Week 6 Lecture Note

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Module 2 - Week 6

Recall

Complex number (Book Appendix C)

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sol: $+i, -i$.

Solution of $x^2 + 1 = 0$: No 'real' solution, we define an 'imaginary' number.

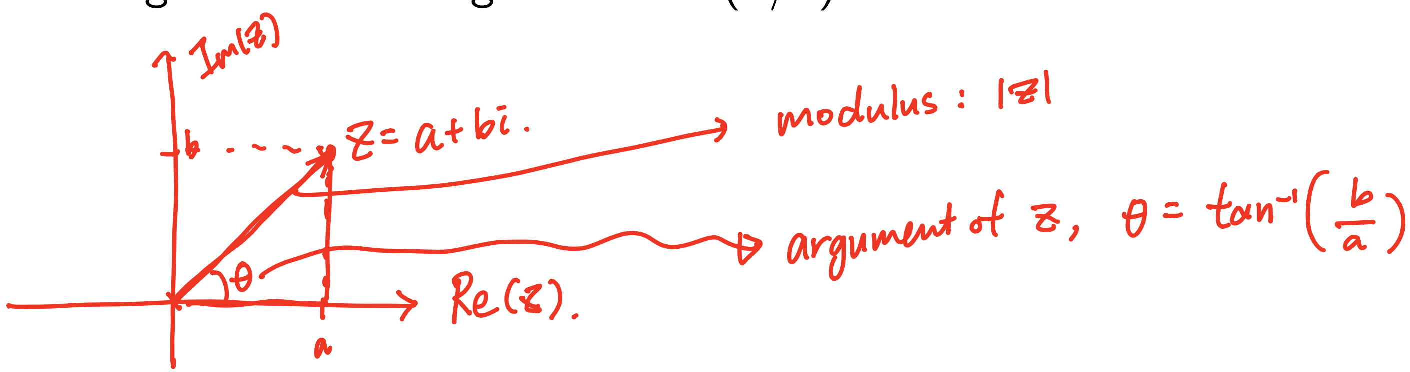
$$i^2 = -1$$

real part

Complex number form: $z = a + bi$, a, b are real numbers.

Imaginary part.

- Modulus of z: $|z| = \sqrt{a^2 + b^2}$
- Argument of z: $\arg z = \arctan(b/a)$.



ARMA(p,q): Autoregressive Moving Average model

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A time series $\{X_t\}$ is following ARMA(p,q) if

1. it is $\boxed{\text{stationary,}}$
2. $$X_t = \underbrace{\phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p}}_{\text{AR part}} + \underbrace{W_t + \theta_1 W_{t-1} + \ldots + \theta_q W_{t-q}}_{\text{MA part.}},$$
   - $\phi_p \neq 0$, $\theta_q \neq 0$,
   - $W_t \sim WN(0, \sigma_W^2)$
3. it is causal, invertible process.

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}.$$

$$X_t - \phi_1 B X_t - \phi_2 B^2 X_t \cdots - \phi_p B^p X_t = W_t + \theta_1 B W_t + \cdots + \theta_q B^q W_t.$$

$$\underbrace{(1 - \phi_1 B - \phi_2 B^2 \cdots - \phi_p B^p)}_{\text{AR operator. } \boxed{\phi(B)}} X_t = \underbrace{(1 + \theta_1 B + \theta_2 B^2 \cdots + \theta_q B^q)}_{\text{MA operator. } \theta(B).} W_t$$

AR, MA operators, parameter redundancy

$$\phi(B) X_t = \theta(B) W_t.$$

AR, MA operators, parameter redundancy

AR operator $\phi(B) := (1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p)$

MA operator. $\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$

$$\phi(B)X_t = \theta(B)W_t$$

- $q = 0$: AR(p) ~ ARMA(p, q=0)
- $p = 0$: MA(q) ~ ARMA(p=0, q).
- **Parameter Redundancy** : We require $\phi(B)$, $\theta(B)$ to have no common factors to avoid parameter redundancy.

$$\cancel{(1+0.5B)}(1-0.9B)X_t = \cancel{(1+0.5)}W_t.$$

$\rightarrow$ ARMA(2,1) ? $\rightarrow$ NO.

