Week 5 Lecture Note

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Linear Models: ARMA

Topics List

- AR model, causality
- MA model, invertibility
- ARMA, parameter redundancy

Autoregressive (AR) Models

p-th order Autoregressive process (AR(p))

The process $\{X_t\}$ satisfies the equation:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + W_t$$

where X_t is *stationary* and W_t is white noise.

First-order Autoregressive process (AR(1))

$$X_t = \phi X_{t-1} + W_t$$

Initially we assume that the process mean is zero. W_t is a white noise.

AR(1) model as a General Linear Process

Recursive substitution of the AR(1) model:

$$\begin{aligned} X_t &= \phi(\phi X_{t-2} + W_{t-1}) + W_t \\ &= W_t + \phi W_{t-1} + \phi^2 X_{t-2} \end{aligned}$$

Assuming $|\phi| < 1$ and k increasing to infinity we get:

$$\begin{aligned} X_t &= W_t + \phi W_{t-1} + \phi^2 W_{t-2} + \phi^3 W_{t-3} + \dots \\ &= \sum_{j=0}^{\infty} \phi^j W_{t-j} \end{aligned}$$

Causal Representation:

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

is called the *causal solution* of the model. The term *causal* refers to the fact X_t does not depend on the future.

Definition [Causality]: A time series X_t is said to be *causal* if it can be written as

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j W_{t-j}$$

for constants ψ_j satisfying $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

AR(1) is causal if $|\phi| < 1$.

AR(1) properties

Variance

$$\gamma(0) = \frac{\sigma_W^2}{1-\phi^2}$$

Autocovariance function

$$\gamma(h) = \phi^h \frac{\sigma_W^2}{1-\phi^2}$$

Autocorrelation function

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h \quad \text{for} \quad h = 1, 2, 3, \dots$$

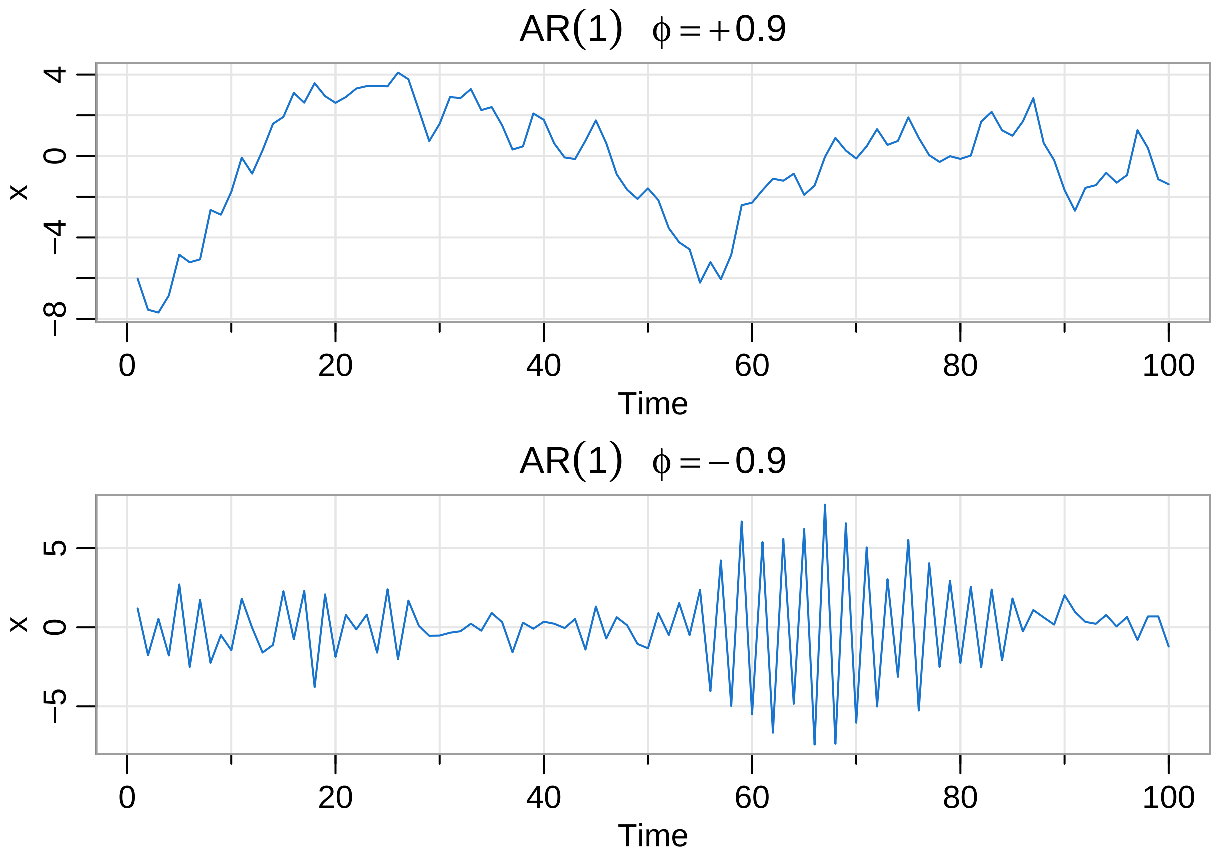
properties of AR(1) ACF:

