Week ~~12~~ 10 Lecture Note

Hyoeun Lee

Module ~~3~~ - Week ~~12~~ 10

Time Series Regression with Autocorrelated Errors

Recall: Time Series Regression

Dep. Var.

Time Series X_t, $t = 1, \ldots, n$ is possibly influenced by $Z_{t1}, Z_{t2}, \ldots, Z_{tq}$. ← Indep. Variables

We express the general relation through the *linear regression model*

$$X_t = \beta_0 + \beta_1 Z_{t1} + \beta_2 Z_{t2} + \cdots + \beta_q Z_{tq} + \boxed{W_t,}$$

- $\beta_0, \ldots, \beta_q$: unknown fixed regression coefficients
- $\{W_t\}$ is white noise (normally distributed) with variance σ_W^2.

Problem: We often see residuals which does not look like white noise.

Time Series Regression with Autocorrelated Errors

$Y_1, \dots, Y_n$.

Time Series Y_t, $t = 1, \dots, n$

error process.

$$Y_t = \underbrace{\beta_1 Z_{t1} + \beta_2 Z_{t2} + \cdots + \beta_r Z_{tr}} + \boxed{X_t} = \sum_{j=1}^{r} \beta_j Z_{tj} + X_t$$

Error.

- $Z_{t1}, Z_{t2}, \dots, Z_{tr}$: Independent variables
- X_t : error process, process with some covariance function $\gamma_X(s, t)$

If possible to assume stationary covariance structure, for X_t → we can find ARMA structure for X_t.

AR(p) Error: $\boxed{X_t}$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + W_t.$$

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) X_t = W_t.$$

- $\phi(B)X_t = W_t$, W_t: white noise

$$B X_t = X_{t-1}$$

$$\vdots$$

$$B^p X_t = X_{t-p}.$$

$$Y_t = \sum_{j=1}^{r} \beta_j Z_{tj} + X_t$$

$$\underbrace{\phi(B)Y_t}_{Y_t^*} = \sum_{j=1}^{r} \beta_j \underbrace{\phi(B)Z_{tj}}_{\text{transformed } Z_{tj}^*} + \underbrace{\phi(B)X_t}_{W_t.}$$

$$Y_t^* - \sum_{j=1}^{r} \beta_j Z_{tj}^* = \boxed{W_t.}$$

β: same as original.

$$S(\phi, \beta) = \sum_{t=1}^{n} W_t^2 = \sum_{t=1}^{n} \left[\phi(B)Y_t - \sum_{j=1}^{r} \beta_j \phi(B) Z_{tj} \right]^2$$

- $\phi = \{\phi_1, \ldots \phi_p\}$, $\beta = \{\beta_1, \ldots, \beta_r\}$

ARMA(p,q) and MA(q) series

$X_t = \theta(B) W_t.$

- $\phi(B)X_t = \theta(B)W_t$, W_t: white noise
- transform to $\boxed{\pi(B)X_t = W_t}$ (AR(∞))

$X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \cdots\cdots = W_t.$

$$Y_t = \sum_{j=1}^{r} \beta_j Z_{tj} + X_t$$

$$\underbrace{\pi(B)Y_t}_{Y_t^*} = \sum_{j=1}^{r} \beta_j \underbrace{\pi(B)Z_{tj}}_{Z_{tj}^*} + \underbrace{\pi(B)X_t}_{W_t}$$

$$S(\phi,\theta,\beta) = \sum_{t=1}^{n} W_t^2 = \sum_{t=1}^{n} \left[\pi(B)Y_t - \sum_{j=1}^{r} \beta_j \pi(B) Z_{tj} \right]^2$$

* $\phi = \{\phi_1, \ldots \phi_p\}$, $\theta = \{\theta_1, \ldots \theta_q\}$, $\beta = \{\beta_1, \ldots, \beta_r\}$

Steps

P/ACF tail off → ARMA. → start with ARMA(1,1)

ACF cutoff (q), PACF tails off → MA(q).

