ARCH & GARCH models

Autoregressive Conditionally Heteroscedastic (1982, Engle)

generalized ARCH (GARCH), 1986, Bollerslev

volatility (variability) of a time series.

ARMA → model conditional mean when conditional variance was constant.

AR(1): $E[X_t \mid X_{t-1}, X_{t-2}, \cdots] = \phi X_{t-1}$

$Var(X_t \mid X_{t-1}, \cdots) = Var(W_t) = \sigma_w^2$.

X_t: asset price at time t.

return $r_t = \dfrac{X_t - X_{t-1}}{X_{t-1}}$ $\sim$ $X_t = (1+r_t) X_{t-1}$

→ percentage change. $\nabla \log(X_t) \approx r_t$

$\boxed{\text{ARCH}(1)}$

$\sigma_t := Var(X_t \mid X_{t-1})$

$$\boxed{\begin{aligned} r_t &= \sigma_t W_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 r_{t-1}^2 \end{aligned}}$$

, $W_t \sim WN$, iid $N(0,1)$

$\alpha_0, \alpha_1 \geq 0$

$r_t \mid r_{t-1} \sim N(0, \alpha_0 + \alpha_1 r_{t-1}^2)$.

$$\left.\begin{aligned} r_t^2 &= \sigma_t^2 W_t^2 \\ \sigma_t^2 &= \alpha_0 + \alpha_1 r_{t-1}^2 \end{aligned}\right\} \rightarrow \quad r_t^2 - (\alpha_0 + \alpha_1 r_{t-1}^2) = \boxed{\overbrace{\sigma_t^2 (W_t^2 - 1)}^{v_t}}$$

$$r_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + v_t$$

$E[r_t | r_{t-1}, \cdots] = 0 \longrightarrow E[r_t] = E[E[r_t | r_{t-1}, \cdots]] = 0.$

$(h>0)$ $Cov(r_{t+h}, r_t) = E[r_{t+h} r_t]$

$$= E[E[r_{t+h} r_t | r_{t+h-1}, \cdots]]$$

$$= E[r_t E[r_{t+h} | r_{t+h-1}, \cdots]]$$

$$= E[r_t \cdot 0] = 0$$

r_t^2 AR structure $\Rightarrow$ r_t ARCH model.

Conditions: $3\alpha_1^2 < 1 \Rightarrow r_t^2$ is following causal AR(1) model.

ACF: $\rho(h) = \alpha_1^h \geq 0.$

$\alpha_1 < 1$, but $3\alpha_1^2 \geq 1$

r_t^2: stationary, but with infinite variance.

ARCH(m) : $\sigma_t := Var(X_t | X_{t-1}, \cdots, X_{t-m})$

$$r_t = \sigma_t W_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \cdots + \alpha_m r_{t-m}^2$$

$\rightarrow$ r_t^2 following AR(m).

GARCH(1,1) : $\sigma_t = Var(X_t | X_{t-1})$

$$r_t = \sigma_t W_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Condition: $\alpha_1 + \beta_1 < 1 \Rightarrow r_t \sim GARCH(1,1) \underset{equiv.}{\sim} r_t^2 \sim ARMA(1,1).$

GARCH(m, r)

$$\sigma_t = Var(x_t \mid x_{t-1}, \cdots, x_{t-m})$$

$$r_t = \sigma_t w_t$$

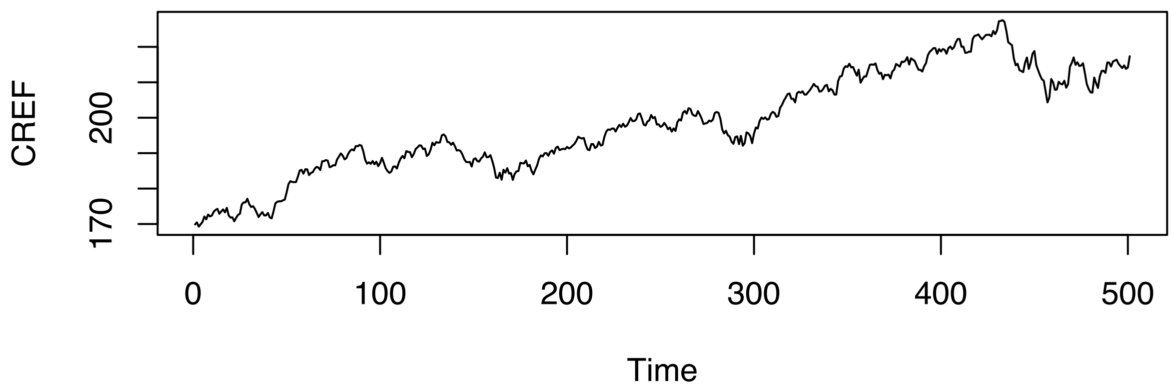
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \cdots + \alpha_m r_{t-m}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_r \sigma_{t-r}^2$$