$(1 - 0.9B)x_t = w_t$. AR(1)

ARMA-Causality, Invertibility

Causality of ARMA(p,q)

Causality of ARMA(p,q)

An ARMA (p,q) model is causal if $\{X_t\}$ can be written as one-sided linear process, (MA(∞) representation)

$$= W_t + \psi_1 W_{t-1} + \psi_2 W_{t-2} + \cdots\cdots$$

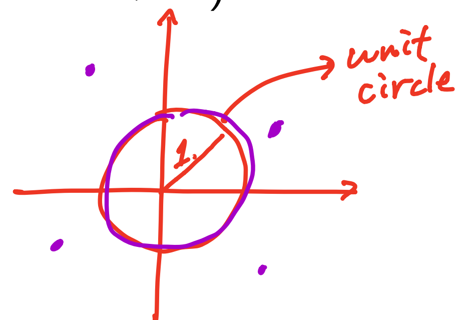
$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t$$

$$= \boxed{(1 + \psi_1 B + \psi_2 B^2 + \cdots\cdots)} W_t$$

where $\psi(B) := \sum_{j=0}^{\infty} \psi_j B^j$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

An ARMA(p,q) model is causal if and only if *all roots of the equation* $\boxed{\phi(z) = 0}$ *lie outside the unit circle* (modulus > 1).

- equivalent to: $\boxed{\phi(z) \neq 0}$ for all $|z| \leq 1$.



The coefficients can be determined by solving

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \boxed{\frac{\theta(z)}{\phi(z)}}, \quad |z| \leq 1$$

Invertibility of ARMA(p,q)

Invertibility of ARMA(p,q)

An ARMA (p,q) model is invertible if $\{W_t\}$ can be written as (Ar(∞) representation)

AR.

$= X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \cdots \cdots$

$$W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = \pi(B) X_t$$

$= (1 + \pi_1 B + \pi_2 B^2 + \cdots\cdots) X_t.$

where $\pi(B) := \sum_{j=0}^{\infty} \pi_j B^j$, $\sum_{j=0}^{\infty} \pi_j^2 < \infty$.

An ARMA(p,q) model is invertible if and only if all roots of the equation $\theta(z) = 0$ lie outside the unit circle (modulus > 1).

- equivalent to: $\theta(z) \neq 0$ for all $|z| \leq 1$.

The coefficients can be determined by solving

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \boxed{\frac{\phi(z)}{\theta(z)}}, \quad |z| \leq 1$$

Example: ARMA(1,1)

Example: ARMA(1,1)

$(1-0.9B)X_t = (1+0.5B)W_t.$

$\phi(B) = 1-0.9B$

$\theta(B) = 1+0.5B.$

$$X_t = 0.9X_{t-1} + W_t + 0.5W_{t-1}$$

causal.

Invertible.

Check causality and invertibility of this process, and find the causal representation and invertible representation. (by hand)

Causal? $\phi(z) = 1-0.9z = 0$

$z = \frac{10}{9}$ $\quad \left|\frac{10}{9}\right| > 1 \Rightarrow$ yes, causal.

Invertible? $\theta(z) = 1+0.5z = 0.$

$z = -2.$ $\quad |-2| = 2 > 1$

$\Rightarrow$ yes. Invertible.

```r
ARMAacf(ar=c(0.9),ma=c(0.5),lag.max=20)
```

```
##         0         1         2         3         4         5         6         7
## 1.0000000 0.9441860 0.8497674 0.7647907 0.6883116 0.6194805 0.5575324 0.5017792
##         8         9        10        11        12        13        14        15
## 0.4516013 0.4064411 0.3657970 0.3292173 0.2962956 0.2666660 0.2399994 0.2159995
##        16        17        18        19        20
## 0.1943995 0.1749596 0.1574636 0.1417173 0.1275455
```

$$X_t = 0.9\,X_{t-1} + W_t + 0.5W_{t-1},$$

$$\phi(B) = 1 - 0.9B \qquad \theta(B) = 1 + 0.5B.$$

$$X_t = W_t + \psi_1 W_{t-1} + \psi_2 W_{t-2} + \cdots$$

Causal Representation

$MA(\infty)$ Representation.

$$\boxed{\psi(z)} = \boxed{\frac{\theta(z)}{\phi(z)}}.$$

$$\phi(B)X_t = \theta(B)W_t. \Rightarrow X_t = \boxed{\frac{\theta(B)}{\phi(B)}}W_t.$$

$$\frac{1+0.5z}{1-0.9z}\ ? \qquad \frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots\cdots$$

$$(1+0.5z)\left(1 + 0.9z + 0.9^2z^2 + 0.9^3z^3 + \cdots\cdots\right)$$

$$= 1 + (0.5 + 0.9)z + (0.5 \cdot 0.9 + 0.9^2)z^2 + (0.5 \cdot 0.9^2 + 0.9^3)z^3 + \cdots$$