PACF cutoff (p), ACF tails off → AR(p)

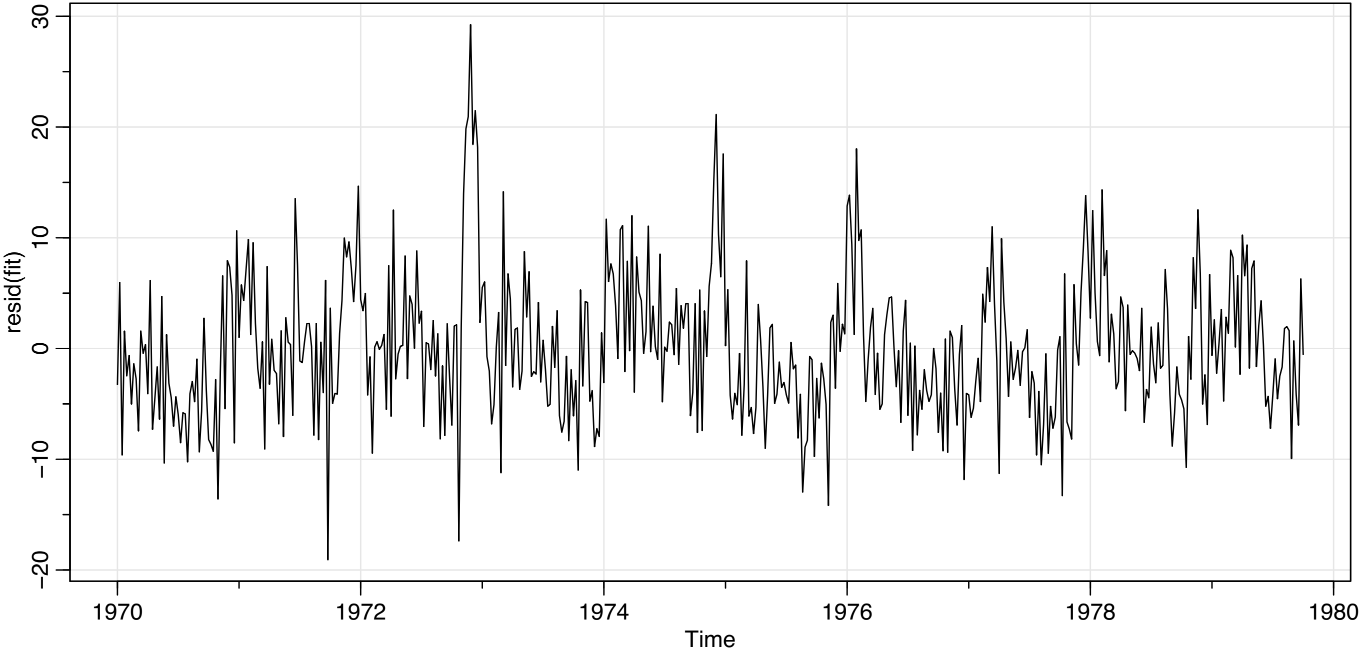
1. Run ordinary regression of Y_t on $Z_{t1}, Z_{t2}, \ldots, Z_{tr}$ acting as if the errors are uncorrelated.
    - Retain the residuals, $\hat{X}_t = Y_t - \sum_{j=1}^{r} \hat{\beta}_j Z_{tj}$.
2. Identify an ARMA model for the residuals $\hat{X}_t$. P/ACF
3. Run MLE on the regression models with autocorrelated errors using the models specified in step 2.
4. Inspect the residuals $\hat{W}_t$ for whiteness, and adjust the model if necessary.

$$\underbrace{M_t}_{\text{Cardiovascular mortality rate}} = \beta_0 + \beta_1 \underbrace{t}_{\text{trend}} + \underbrace{\beta_2 T_t + \beta_3 T_t^2}_{\text{(centered) Temp \& Temp}^2} + \beta_4 \underbrace{P_t}_{\text{pollutant level.}} + \underbrace{x_t}_{\text{Error.}}.$$