$$= \alpha_0 + \sum_{j=1}^{m} \alpha_j r_{t-j}^2 + \sum_{i=1}^{r} \beta_i \sigma_{t-i}^2 .$$

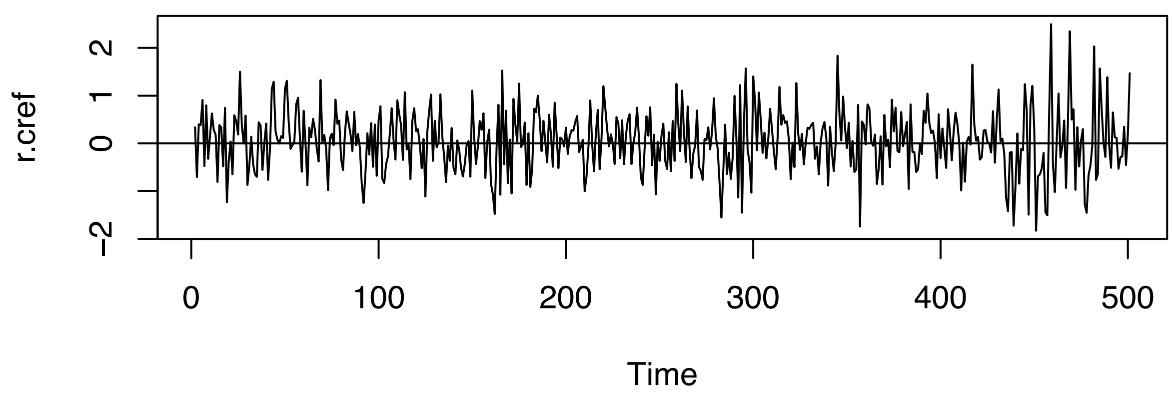
ARCH and GARCH

```
library(TSA)
library(tseries)
```