$$= 1 + 1.4z + 1.4 \cdot 0.9 z^2 + 1.4 \cdot 0.9^2 z^2 + \cdots$$

$$\to \psi_j = 1.4 \cdot 0.9^{j-1}$$

$$(1 + 0.5z)(1 + 0.9z + 0.9^2 z^2 + \cdots).$$

coefficient of z^j $\Rightarrow$ $1 \cdot 0.9^j + 0.5 \cdot 0.9^{j-1}$

$$= 0.9 \cdot 0.9^{j-1} + 0.5 \cdot 0.9^{j-1}$$

$$= \boxed{1.4 \cdot 0.9^{j-1}}$$

Causal Representation:

$$X_t = W_t + 1.4 \sum_{j=1}^{\infty} 0.9^{j-1} W_{t-j}.$$

Invertible Representation?

$$(1 - 0.9B)X_t = (1 + 0.5B)W_t.$$

$$W_t = \frac{1 - 0.9B}{1 + 0.5B} X_t$$

$$= \frac{\phi(B)}{\theta(B)} X_t.$$

$$\pi(z) = \frac{\phi(z)}{\theta(z)}$$

$$\frac{1-0.9z}{1+0.5z} = (1-0.9z)\left(1+(-0.5z)+(-0.5z)^2+\cdots\right)$$

$$= 1 + (-0.9-0.5)z$$

$$+ \left((-0.5)^2 + (-0.9)(-0.5)\right) z^2$$

$$+ \left((-0.5)^3 + (-0.9)(-0.5)^2\right) z^3$$

$$\vdots$$

coeffi. for z^j $\rightarrow$ $(-0.5)^j + (-0.9)(-0.5)^{j-1}$

$$= (-0.5)(-0.5)^{j-1} + (-0.9)(-0.5)^{j-1}$$

$$= -1.4(-0.5)^{j-1}.$$

$$\boxed{W_t = X_t - 1.4 \sum_{j=1}^{\infty} (-0.5)^{j-1} X_{t-j}}$$

Invertible Representation.

$$\boxed{X_t = 1.4 \sum_{j=1}^{\infty} (-0.5)^{j-1} X_{t-j} + W_t}$$

Invertible Rep. AR(∞).

R. ARMA to AR. $\rightarrow$ Invertible Rep. AR(∞).

ARMA to MA $\rightarrow$ Causal Rep. MA(∞).

Correlation functions (ACF and PACF)

ACF of MA(q)

$$\gamma(h) = \begin{cases} \sigma_W^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} & 0 \leq h \leq q \\ 0 & h > q \end{cases}$$

$\gamma(h)/\gamma(0)$

$$\boxed{\rho(h)} = \begin{cases} \frac{\sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{1+\theta_1^2+\ldots+\theta_q^2} & 0 \leq h \leq q \\ 0 & h > q \end{cases}$$

ACF for MA(q).

$h = 0$.

$$\gamma(0) = (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma_w^2.$$

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \cdots + \theta_q W_{t-q}.$$

$\mathrm{Cov}(X_{t+h}, X_t)$

$$= \mathrm{Cov}(W_{t+h} + \theta_1 W_{t+h-1} + \cdots + \theta_q W_{t+h-q},$$
$$W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \cdots + \theta_q W_{t-q}).$$

If $h > q. \to \boxed{t+h-q > t} \Leftrightarrow \cancel{t}+h > \cancel{t}+q$

$\hookrightarrow \mathrm{Cov} = 0.$

$h \le q.$ $\qquad t+h-h = t$

$$= \mathrm{Cov}(W_{t+h} + \theta_1 W_{t+h-1} + \cdots + \theta_q W_{t+h-q},$$
$$W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \cdots + \theta_q W_{t-q}).$$

$$= \mathrm{Cov}(W_{t+h} + \theta_1 W_{t+h-1} \cdots + \theta_{h-1} W_{t+1} + \theta_h W_t + \cdots + \theta_q W_{t+h-q},$$
$$W_t + \theta_1 W_{t-1} \cdots + W_{t+h-q} + \cdots + \theta_q W_{t-q}).$$

$\to \theta_{q-h} W_{t-(q-h)}$

$\hookrightarrow \theta_{q-h}.$

$$= \mathrm{Var}(W_t) \cdot (1 \cdot \theta_h + \theta_{h+1} \cdot \theta_1 + \theta_{h+2} \cdot \theta_2 + \cdots + \theta_q \cdot \theta_{q-h})$$

$$= \sigma_w^2 (1 \cdot \theta_h + \theta_1 \cdot \theta_{h+1} + \theta_2 \cdot \theta_{h+2} + \cdots + \theta_{q-h} \cdot \theta_q)$$

$$= \boxed{\sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}} \rightsquigarrow \gamma(h). \qquad (\theta_0 = 1)$$

ACF of AR(p) or ARMA(p,q)

$MA(q) \Rightarrow \gamma(h) = \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}.$

(MA(∞) representation)

$$\boxed{X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}}$$

$$\gamma(h) = \sigma_W^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}, \quad h \geq 0$$

$$\frac{\gamma(h)}{\gamma(0)} = \rho(h) = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+h}}{\sum_{j=0}^{\infty} \psi_j^2}, \quad h \geq 0$$

ARMA acf.