- Since $|\phi| < 1$, the autocorrelation function decreases exponentially with the number of lags
- If $0 < \phi < 1$ all autocorrelations are positive
- If $-1 < \phi < 0$ lag one autocorrelation is negative and the signs of the next autocorrelations will alternate from negative to positive

R functions arima.sim

```
arima.sim(model, n, rand.gen = rnorm, innov = rand.gen(n, ...),
          n.start = NA, start.innov = rand.gen(n.start, ...),
          ...)
```

```
par(mfrow=c(2,1))
tsplot(arima.sim(list(order=c(1,0,0), ar=.9), n=100), ylab="x", col=4, main=expression(AR(1)~~~phi==+.9
tsplot(arima.sim(list(order=c(1,0,0),ar=-.9), n=100), ylab="x", col=4, main=expression(AR(1)~~~phi==-.9
```



AR(1) ϕ = +0.9
x
4
0
−4
−8
0
20
40
60
80
100
Time
AR(1) ϕ = −0.9
x
5
0
−5
0
20
40
60
80
100
Time

General Autoregressive process (AR(p))

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t$$

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} \ldots - \phi_p X_{t-p} = W_t$$
$$(1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) X_t = W_t$$

We denote $\phi(B) := (1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p)$ as the autogregressive operator, and we write $\phi(B) X_t = W_t$.

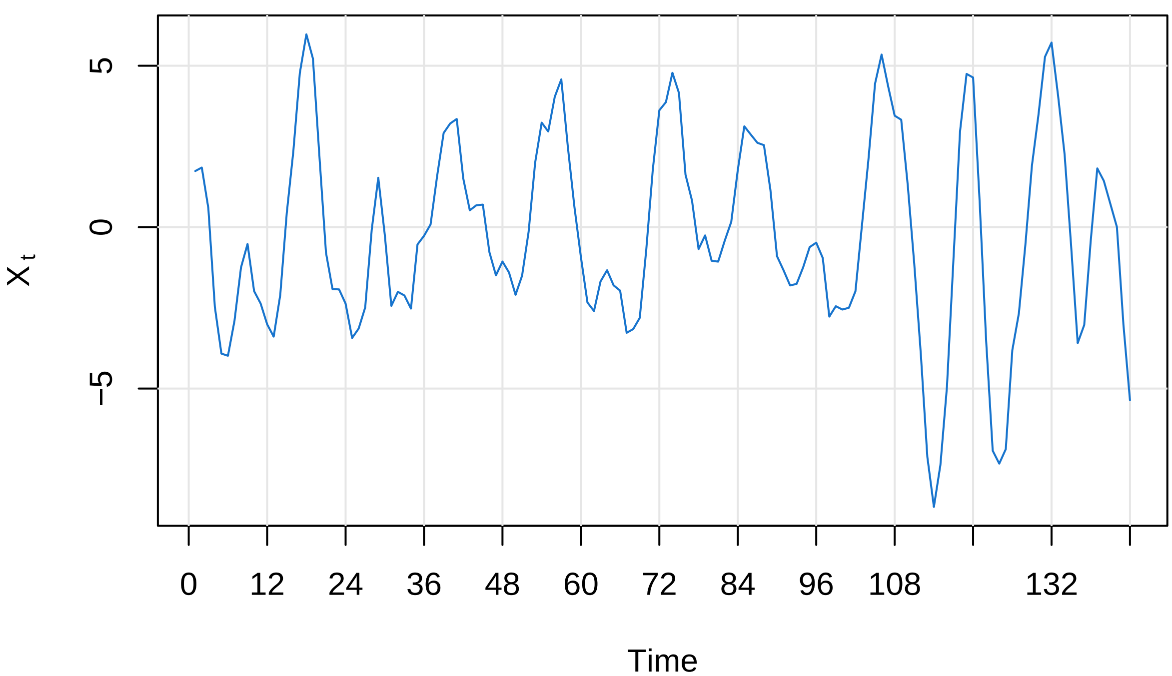
Characteristic equation: $(1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) = 0$