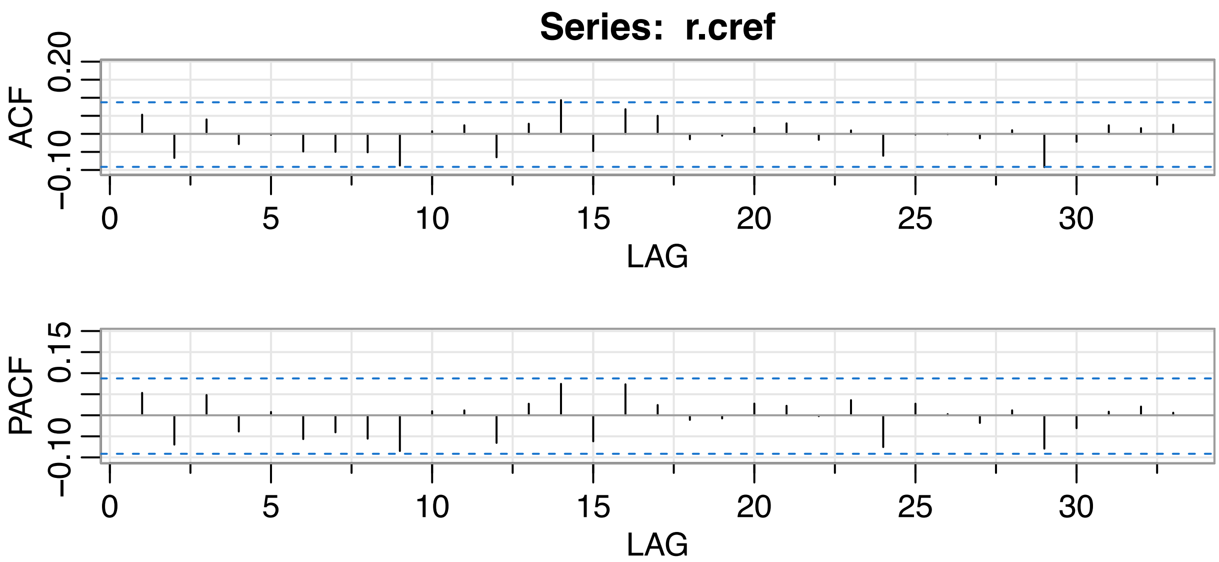
```
data(CREF);
plot(CREF)
```



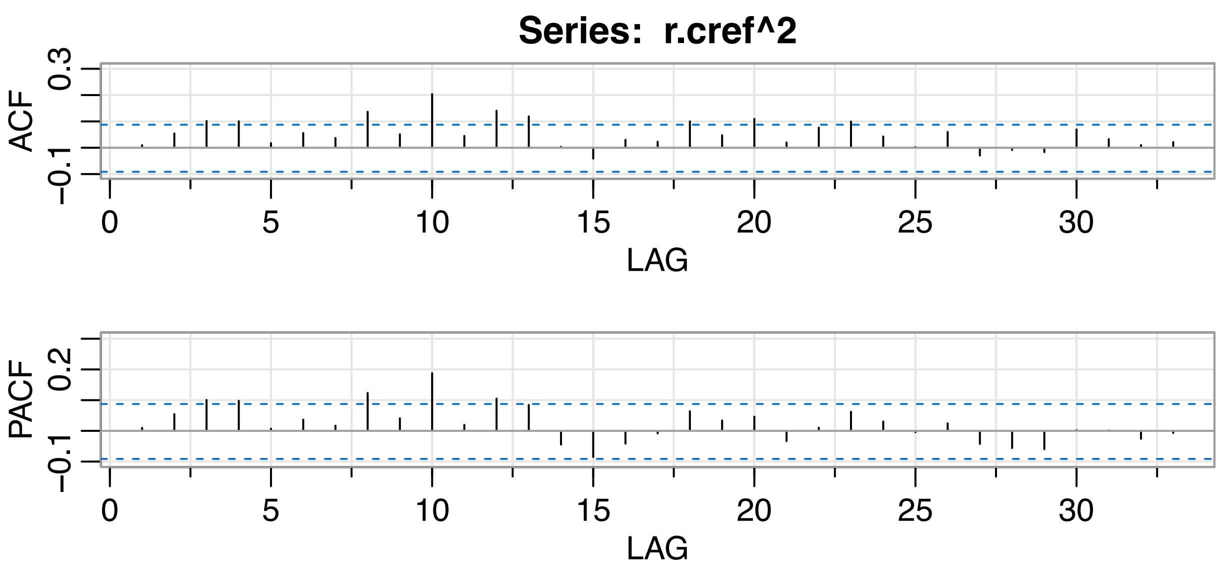

```
r.cref=diff(log(CREF)*100)
plot(r.cref);abline(h=0)
```



```r
acf2(r.cref)
```




```r
acf2(r.cref^2)
```



