Week 2 Lecture note: Measure of dependence

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Module 1 - Week 2

Properties of the Mean, Variance and Covariance (Lecture 1.2.1)

- $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$
- $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $Var(a + bX) = b^2 Var(X)$
- If X and Y are independent $Var(X + Y) = Var(X) + Var(Y)$
- $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
- $Cov(a + bX, c + dY) = bdCov(X, Y)$
- $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$
- $Cov(X, X) = Var(X)$
- $Cov(X, Y) = Cov(Y, X)$
- $Cov(X, Y) \leq \sqrt{(Var(X)Var(Y))}$
- $Corr(X, Y) = \frac{Cov(X,Y)}{\sqrt{(Var(X)Var(Y))}}$

Mean, Variance, Auto-covariance, Auto-correlation (Lecture 1.2.2)

Mean Function

$$\mu_X(t) = \mathbb{E}[X_t]$$

1. Mean function of White Noise:
$$\mu_W(t) = \mathbb{E}[W_t] = 0$$
2. $X_t = \frac{1}{2}(W_{t-1} + W_t)$

   $\mu_X(t) = 0$
3. $X_t = \delta t + \sum_{j=1}^{t} W_j$

   $\mu_X(t) = \delta t$
4. $X_t = 2\cos(2\pi \frac{t+15}{50}) + W_t$

   $\mu_X(t) = 2\cos(2\pi \frac{t+15}{50})$

Auto-covariance function

$$\begin{aligned}\gamma_X(t,s) = Cov(X_t, X_s) =& \mathbb{E}[(X_t - \mu_X(t))(X_s - \mu_X(s))] \\ =& \mathbb{E}[X_t X_s] - \mu_X(t)\mu_X(s)\end{aligned}$$

1. Autocovariance of White Noise:
$$\gamma_W(t,s) = Cov(W_t, W_s) = \begin{cases} \sigma_W^2 & s = t \\ 0 & s \neq t. \end{cases}$$
2. $X_t = \frac{1}{2}(W_{t-1} + W_t)$
$$\gamma_X(t,s) = \begin{cases} \frac{1}{2}\sigma_W^2 & s = t, \\ \frac{1}{4}\sigma_W^2 & |t-s| = 1, \\ 0 & |t-s| > 1. \end{cases}$$
3. $X_t = \delta t + \sum_{j=1}^{t} W_j$
$$\gamma_X(t,s) = Cov\left(\delta t + \sum_{j=1}^{t} W_j, \delta s + \sum_{j=1}^{s} W_j\right) = \min(t,s)\sigma_W^2$$
4. $X_t = 2\cos(2\pi \frac{t+15}{50}) + W_t$
$$\gamma_X(t,s) = \begin{cases} \sigma_W^2 & s = t \\ 0 & s \neq t. \end{cases}$$

Auto-correlation function (ACF)

$$\rho_X(t,s) = Corr(X_t, X_s)$$

where

$$Corr(X_t, X_s) = \frac{Cov(X_t, X_s)}{\sqrt{Var(X_t)Var(X_s)}} = \frac{\gamma_X(t,s)}{\sqrt{\gamma_X(t,t)\gamma_X(s,s)}}$$

Covariance and correlation are a measure of **linear** relationship between two random variables.

1. Autocorrelation of White Noise:

$$\rho_W(t,s) = Corr(W_t, W_s) = \begin{cases} 1 & s = t \\ 0 & s \neq t. \end{cases}$$

2. $X_t = \frac{1}{2}(W_{t-1} + W_t)$

$$\rho_X(t,s) = \begin{cases} 1 & s = t, \\ \frac{1}{2} & |t-s| = 1, \\ 0 & |t-s| > 1. \end{cases}$$

3. $X_t = \delta t + \sum_{j=1}^{t} W_j$

$$\rho_X(t,s) = \frac{\min(t,s)}{\sqrt{ts}}$$

4. $X_t = 2\cos(2\pi \frac{t+15}{50}) + W_t$

$$\gamma_X(t,s) = \begin{cases} 1 & s = t \\ 0 & s \neq t. \end{cases}$$

Properties of Auto-covariance and Auto-correlation

- $\gamma_X(t,t) = Var(X_t)$
- $\rho_X(t,t) = 1$
- $\gamma_X(t,s) = \gamma_X(s,t)$

- $\rho_X(t,s) = \rho_X(s,t)$
- $|\gamma_X(t,s)| \leq \sqrt{\gamma_X(t,t)\gamma_X(s,s)}$
- $|\rho_X(t,s)| \leq 1$