HW 11

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Due: 11/10/2022 11:59pm

Question 1.

Find A and B so that
$$3\cos(2\pi\omega t + 0.4) = A\cos(2\pi\omega t) + B\sin(2\pi\omega t).$$

$$\begin{aligned}
3\cos(2\pi\omega t + 0.4) &= 3\cos(2\pi\omega t)\cos(0.4) - 3\sin(2\pi\omega t)\sin(0.4) \\
&= 3\cos(0.4)\cos(2\pi\omega t) - 2\sin(0.4)\sin(2\pi\omega t) \\
&\implies \boxed{A = 3\cos(0.4),\ B = -3\sin(0.4)}
\end{aligned}$$

Question 2.

Find R and Φ so that
$$R\cos(2\pi\omega t + \Phi) = \cos(2\pi\omega t) + 3\sin(2\pi\omega t).$$
$$= R\cos(2\pi\omega t) cos(\Phi) - R\sin(2\pi\omega t)\sin(\Phi)$$
$$R\cos(\Phi) = 1 \text{ and } -R\sin(\Phi) = 3$$
$$\implies R = \frac{1}{\cos(\Phi)} \implies -\frac{\sin(\Phi)}{\cos(\Phi)} = 3 \implies -\tan(\Phi) = 3 \implies \Phi = \tan^{-1}(-3) \implies R = \sec(\tan^{-1}(-3))$$
$$\boxed{\Phi = \tan^{-1}(-3),\ R = \sec(\tan^{-1}(-3))}$$

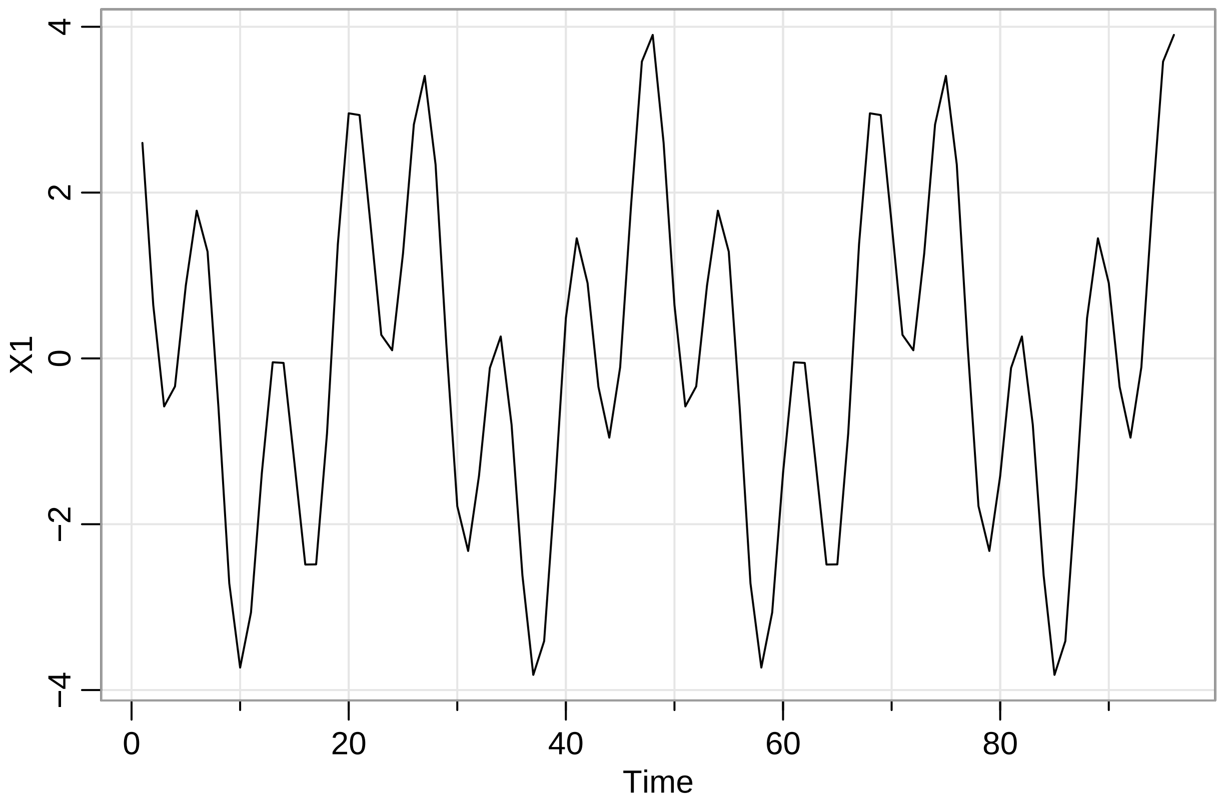
Question 3.

