Week 5 Lecture Note 2

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Week 5 part 2 lecture note

Recall

Complex number (Book Appendix C)

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Solution of $x^2 + 1 = 0$: No 'real' solution, we define an 'imaginary' number.

$$i^2 = -1$$

Complex number form: $z = a + bi$, a, b are real numbers.

- Modulus of z: $|z| = \sqrt{a^2 + b^2}$
- Argument of z: $\arg z = \arctan(b/a)$.

ARMA(p,q): Autoregressive Moving Average model

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A time series $\{X_t\}$ is following ARMA(p,q) if

1. it is stationary,
2.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \ldots + \theta_q W_{t-q},$$

- $\phi_p \neq 0$, $\theta_q \neq 0$,
- $W_t \sim WN(0, \sigma_W^2)$

3. it is causal, invertible process.

AR, MA operators, parameter redundancy

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$$\phi(B) := (1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p)$$
$$\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$$

$$\phi(B) X_t = \theta(B) W_t$$

- $q = 0$: AR(p)
- $p = 0$: MA(q)
- **Parameter Redundancy** : We require $\phi(B), \theta(B)$ to have no common factors to avoid parameter redundancy.

ARMA-Causality, Invertibility

Causality of ARMA(p,q)

Causality of ARMA(p,q)

An ARMA (p,q) model is causal if $\{X_t\}$ can be written as one-sided linear process, (MA(∞) representation)

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t$$

where $\psi(B) := \sum_{j=0}^{\infty} \psi_j B^j$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

An ARMA(p,q) model is causal if and only if *all roots of the equation* $\phi(z) = 0$ *lie outside the unit circle* (modulus > 1).

- equivalent to: $\phi(z) \neq 0$ for all $|z| \leq 1$.

The coefficients can be determined by solving

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}, \quad |z| \leq 1$$

Invertibility of ARMA(p,q)

Invertibility of ARMA(p,q)

An ARMA (p,q) model is invertible if $\{W_t\}$ can be written as (Ar(∞) representation)

$$W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = \pi(B) X_t$$

where $\pi(B) := \sum_{j=0}^{\infty} \pi_j B^j$, $\sum_{j=0}^{\infty} \pi_j^2 < \infty$.

An ARMA(p,q) model is invertible if and only if all roots of the equation $\theta(z) = 0$ lie outside the unit circle (modulus > 1).

- equivalent to: $\theta(z) \neq 0$ for all $|z| \leq 1$.

The coefficients can be determined by solving

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)}, \quad |z| \leq 1$$

Example: ARMA(1,1)

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$$X_t = 0.9X_{t-1} + W_t + 0.5W_{t-1}$$

Check causality and invertibility of this process, and find the causal representation and invertible representation. (by hand)

Correlation functions (ACF and PACF)

ACF

ACF of MA(q)

$$\gamma(h) = \begin{cases} \sigma_W^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} & 0 \leq h \leq q \\ 0 & h > q \end{cases}$$

$$\rho(h) = \begin{cases} \frac{\sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{1+\theta_1^2+\ldots+\theta_q^2} & 0 \leq h \leq q \\ 0 & h > q \end{cases}$$

ACF of AR(p) or ARMA(p,q)

(MA(∞) representation)

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$$

$$\gamma(h) = \sigma_W^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}, \quad h \geq 0$$

$$\rho(h) = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+h}}{\sum_{j=0}^{\infty} \psi_j^2}, \quad h \geq 0$$

Example: AR(2)

$$X_t = 1.5X_{t-1} - 0.75X_{t-2} + W_t$$

ACF of ARMA(1,1)

$$X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$$

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{\phi(1+2\theta\phi+\theta^2)}\phi^h, \quad h \geq 1$$

Example:

$$X_t = 0.9X_{t-1} + W_t + 0.5W_{t-1}$$

```
ARMAacf(ar = numeric(), ma = numeric(), lag.max = r,
 pacf = FALSE)
```