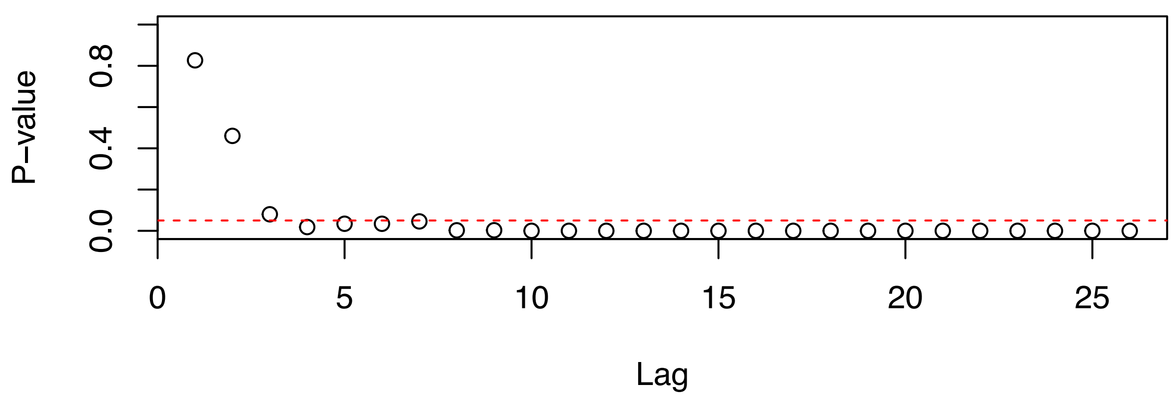

```r
Box.test(r.cref,type="Ljung-Box",lag=20)
```

```
##
##  Box-Ljung test
##
## data:  r.cref
## X-squared = 25.207, df = 20, p-value = 0.1936
```

```
Box.test(r.cref^2,type="Ljung-Box",lag=20)
```

```
##
##  Box-Ljung test
##
## data:  r.cref^2
## X-squared = 79.398, df = 20, p-value = 4.967e-09
```

```
McLeod.Li.test(y=r.cref)
```




```
set.seed(12345678)
wn=rnorm(500)
Box.test(wn,lag=20)
```

```
##
##  Box-Pierce test
##
## data:  wn
## X-squared = 19.464, df = 20, p-value = 0.4919
```

```
Box.test(wn^2,lag=20)
```

```
##
##  Box-Pierce test
##
## data:  wn^2
## X-squared = 19.013, df = 20, p-value = 0.521
```

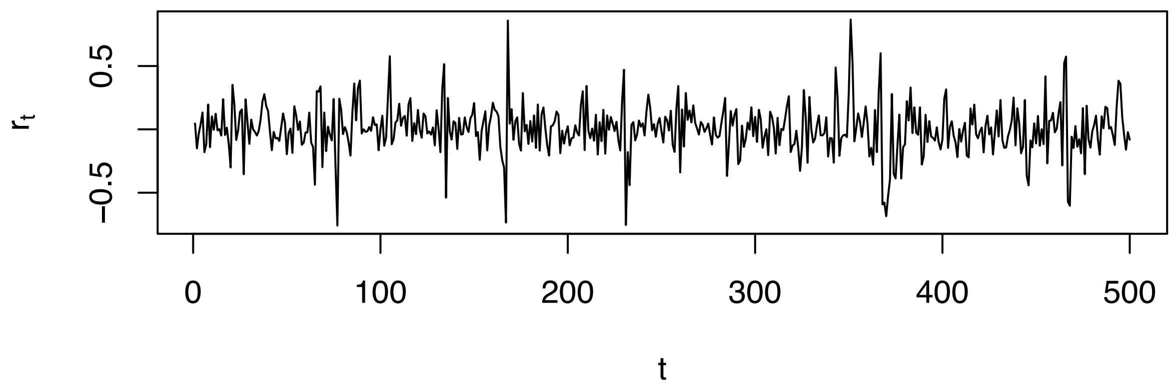
ARCH(1)

