HW 03

Name: Paul Holaway, NetID: paulch2

Due: 9/15/2022 11:59pm

- Unless stated otherwise, W_t is a white noise process with variance σ_w^2.
  - (W_t are independent with zero means and variance σ_w^2.)
- Show your full work to receive full credit.
- For question 1 ~ question 3, you may use your answers from HW 02 directly.

Question 1

Consider the time series

$$X_t = \beta_1 + \beta_2 t + W_t,$$

where β_1 and $\beta_2 (\neq 0)$ are known constants.

(a) Is X_t stationary? Why or why not?

Recall:

$$\mu_X(t) = \beta_1 + \beta_2 t$$

$$\gamma_X(s,t) = \begin{cases} \sigma_w^2 & ; s = t \\ 0 & ; s \neq t \end{cases}$$

X_t is not stationary as the mean is not constant. It depends on t.

(b) Is $Y_t = X_t - X_{t-1}$ stationary? Support your answer.

Recall:

$$\mu_Y(t) = \beta_2$$

$$\gamma_Y(s,t) = \begin{cases} 2\sigma_w^2 & ; s = t \\ -\sigma_w^2 & ; |s-t| = 1 \\ 0 & ; |s-t| \geq 2 \end{cases}$$

Y_t is stationary as the mean is constant (does not depend on t) and the autocovariance function only depends on the lag (time difference between s and t).

Question 2

For a moving average process of the form

$$X_t = 0.1W_{t-2} + 0.2W_{t-1} + W_t,$$

is this process stationary? Support your answer.

Recall:

$$\mu_X(t) = 0$$

$$\gamma_X(s,t) = \begin{cases} 1.05\sigma_w^2 & ; s = t \\ 0.22\sigma_w^2 & ; |s-t| = 1 \\ 0.1\sigma_w^2 & ; |s-t| = 2 \\ 0 & ; |s-t| \geq 3 \end{cases}$$

X_t is stationary as the mean is constant (does not depend on t) and the autocovariance function only depends on the lag (time difference between s and t).

Question 3

A time series with a periodic component can be constructed from

$$X_t = U_1 \sin(2\pi\omega_0 t) + U_2 \cos(2\pi\omega_0 t),$$

where U_1 and U_2 are independent random variables with zero means and variances σ^2. The constant ω_0 determines the period or time it takes the process to make one complete cycle. Is this process stationary? Why or why not?

Recall:

$$\mu_X(t) = 0$$

$$\gamma_X(t) = \begin{cases} \sigma^2 & ; s = t \\ \sigma^2 cos(2\pi\omega_0(s-t)) & ; s \neq t \end{cases}$$

X_t is stationary as the mean is constant (does not depend on t) and the autocovariance function only depends on the lag (time difference between s and t).

Question 4

Suppose that U is a random variable with zero mean with finite variance σ^2. Define a time series by $X_t = (-1)^t U$. Find the mean function and autocovariance function of X_t. Is X_t stationary? Why or why not?

$$\mu_X(t) = \mathbb{E}((-1)^t U) = (-1)^t \mathbb{E}(U) = (-1)^t * 0$$

$$\Longrightarrow \boxed{\mu_X(t) = 0}$$

$$\gamma_X(s,t) = Cov(X_s, X_t) = Cov((-1)^s U, (-1)^t U) = (-1)^{s+t} Cov(U,U) = (-1)^{s+t} Var(U) = (-1)^{s+t}\sigma^2$$