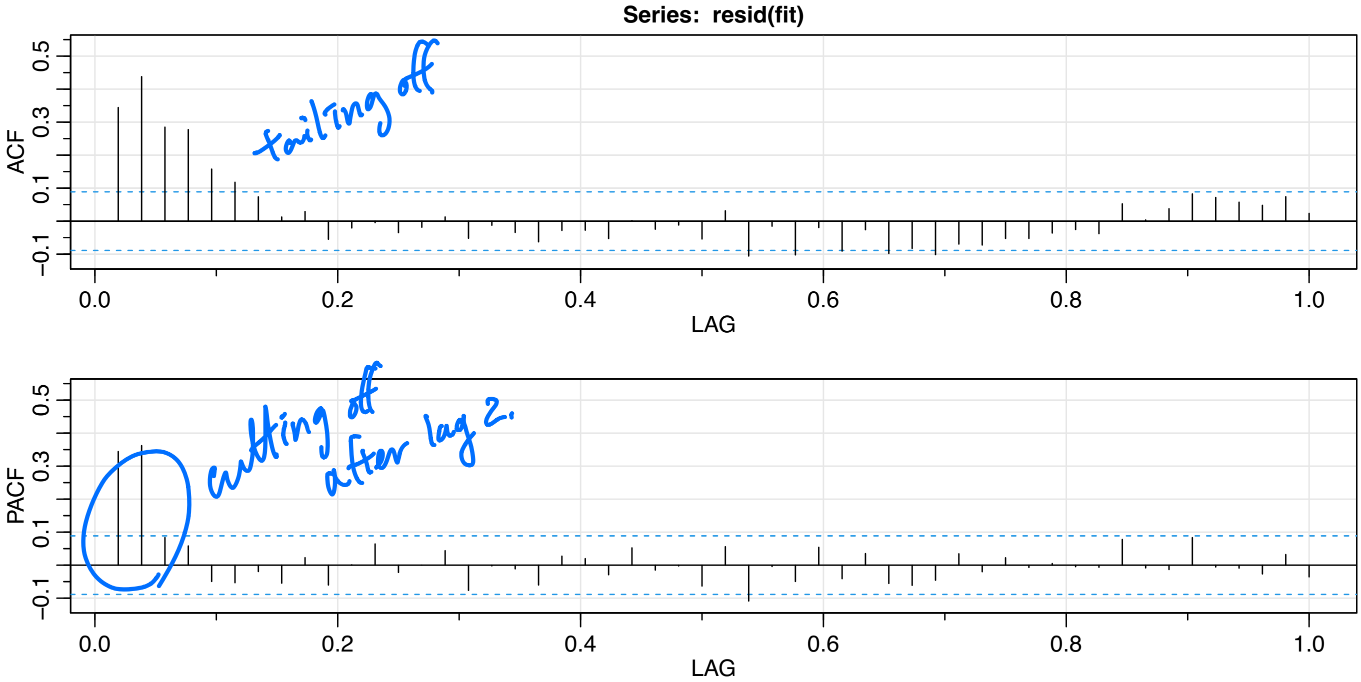
Example 5.16

Example 5.16

```r
trend = time(cmort)
temp = tempr - mean(tempr)
temp2 = temp^2
fit = lm(cmort ~ trend + temp + temp2 + part, na.action = N
tsplot(resid(fit))
```



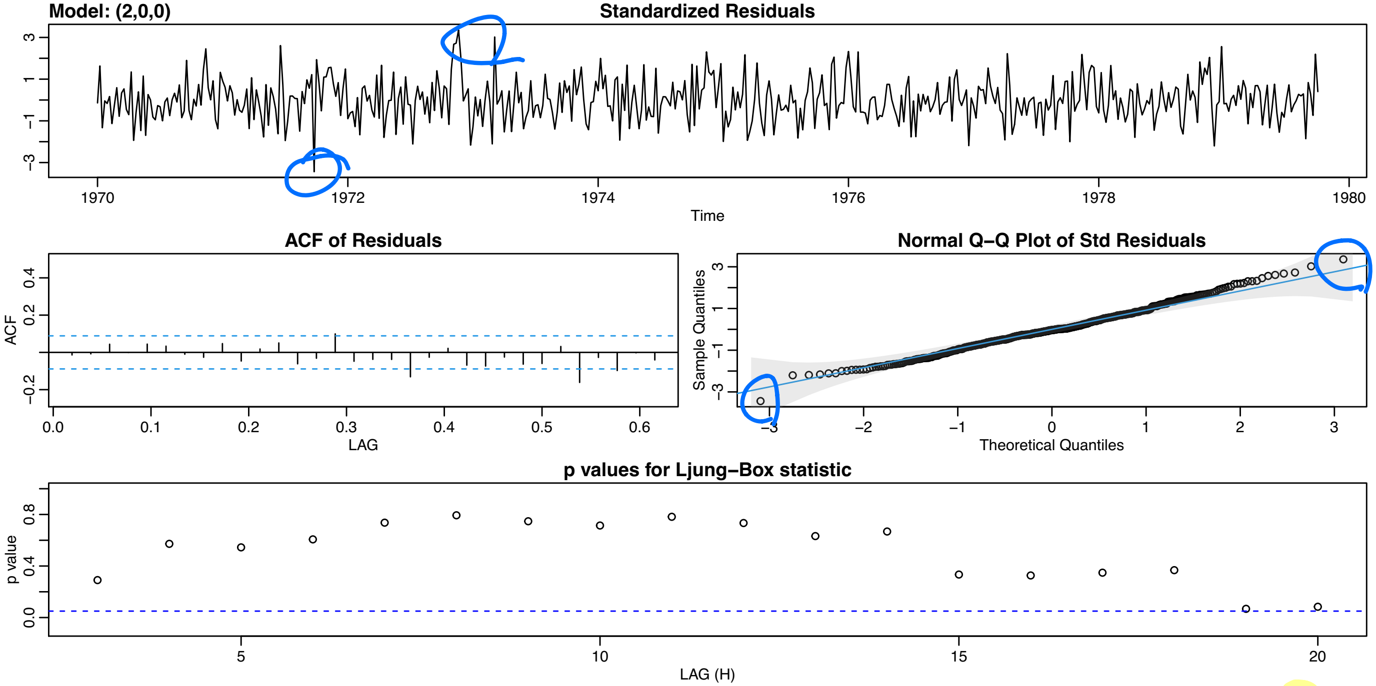
```r
acf2(resid(fit), 52)   # implies AR2
```