A causal solutions of the AR(p) process exists if and only the p roots of the characteristic equation lie outside the unit circle (modulus greater than 1).

Example:

$$X_t = 1.5 X_{t-1} - 0.75 X_{t-2} + W_t$$

```
set.seed(8675309)
simulation = arima.sim(list(order=c(2,0,0), ar=c(1.5,-.75)), n=144)
plot(simulation, xaxp=c(0,144,12), type="n", ylab=expression(X[~t]))
abline(v=seq(0,144,by=12), h=c(-5,0,5), col=gray(.9))
lines(simulation, col=4)
```



can be written in its causal form.

How to check causality?

```
AR=c(1,-1.5,0.75)
polyroot(AR)
```

```
## [1] 1+0.57735i 1-0.57735i
```

```
Mod(polyroot(AR))
```

```
## [1] 1.154701 1.154701
```

Causal form:

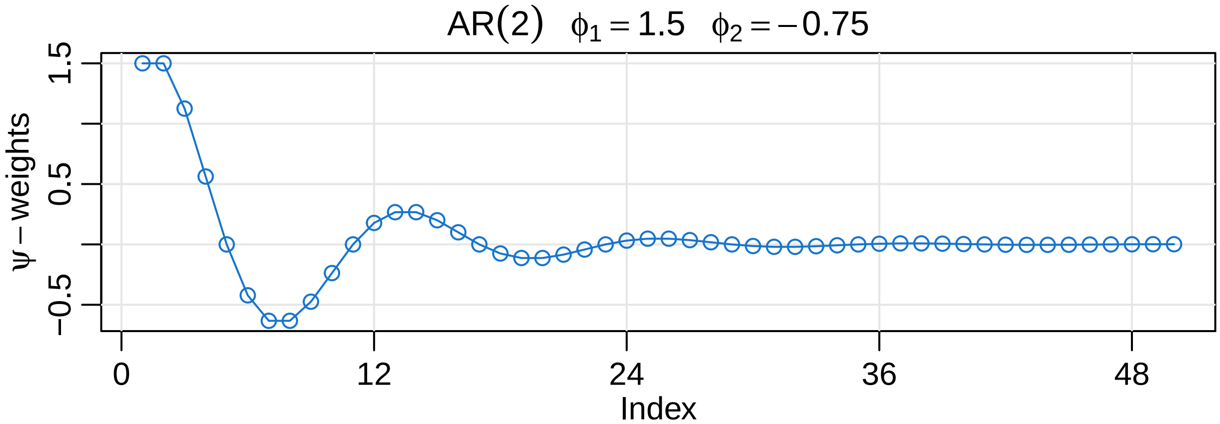
$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$, where $\psi_0 = 1$ and

