2\cos(2\pi\omega h)$$

Seasonality is built in here.

Increase h does not necessarily mean correlation decrease in h.

- $\gamma(0) = \sigma^2$
- $\rho(h) = \cos(2\pi\omega h)$
- Amplitude: $A = \sqrt{U_1^2 + U_2^2}$

General model for periodic series

General model for periodic series

$$X_t = \sum_{k=1}^{q} U_{1,k} \cos(2\pi\omega_k t) + U_{2,k} \sin(2\pi\omega_k t)$$

- estimate?
  - underlying parameters of $U_{1,k}, U_{2,k}$
  - how many q? $\omega_1, \dots, \omega_q$?
- From data, we deduce a model.
- Fast Fourier Transform: we can figure out all parameters. (discuss later)

Assumptions for general model

For $k = 1, \ldots, q$,

1. $U_{1,k}, U_{2,k}$, are uncorrelated
2. $E[U_{1,k}] = E[U_{2,k}] = 0$
3. $Var(U_{1,k}) = Var(U_{2,k}) = \sigma_k^2$

Mean function: $E[X_t] = 0$

Autocovaraince:

$$\gamma(t, s) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k(t - s))$$