HW 05

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- Unless stated otherwise, W_t is a white noise process with variance σ_w^2.
  - (W_t are independent with zero means and variance σ_w^2.)
- Show your full work to receive full credit.

Question 1.

For an MA(1), $X_t = W_t + \theta W_{t-1}$, show that $|\rho_x(1)| \leq 1/2$ for any number θ. For which values of θ does $\rho_x(1)$ attain its maximum and minimum?

$$\gamma_X(s,t) = Cov(W_s + \theta W_{s-1}, W_t + \theta W_{t-1})$$
$$= Cov(W_s, W_t) + \theta Cov(W_{s-1}, W_t) + \theta Cov(W_s, W_{t-1}) + \theta^2 Cov(W_{s-1}, W_{t-1})$$

1. $h = 0 \implies \gamma_X(h) = \sigma_w^2 + 0 + 0 + \theta\sigma_w^2 = \sigma_w^2(1+\theta^2)$
2. $h = 1 \implies \gamma_X(h) = 0 + \theta\sigma_w^2 + 0 + 0 = \theta\sigma_w^2$
3. $h = 2 \implies \gamma_X(h) = 0 + 0 + 0 + 0 = 0$

$$\gamma_X(h) = \begin{cases} \sigma_w^2(1+\theta^2) & ; h = 0 \\ \theta\sigma_w^2 & ; h = 1 \\ 0 & ; h \geq 2 \end{cases}$$

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} \implies \rho_X(h) = \begin{cases} 1 & ; h = 0 \\ \frac{\theta}{1+\theta^2} & ; h = 1 \\ 0 & ; h \geq 2 \end{cases}$$

$$\rho_X(1) = \frac{\theta}{1+\theta^2} \to \rho_X^{'} = \frac{1-\theta^2}{(1+\theta^2)^2}$$

There are no domain restrictions in the denominator as it can never be 0.

$$1 - \theta^2 = 0 \implies \theta^2 = 1 \implies \theta = \pm 1$$

1. $\theta = 1$

$$\rho_X(1) = \frac{1}{1+1} = \frac{1}{2}$$

2. $\theta = -1$

$$\rho_X(1) = \frac{-1}{1+1} = -\frac{1}{2}$$

When setting the lag (h) equal to 1 in $\rho_X(h)$ and taking the derivative, we find that there are two critical points of the function at ± 1. When plugging those two values into $\rho_X(h)$ we get $\frac{1}{2}$ and $-\frac{1}{2}$. With having no domain restrictions for θ as the denominator for $\rho_X(1)$ cannot be 0, we can conclude that $|\rho_X(h)| \leq \frac{1}{2}\ \forall\theta$.

- Maximum of $\rho_X(1) = \frac{1}{2}$ at $\theta = 1$.
- Minimum of $\rho_X(1) = -\frac{1}{2}$ at $\theta = -1$.

Question 2.

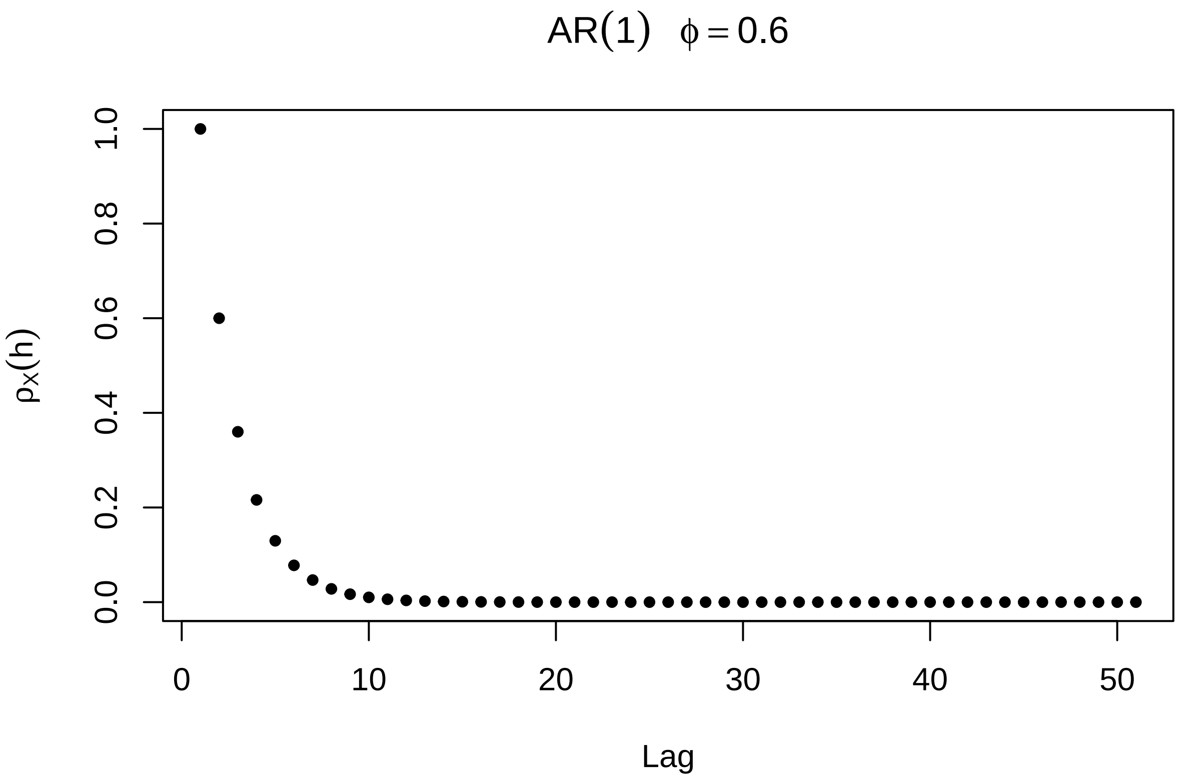
Calculate and sketch the autocorrelation functions for each of the following AR(1) models. Plot for sufficient lags that the autocorrelation function has nearly died out.