Example: AR(2)

ACF

MA(q)	ARMA(p,q)	AR(p)
cut off after q	tail off	tail off.

$$X_t = 1.5X_{t-1} - 0.75X_{t-2} + W_t$$

ACF of ARMA(1,1)

$$X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$$

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{\phi(1+2\theta\phi+\theta^2)}\phi^h, \quad h \geq 1$$

```
ARMAacf(ar = numeric(), ma = numeric(), lag.max = r,
 pacf = FALSE)
```

Arguments

ar: numeric vector of AR coefficients
ma: numeric vector of MA coefficients
lag.max: integer. Maximum lag required. Defaults to max(p, q+1), where p, q are the numbers of AR and MA terms respectively.
pacf: logical. Should the partial autocorrelations be returned?

$X_t = 1.5X_{t-1} - 0.75X_{t-1} + W_t.$

```
ARMAacf(ar=c(1.5,-0.75), lag.max=10)
```

ma=0,

```
##      ρ(0)   0          ρ(1)   1          ρ(2)   2                3
##   1.00000000  0.85714286  0.53571429  0.16071429 -0.1607
##            6           7           8           9
## -0.42187500 -0.36160714 -0.22600446 -0.06780134  0.0678
```

```
ARMAacf(ar=c(0.9),ma=c(0.5), lag.max=10)
```

```
##    ρ(0)   0     ρ(1)   1     ρ(2)   2          3          4
## 1.0000000 0.9441860 0.8497674 0.7647907 0.6883116 0.619
##         8         9        10
## 0.4516013 0.4064411 0.3657970
```

$X_t = 0.9X_{t-1} + W_t + 0.5W_{t-1}.$

ACF of MA, AR, ARMA

- MA(q) models: the ACF will be zero for lags greater than q.
  - ACF provides a considerable amount of information when the process is MA.
- AR(p), ARMA(p,q): ACF alone tells us little about the orders of dependence
  - We introduce PACF (partial autocorrelation function)

PACF

Partial autocorrelation

$\rho_{XY|Z}$

X, Y, Z random variables.

Partial correlation between X and Y given Z: obtained by + regressing X on Z to obtain $\hat{X}$ + regressing Y on Z to obtain $\hat{Y}$, and then

$$\rho_{XY|Z} = corr\{X - \hat{X}, Y - \hat{Y}\}$$

$X_t = \phi X_{t-1} + W_t.$ Regress X_t on X_{t-1} ?

$\hat{X}_t \overset{?}{=} \phi X_{t-1}.$

Definition: Partial autocorrelation function (PACF)

Partial autocorrelation function (PACF) of a stationary process $\{X_t\}$, denoted ϕ_{hh}, for $h = 1, 2, \ldots$ is

$X_t.$

$$\phi_{11} = \underline{corr(X_{t+1}, X_t)} = \rho(1).$$

and $X_t \quad X_{t+1} \quad \cdots \quad X_{t+h-1} \quad X_{t+h}.$

$$\phi_{hh} = \boxed{corr(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t), h \geq 2}$$

where $\hat{X}_{t+h}$ is the regression of X_{t+h} on $\boxed{\{X_{t+h-1}, \ldots X_{t+1}\}}$, and $\hat{X}_t$ is the regression of X_t on $\{X_{t+h-1}, \ldots X_{t+1}\}$

Partial autocorrelation of the AR(1)

$X_t = \phi X_{t-1} + W_t,$ $\rightarrow$ $\hat{X}_t$ on X_{t-1} is $\phi X_{t-1}.$

$$\gamma_X(2) = Cov(X_{t+2}, X_t) = Cov(\phi X_{t+1} + W_{t+2}, X_t)$$
$$= Cov(\phi X_{t+1}, X_t) + \underbrace{Cov(W_{t+2}, X_t)}_{0}$$
$$= \phi \gamma_X(1).$$

$$X_t = \phi X_{t-1} + W_t$$

Break the chain of dependence by removing the effect of X_{t+1}.

with $|\phi| < 1$.

By definition $\phi_{11} = Corr(X_t, X_{t-1}) = \phi = \rho_1$.