Arguments

ar: numeric vector of AR coefficients
ma: numeric vector of MA coefficients
lag.max: integer. Maximum lag required. Defaults to max(p, q+1), where p, q are the numbers of AR and MA terms respectively.
pacf: logical. Should the partial autocorrelations be returned?

```r
ARMAacf(ar=c(1.5,-0.75), lag.max=10)
```

```
##           0           1           2           3           4           5 
##  1.00000000  0.85714286  0.53571429  0.16071429 -0.16071429 -0.36160714 
##           6           7           8           9          10 
## -0.42187500 -0.36160714 -0.22600446 -0.06780134  0.06780134
```

```r
ARMAacf(ar=c(0.9),ma=c(0.5), lag.max=10)
```

```
##         0         1         2         3         4         5         6         7 
## 1.0000000 0.9441860 0.8497674 0.7647907 0.6883116 0.6194805 0.5575324 0.5017792 
##         8         9        10 
## 0.4516013 0.4064411 0.3657970
```

ACF of MA, AR, ARMA

- MA(q) models: the ACF will be zero for lags greater than q.
  - ACF provides a considerable amount of information when the process is MA.
- AR(p), ARMA(p,q): ACF alone tells us little about the orders of dependence
  - We introduce PACF (partial autocorrelation function)

PACF

Partial autocorrelation

X, Y, Z random variables.

- Partial correlation between X and Y given Z: obtained by
  - regressing X on Z to obtain $\hat{X}$
  - regressing Y on Z to obtain $\hat{Y}$, and then

$$\rho_{XY|Z} = corr\{X - \hat{X}, Y - \hat{Y}\}$$

Definition: Partial autocorrelation function (PACF)

Partial autocorrelation function (PACF) of a stationary process $\{X_t\}$, denoted ϕ_{hh}, for $h = 1, 2, \ldots$ is

$$\phi_{11} = corr(X_{t+1}, X_t) = \phi(1)$$

and

$$\phi_{hh} = corr(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t), h \geq 2$$

- where
  - $\hat{X}_{t+h}$ is the regression of X_{t+h} on $\{X_{t+h-1}, \ldots X_{t+1}\}$, and
  - $\hat{X}_t$ is the regression of X_t on $\{X_{t+h-1}, \ldots X_{t+1}\}$

Partial autocorrelation of the AR(1)

$$X_t = \phi X_{t-1} + W_t$$

with $|\phi| < 1$.

By definition $\phi_{11} = Corr(X_t, X_{t-1}) = \phi = \rho_1$.

We can get ϕ_{22} by calculating:

$$Cov(X_t - \rho_1 X_{t-1}, X_{t-2} - \rho_1 X_{t-1})$$

$$= \gamma_0(\rho_2 - \rho_1^2)$$

And

$$Var(X_t - \rho_1 X_{t-1}) = Var(X_{t-2} - \rho_1 X_{t-1})$$
$$= \gamma_0(1 - \rho_1^2)$$

Then

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\phi^2 - \rho_1^2}{1 - \rho_1^2} = 0$$

Since $\rho_k = \phi^k$ for $k \geq 0$

It can be seen that $\phi_{kk} = 0$ for $k > 1$

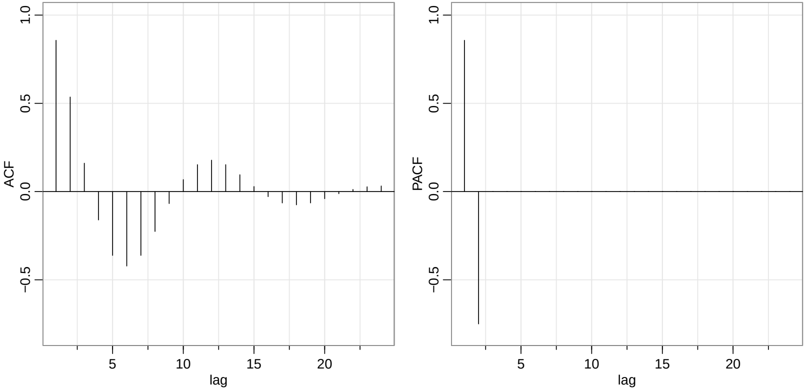
Partial autocorrelation of the AR(p)

$$\phi_{kk} = 0$$

for $k > p$

ϕ_{kk} is **not zero** for $k \leq p$ and ϕ_{pp}=ϕ_p.

```r
ACF = ARMAacf(ar=c(1.5,-.75), ma=0, 24)[-1]
PACF = ARMAacf(ar=c(1.5,-.75), ma=0, 24, pacf=TRUE)
par(mfrow=1:2)
tsplot(ACF, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
tsplot(PACF, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
```



Lare sample distribution of the PACF

If a time series is an AR(p), and the sample size n is large, then for $h > p$, the $\hat{\phi}_{hh}$ are approximately independent normal with mean 0 and standard deviation $1/\sqrt{n}$. This result also holds for $p = 0$, wherein the process is white noise.