$$\begin{aligned}
\rho_X(h) = Cov(X_{t-h}, X_t) &= Cov(\sum_{j=0}^{\infty} \phi^j, W_{t+h-j}, \sum_{k=0}^{\infty} \phi^k W_{t-k}) \\
&= Cov(\phi^h W_t + \phi^{h+1} W_{t-1} + \ldots, W_t + \phi W_{t-1} + \ldots) \\
&= \phi^h Var(W_t) + \phi^{h+1}\phi Var(W_{t-1}) + \phi^{h+2}\phi^2 Var(W_{t-2}) + \ldots \\
&= (\phi^h + \phi^{h+2} + \phi^{h+4} + \ldots)\sigma_w^2 = \phi^h(1 + \phi^2 + \phi^4 + \ldots)\sigma_w^2 \\
&\implies \gamma_X(h) = \phi^h \frac{\sigma_w^2}{1-\phi^2} \\
\rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)} = \phi^h
\end{aligned}$$

(a) $\phi = 0.6$

$$\boxed{\rho_X(h) = 0.6^h}$$

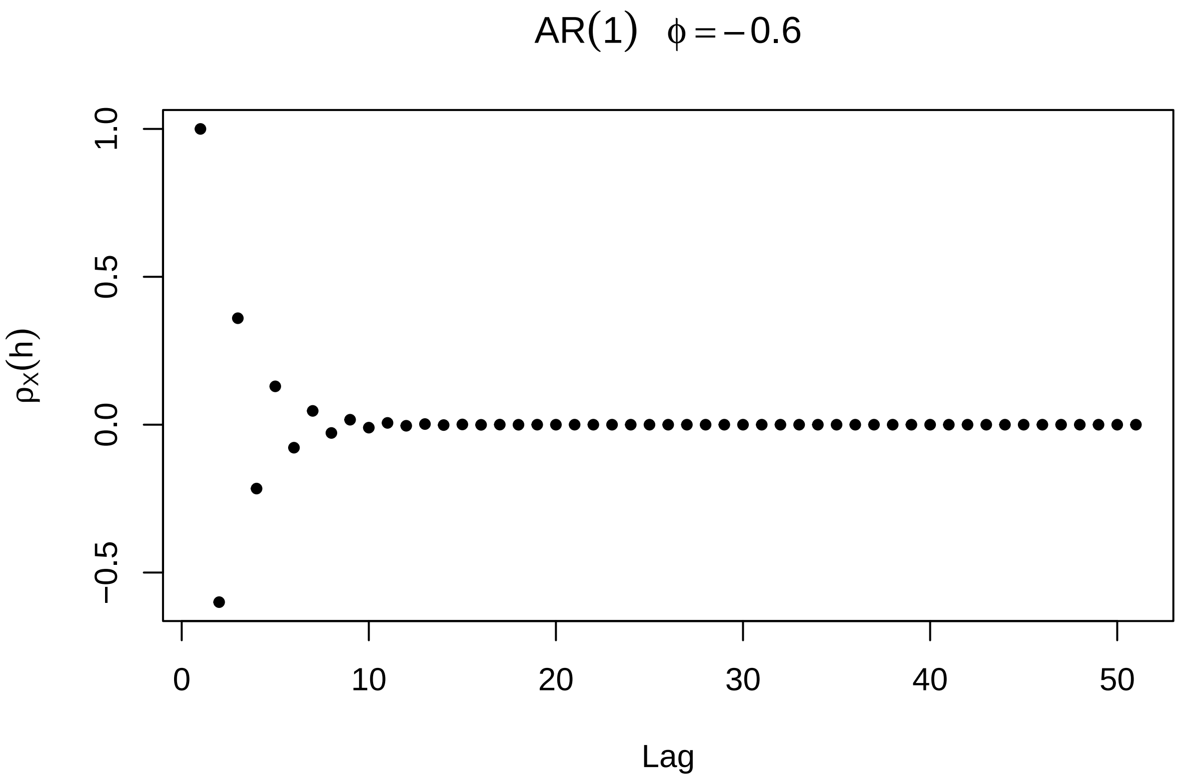
```
set.seed(314439)
data2a = rep(0,51)
for(i in 1:51){
  data2a[i] = (0.6)^(i-1)
}
plot(data2a, pch = 20, xlab = "Lag", ylab = expression(rho[X](h)),
      main = expression(AR(1)~~~phi==0.6))
```