$\rho(1)$ for AR(1) $= \phi$.

ϕ_{22}?

Regression of X_{t+2} on X_{t+1}. $\hat{X}_{t+2} = \beta X_{t+1}$

$E[(X_{t+2} - \hat{X}_{t+2})^2]$ minimize

$= E[(X_{t+2} - \hat{X}_{t+2}, X_{t+2} - \hat{X}_{t+2})$.

$= Cov(X_{t+2} - \hat{X}_{t+2}, X_{t+2} - \hat{X}_{t+2})$

$= Cov(X_{t+2} - \beta X_{t+1}, X_{t+2} - \beta X_{t+1})$

$= \gamma(0) - 2\beta\gamma(1) + \beta^2\gamma(0)$. ← want to find β which minimize this.

$\xrightarrow{\frac{\partial}{\partial\beta}}$ $-2\gamma(1) + 2\beta\gamma(0) = 0$.

$$\beta = \frac{\gamma(1)}{\gamma(0)} = \rho(1) = \phi.$$

Regression of X_t on X_{t+1}. $\hat{X}_t = \beta X_{t+1}$

minimize.

$E[(X_t - \beta X_{t+1})^2]$.

$= Cov(X_t - \beta X_{t+1}, X_t - \beta X_{t+1})$

$= \gamma(0) - 2\beta\gamma(1) + \beta^2\gamma(0)$

$\xrightarrow{\frac{\partial}{\partial\beta}}$ $-2\gamma(1) + 2\beta\gamma(0) = 0$, $\Rightarrow \beta = \frac{\gamma(1)}{\gamma(0)} = \rho(1)$.

X_{t+2} regression on X_{t+1} : ϕX_{t+1}, X_t regression of X_{t+1} : ϕX_{t+1}.

$Cov(X_t - \hat{X}_t, X_{t-2} - \hat{X}_{t-2})$.

We can get ϕ_{22} by calculating:

$$Cov(X_t - \overset{\phi_1}{\rho_1} X_{t-1}, X_{t-2} - \boxed{\overset{\phi_1}{\rho_1} X_{t-1}})$$

$$= \gamma_0(\rho_2 - \rho_1^2)$$

$Cov(X_t, X_{t-2})$
$- \rho_1 Cov(X_{t-1}, X_{t-2})$
$- \rho_1 Cov(X_t, X_{t-1})$
$+ \rho_1^2 Cov(X_{t-1}, X_{t-1})$
$= \gamma(2) - 2\rho_1\gamma(1) + \rho_1^2\gamma(0)$
$= \gamma(0)\rho(2) - 2\rho_1^2\gamma(0) + \rho_1^2\gamma(0)$
$= \gamma(0)(\rho(2) - \rho(1)^2)$.

And

$$Var(X_t - \rho_1 X_{t-1}) = Var(X_{t-2} - \rho_1 X_{t-1})$$

$$= \gamma_0(1 - \rho_1^2) = \gamma(0)(1 - \rho(1)^2).$$

Then

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\phi^2 - \rho_1^2}{1 - \rho_1^2} = 0$$

Since $\rho_k = \phi^k$ for $k \geq 0$

It can be seen that $\phi_{kk} = 0$ for $k > 1$

$\rho(2)$ of AR(1)? $\to \phi^2$.

Partial autocorrelation of the AR(p)

$\phi_{11} = \rho(1)$, ----- $\phi_{pp} = \boxed{\phi_p}$ last parameter in the model.