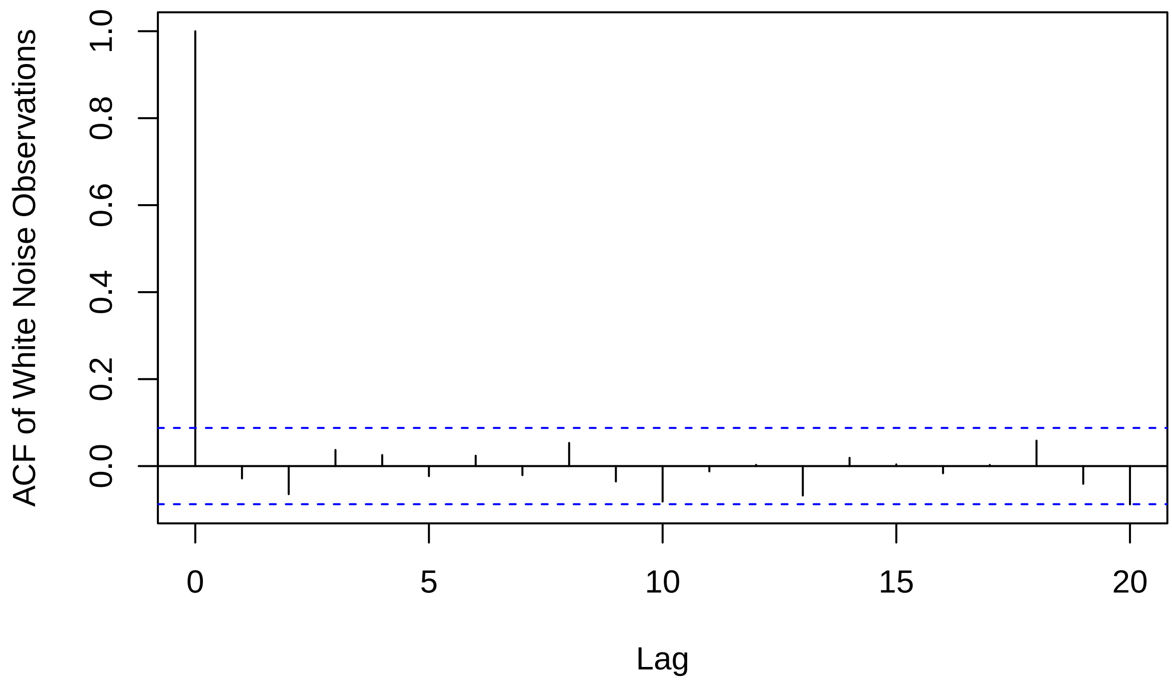
$$\Longrightarrow \boxed{\gamma_X(s,t) = \begin{cases} \sigma^2 & ; |s-t| = Even \\ -\sigma^2 & ; |s-t| = Odd \end{cases}}$$

X_t is stationary as the mean is constant (does not depend on t) and the autocovariance function only depends on the lag (time difference between s and t).

Question 5

(a) Simulate a series of $n = 500$ Gaussian white noise observations and compute the sample ACF, $\hat{\rho}(h)$, to lag 20. Compare the sample ACF you obtain to the actual ACF, $\rho(h)$.

```
set.seed(143572)
w1 = rnorm(500, mean = 0, sd = 1)
acf(w1, ylab = "ACF of White Noise Observations", lag.max = 20, main = "")
```

