Week 3 Lecture Note

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Module 1 - Week 3

Stationarity

Stationary Time Series

a finite variance process ($Var(X_t) < \infty$) where

1. $\mu_X(t) = \mathbb{E}[X_t]$ is constant, does not depend on time t
2. $\gamma_X(t, s)$ depends on times s and t only through their time difference

auto-covariance function of a stationary time series

$$\gamma(h) := \gamma_X(t+h, t) = Cov(X_{t+h}, X_t) = \mathbb{E}[(X_{t+h} - \mu_X(t+h))(X_t - \mu_X(t))]$$

h: time shift or lag
(We may use γ instead of γ_X and ρ instead of ρ_X for simplicity)

auto-correlation function (ACF) of a stationary time series

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

General Properties of auto-covariance and ACF

$$\begin{aligned} \gamma(0) &= Var(X_t) & \qquad \rho(0) &= 1 \\ \gamma(h) &= \gamma(-h) & \qquad \rho(h) &= \rho(-h) \\ |\gamma(h)| &\le \gamma(0) & \qquad |\rho(h)| &\le 1 \end{aligned}$$

Example: Is Random Walk stationary?

$$X_t = \delta t + \sum_{j=1}^{t} W_j$$

Recall:

$$\mu_X(t) = \delta t$$

$$\gamma_X(t,s) = Cov\left(\delta t + \sum_{j=1}^{t} W_j, \delta s + \sum_{j=1}^{s} W_j\right) = \min(t,s)\sigma_W^2$$

`Not stationary!`

Trend and Trend Stationarity

- The mean function of a Stochastic Process is an arbitrary function of time
- If the process is stationary, the mean function is constant with time
- We are going to consider Stochastic processes with a mean function which is not constant in time. We will call these functions a Trend.

A time series that has stationary behavior around a trend : called trend stationary.

Example: When Y_t is stationary ,

$$X_t = \beta t + Y_t,$$

X_t is called *trend stationary.*

Autoregressive Model

$$X_t = \phi X_{t-1} + W_t$$

1. Mean function

$$\begin{aligned}
\mathbb{E}[X_t] = \phi\mathbb{E}[X_{t-1}] + \mathbb{E}[W_t] = \phi\mathbb{E}[X_{t-1}] \\
\mu_X(t) = \phi\mu_X(t-1) = \phi^2\mu_X(t-2)\cdots = \phi^t\mu_X(0)
\end{aligned}$$

2. Covariance function

$$\begin{aligned}
X_{t+h} =& \phi X_{t+h-1} + W_{t+h} \\
=& \phi(\phi X_{t+h-2} + W_{t+h-1}) + W_{t+h} \\
=& \phi^2 X_{t+h-2} + \phi W_{t+h-1} + W_{t+h} \\
=& \phi^h X_t + \phi^{h-1} W_{t+1} \ldots \phi W_{t_h-1} + W_{t+h} \\
=& \phi^h X_t + \sum_{i=1}^{h} \phi^{h-i} W_{t+i}
\end{aligned}$$

$$\begin{aligned}
Cov(X_{t+h}, X_t) =& Cov(\phi^h X_t + \sum_{i=1}^{h} \phi^{h-i} W_{t+i}, X_t) \\
=& Cov(\phi^h X_t, X_t) + Cov(\sum_{i=1}^{h} \phi^{h-i} W_{t+i}, X_t) \\
=& \phi^h Cov(X_t, X_t) + \sum_{i=1}^{h} \phi^{h-i} Cov(W_{t+i}, X_t)
\end{aligned}$$

Assuming X_t and W_{t+i} $(i \geq 1)$ uncorrelated, $Cov(W_{t+i}, X_t) = 0$,

$$Cov(X_{t+h}, X_t) = \phi^h Cov(X_t, X_t) = \phi^h Var(X_t).$$

$$Var(X_t) = Var(\phi X_{t-1} + W_t) = \phi^2 Var(X_{t-1} + W_t) = \phi^2 Var(X_{t-1}) + \sigma_W^2$$

When is this model stationary?

$\mu_X(t)$, $Var(X_t)$ should not depend on t (constant), so

$$\begin{aligned} Var(X_t) =& \phi^2 Var(X_{t-1}) + \sigma_W^2 \\ =& \phi^2 Var(X_t) + \sigma_W^2 \\ Var(X_t) =& \frac{\sigma_W^2}{(1-\phi^2)} \end{aligned}$$

So we need $|\phi| < 1$

Wold Decomposition

Any stationary time series X_t can be written as a filter (linear combination) of white noise.

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j W_{t-j}$$

where ψs are numbers satisfying $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and $\psi_0 = 1$.

$$\begin{aligned} \mathbb{E}[X_t] =& \mu \\ \gamma(h) =& \sigma_W^2 \sum_{j=0}^{\infty} \psi_{j+h} \psi_j \end{aligned}$$

Estimation of Correlation

- We have sample values $X_1, X_2, \dots X_n$.
- We estimate:
  - mean
  - autocovariance
  - autocorrelation function
- We assume that the process is stationary.