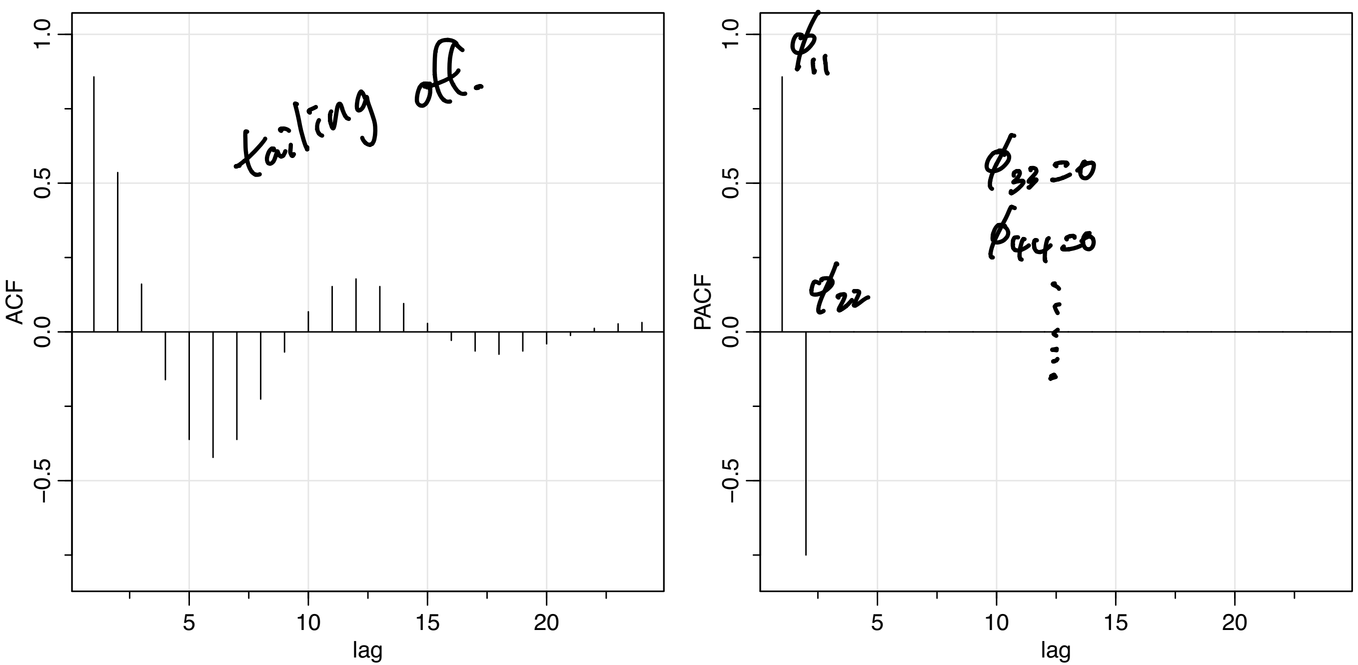
$\phi_{hh} = 0$ for $h > p$

In general:

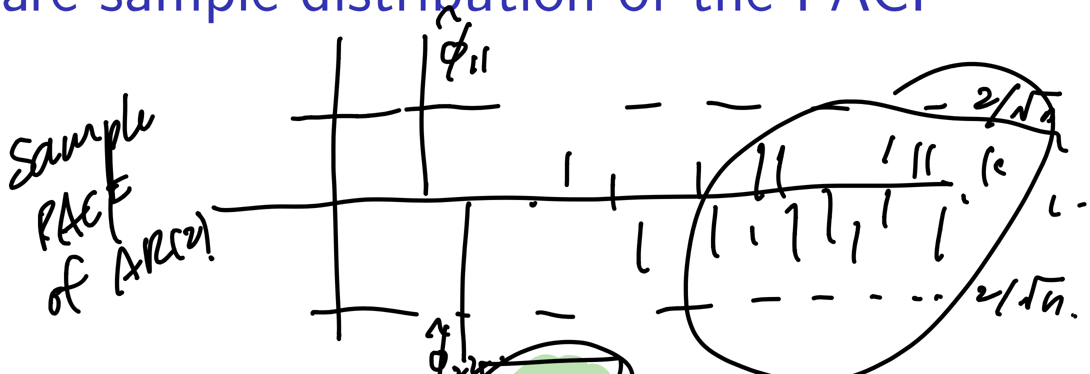
$$\phi_{kk} = Corr(X_t - \beta_1 X_{t-1} - \beta_2 X_{t-2} - \ldots - \beta_{k-1} X_{t-k+1},$$

$$X_{t-k} - \beta_1 X_{t-k+1} - \beta_2 X_{t-k+2} - \ldots - \beta_{k-1} X_{t-1})$$

```
ACF = ARMAacf(ar=c(1.5,-.75), ma=0, 24)[-1]
PACF = ARMAacf(ar=c(1.5,-.75), ma=0, 24, pacf=TRUE)
par(mfrow=1:2)
tsplot(ACF, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
tsplot(PACF, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
```



Lare sample distribution of the PACF



If a time series is an AR(p), and the sample size n is large, then for $h > p$, the $\hat{\phi}_{hh}$ are approximately independent normal with mean 0 and standard deviation $1/\sqrt{n}$. This result also holds for $p = 0$, wherein the process is white noise.

normal

WN large sample distribution

ACF : → normal with mean 0, sd $1/\sqrt{n}$.

PACF : normal with mean 0, sd $1/\sqrt{n}$.

PACF of MA(q) and ARMA(p,q)

An MA(q) or ARMA(p,q) are invertible, so both have AR(∞) representation,

$$W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = \pi(B) X_t,$$

or

$$X_t = -\sum_{j=1}^{\infty} \pi_j X_{t-j} + W_t$$

- No finite representation exists
- PACF will never cut off

failing off, but not cut off.

