HW 07

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Due: 10/13/2022 11:59pm

1.

You have a zero-mean $AR(2)$ process with $n = 144$ sample size. You have
$\hat{\gamma}(0) = 9$, $\hat{\rho}(1) = 0.8$, $\hat{\rho}(2) = 0.6$.

Find the Yule-Walker estimates of ϕ_1, ϕ_2, and $\hat{\sigma_W}^2$.

$$\begin{cases} \hat{\rho}(1) = \hat{\phi}_1 \hat{\rho}(0) + \hat{\phi}_2 \hat{\rho}(1) \\ \hat{\rho}(2) = \hat{\phi}_1 \hat{\rho}(1) + \hat{\phi}_2 \hat{\rho}(0) \end{cases}$$

$$\implies \begin{cases} 0.8 = \hat{\phi}_1 + 0.8\hat{\phi}_2 \\ 0.6 = 0.8\hat{\phi}_1 + \hat{\phi}_2 \end{cases}$$

$$\implies \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}$$

$$\implies \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$\implies \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{25}{9} & -\frac{20}{9} \\ -\frac{20}{9} & \frac{25}{9} \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$\implies \boxed{\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} \\ -\frac{1}{9} \end{bmatrix}}$$

$$\hat{\sigma}_w^2 = \hat{\gamma}(0)(1 - \hat{\phi}_1 \hat{\rho}(1) - \hat{\phi}_2 \hat{\rho}(2))$$

$$= 9(1 - \frac{8}{9}(0.8) + \frac{1}{9}(0.6))$$

$$\boxed{\hat{\sigma}_w^2 = 3.2}$$

Question 2.

Read the 'ar1_stat429_fa22.txt' file. This is a simulated AR(1) process. Find Yule-Walker Estimation method (use R) to find ϕ and σ_W^2 of the process.

```
Q2 <- read.csv("~/Classes/STAT429 (UIUC)/Data/ar1_stat429_fa22.txt", sep="",
               stringsAsFactors=TRUE)
Q2_yw = ar.yw(Q2, order = 1)
#Phi Parameter Estimate
phi.yw = Q2_yw$ar
#Sigma^2 Estimate
mu = Q2_yw$x.mean
v = mean((Q2$x - mu)^2)
s2 = v * (1 - Q2_yw$ar^2)
```

- $\boxed{\hat{\phi} \approx 0.6281}$
- $\boxed{\hat{\sigma}_w^2 \approx 0.9284}$

Question 3.

Read the 'ma1_stat429_fa22.txt' file. This is a simulated MA(1) process.

```
Q3 <- read.csv("~/Classes/STAT429 (UIUC)/Data/ma1_stat429_fa22.txt", sep="",
               stringsAsFactors=TRUE)
```

(a) Compute Method of Moment estimate of $\hat{\theta}$

```
p = acf(Q3, plot = FALSE)
p1 = p$acf[2] #Lag 1
```

$$\hat{\rho}(1) = \frac{\hat{\theta}}{1 + \hat{\theta}^2} \implies (1 + \hat{\theta}^2)\hat{\rho}(1) = \hat{\theta} \implies \hat{\rho}(1) + \hat{\rho}(1)\hat{\theta}^2 = \hat{\theta} \implies \hat{\rho}(1)\hat{\theta}^2 - \hat{\theta} + \hat{\rho}(1) = 0$$

$$\hat{\theta} = \frac{1 \pm \sqrt{1 - 4(\hat{\rho}(1))^2}}{2\hat{\rho}(1)}$$

```
#Calculating theta hat
theta.mom = (1 + c(-1,1) * sqrt(1 - 4 * p1^2)) / (2 * p1)
```

We have two solutions for $\hat{\theta}$, so we will pick the one that makes the MA(1) process invertible. Therefore our Method of Moments estimate of $\hat{\theta}$ will be $\boxed{\hat{\theta} \approx 0.8029}$ as for that value of $\hat{\theta}$, the root of the characteristic polynomial will be outside the unit circle.