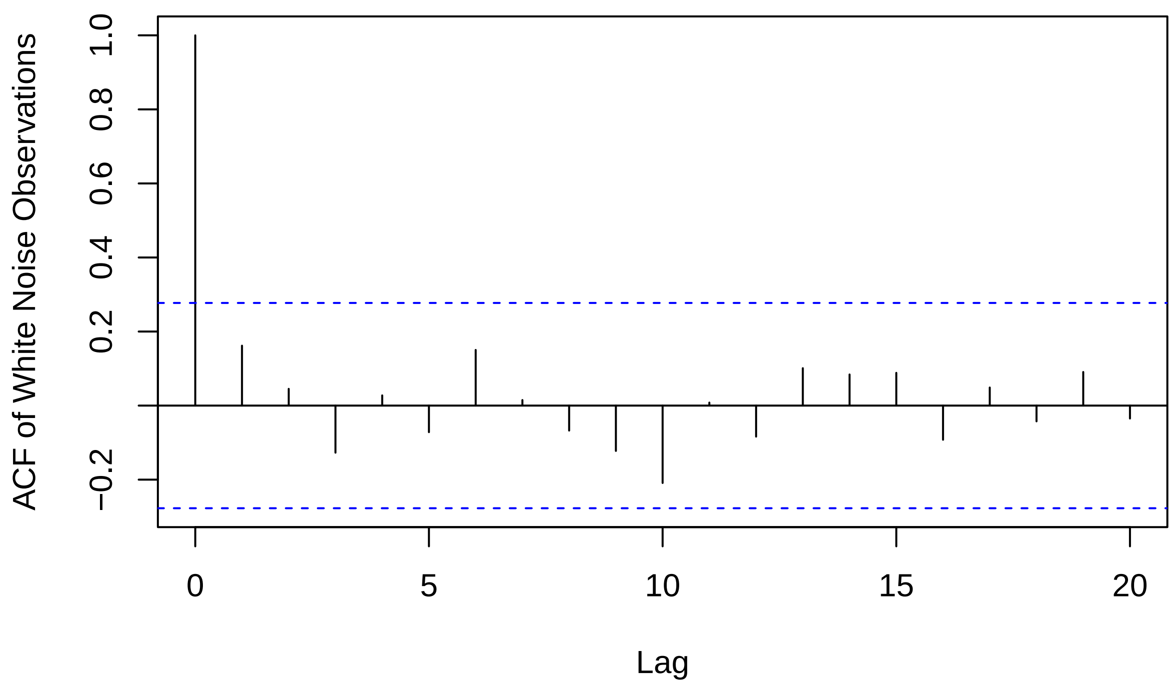


$$\rho_X(s,t) = \begin{cases} 1 & ; s = t \\ 0 & ; s \neq t \end{cases}$$

From the simulated white noise we can see that the sample ACF matches the actual ACF. When the lag $h = 0$, we see that the sample ACF $\hat{\rho}_X(0) = 1$ which matches what the actual ACF $\rho_X(0) = 1$. Then from the plot we can see that for all $h \neq 0$ the sample ACF is statistically no different than 0, which matches what the actual ACF should be.

(b) Repeat part (a) using only $n = 50$. Does changing n affect the results?

```
set.seed(143572)
w2 = rnorm(50, mean = 0, sd = 1)
acf(w2, ylab = "ACF of White Noise Observations", lag.max = 20, main = "")
```



Changing n does not affect the results. $\hat{\rho}_X(0) = \rho_X(0) = 1$ and for all other $\rho_X(h)$ with $h \neq 0$ are still statistically no different than 0, so $\hat{\rho}_X(0) = \rho_X(0) = 0$. We would most likely not see any differences unless we made n really small.

Question 6

Suppose that $X_t = \mu + W_t + \theta W_{t-1}$, where $W_t \sim WN(0, \sigma_w^2)$.

(a) Show the mean function is μ.

$$\mu_X(t) = \mathbb{E}(X_t) = \mathbb{E}(\mu + W_t + \theta W_{t-1}) = \mathbb{E}(\mu) + \mathbb{E}(W_t) + \mathbb{E}(\theta W_{t-1}) = \mu + 0 + 0$$
$$\implies \boxed{\mu_X(t) = \mu}$$

(b) Show that the autocovariance function of X_t is given by $\gamma_x(t,t) = \sigma_w^2(1+\theta^2)$, $\gamma_x(t \pm 1, t) = \sigma_w^2\theta$, and $\gamma_x(t+h,t) = 0$ otherwise.

$$\gamma_X(s,t) = Cov(X_s, X_t) = Cov(\mu + W_s + \theta W_{s-1}, \mu + W_s + \theta W_{t-1}) = Cov(W_s + \theta W_{s-1}, W_t + \theta W_{t-1})$$

$$= Cov(W_s, W_t) + \theta Cov(W_{s-1}, W_t) + \theta Cov(W_s, W_{t-1}) + \theta^2 Cov(W_{s-1}, W_{t-1})$$

1. $s = t \implies \gamma_X(s,t) = \sigma_w^2 + 0 + 0 + \theta^2\sigma_w^2 = (1+\theta^2)\sigma_w^2$
2. $s = t+1 \implies \gamma_X(s,t) = 0 + \theta\sigma_w^2 + 0 + 0 = \theta\sigma_w^2$
3. $s = t-1 \implies \gamma_X(s,t) = 0 + 0 + \theta\sigma_w^2 + 0 = \theta\sigma_w^2$
4. $s = t+2 \implies \gamma_X(s,t) = 0 + 0 + 0 + 0 = 0$
5. $s = t-2 \implies \gamma_X(s,t) = 0 + 0 + 0 + 0 = 0$

$$\implies \boxed{\gamma_X(s,t) = \begin{cases} (1+\theta^2)\sigma_w^2 & ; s = t \\ \theta\sigma_w^2 & ; |s-t| = 1 \\ 0 & ; |s-t| \geq 2 \end{cases}}$$

(c) Show that x_t is stationary for all values of $\theta \in \mathbb{R}$.

First, the mean function, $\mu_X(t) = \mu$. Since the mean function does not depend on the value of θ,it will be constant. Second, the autocovariance function...

1. If $\theta = 0$

$$\gamma_X(s,t) = \begin{cases} \sigma_w^2 & ; s = t \\ 0 & ; |s-t| \geq 1 \end{cases}$$

2. If $\theta > 0$

$$\gamma_X(s,t) = \begin{cases} (1+\theta^2)\sigma_w^2 > 0 & ; s = t \\ \theta\sigma_w^2 > 0 & ; |s-t| = 1 \\ 0 & ; |s-t| \geq 2 \end{cases}$$

3. If $\theta < 0$

$$\gamma_X(s,t) = \begin{cases} (1+\theta^2)\sigma_w^2 > 0 & ; s = t \\ \theta\sigma_w^2 < 0 & ; |s-t| = 1 \\ 0 & ; |s-t| \geq 2 \end{cases}$$

For each of the three cases we can see that regardless of what value of θ is chosen, the autocovariance function still just depends on the lag. Also, there is no value of θ that can make $\gamma_X(s,t) < 0$ when $s = t$ nor a value that would make $\gamma_X(s,t)$ undefined. Therefore X_t is stationary for all values of $\theta \in \mathbb{R}$.