For an invertible MA(1):

tailing off, Not cutting off.

$$\phi_{hh} = \frac{(-\theta)^h(1-\theta^2)}{1-\theta^{2(h+1)}}, \quad h \geq 1$$

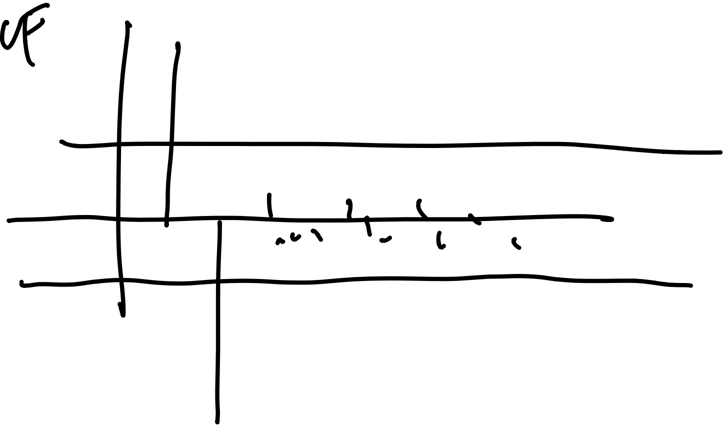
This function is never zero and decays to zero exponentially as lag increases (like the autocorrelation function of an AR(q))

If you have stationary process, by looking at ACF / PACF, ARMA(p, q)?

☆.

Model	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

ACF



→ MA(2).

PACF

AR(3)

