ARMA(1,1)

ex) $x_t = \phi x_{t-1} + \theta w_{t-1} + w_t$

$p+1 = 2.$ set $w_1 = 0$

$w_2 = x_2 - \phi x_1 - \theta w_1 = x_2 - \phi x_1$

$w_3 = x_3 - \phi x_2 - \theta w_2$

$\vdots$

$w_n = x_n - \phi x_{n-1} - \theta w_{n-1}.$

now we can estimate errors at any values of the parameters.

let $\beta_{(0)} = (\phi_{(0)}, \theta_{(0)})$: Initial estimate.

$$Z_t(\beta_{(0)}) = \left.\begin{bmatrix} -\dfrac{\partial w_t(\phi,\theta)}{\partial \phi} \\ \\ -\dfrac{\partial w_t(\phi,\theta)}{\partial \theta} \end{bmatrix}\right|_{\beta=\beta_{(0)}}$$

$w_t = x_t - \phi x_{t-1} - \theta w_{t-1}, \quad t=2,\cdots n,$

$$\frac{\partial w_t}{\partial \phi} = -x_{t-1} - \theta \frac{\partial w_{t-1}}{\partial \phi} \qquad \frac{\partial w_t}{\partial \theta} = -w_{t-1} - \theta \frac{\partial w_{t-1}}{\partial \theta}$$

$$Z_t(\beta_{(0)}) = \begin{bmatrix} \underbrace{x_{t-1} - \theta \cdot \left(-\frac{\partial w_{t-1}}{\partial \phi}\right)}_{\text{Denote } Z_t[1]} \\ \underbrace{w_{t-1} - \theta\left(\frac{-\partial}{\partial \theta} w_{t-1}\right)}_{Z_t[2]} \end{bmatrix}$$

Recursive. ...

Set $\frac{\partial}{\partial \theta} w_1 = 0$, $\frac{\partial w_1}{\partial \phi} = 0$,

$$Z_2(\beta(0)) = \begin{bmatrix} \boxed{x_1 - 0} \\ w_1 - 0 \end{bmatrix}, \quad Z_3(\beta(0)) = \begin{bmatrix} x_2 - \theta \boxed{} \\ w_2 - \theta \boxed{} \end{bmatrix} \quad \ldots$$

$Z_{t-1}[1]$

$Z_{t-1}[2]$

$$Q(\beta) = \sum_{t=2}^{n} \left(w_t(\beta_0) - (\beta - \beta_{(0)})^T Z_t(\beta_0) \right)^2$$

$$\hat{\beta} = \beta(0) + \underbrace{\left(\sum_{t=2}^{n} Z_t(\beta(0)) \, Z_t(\beta(0))^T \right)^{-1}}_{2 \times 2 \text{ matrix}} \underbrace{\sum_{t=2}^{n} Z_t(\beta(0)) \, w_t(\beta(0))}_{2 \times 1 \text{ vector.}}$$

Next step: Iterate by replacing $\beta(0)$ with $\beta(1)$.

$$w_t = x_t - \phi x_{t-1} - \theta w_{t-1}, \quad t=2, \cdots n,$$

$$Z_t(\beta_j) = \begin{bmatrix} x_{t-1} - \theta \cdot Z_{t-1}(\beta_j)[1] \\ \\ w_{t-1} - \theta \; Z_{t-1}(\beta_j)[2] \end{bmatrix}$$

$$\beta_{(j)} = \beta_{j-1} + \underbrace{\left(\sum_{t=2}^{n} Z_t(\beta_{j-1}) \, Z_t(\beta_{j-1})^T \right)^{-1}}_{2\times 2 \text{ matrix}} \underbrace{\sum_{t=2}^{n} Z_t(\beta_{j-1}) \, w_t(\beta_{j-1})}_{2\times 1 \text{ vector.}}$$