(b) Use the Gauss-Newton Algorithm to find the Conditional LS estimate, $\hat{\theta}$ and $\hat{\sigma_W}^2$. For the initial step, use answer from part (a). (You may use R and example 3.32 of [TSA4], but you need to explain your steps.)

This chunk of code uses the formula from example 3.32. We first start by initializing our $W_t(\theta)$ and $Z_t(\theta)$. Using our initial value we then iterate it using the formulas and then updating our $\hat{\theta}$. After updating our $\hat{\theta}$, we then check to see if the $|\hat{\theta}_{j+1} - \hat{\theta}_j|<$tolerance. Here I chose the value of 0.0001 for the tolerance. After that, we then use the conditional sum of squares for that value of $\hat{\theta}$ to calculate our $\hat{\sigma}_w^2$.

```r
#Number of Observations in Data
n = nrow(Q3)
#Initializing W(theta), Z(theta), and Theta Vectors
w = z = theta = rep(0,n)
#Starting Value of Theta using MoM Estimate
theta[1] = (1 - sqrt(1 - 4 * p1^2)) / (2 * p1)
#Setting Up Counter for Value of Theta in Formula
t = 1

repeat{
  for(i in 2:n){
    #Calculation of all W
    w[i] = Q3[i,1] - theta[t]*w[i-1]
    #Calculation of all Z
    z[i] = w[i-1] - theta[t]*z[i-1]
  }
  #Calculating the conditional error.
  Sc = sum(w^2)
  #Calculating numerator of update
  Szw = sum(z*w)
  #Calculating denominator of update
  Sz = sum(z^2)
  #Calculating new theta
  theta[t+1] = theta[t] + Szw / Sz
  #Test to see if we may stop iterating.
  #We stop if the absolute difference between the current and previous theta is
  #less than 0.0001.
  if(t > 1){
    if(abs(theta[t] - theta[t-1]) < 0.0001){
      break
    } else {
      #Updating t for next iteration.
      t = t + 1
    }
  } else {
    #Updating t for next iteration.
    t = t + 1
  }
}
theta = data.frame(theta)
theta = theta %>% filter(theta != 0)
#theta.hat using Gauss-Newton Algorithm
theta.hat = theta[t,1]
#sigma^2.hat using Gauss-Newton Algorithm
sigma2.hat = Sc / (nrow(Q3) - 1)
```

- $\boxed{\hat{\theta} \approx 0.7727}$
- $\boxed{\hat{\sigma}_w^2 \approx 0.9751}$

(c) Find the Unconditional LS estimate (MLE) of $\hat{\theta}$ and $\hat{\sigma_W}^2$. Use R.

```
uncond = sarima(Q3$x, p = 0, d = 0, q = 1, details = FALSE)
#Finding the Unconditional LS Estimate of theta.hat
theta.ls = uncond$fit$coef[1]
#Finding the Unconditional LE Estimate of sigma^2.hat
sigma2.ls = uncond$fit$sigma2
```

- $\boxed{\hat{\theta} \approx 0.776}$
- $\boxed{\hat{\sigma}_w^2 \approx 0.969}$

Question 4.

Read the 'arma11_stat429_fa22.txt' file. This is a simulated ARMA(1,1) process.

```
Q4 <- read.csv("~/Classes/STAT429 (UIUC)/Data/arma11_stat429_fa22.txt", sep="",
               stringsAsFactors=TRUE)
```

(a) Compute Method of Moment estimates of $\hat{\theta}$ and $\hat{\phi}$.