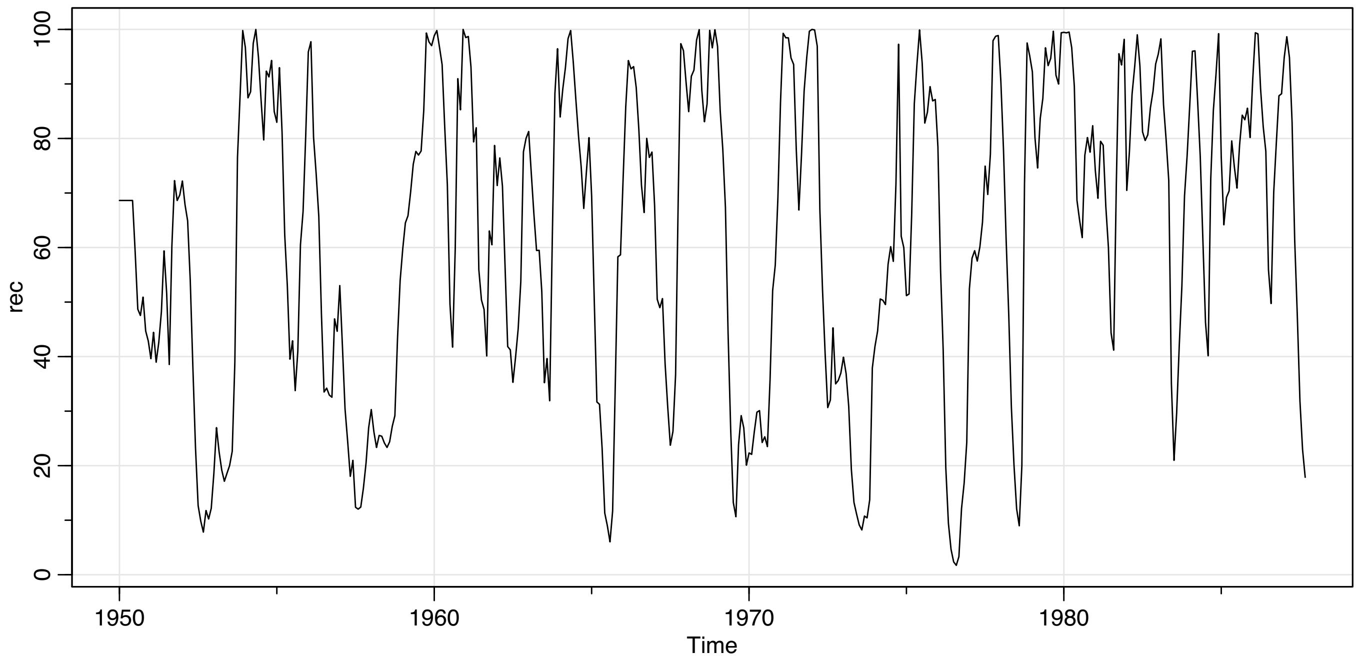
Example

Example

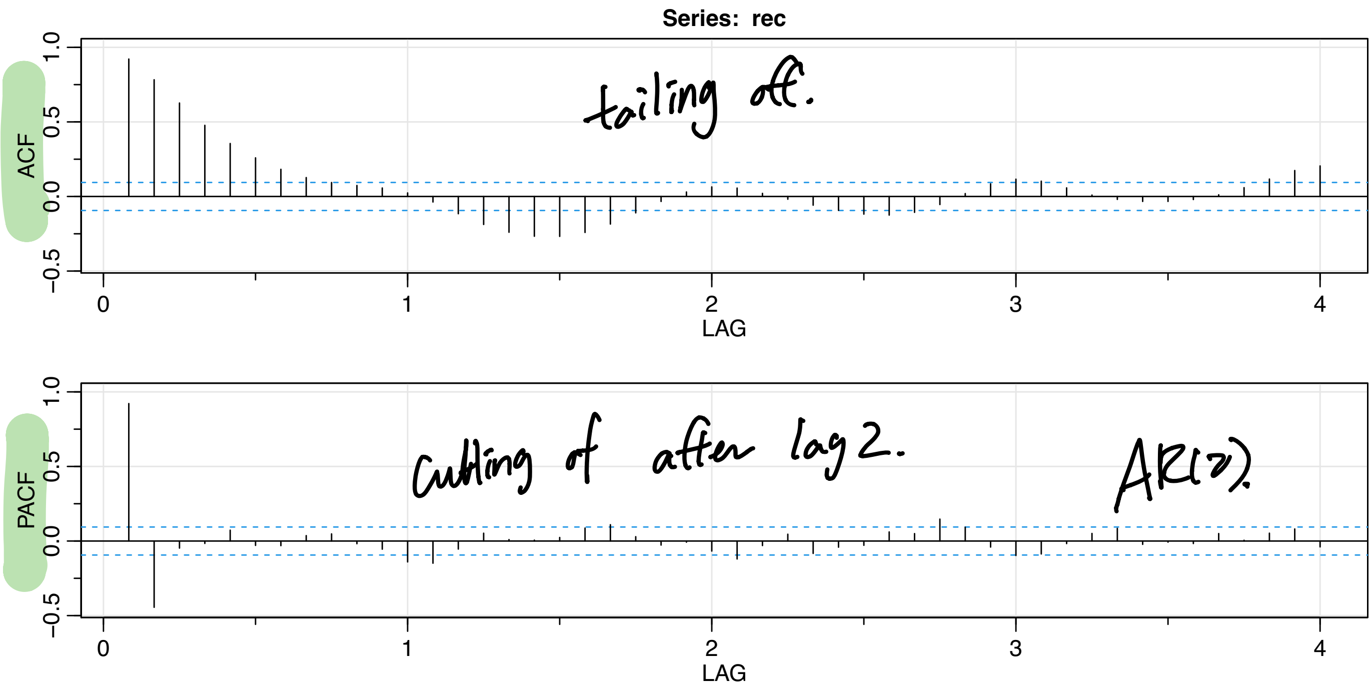
rec.

ACF, PACF of recruitment series

```
tsplot(rec)
```



```r
acf2(rec, 48) # will produce values and a graphic
```




```
##       [,1]  [,2]  [,3]  [,4] [,5]  [,6]  [,7] [,8] [,9] [,10] [,11] [,12] [,13]
## ACF  0.92  0.78  0.63  0.48 0.36  0.26  0.18 0.13 0.09  0.07  0.06  0.02 -0.04
## PACF 0.92 -0.44 -0.05 -0.02 0.07 -0.03 -0.03 0.04 0.05 -0.02 -0.05 -0.14 -0.15
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
## ACF  -0.12 -0.19 -0.24 -0.27 -0.27 -0.24 -0.19 -0.11 -0.03  0.03  0.06  0.06
## PACF -0.05  0.05  0.01  0.01  0.02  0.09  0.11  0.03 -0.03 -0.01 -0.07 -0.12
##      [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
## ACF   0.02 -0.02 -0.06 -0.09 -0.12 -0.13 -0.11 -0.05  0.02  0.08  0.12  0.10
## PACF -0.03  0.05 -0.08 -0.04 -0.03  0.06  0.05  0.15  0.09 -0.04 -0.10 -0.09
##      [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
## ACF   0.06  0.01 -0.02 -0.03 -0.03 -0.02  0.01  0.06  0.12  0.17  0.20
## PACF -0.02  0.05  0.08 -0.02 -0.01 -0.02  0.05  0.01  0.05  0.08 -0.04
```