Let

$$X_t = 2\cos\left(2\pi t \frac{4}{96}\right) + 2\sin\left(2\pi\left(t\frac{14}{96} + 0.3\right)\right).$$

(a) Make a time series plot of the time series, for $t = 1, \ldots, 96$.

```
X1 = 2*cos(2*pi*1:96*(4/96)) + 2*sin(2*pi*(1:96*(14/96)+0.3))
tsplot(X1, main="")
```



(b) Conduct the regression of X_t on $\cos(2\pi\omega t)$ and $\sin(2\pi\omega t)$ for $\omega = \frac{4}{96}$. Use R. Verify that they are perfect estimates (no error/noise term).

```
#w = 4/96
Z1 = cos(2*pi*(4/96)*1:96)
Z2 = sin(2*pi*(4/96)*1:96)
#Regression
m1 = lm(X1 ~ Z1 + Z2)
#Model
summary(m1)
```

```
##
## Call:
## lm(formula = X1 ~ Z1 + Z2)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -1.997 -1.375  0.000  1.375  1.997
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -7.705e-16  1.466e-01   0.000        1
## Z1           2.000e+00  2.074e-01   9.644 1.16e-15 ***
## Z2          -3.205e-16  2.074e-01   0.000        1
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.437 on 93 degrees of freedom
## Multiple R-squared:    0.5,  Adjusted R-squared:  0.4892
## F-statistic:  46.5 on 2 and 93 DF,  p-value: 1.005e-14
```

Yes, we have perfect estimates. We notice that the R^2 is exactly 0.5, our `Z1` coefficient is 2, and our `Z2` coefficient is 0. This means that we are perfectly explaining the first part of X_t when $\omega = \frac{4}{96}$.

(c) Conduct the regression of X_t on $\cos(2\pi\omega t)$ and $\sin(2\pi\omega t)$ for $\omega = \frac{14}{96}$. Use R. Verify that they are perfect estimates (no error/noise term).

```
#w = 14/96
Z3 = cos(2*pi*(14/96)*1:96)
Z4 = sin(2*pi*(14/96)*1:96)
#Regression
m2 = lm(X1 ~ Z3 + Z4)
#Model
summary(m2)
```

```
##
## Call:
## lm(formula = X1 ~ Z3 + Z4)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -2.000 -1.414  0.000  1.414  2.000
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -7.821e-16  1.466e-01   0.000  1.00000
## Z3           1.902e+00  2.074e-01   9.172 1.16e-14 ***
## Z4          -6.180e-01  2.074e-01  -2.980  0.00368 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.437 on 93 degrees of freedom
## Multiple R-squared:    0.5,  Adjusted R-squared:  0.4892
## F-statistic:  46.5 on 2 and 93 DF,  p-value: 1.005e-14
```

Yes, we have perfect estimates. We notice that the R^2 is exactly 0.5, our `Z3` coefficient is 1.902, and our `Z4` coefficient is -0.618. This means that we are perfectly explaining the second part of X_t when $\omega = \frac{14}{96}$.