From the autocovariance function of ARMA(1,1) we know...

$$\gamma_X(h) = \phi\gamma_X(h) \implies \gamma_X(h) - \phi\gamma_X(h) = 0 \implies \gamma_X(h) = C\phi^h; h = 1, 2, \ldots$$

To solve for c...

$$\text{Let h} = 1 \implies \gamma_X(1) = C\phi \implies C = \frac{\gamma_X(1)}{\phi}$$

Using equation 3.48 from TSA4...

$$\gamma_X(0) = \phi\gamma_X(1) + \sigma_w^2(1 + \theta\phi + \theta^2) \text{ and } \gamma_X(1) = \phi\gamma_X(0) + \sigma_w^2\theta$$

By plugging $\gamma_X(1) into \gamma_X(0)$ we get...

$$\gamma_X(0) = \phi(\phi\gamma_X(0) + \sigma_w^2\theta) + \sigma_w^2(1 + \theta\phi + \theta^2) = \phi^2\gamma_X(0) + \sigma_w^2\theta\phi + \sigma_w^2(1 + \theta\phi + \theta^2)$$

$$\implies \gamma_X(0) = \sigma_w^2(\frac{1 + 2\theta\phi + \theta^2}{1 - \phi^2})$$

By plugging $\gamma_X(0) into \gamma_X(1)$ we get...

$$\gamma_X(1) = \phi(\phi\gamma_X(1) + \sigma_w^2(1 + \theta\phi + \theta^2)) + \sigma_w^2\theta$$

$$\gamma_X(1) = \phi^2\gamma_X(1) + \sigma_w^2(\phi + \theta\phi^2 + \theta^2\phi + \theta)$$

$$\implies \gamma_X(1) = \sigma_w^2(\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2})$$

$$\gamma_X(h) = \frac{\gamma_X(1)}{\phi}\phi^h = \sigma_w^2\left(\frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}\right)\phi^{h-1}; h \geq 1$$

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \left(\frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\right)\phi^{h-1}; h \geq 1$$

For $\hat{\theta}$...

$$\hat{\rho}_X(2) = \phi\hat{\rho}_X(1) \implies \hat{\phi} = \frac{\hat{\rho}_X(2)}{\hat{\rho}_X(1)}$$

For $\hat{\theta}$ we can use the value we found for $\hat{\theta}$ and the quadratic formula...

$$\hat{\rho}_X(1) = \frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2} \implies (1+2\theta\phi+\theta^2)\hat{\rho}_X(1) = (1+\theta\phi)(\phi+\theta)$$

$$\implies \hat{\rho}_X(1) + 2\phi\hat{\rho}_X(1)\theta + \hat{\rho}_X(1)\theta^2 = \phi + \theta + \phi^2\theta + \phi\theta^2$$

$$\implies (\phi - \hat{\rho}_X(1))\theta^2 + (\phi^2 + 1 - 2\hat{\rho}_X(1))\theta + \phi - \hat{\rho}_X(1) = 0$$

By using the quadratic formula...

$$\hat{\theta} = \frac{-(\hat{\phi}^2 + 1 - 2\hat{\phi}\hat{\rho}_X(1)) \pm \sqrt{(\hat{\phi}^2 + 1 - 2\hat{\phi}\hat{\rho}_X(1))^2 - 4(\hat{\phi} - \hat{\rho}_X(1))(\hat{\phi} - \hat{\rho}_X(1))}}{2(\hat{\phi} - \hat{\rho}_X(1))}$$

```
r = acf(Q4$x, plot = FALSE)
r1 = r$acf[2] #Lag 1
r2 = r$acf[3] #Lag 2
#MoM Est for phi.hat
phi.hat = r2/r1
#MoM Est for theta.hat
a = phi.hat - r1
b = phi.hat^2 - 2*phi.hat*r1 + 1
c = phi.hat-r1
theta.hat = (-b + c(-1,1)*sqrt(b^2 - 4*a*c))/(2*a)
```

As we can see, there will be two values given to us from solving the equation above, so we will choose the value of $\hat{\theta}$ that will make the ARMA(1,1) model invertible (the root of the characteristic polynomial will be outside the unit circle).