$\alpha_0 = 0.01, \quad \alpha_1 = 0.9$

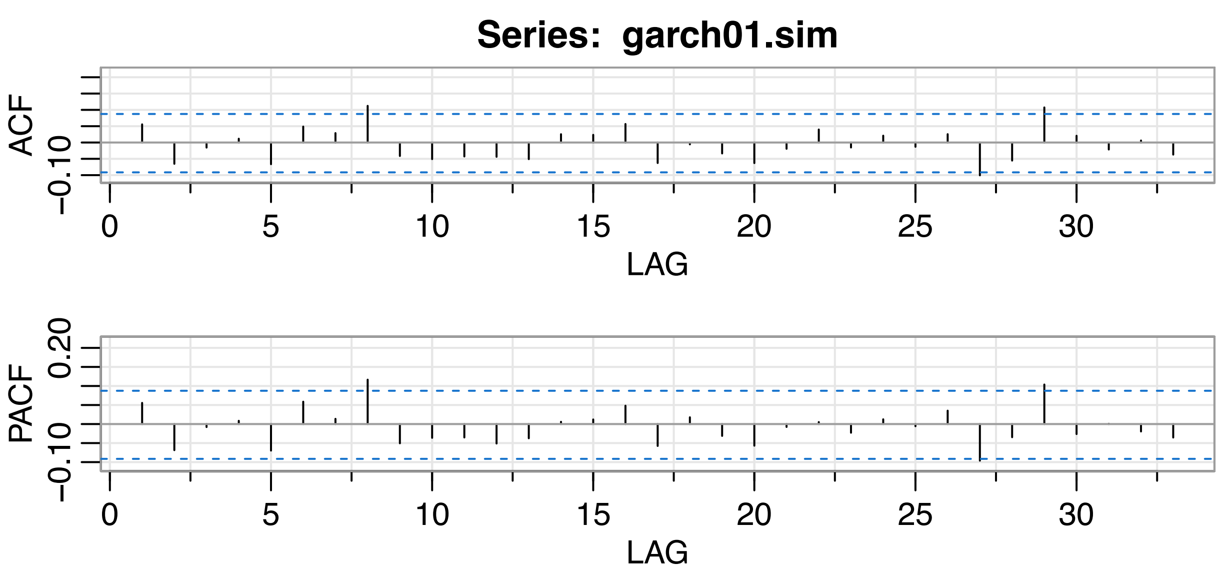
```
set.seed(1235678)
garch01.sim=garch.sim(alpha=c(.01,.9),n=500)
plot(garch01.sim,type='l',ylab=expression(r[t]), xlab='t')
```

$r_t = \sigma_t w_t$

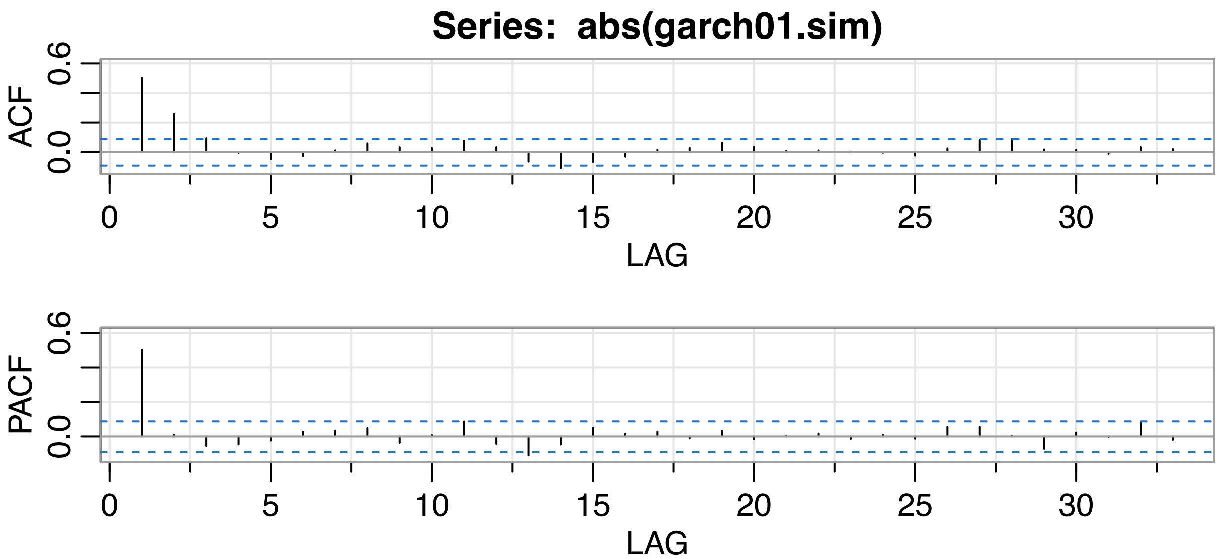
$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2$




```
acf2(garch01.sim)
```