Mean and its standard error

If time series is stationary, mean function is constant:

$$\mu_X(t) = \mu$$

so, we can estimate it by the sample mean

$$\bar{X} := \frac{1}{n} \sum_{t=1}^{n} X_t$$

$E[\bar{X}] = \mu$: unbiased

standard error of the estimate:

$$Var(\bar{X}) = \frac{1}{n} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)$$

Autocovariance estimate:

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(X_t - \bar{X})$$

for $h = 0, 1, \dots, n-1$.

note:

- This runs over restricted range because X_{t+h} is not available for $t + h > n$.
- We divide by n even though there are only $n - h$ pairs of observations at lag h.

Autocorrelation estimate (sample ACF)

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = \frac{\sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(X_t - \bar{X})}{\sum_{t=1}^{n} (X_t - \bar{X})^2}$$

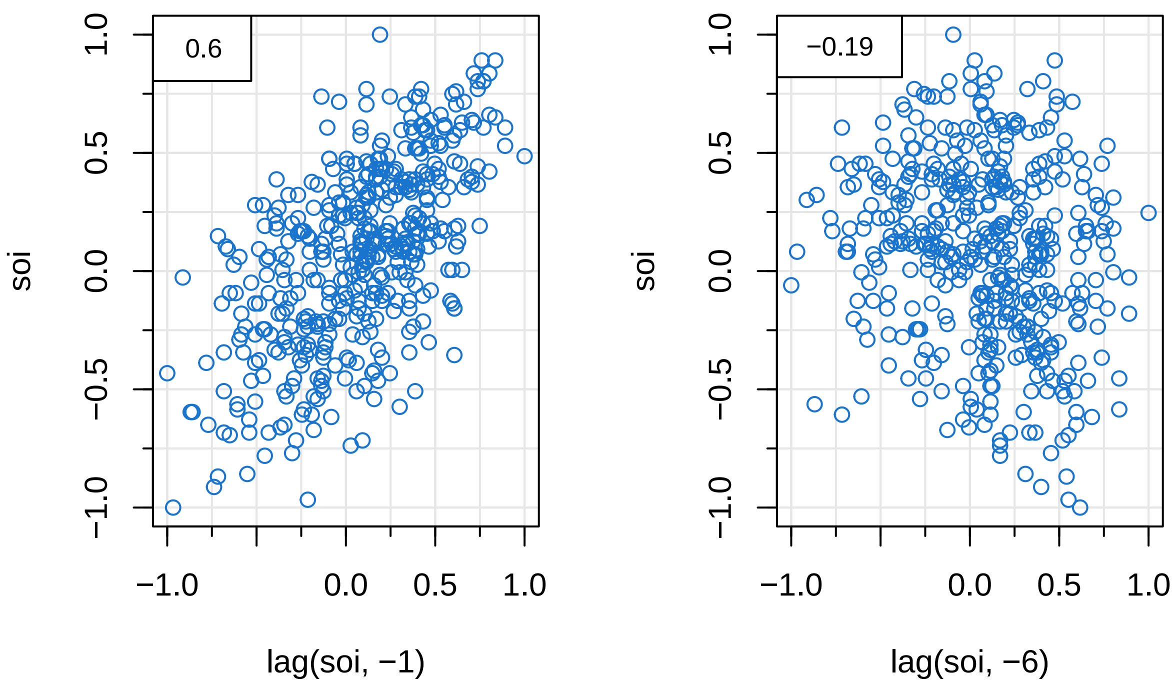
for $h = 0, 1, \dots, n-1$.

Example 2.27

```r
(r = acf1(soi, 6, plot=FALSE)) # sample acf values
```

```
## [1]  0.60410089  0.37379533  0.21412447  0.05013659 -0.10703704 -0.18698742
```

```r
# [1] 0.60 0.37 0.21 0.05 -0.11 -0.19
par(mfrow=c(1,2))
plot(lag(soi,-1), soi, col="dodgerblue3", panel.first=Grid())
legend("topleft", legend=0.60, bg="white", adj=.45, cex = 0.85)
plot(lag(soi,-6), soi, col="dodgerblue3", panel.first=Grid())
legend("topleft", legend= -0.19, bg="white", adj=.25, cex = 0.8)
```



Large-sample Distribution of the ACF

If $X_t = W_t$, then $\hat{\rho}_X(h)$ is approximately normally distributed (AN):

$$\hat{\rho}_X(h) \sim AN(0, 1/n)$$

```
w=rnorm(500)
par(mfrow=c(1,2))
acf(soi,ylab='ACF of soi series')
acf(w,ylab='ACF of white noise')
```