AR(2).

Indep. Variables

```
sarima(cmort, 2, 0, 0, xreg = cbind(trend, temp, temp2, part))
```



$$M_t = \beta_0 + \beta_1 t + \beta_2 T_t + \beta_3 T_t^2 + \beta_4 P_t + X_t, \quad X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t.$$

```
$ttable
           Estimate       SE t.value p.value
ar1          0.3848   0.0436  8.8329  0.0000
ar2          0.4326   0.0400 10.8062  0.0000
intercept 3075.1482 834.7157  3.6841  0.0003
trend       -1.5165   0.4226 -3.5882  0.0004
temp        -0.0190   0.0495 -0.3837  0.7014
temp2        0.0154   0.0020  7.6117  0.0000
part         0.1545   0.0272  5.6803  0.0000

$AIC
[1] 6.130066

$AICc
[1] 6.130507

$BIC
[1] 6.196687
```

$\hat{\phi}_1$

$\hat{\phi}_2$

$\hat{\beta}_0$

$\hat{\beta}_1$

$\hat{\beta}_2$

$\hat{\beta}_3$

$\hat{\beta}_4$

Example 5.17: Lagged Regression

Example 5.17: Lagged Regression

predator

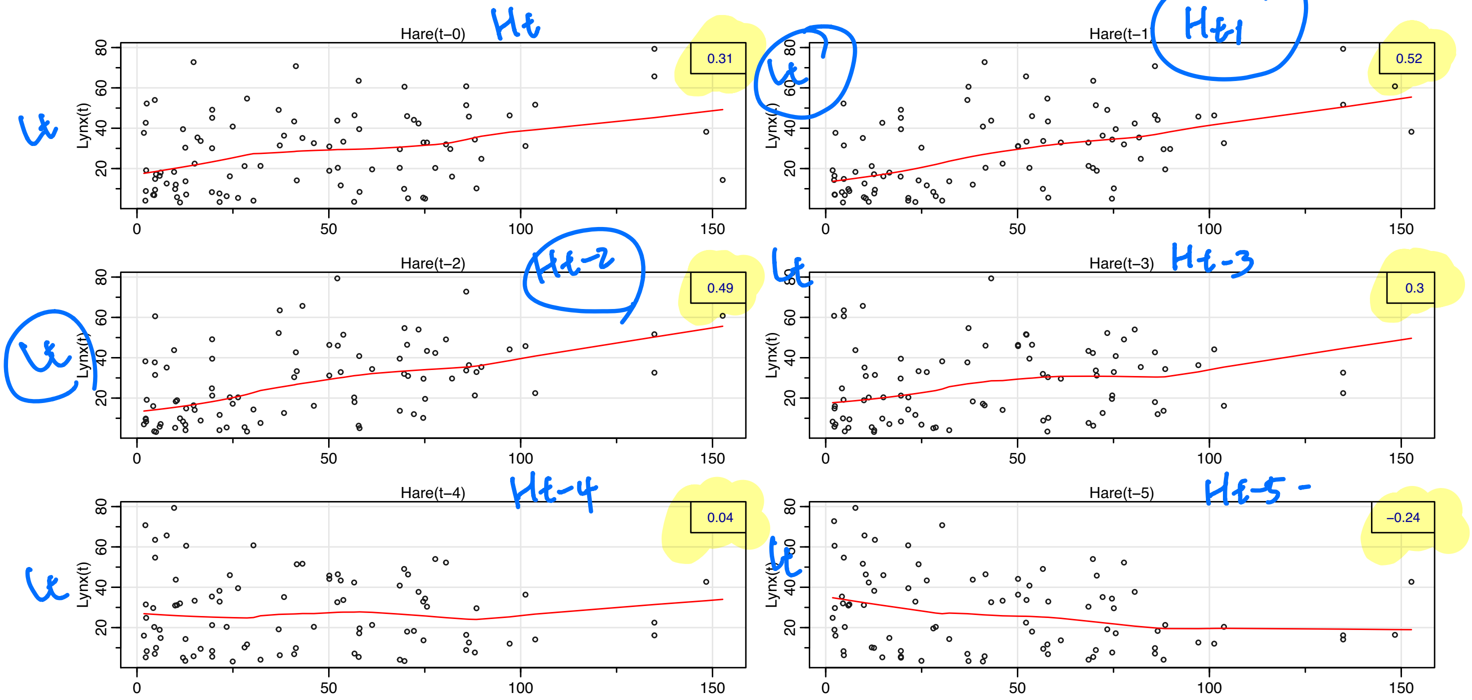
Lynx ↔ prey.