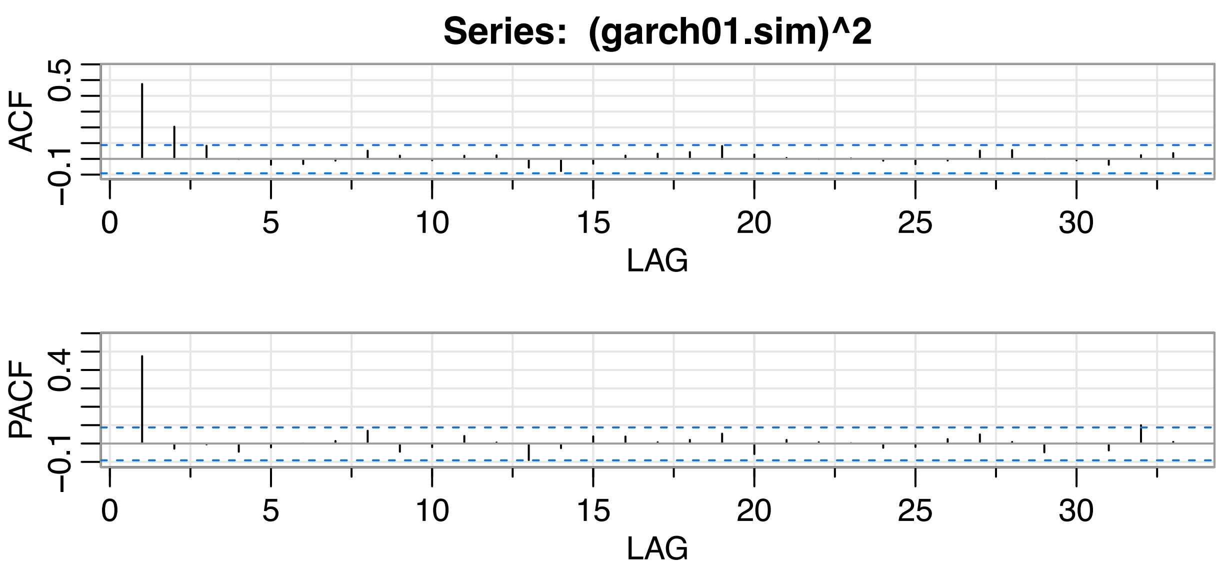



```
acf2(abs(garch01.sim))
```




```
acf2((garch01.sim)^2)
```

r_t^2: ACF/PACF of AR(1). $\Rightarrow$ r_t : ARCH.




```
Box.test((garch01.sim)^2,lag=20)
```

r ARCH(m)

```
m1=garch(x=garch01.sim,order=c(0,1))
```

```
summary(m1)
```

garch() in R

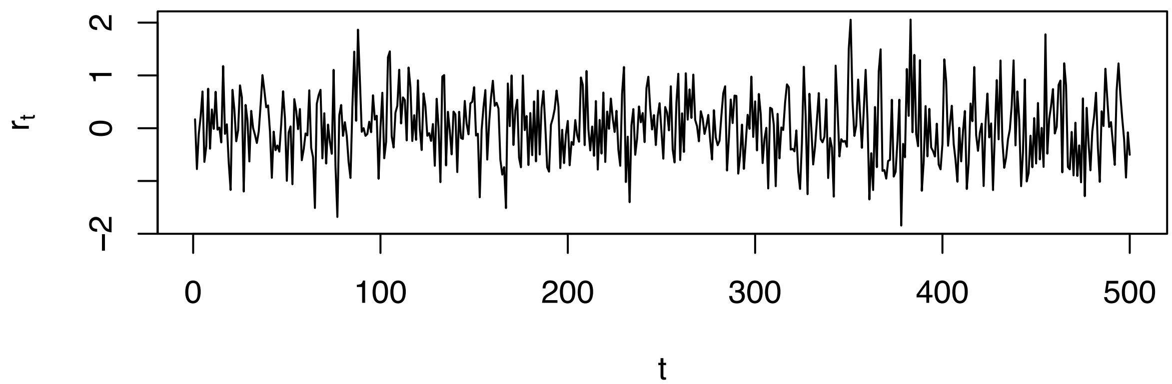
```
##
## Call:
## garch(x = garch01.sim, order = c(0, 1))
##
## Model:
## GARCH(0,1)
```

```
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.71843 -0.68166 -0.02012  0.67266  3.11395
##
## Coefficient(s):
##     Estimate  Std. Error  t value Pr(>|t|)
## a0  0.011440    0.001491    7.672 1.69e-14 ***
## a1  0.813664    0.107394    7.576 3.55e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Diagnostic Tests:
##  Jarque Bera Test
##
## data:  Residuals
## X-squared = 0.99378, df = 2, p-value = 0.6084
##
##
##  Box-Ljung test
##
## data:  Squared.Residuals
## X-squared = 0.0068806, df = 1, p-value = 0.9339
```