(d) Conduct the regression of X_t on $\cos(2\pi\omega t)$ and $\sin(2\pi\omega t)$ for $\omega = \frac{4}{96}$ and $\omega = \frac{14}{96}$ together. Use R. Verify that they are perfect estimates (no error/noise term).

```
#w = 4/96 and 14/96 together
#Regression
m3 = lm(X1 ~ Z1 + Z2 + Z3 + Z4)
#Model
summary(m3)
```

```
##
## Call:
## lm(formula = X1 ~ Z1 + Z2 + Z3 + Z4)
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -2.056e-14 -4.292e-15 -4.716e-16  3.532e-15  2.135e-14
##
## Coefficients:
##               Estimate Std. Error    t value Pr(>|t|)
## (Intercept) -6.008e-16  7.309e-16 -8.220e-01    0.413
## Z1           2.000e+00  1.034e-15  1.935e+15   <2e-16 ***
## Z2           4.094e-16  1.034e-15  3.960e-01    0.693
## Z3           1.902e+00  1.034e-15  1.840e+15   <2e-16 ***
## Z4          -6.180e-01  1.034e-15 -5.979e+14   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 7.161e-15 on 91 degrees of freedom
## Multiple R-squared:      1,  Adjusted R-squared:      1
## F-statistic: 1.872e+30 on 4 and 91 DF,  p-value: < 2.2e-16
```

Yes, we have perfect estimates. We notice that the R^2 is exactly 1, which means that we are perfectly explaining both parts of X_t.

(e) Conduct the regression of X_t on $\cos(2\pi\omega t)$ and $\sin(2\pi\omega t)$ for $\omega = \frac{3}{96}$ and $\omega = \frac{13}{96}$ together. Use R. Are those estimates still perfect?

```
#w = 3/96 and 13/96 together
Z5 = cos(2*pi*(3/96)*1:96)
Z6 = sin(2*pi*(3/96)*1:96)
Z7 = cos(2*pi*(13/96)*1:96)
Z8 = sin(2*pi*(13/96)*1:96)
#Regression
m4 = lm(X1 ~ Z5 + Z6 + Z7 + Z8)
#Model
summary(m4)
```

```
##
## Call:
## lm(formula = X1 ~ Z5 + Z6 + Z7 + Z8)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.8171 -1.3904 -0.0506  1.3962  3.9021
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -9.518e-16  2.097e-01       0        1
## Z5          -6.410e-17  2.965e-01       0        1
## Z6          -4.807e-17  2.965e-01       0        1
## Z7           6.330e-15  2.965e-01       0        1
## Z8          -2.035e-15  2.965e-01       0        1
##
## Residual standard error: 2.054 on 91 degrees of freedom
## Multiple R-squared:  5.69e-30,   Adjusted R-squared:  -0.04396
## F-statistic: 1.294e-28 on 4 and 91 DF,  p-value: 1
```

No, the estimates are no longer perfect. We can see that the R^2 is practically 0 and all of the coefficient estimates are also practically 0. Note also how none of them are insignificant.

Question 4.

Generate any series of length n = 10. Show that the series may be fit exactly by a linear combination of enough cosine-sine curves at the Fourier frequencies, $\omega = 1/10, \ldots, 5/10$. You may use R and conduct regression for this problem.