(b) $\phi = -0.6$

$$\boxed{\rho_X(h) = (-0.6)^h}$$

```r
set.seed(314439)
data2b = rep(0,51)
for(i in 1:51){
  data2b[i] = (-0.6)^(i-1)
}
plot(data2b, pch = 20, xlab = "Lag", ylab = expression(rho[X](h)),
      main = expression(AR(1)~~phi==-0.6))
```



Question 3.

Suppose that $\{Y_t\}$ is an AR(1) process with $-1 < \phi < +1$.

(a) Find the autocovariance function for $V_t = \nabla Y_t = Y_t - Y_{t-1}$ in terms of ϕ and σ_W^2.

$$\gamma_V(s,t) = Cov(Y_s - Y_{s-1}, Y_t - Y_{t-1})$$

$$= Cov(Y_s, Y_t) - Cov(Y_{s-1}, Y_t) - Cov(Y_s, Y_{t-1}) + Cov(Y_{s-1}, Y_{t-1})$$

$$= \frac{\phi^{|s-t|}\sigma_w^2}{1-\phi^2} + \frac{\phi^{|s-t+1|}\sigma_w^2}{1-\phi^2} + \frac{\phi^{|s-t-1|}\sigma_w^2}{1-\phi^2} + \frac{\phi^{|s-t|}\sigma_w^2}{1-\phi^2} = \frac{\phi^h\sigma_w^2 - \phi^{|h+1|}\sigma_w^2 - \phi^{|h-1|}\sigma_w^2 + \phi^h\sigma_w^2}{1-\phi^2}$$

$$\implies \boxed{\gamma_V(h) = \frac{\sigma_w^2(2\phi^h - \phi^{|h+1|} - \phi^{|h-1|})}{1-\phi^2}}$$

(b) In particular, show that $Var(V_t) = 2\sigma_W^2/(1+\phi)$.

$$Var(V_t) = \frac{\sigma_w^2(2\phi^0 - \phi - \phi)}{1-\phi^2} = \frac{2\sigma_w^2(1-\phi)}{(1+\phi)(1-\phi)}$$
$$\implies \boxed{Var(V_t) = \frac{2\sigma_w^2}{1+\phi}}$$

Question 4.

Let $|\phi| < 1$ be a constant. Consider the process $X_0 = W_0$, and

$$X_t = \phi X_{t-1} + W_t, \quad t = 1, 2, \ldots$$

(a) Show that $X_t = \sum_{j=0}^{t} \phi^j W_{t-j}$ for any $t = 0, 1, \ldots$.

$$X_t = \phi(\phi X_{t-2} + W_{t-1}) + W_t = \phi^2 X_{t-2} + \phi W_{t-1} + W_t$$
$$X_t = \phi^2(\phi X_{t-3} + W_{t-2}) + \phi W_{t-1} + W_t = \phi^3 X_{t-3} + \phi^2 W_{t-2} + \phi W_{t-1} + W_t$$
$$X_t = \phi^3(\phi X_{t-4} + W_{t-3}) + \phi^2 W_{t-2} + \phi W_{t-1} + W_t = \phi^4 X_{t-4} + \phi^3 W_{t-3} + \phi^2 W_{t-2} + \phi W_{t-1} + W_t$$
$$\ldots \implies \boxed{X_t = \sum_{j=0}^{t} \phi^j W_{t-j}}$$

for any $t = 0, 1, \ldots$.