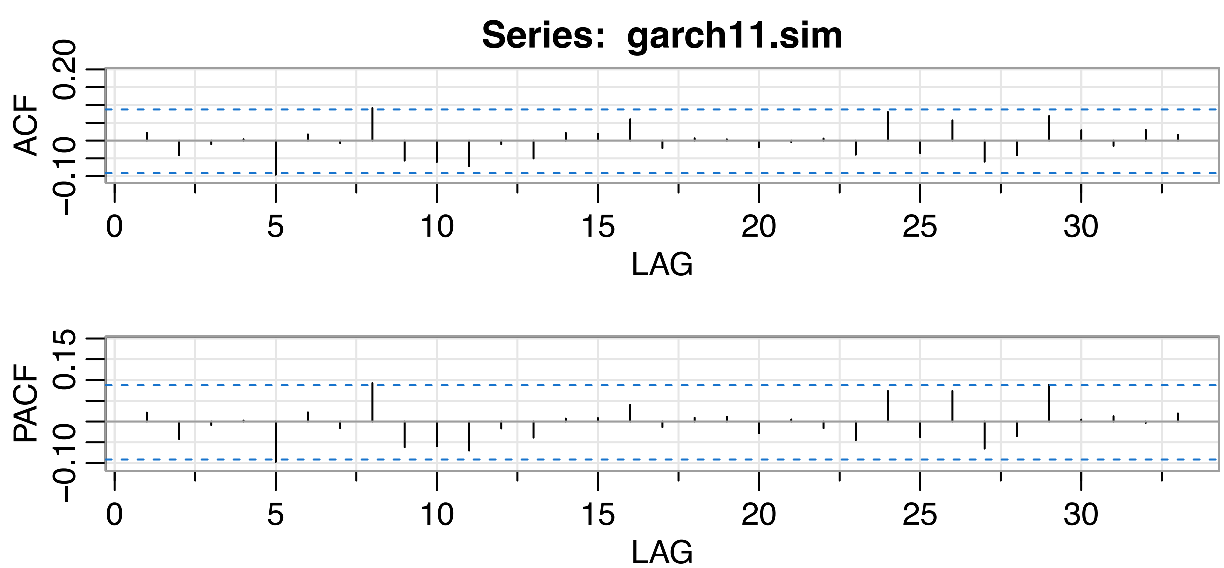
$$r_t = \sigma_t w_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

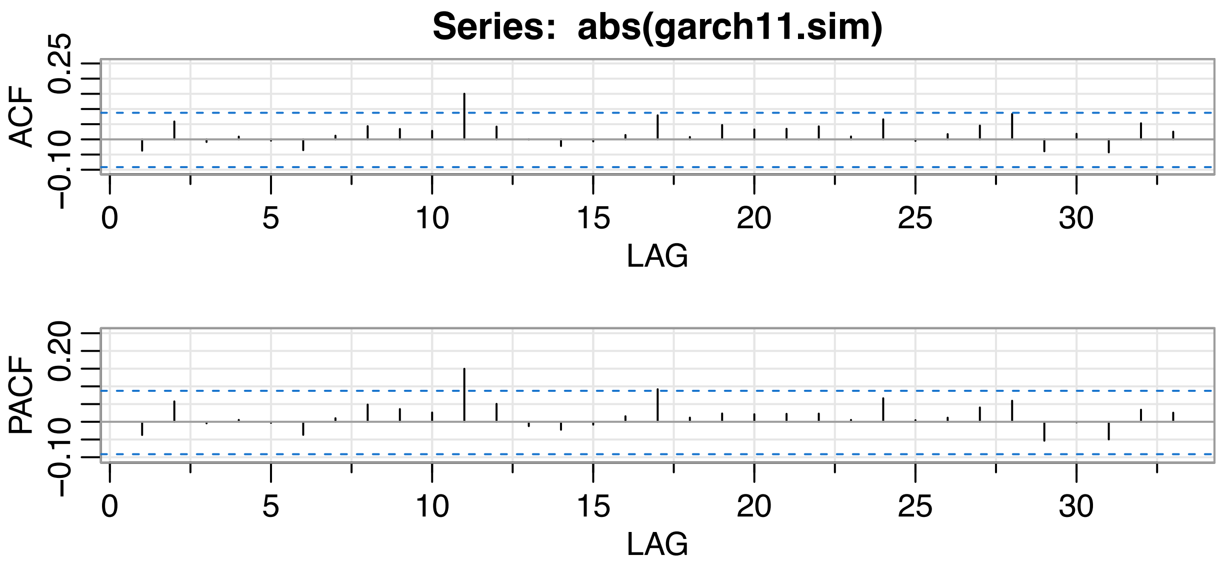
```
set.seed(1235678)
garch11.sim=garch.sim(alpha=c(0.02,0.05),beta=.9,n=500)
plot(garch11.sim,type='l',ylab=expression(r[t]), xlab='t')
```