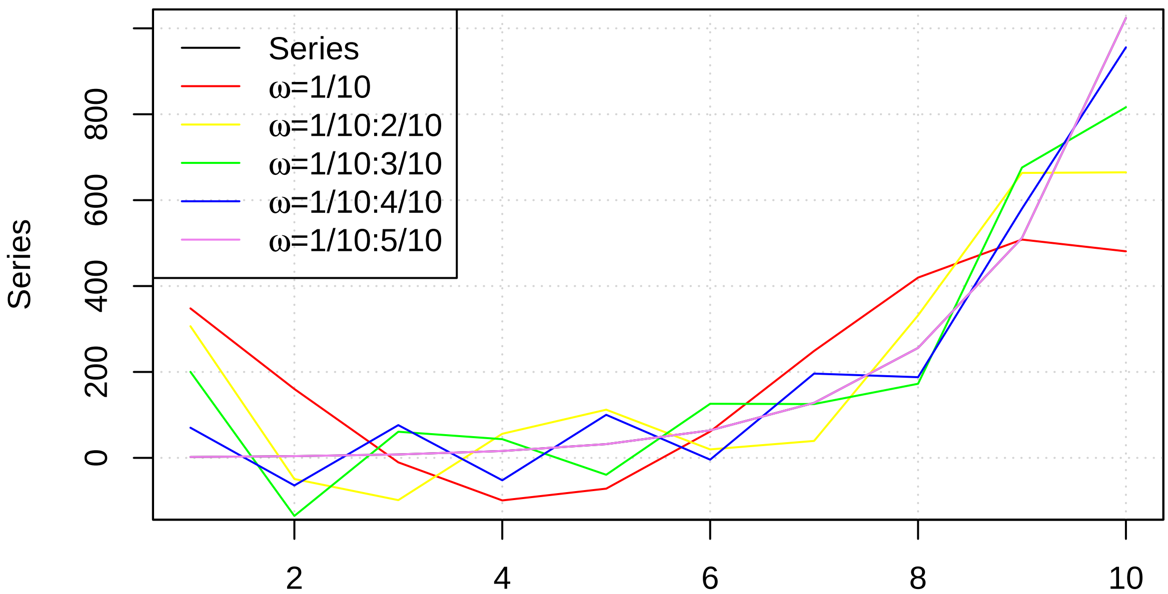
```r
#Creating Sequence
myseq = rep(0,10)

for(i in 1:10){
  myseq[i] = 2^i
}

#Creating Frequency Terms
F1 = cos(2*pi*(1/10)*1:10)
F2 = sin(2*pi*(1/10)*1:10)
F3 = cos(2*pi*(2/10)*1:10)
F4 = sin(2*pi*(2/10)*1:10)
F5 = cos(2*pi*(3/10)*1:10)
F6 = sin(2*pi*(3/10)*1:10)
F7 = cos(2*pi*(4/10)*1:10)
F8 = sin(2*pi*(4/10)*1:10)
F9 = cos(2*pi*(5/10)*1:10)
F10 = sin(2*pi*(5/10)*1:10)

#Creating models
M1 = lm(myseq ~ F1 + F2)
M2 = lm(myseq ~ F1 + F2 + F3 + F4)
M3 = lm(myseq ~ F1 + F2 + F3 + F4 + F5 + F6)
M4 = lm(myseq ~ F1 + F2 + F3 + F4 + F5 + F6 + F7 + F8)
M5 = lm(myseq ~ F1 + F2 + F3 + F4 + F5 + F6 + F7 + F8 + F9 + F10)

#Plotting
plot(1:10, myseq, col = "white", xlab = "", ylab = "Series", ylim = c(-100,1000))
grid()
box()
lines(1:10, myseq)
lines(1:10, M1$fitted.values, col = "red")
lines(1:10, M2$fitted.values, col = "yellow")
lines(1:10, M3$fitted.values, col = "green")
lines(1:10, M4$fitted.values, col = "blue")
lines(1:10, M5$fitted.values, col = "violet")
legend("topleft", legend = c("Series",TeX(r"($\omega$=1/10)"),TeX(r"($\omega$=1/10:2/10)"),
       TeX(r"($\omega$=1/10:3/10)"), TeX(r"($\omega$=1/10:4/10)"),TeX(r"($\omega$=1/10:5/10)")),
       col = c("black","red","yellow","green","blue","violet"), lty = 1)
```



As we can see, once we use all five of the Fourier frequencies ($\omega = 1/10, \ldots, 5/10$), the series of 2^x will be fit exactly by a linear combination of the cos-sin curves.

Question 5. [GR-only]

For simplicity, let's assume that n is an even integer.

Hint: Use $\cos(b) = \frac{e^{ib}+e^{-ib}}{2}$, $\sin(b) = \frac{e^{ib}-e^{-ib}}{2i}$, $e^{ib} = \cos(b) + i\sin(b)$.