PACF of MA(q) and ARMA(p,q)

An MA(q) or ARMA(p,q) are invertible, so both have AR(∞) representation,

$$W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = \pi(B) X_t,$$

or

$$X_t = -\sum_{j=1}^{\infty} \pi_j X_{t-j} + W_t$$

- No finite representation exists
- PACF will never cut off

For an invertible MA(1):

$$\phi_{hh} = \frac{(-\theta)^h(1-\theta^2)}{1-\theta^{2(h+1)}}, \quad h \geq 1$$

This function is never zero and decays to zero exponentially as lag increases (like the autocorrelation function of an AR(q))

Model	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

Example: ACF, PACF of recruitment series

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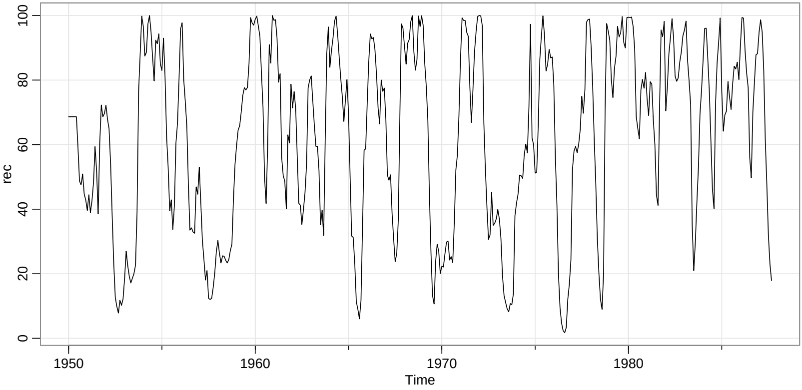
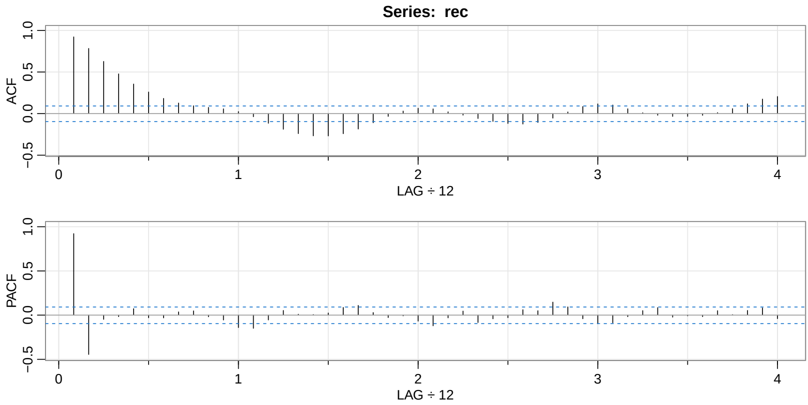
```
tsplot(rec)
```



```r
acf2(rec, 48) # will produce values and a graphic
```




```
##      [,1]  [,2]  [,3]  [,4] [,5]  [,6]  [,7] [,8] [,9] [,10] [,11] [,12] [,13]
## ACF  0.92  0.78  0.63  0.48 0.36  0.26  0.18 0.13 0.09  0.07  0.06  0.02 -0.04
## PACF 0.92 -0.44 -0.05 -0.02 0.07 -0.03 -0.03 0.04 0.05 -0.02 -0.05 -0.14 -0.15
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
## ACF  -0.12 -0.19 -0.24 -0.27 -0.27 -0.24 -0.19 -0.11 -0.03  0.03  0.06  0.06
## PACF -0.05  0.05  0.01  0.01  0.02  0.09  0.11  0.03 -0.03 -0.01 -0.07 -0.12
##      [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
## ACF   0.02 -0.02 -0.06 -0.09 -0.12 -0.13 -0.11 -0.05  0.02  0.08  0.12  0.10
## PACF -0.03  0.05 -0.08 -0.04 -0.03  0.06  0.05  0.15  0.09 -0.04 -0.10 -0.09
##      [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
## ACF   0.06  0.01 -0.02 -0.03 -0.03 -0.02  0.01  0.06  0.12  0.17  0.20
## PACF -0.02  0.05  0.08 -0.02 -0.01 -0.02  0.05  0.01  0.05  0.08 -0.04
```