$$\psi_j = 2\left(\frac{\sqrt{3}}{2}\right)^j cos\left(\frac{2\pi(j-2)}{12}\right), \quad j = 1, 2, \ldots$$

```
#Example 4.3
psi = ARMAtoMA(ar = c(1.5, -.75), ma = 0, 50)
round(psi,2)
```

```
##  [1]  1.50  1.50  1.12  0.56  0.00 -0.42 -0.63 -0.63 -0.47 -0.24  0.00  0.18
## [13]  0.27  0.27  0.20  0.10  0.00 -0.08 -0.11 -0.11 -0.08 -0.04  0.00  0.03
## [25]  0.05  0.05  0.04  0.02  0.00 -0.01 -0.02 -0.02 -0.02 -0.01  0.00  0.01
## [37]  0.01  0.01  0.01  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
## [49]  0.00  0.00
```

```
par(mfrow=c(2,1), mar=c(2,2.5,1,0)+.5, mgp=c(1.5,.6,0), cex.main=1.1)
plot(psi, xaxp=c(0,144,12), type="n", col=4, ylab=expression(psi-weights), main=expression(AR(2)~~~phi[
abline(v=seq(0,48,by=12), h=seq(-.5,1.5,.5), col=gray(.9))
lines(psi, type="o", col=4)
```



AR(2) $\phi_1 = 1.5$ $\phi_2 = -0.75$
ψ − weights
1.5
0.5
−0.5
0
12
24
36
48
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Moving Average (MA) models

Moving Average process of order q (MA(q))

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$$

As an alternative to AR, think of W_t as a *shock* to the process at time t.

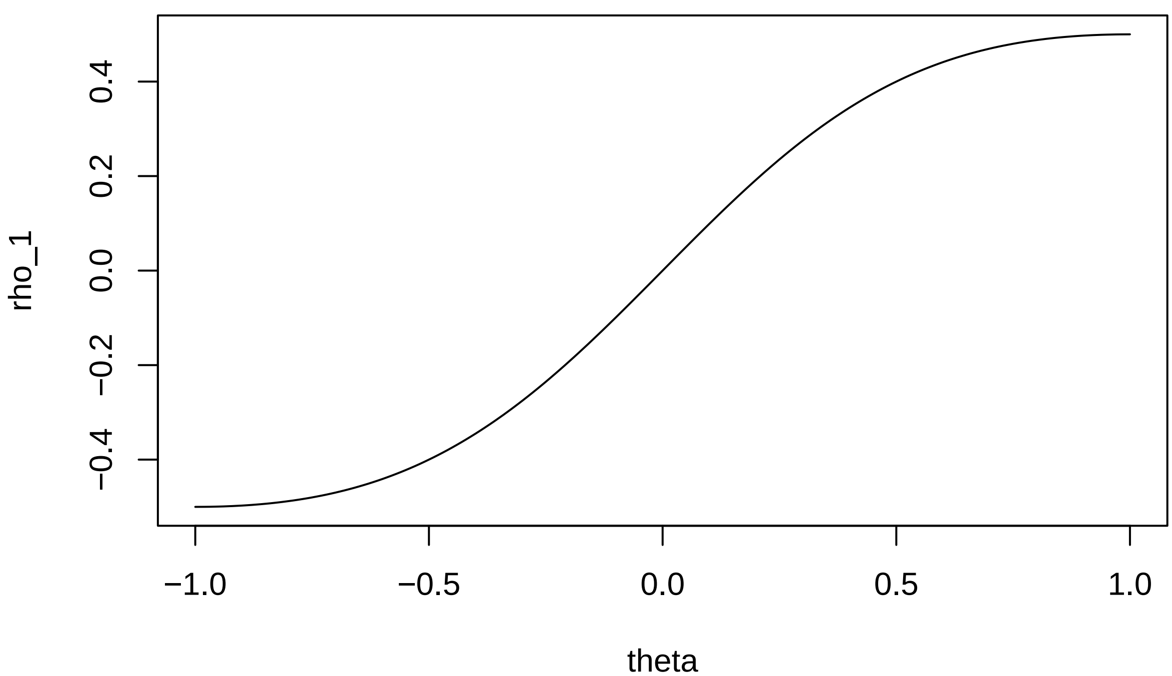
The stochastic process X_t is obtained by applying weights $1, \theta_1, \theta_2, \ldots, \theta_q$ to $W_t, W_{t-1}, \ldots, W_{t-q}$