Verify that for any positive integer n and $j, k = 0, 1, \dots n/2$:

(a) Except for $j = 0$ or $j = n/2$,

$$\sum_{t=1}^{n} \cos^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} \sin^2(2\pi t \frac{j}{n}) = \frac{n}{2}.$$

$$\sum_{t=1}^{n} \cos^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} (\frac{e^{2\pi t \frac{j}{n} i} + e^{-2\pi t \frac{j}{n} i}}{2})^2 = \sum_{t=1}^{n} (\frac{e^{4\pi t \frac{j}{n} i} + e^{-4\pi t \frac{j}{n} i} + 2}{4})$$

$$= \sum_{t=1}^{n} [\frac{1}{2} + \frac{\cos(4\pi t \frac{j}{n})}{4} + \frac{i\sin(4\pi t \frac{j}{n})}{4} + \frac{\cos(-4\pi t \frac{j}{n})}{4} + \frac{i\sin(-4\pi t \frac{j}{n})}{4}]$$

$$= \sum_{t=1}^{n} [\frac{1}{2} + \frac{\cos(4\pi t \frac{j}{n})}{4} + \frac{i\sin(4\pi t \frac{j}{n})}{4} + \frac{\cos(4\pi t \frac{j}{n})}{4} - \frac{i\sin(4\pi t \frac{j}{n})}{4}]$$

$$= \frac{n}{2} + \frac{1}{2} \sum_{t=1}^{n} \cos(4\pi t \frac{j}{n})$$

$$\sum_{t=1}^{n} \cos(4\pi t \frac{j}{n}) = 0 \text{ as } n \text{ is even.}$$

$$\implies \sum_{t=1}^{n} \cos^2(2\pi t \frac{j}{n}) = \frac{n}{2}$$

$$\sum_{t=1}^{n} \sin^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} (\frac{e^{2\pi t \frac{j}{n} i} - e^{-2\pi t \frac{j}{n} i}}{2i})^2 = \sum_{t=1}^{n} (\frac{e^{4\pi t \frac{j}{n} i} + e^{-4\pi t \frac{j}{n} i} - 2}{-4})$$

$$= \sum_{t=1}^{n} [\frac{1}{2} - \frac{\cos(4\pi t \frac{j}{n})}{4} - \frac{i\sin(4\pi t \frac{j}{n})}{4} - \frac{\cos(-4\pi t \frac{j}{n})}{4} - \frac{i\sin(-4\pi t \frac{j}{n})}{4}]$$

$$= \sum_{t=1}^{n} [\frac{1}{2} - \frac{\cos(4\pi t \frac{j}{n})}{4} - \frac{i\sin(4\pi t \frac{j}{n})}{4} - \frac{\cos(4\pi t \frac{j}{n})}{4} + \frac{i\sin(4\pi t \frac{j}{n})}{4}]$$

$$= \frac{n}{2} - \frac{1}{2} \sum_{t=1}^{n} \cos(4\pi t \frac{j}{n})$$

$$\sum_{t=1}^{n} \cos(4\pi t \frac{j}{n}) = 0 \text{ as } n \text{ is even.}$$

$$\implies \sum_{t=1}^{n} \sin^2(2\pi t \frac{j}{n}) = \frac{n}{2}$$

(b) When $j = 0$ or $j = n/2$,

$$\sum_{t=1}^{n} \cos^2(2\pi t \frac{j}{n}) = n, \quad \sum_{t=1}^{n} \sin^2(2\pi t \frac{j}{n}) = 0.$$

- $j = 0$

$$\sum_{t=1}^{n} \cos^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} \cos^2(0) = \sum_{t=1}^{n} 1 = n$$
$$\sum_{t=1}^{n} \sin^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} \sin^2(0) = \sum_{t=1}^{n} 0 = 0$$

- $j = \frac{n}{2}$

$$\sum_{t=1}^{n} \cos^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} \cos^2(\pi t) = \sum_{t=1}^{n} 1 = n$$
$$\sum_{t=1}^{n} \sin^2(2\pi t \frac{j}{n}) = \sum_{t=1}^{n} \sin^2(\pi t) = \sum_{t=1}^{n} 0 = 0$$