Hare.

L_t H_t

Figure 1: source:
https://www.nps.gov/articles/netn-species-spotlight-snowshoe-hare.htm

```
# Example 5.17
library(zoo)
lag2.plot(Hare, Lynx, 5)  # lead-lag relationship
```



```
pp = as.zoo(ts.intersect(Lynx, HareL1 = lag(Hare, -1)))
summary(reg <- lm(pp$Lynx ~ pp$HareL1))
```

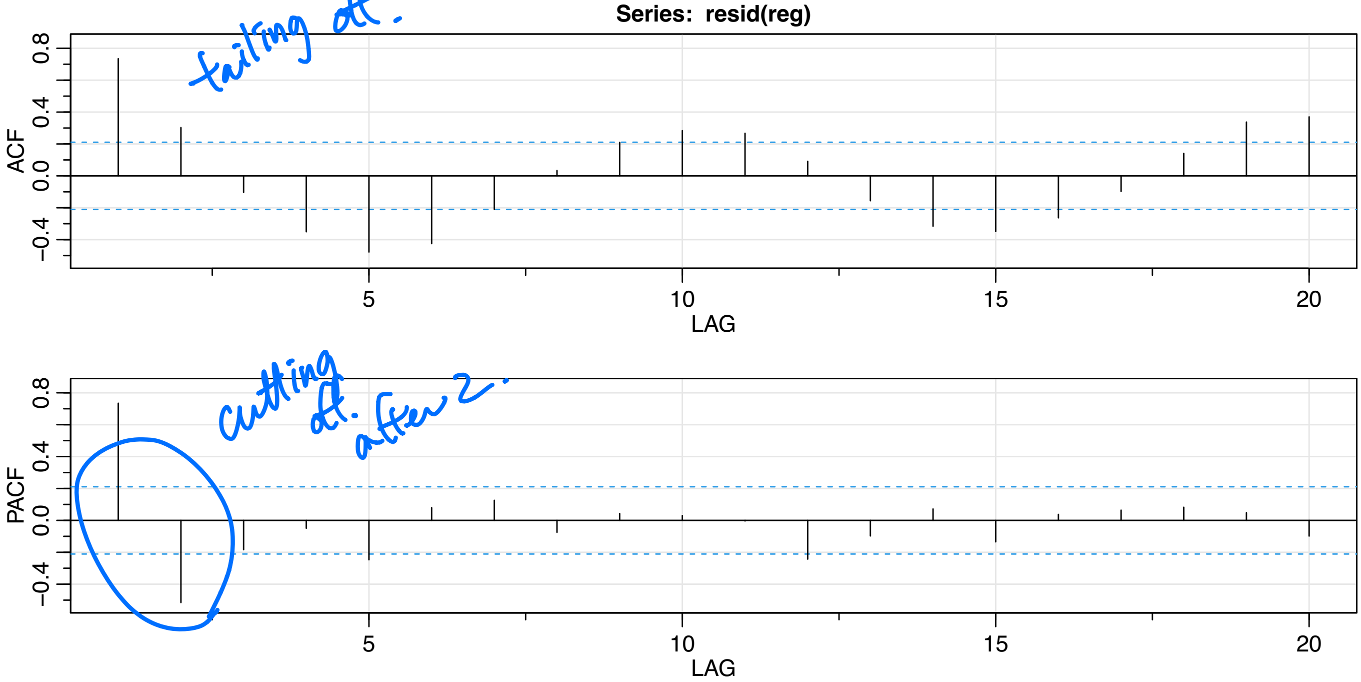
$$L_t = \beta_0 + \beta_1 H_{t-1} + X_t.$$

```
##
## Call:
```