First order Moving Average MA(1)

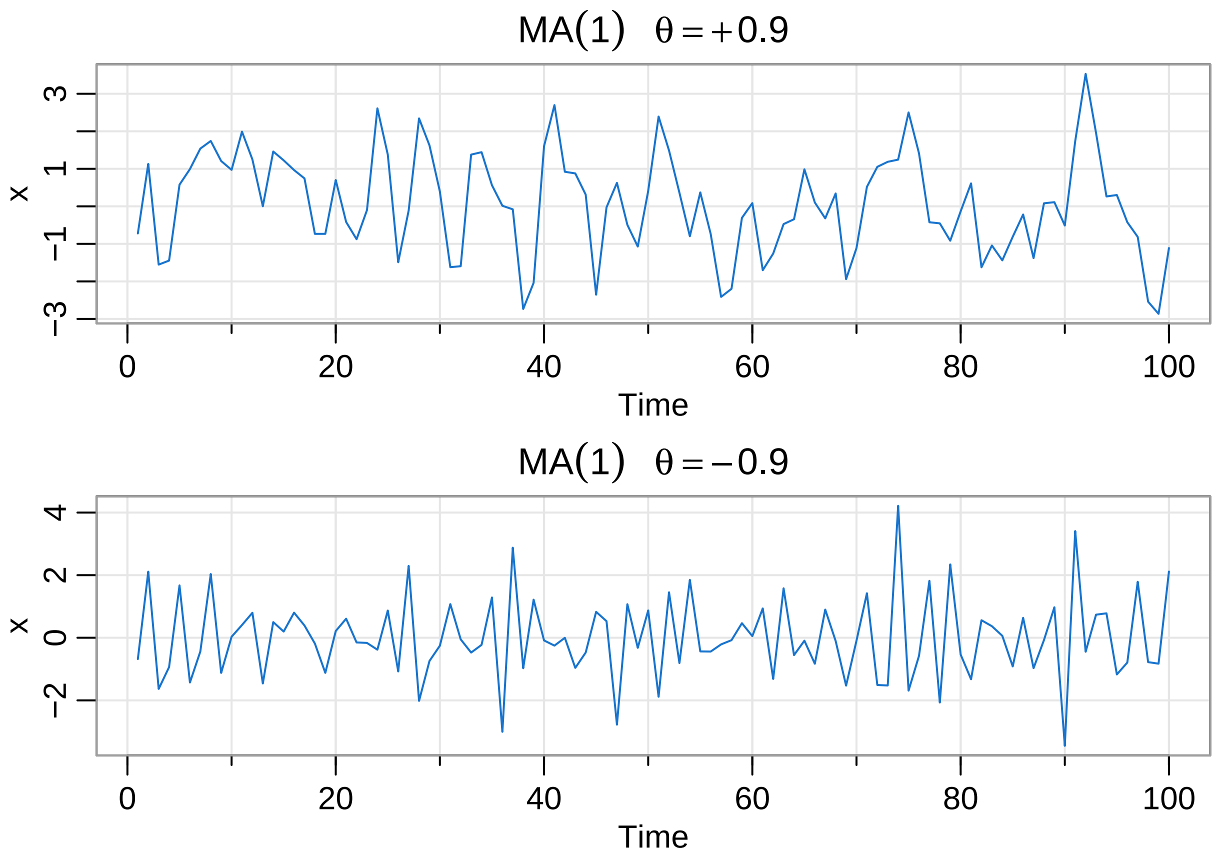
$$X_t = W_t + \theta W_{t-1}$$

$$\begin{aligned} E(X_t) &= 0 \\ \gamma(0) &= Var(X_t) = \sigma_W^2(1+\theta^2) \\ \gamma(1) &= \theta\sigma_W^2 \\ \rho(1) &= (\theta)/(1+\theta^2) \\ \gamma(h) &= \rho(h) = 0 \ \text{ for } h \geq 2 \end{aligned}$$

```
#rho1 vs. theta for MA(1)
theta=seq(-1,1,0.01)
rho1=theta/(1+theta^2)
plot(theta,rho1,type='l',ylab=expression(rho_1))
```



```r
par(mfrow = c(2,1))
tsplot(arima.sim(list(order=c(0,0,1), ma=.9),  n=100), col=4, ylab="x", main=expression(MA(1)~~~theta==+
tsplot(arima.sim(list(order=c(0,0,1), ma=-.9), n=100), col=4, ylab="x", main=expression(MA(1)~~~theta==-
```



Non-uniqueness of MA models and invertibility

When θ is replaced by $1/\theta$, the autocorrelation function for MA(1) process does not change.

$$\rho(h) = \begin{cases} 1 & h = 0 \\ \theta/(1+\theta^2) & h = 1, \\ 0 & otherwise \end{cases} \tag{1}$$

This meas you can not distinguish between the autocorrelation function of a MA(1) process with $\theta = 1/5$ and $\theta = 5$.

Thus, the MA(1) processes

$$X_t = W_t + \frac{1}{5}W_t, \quad W_t \sim \text{iid } \; N(0, 25)$$
$$Y_t = V_t + 5V_t, \quad V_t \sim \text{iid } \; N(0, 1)$$

are stochastically the same, we cannot distinguish between the models.

We choose the model with an infinite AR representation, an *invertible* process.

Definition [Invertibility] A time series X_t is said to be invertible if it can be written as

$$W_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$$

for constants π_j satisfying $\sum_{j=0}^{\infty} \pi_j^2 < \infty$.

General MA(q) model