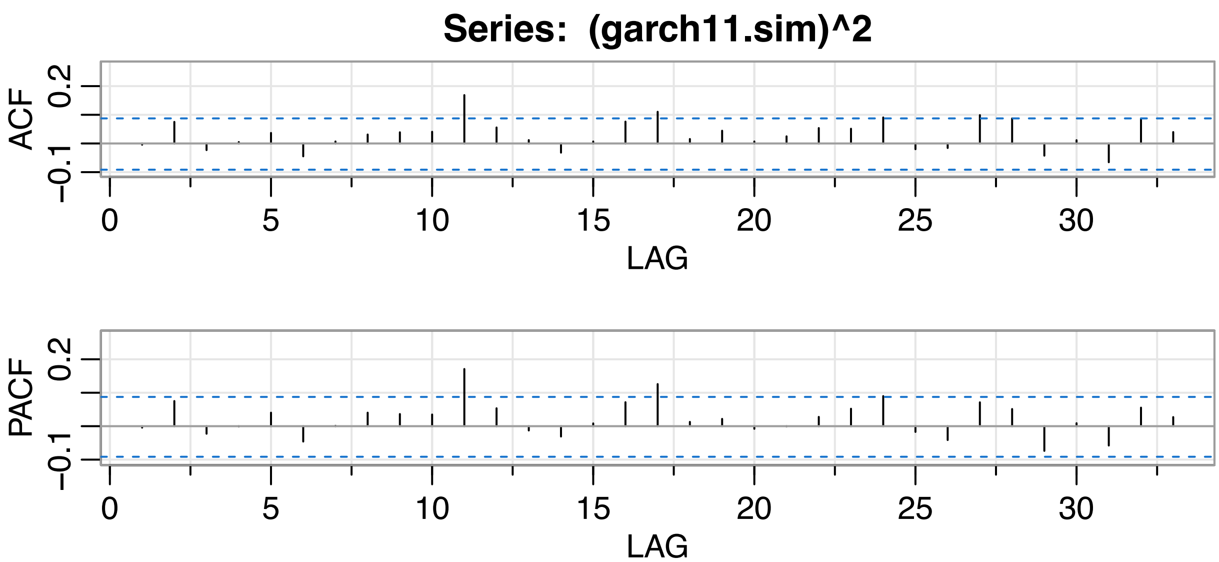

```
acf2(garch11.sim)
```




```
acf2(abs(garch11.sim))
```




```
acf2((garch11.sim)^2)
```




```
m2=garch(x=garch11.sim,order=c(1,1))
```

```
summary(m2)
```

```
##
## Call:
## garch(x = garch11.sim, order = c(1, 1))
##
## Model:
## GARCH(1,1)
##
## Residuals:
```

```
##      Min       1Q   Median       3Q      Max
## -2.87179 -0.65553 -0.01878  0.66211  3.20455
##
## Coefficient(s):
##     Estimate  Std. Error  t value Pr(>|t|)
## a0 3.871e-01   2.676e+01    0.014    0.988
## a1 1.899e-04   3.817e-02    0.005    0.996
## b1 6.116e-02   6.487e+01    0.001    0.999
##
## Diagnostic Tests:
##  Jarque Bera Test
##
## data:  Residuals
## X-squared = 1.7507, df = 2, p-value = 0.4167
##
##
##  Box-Ljung test
##
## data:  Squared.Residuals
## X-squared = 0.0082792, df = 1, p-value = 0.9275
```