(c) When $j \neq k$,

$$\sum_{t=1}^{n} \cos(2\pi t \frac{j}{n}) \cos(2\pi t \frac{k}{n}) = \sum_{t=1}^{n} \sin(2\pi t \frac{j}{n}) \sin(2\pi t \frac{k}{n}) = 0.$$

$$\sum_{t=1}^{n} \cos(2\pi t \frac{j}{n}) \cos(2\pi t \frac{k}{n}) = \sum_{t=1}^{n} (\frac{e^{2\pi t \frac{j}{n} i} + e^{-2\pi t \frac{j}{n} i}}{2})(\frac{e^{2\pi t \frac{k}{n} i} + e^{-2\pi t \frac{k}{n} i}}{2})$$
$$= \frac{1}{4} \sum_{t=1}^{n} e^{\frac{2\pi t}{n}(j+k)i} + e^{\frac{2\pi t}{n}(k-j)i} + e^{\frac{2\pi t}{n}(j-k)i} + e^{\frac{2\pi t}{n}(-j-k)i}$$
$$= \frac{1}{4} \sum_{t=1}^{n} [\cos(\frac{2\pi t}{n}(j+k)) + i \sin(\frac{2\pi t}{n}(j+k)) + \cos(\frac{2\pi t}{n}(k-j)) + i \sin(\frac{2\pi t}{n}(k-j))$$
$$+ \cos(\frac{2\pi t}{n}(j-k)) + i \sin(\frac{2\pi t}{n}(j-k)) + \cos(\frac{2\pi t}{n}(-j-k)) + i \sin(\frac{2\pi t}{n}(-j-k))]$$
$$= \frac{1}{4} \sum_{t=1}^{n} [\cos(\frac{2\pi t}{n}(j+k)) + i \sin(\frac{2\pi t}{n}(j+k)) + \cos(\frac{2\pi t}{n}(j-k)) - i \sin(\frac{2\pi t}{n}(j-k))$$
$$+ \cos(\frac{2\pi t}{n}(j-k)) + i \sin(\frac{2\pi t}{n}(j-k)) + \cos(\frac{2\pi t}{n}(j+k)) - i \sin(\frac{2\pi t}{n}(j+k))]$$
$$= \frac{1}{2} \sum_{t=1}^{n} \cos(\frac{2\pi t}{n}(j+k)) + \cos(\frac{2\pi t}{n}(j-k))$$
$$\sum_{t=1}^{n} \cos(\frac{2\pi t}{n}(j+k)) + \cos(\frac{2\pi t}{n}(j-k)) = 0 \text{ as } n \text{ is even.}$$
$$\sum_{t=1}^{n} \cos(2\pi t \frac{j}{n}) \cos(2\pi t \frac{k}{n}) = 0$$
$$\sum_{t=1}^{n} \sin(2\pi t \frac{j}{n}) \sin(2\pi t \frac{k}{n}) = \sum_{t=1}^{n} (\frac{e^{2\pi t \frac{j}{n} i} - e^{-2\pi t \frac{j}{n} i}}{2i})(\frac{e^{2\pi t \frac{k}{n} i} - e^{-2\pi t \frac{k}{n} i}}{2i})$$

$$= -\frac{1}{4}\sum_{t=1}^{n} e^{\frac{2\pi t}{n}(j+k)i} - e^{\frac{2\pi t}{n}(k-j)i} - e^{\frac{2\pi t}{n}(j-k)i} + e^{\frac{2\pi t}{n}(-j-k)i}$$

$$= -\frac{1}{4}\sum_{t=1}^{n}[\cos(\frac{2\pi t}{n}(j+k)) + i\sin(\frac{2\pi t}{n}(j+k)) - \cos(\frac{2\pi t}{n}(k-j)) - i\sin(\frac{2\pi t}{n}(k-j))$$