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$$

$$X_t = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q) W_t$$

We denote $\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$ as a moving average operator, and we write MA(q) model as $X_t = \theta(B) W_t$.

Example: MA(2), how to check invertibility?

$$X_t = W_t - 0.5 W_{t-1} + 0.8 W_{t-2}$$

```
MA=c(1,-0.5,0.8)
polyroot(MA)
```

```
## [1] 0.3125+1.073473i 0.3125-1.073473i
```

```
Mod(polyroot(MA))
```

```
## [1] 1.118034 1.118034
```

The Mixed Autoregressive Moving Average model (ARMA(p,q))

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \ldots + \theta_q W_{t-q}$$

Using autoregressive operator and moving average operator,

$\phi(B)X_t = \theta(B)W_t$.

Parameter redundancy

Given an ARMA (p,q) model, we can unnecessarily complicate the model by multiplying both sides of another operator.

Example: Checking for Parameter Redundancy

$$X_t = 0.3X_{t-1} + 0.4X_{t-2} + W_t + 0.5W_{t-1}$$

```
#Example 4.11
AR = c(1, -.3, -.4) # original AR coefs on the left
polyroot(AR)
```

```
## [1]  1.25-0i -2.00+0i
```

```
MA = c(1, .5) # original MA coefs on the right
polyroot(MA)
```

```
## [1] -2+0i
```

Example: Causal and Invertible ARMA

$$X_t = 0.8X_{t-1} + W_t - 0.5W_{t-1}$$

```
#Example 4.12
round( ARMAtoMA(ar=.8, ma=-.5, 10), 2) # first 10 psi-weights
```

```
##  [1] 0.30 0.24 0.19 0.15 0.12 0.10 0.08 0.06 0.05 0.04
```

```
round( ARMAtoAR(ar=.8, ma=-.5, 10), 2) # first 10 pi-weights
```

```
##  [1] -0.30 -0.15 -0.08 -0.04 -0.02 -0.01  0.00  0.00  0.00  0.00
```

```
ARMAtoMA(ar=1, ma=0, 20)
```

```
##  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```