(d) Note that

$$Var(\bar{X}) = \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma_x(h).$$

Using this, calculate $Var(\bar{X})$ for estimating μ when $\theta = 1$, $\theta = 0$, and $\theta = -1$.

$$Var(\bar{X}) = \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma_x(h) = \frac{\gamma_X(0)}{n} + \frac{2}{n}\sum_{h=1}^{n}(1 - \frac{h}{n})\gamma_X(h)$$

- $\theta = 1$

$$Var(\bar{X}) = \frac{(1+(1)^2)\sigma_w^2}{n} + \frac{2}{n}(1 - \frac{1}{n})\sigma_w^2 = \frac{2\sigma_w^2}{n} + \frac{2\sigma_w^2}{n} - \frac{2\sigma_w^2}{n^2}$$
$$\implies \boxed{Var(\bar{X}) = \frac{4\sigma_w^2}{n} - \frac{2\sigma_w^2}{n^2}}$$

- $\theta = 0$

$$Var(\bar{X}) = \frac{(1+(0)^2)\sigma_w^2}{n} + \frac{2}{n}(1 - \frac{1}{n})(0) = \frac{\sigma_w^2}{n}$$
$$\implies \boxed{Var(\bar{X}) = \frac{\sigma_w^2}{n}}$$

- $\theta = -1$

$$Var(\bar{X}) = \frac{(1+(-1)^2)\sigma_w^2}{n} + \frac{2}{n}(1 - \frac{1}{n})(-\sigma_w^2) = \frac{2\sigma_w^2}{n} - \frac{2\sigma_w^2}{n} + \frac{2\sigma_w^2}{n^2}$$
$$\implies \boxed{Var(\bar{X}) = \frac{2\sigma_w^2}{n^2}}$$

(e) (grad-only) In time series, the sample size n is typically large, so that $(n-1)/n \approx 1$. With this as a consideration, comment on the results of (d). In particular, how does accuracy in the estimate of the mean μ change for the three different cases?

In all three cases $Var(\bar{X}) \to 0$ as $n \to \infty$. However, the rate at which $Var(\bar{X}) \to 0$ will be different for the three different cases. We know that any constant over n^2 will approach 0 faster than when over n. Even if the constants are different, the quantity will be smaller when dividing by n^2 than when dividing by n if n is large. We can also say that for large n, $\frac{2\sigma_w^2}{n} - \frac{2\sigma_w^2}{n^2} = \frac{2\sigma_w^2(2n-1)}{n^2} \approx \frac{4\sigma_w^2}{n}$. This means that for large n, if $\theta = -1$ the variance will be the smallest, if $\theta = 0$ the variance will be the next smallest, and if $\theta = 1$ the variance will be the largest. The numerator will be larger for the variance when $\theta = 1$ than when $\theta = 0$. As a result, the accuracy of the estimate of the mean μ will be the best when $\theta = -1$, the next best when $\theta = 0$, and the worst when $\theta = 1$.

Question 7 (Grad-Only)

Consider the time series

$$X_t = \beta_1 + \beta_2 t + W_t,$$

where β_1 and $\beta_2 (\neq 0)$ are known constants. Let

$$V_t = \frac{1}{2q+1} \sum_{j=-q}^{j=q} X_{t-j}$$

Show mean and simplified expression for the autocovariance function of V_t. Is V_t stationary?

$$\mathbb{E}(V_t) = \mathbb{E}(\frac{1}{2q+1} \sum_{j=-q}^{j=q} X_{t-j}) = \frac{1}{2q+1} \sum_{j=-q}^{j=q} \mathbb{E}(X_{t-j}) = \frac{1}{2q+1} (\mathbb{E}(X_t) + \sum_{j=1}^{q} \mathbb{E}(X_{t-j}) + \sum_{j=-1}^{-q} \mathbb{E}(X_{t-j}))$$

$$\mathbb{E}(X_t) = \mathbb{E}(\beta_1 + \beta_2 t + W_t) = \mathbb{E}(\beta_1) + \mathbb{E}(\beta_2 t) + \mathbb{E}(W_t) = \beta_1 + \beta_2 t$$