$$- \cos(\frac{2\pi t}{n}(j-k)) - i\sin(\frac{2\pi t}{n}(j-k)) + \cos(\frac{2\pi t}{n}(-j-k)) + i\sin(\frac{2\pi t}{n}(-j-k))]$$

$$= -\frac{1}{4}\sum_{t=1}^{n}[\cos(\frac{2\pi t}{n}(j+k)) + i\sin(\frac{2\pi t}{n}(j+k)) - \cos(\frac{2\pi t}{n}(j-k)) + i\sin(\frac{2\pi t}{n}(j-k))$$

$$- \cos(\frac{2\pi t}{n}(j-k)) - i\sin(\frac{2\pi t}{n}(j-k)) + \cos(\frac{2\pi t}{n}(j+k)) - i\sin(\frac{2\pi t}{n}(j+k))]$$

$$= \frac{1}{2}\sum_{t=1}^{n}\cos(\frac{2\pi t}{n}(j-k)) - \cos(\frac{2\pi t}{n}(j+k))$$

$$\sum_{t=1}^{n}\cos(\frac{2\pi t}{n}(j-k)) - \cos(\frac{2\pi t}{n}(j+k)) = 0 \text{ as } n \text{ is even.}$$

$$\sum_{t=1}^{n}\sin(2\pi t\frac{j}{n})\sin(2\pi t\frac{k}{n}) = 0$$

(d) For any j and k,

$$\sum_{t=1}^{n}\cos(2\pi t\frac{j}{n})\sin(2\pi t\frac{k}{n}) = 0.$$

$$\sum_{t=1}^{n}\cos(2\pi t\frac{j}{n})\sin(2\pi t\frac{k}{n}) = \sum_{t=1}^{n}(\frac{e^{2\pi t\frac{j}{n}i} + e^{-2\pi t\frac{j}{n}i}}{2})(\frac{e^{2\pi t\frac{k}{n}i} - e^{-2\pi t\frac{k}{n}i}}{2i})$$

$$= \frac{1}{4i}\sum_{t=1}^{n} e^{\frac{2\pi t}{n}(j+k)i} + e^{\frac{2\pi t}{n}(k-j)i} - e^{\frac{2\pi t}{n}(j-k)i} - e^{\frac{2\pi t}{n}(-j-k)i}$$

$$= \frac{1}{4i}\sum_{t=1}^{n}[\cos(\frac{2\pi t}{n}(j+k)) + i\sin(\frac{2\pi t}{n}(j+k)) + \cos(\frac{2\pi t}{n}(k-j)) + i\sin(\frac{2\pi t}{n}(k-j))$$

$$- \cos(\frac{2\pi t}{n}(j-k)) - i\sin(\frac{2\pi t}{n}(j-k)) - \cos(\frac{2\pi t}{n}(-j-k)) - i\sin(\frac{2\pi t}{n}(-j-k))]$$

$$= \frac{1}{4i}\sum_{t=1}^{n}[\cos(\frac{2\pi t}{n}(j+k)) + i\sin(\frac{2\pi t}{n}(j+k)) + \cos(\frac{2\pi t}{n}(j-k)) - i\sin(\frac{2\pi t}{n}(j-k))$$

$$- \cos(\frac{2\pi t}{n}(j-k)) - i\sin(\frac{2\pi t}{n}(j-k)) - \cos(\frac{2\pi t}{n}(j+k)) + i\sin(\frac{2\pi t}{n}(j+k))]$$

$$= \frac{1}{2}\sum_{t=1}^{n}\sin(\frac{2\pi t}{n}(j+k)) - \sin(\frac{2\pi t}{n}(j-k))$$

$$\sum_{t=1}^{n}\sin(\frac{2\pi t}{n}(j+k)) - \sin(\frac{2\pi t}{n}(j-k)) = 0 \text{ as } n \text{ is even.}$$

$$\sum_{t=1}^{n}\cos(2\pi t\frac{j}{n})\sin(2\pi t\frac{k}{n}) = 0.$$