- $\boxed{\hat{\phi} \approx 0.65}$
- $\boxed{\hat{\theta} \approx 0.872}$

(b) Find the Unconditional LS estimate (MLE) of $\hat{\theta}$ and $\hat{\sigma_W}^2$. Use R.

```
MLE = sarima(Q4$x, p = 1, d = 0, q = 1, details = FALSE)
#Finding the Unconditional LS Estimate of theta.hat
coef = MLE$fit$coef
#Finding the Unconditional LS Estimate of sigam^2.hat
s = MLE$fit$sigma2
```

- $\boxed{\hat{\phi} \approx 0.6474}$
- $\boxed{\hat{\theta} \approx 0.8952}$
- $\boxed{\hat{\sigma}_w^2 \approx 0.9115}$

(c) [Grad-only] Use the Gauss-Newton Algorithm to find the Conditional LS estimate, $\hat{\theta}$, $\hat{\phi}$ and $\hat{\sigma_W}^2$. For the initial step, use answer from (a). (You may use R and example 3.32 of [TSA4] (and also check the hints), but you need to explain your steps.)

This chunk of code uses the formula from example 3.32 and the hints on Canvas. We first start by initializing our $\mathbf{W}$ and $\mathbf{Z}$. Using our initial values we then iterate using the formulas from the hints on Canvas and then updating our $\hat{\phi}$ and $\hat{\theta}$. After updating, we then check to see if the $|\hat{\phi}_{j+1} - \hat{\phi}_j|$ and $|\hat{\theta}_{j+1} - \hat{\theta}_j| <$ tolerance. Here I chose the value of 0.0001 for the tolerance. After that, we then use the conditional sum of squares for the values of $\hat{\phi}$ and $\hat{\theta}$ to calculate our $\hat{\sigma}_w^2$.

```
#Number of Observations in Data
N = nrow(Q4)
#Initialization of W(theta), Theta, and Phi Vectors
W = Theta = Phi = rep(0,N)
#Initialization of Z(phi, theta) matrix
Z = matrix(data = rep(0,2*N), nrow = 2, ncol = N, byrow = TRUE)
#Starting Value of Phi and Theta using MoM Estimates
Theta[1] = theta.hat[2]
Phi[1] = phi.hat
#Setting Up Counter for Value of Phi and Theta in Formula
B = 1

repeat{
  for(i in 2:N){
    #Calculation for all W
    W[i] = Q4[i,1] - Phi[B]*Q4[i-1,1] - Theta[B]*W[i-1]
    #Calculation of all Z
    Z[1,i] = Q4[i-1,1] - Theta[B]*Z[1,i-1]
    Z[2,i] = W[i-1] - Theta[B]*Z[2,i-1]
  }
  #Calculating the conditional error.
  SC = sum(W^2)
  #Calculating numerator of update
  SZW = Z %*% W
  #Calculating denominator of update
  SZ = Z %*% t(Z)
  #Calculating Update Vector
  new = inv(SZ) %*% SZW
  #Calculating new Phi and Theta
  Phi[B+1] = Phi[B] + new[1,1]
  Theta[B+1] = Theta[B] + new[2,1]
  #Test to see if we may stop iterating.
  #We stop if the absolute difference between the current and previous theta and
  #phi is less than 0.0001.
  if(B > 1){
    if(abs(Phi[B] - Phi[B-1]) & abs(Theta[B] - Theta[B-1]) < 0.0001){
      break
    } else {
      #Updating B for next iteration.
      B = B + 1
    }
  } else {
    #Updating B for next iteration.
    B = B + 1
```

```
    }
}
Phi = data.frame(Phi)
Phi = Phi %>% filter(Phi != 0)
Theta = data.frame(Theta)
Theta = Theta %>% filter(Theta != 0)
#Phi.hat using Gauss-Newton Algorithm
Phi.hat = Phi[B,1]
#Theta.hat using Gauss-Newton Algorithm
Theta.hat = Theta[B,1]
#Sigma^2.hat using Gauss-Newton Algorithm
Sigma2.hat = SC / (nrow(Q4) - 1)
```

- $\hat{\phi} \approx 0.6509$
- $\hat{\theta} \approx 0.8939$
- $\hat{\sigma}_w^2 \approx 0.9147$