$$\mathbb{E}(X_{t-j}) = \mathbb{E}(\beta_1 + \beta_2 (t-j) + W_{t-j}) = \mathbb{E}(\beta_1) + \mathbb{E}(\beta_2 (t-j)) + \mathbb{E}(W_{t-j}) = \beta_1 + \beta_2 t - \beta_2 j$$

$$\sum_{j=1}^{q} (\beta_1 + \beta_2 t - \beta_2 j = (\beta_1 + \beta_2 t) q - \beta_2 \sum_{j=1}^{q} j)$$

$$\sum_{j=-1}^{-q} (\beta_1 + \beta_2 t - \beta_2 j = (\beta_1 + \beta_2 t)(-q) - \beta_2 \sum_{j=-1}^{-q} j)$$

$$\mathbb{E}(V_t) = \frac{\beta_1 + \beta_2 t + \beta_1 q + \beta_2 t q - \beta_2 \sum_{j=1}^{q} j - \beta_1 q - \beta_2 t q - \sum_{j=-1}^{-q} j}{2q+1}$$

$$\mathbb{E}(V_t) = \frac{\beta_1 + \beta_2 t - \beta_2 (1 + 2 + 3 + \ldots + q) - \beta_2 (-1 - 2 - 3 - \ldots - q)}{2q+1}$$

$$\boxed{\mathbb{E}(V_t) = \frac{\beta_1 + \beta_2 t}{2q+1}}$$

$$\gamma_X(s,t) = Cov(V_s, V_t) = Cov(\frac{1}{2q+1} \sum_{j=-q}^{j=q} X_{s-j}, \frac{1}{2q+1} \sum_{j=-q}^{j=q} X_{t-j}) = \frac{1}{(2q+1)^2} Cov(\sum_{j=-q}^{j=q} X_{s-j}, \sum_{j=-q}^{j=q} X_{t-j})$$

$= \frac{1}{(2q+1)^2} Cov(\beta_1 + \beta_2 s + W_s + \sum_{j=1}^{q} \beta_1 + \beta_2 (s-j) + W_{s-j} + \sum_{j=-1}^{q} \beta_1 + \beta_2 (s-j) + W_{s-j}, \beta_1 + \beta_2 t + W_t + \sum_{j=1}^{q} \beta_1 + \beta_2 (t-j) + W_{t-j} + \sum_{j=-1}^{-q} \beta_1 + \beta_2 (t-j) + W_{t-j})$

$= \frac{1}{(2q+1)^2} Cov(\beta_1 + \beta_2 s + W_s + \beta_1 q + \beta_2 s q - \beta_2 \sum_{j=1}^{q} j + \sum_{j=1}^{q} W_{s-j} - \beta_1 q - \beta_2 s q - \beta_2 \sum_{j=-1}^{-q} j + \sum_{j=1}^{-q} W_{s-j}, \beta_1 + \beta_2 t + W_t + \beta_1 q + \beta_2 t q - \beta_2 \sum_{j=1}^{q} j + \sum_{j=1}^{q} W_{t-j} - \beta_1 q - \beta_2 t q - \beta_2 \sum_{j=-1}^{-q} j + \sum_{j=1}^{-q} W_{t-j})$

$$= \frac{1}{(2q+1)^2} Cov(\beta_1 + \beta_2 s + \sum_{j=-q}^{j=q} W_{s-j}, \beta_1 + \beta_2 t + \sum_{j=-q}^{j=q} W_{t-j})$$

$$= \frac{1}{(2q+1)^2} Cov(\beta_2 s + \sum_{j=-q}^{j=q} W_{s-j}, \beta_2 t + \sum_{j=-q}^{j=q} W_{t-j})$$

$$= \frac{1}{(2q+1)^2} Cov(\sum_{j=-q}^{j=q} W_{s-j}, \sum_{j=-q}^{j=q} W_{t-j})$$

$$= \frac{1}{(2q+1)^2} \sum_{j=-q}^{j=q} Cov(W_{s-j}, W_{t-j})$$

$$\forall j \to Cov(W_{s-j}, W_{t-j}) = \begin{cases} \sigma_w^2 & ; s = t \\ 0 & ; s \neq t \end{cases}$$

$$= \begin{cases} \frac{1}{(2q+1)^2} \sum_{j=-q}^{j=q} \sigma_w^2 & ; s = t \\ 0 & ; s \neq t \end{cases}$$

$$\boxed{\gamma_x(s,t) = \begin{cases} \frac{\sigma_w^2}{2q+1} & ; s = t \\ 0 & ; s \neq t \end{cases}}$$

V_t is not stationary as the mean is not constant. It depends on t.