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t.$

```
acf2(resid(reg))   # in Figure 5.11
```

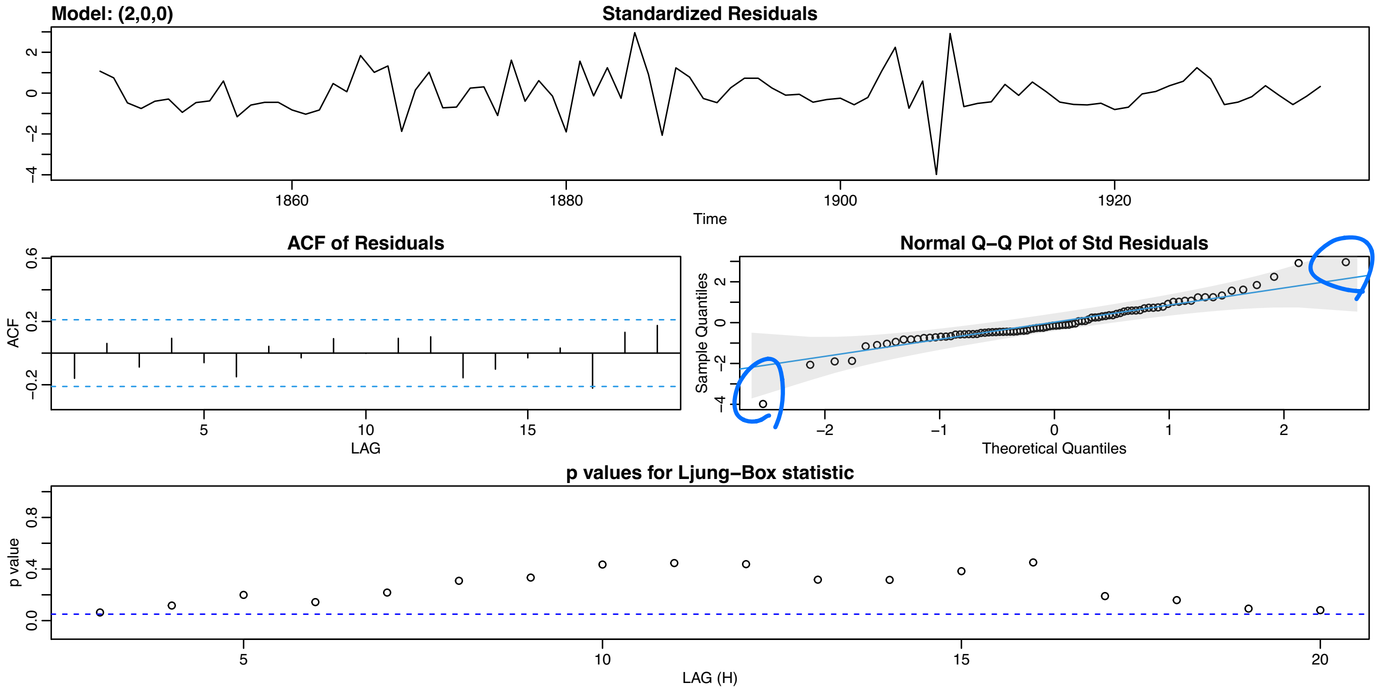
AR(2) X_t.



tailing off.

cutting off. after 2.

```
sarima(pp$Lynx, 2, 0, 0, xreg = pp$HareL1)
```



```
$ttable
          Estimate     SE t.value p.value
ar1         1.3258 0.0732 18.1184  0.0000
ar2        -0.7143 0.0731 -9.7689  0.0000
intercept  25.1319 2.5469  9.8676  0.0000
xreg        0.0692 0.0318  2.1727  0.0326

$AIC
[1] 7.062148

$AICc
[1] 7.067377

$BIC
[1] 7.201026
```

$\hat{\phi}_1$

$\hat{\phi}_2$

$\hat{\beta}_0$

$\hat{\beta}_1$

$L_t = \beta_0 + \beta_1 H_{t-1} + X_t.$

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t.$