(b) Find the $E[X_t]$.

$$\mathbb{E}(X_t) = \mathbb{E}(\sum_{j=0}^{t} \phi^j W_{t-j}) = \sum_{j=0}^{t} \mathbb{E}(\phi^j W_{t-j}) = \sum_{j=0}^{t} \phi^j \mathbb{E}(W_{t-j})$$
$$\mathbb{E}(W_{t-j}) = 0 \quad \forall j \implies \boxed{\mathbb{E}(X_t) = 0}$$

(c) Show that, for $t = 0, 1, \ldots$,

$$Var(X_t) = \frac{\sigma_w^2}{1-\phi^2}(1 - \phi^{2(t+1)})$$

$$Var(X_t) = Var(\sum_{j=0}^{t} \phi^j W_{t-j}) = \sum_{j=0}^{t} Var(\phi^j W_{t-j})$$
$$= \sigma_w^2(1 + \phi^2 + \phi^4 + \phi^6 + \ldots + \phi^t)$$
$$\implies \textit{Partial} \quad \textit{Geometric} \quad \textit{Series} \implies (1 + x + x^2 + \ldots + x^n) = \frac{1-x^{n+1}}{1-x}$$
$$\implies \boxed{Var(X_t) = \frac{\sigma_w^2}{1-\phi^2}(1 - \phi^{2(t+1)})}$$

(d) Show that, for $h \geq 0$,

$$Cov(X_{t+h}, X_t) = \phi^h Var(X_t)$$

$$\gamma_X(h) = Cov(X_{t+h}, X_t) = Cov(\sum_{j=0}^{t} \phi^j W_{t+h-j}, \sum_{k=0}^{t} \phi^k W_{t-k})$$

$$= Cov(\phi^h W_t + \phi^{h+1} W_{t-1} + \phi^{h+2} W_{t-2} + \ldots + \phi^{h+t} W_h, W_t + \phi W_{t-1} + \phi^2 W_{t-2} + \ldots + \phi^t W_0)$$

$$= (\phi^h + \phi^{h+2} + \phi^{h+4} + \ldots + \phi^{h+t})\sigma_w^2$$

$$= \phi^h \frac{\sigma_w^2}{1-\phi^2}(1 - \phi^{2(t+1)}) = \phi^h Var(X_t)$$

$$\implies \boxed{Cov(X_{t+h}, X_t) = \phi^h Var(X_t)}$$

(e) Is X_t stationary?

X_t is **NOT** stationary as the autocovariance function depends on the time t.

(f) Now suppose $X_0 = \frac{W_0}{\sqrt{1-\phi^2}}$. Is this process stationary?

No, X_t is still not stationary. The only difference would be that the value of $Var(X_t)$ at $t = 0$ would change, but for $t = 1, 2, \ldots$ the $Var(X_t)$ would be the same as part c. We are just scaling the initial data point. Therefore $Var(X_t)$ would still depend on the time t, meaning the process is not stationary.

Question 5. (Grad only)

Show that a time series model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$$

is causal if and only if

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1, \quad |\phi_2| < 1.$$

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = W_t$$

$$(1 - \phi_2 B - \phi_2 B^2) X_t = W_t$$

$$1 - \phi_2 B - \phi_2 B^2 = 0$$

$$\implies B = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

1.

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} > 1$$
$$\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < -2\phi_2$$
$$\sqrt{\phi_1^2 + 4\phi_2} < -2\phi_2 - \phi_1$$
$$\phi_1^2 + 4\phi_2 > 4\phi_2^2 + 4\phi_1\phi_2 + \phi_1^2$$
$$\implies \boxed{\phi_1 + \phi_2 < 1}$$

2.

$$\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} < -1$$
$$\phi_1 - \sqrt{\phi_1^2 + 4\phi_2} > 2\phi_2$$
$$\phi_1 - 2\phi_2 > \sqrt{\phi_1^2 + 4\phi_2}$$
$$4\phi_2^2 - 4\phi_1\phi_2 + \phi_1^2 < \phi_1^2 + 4\phi_2$$
$$\implies \boxed{\phi_2 - \phi_1 < 1}$$

3.

$$\begin{cases} \phi_1 + \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \end{cases}$$

- $\phi_2 > 0$

$$\implies 2\phi_2 < 2 \implies \phi_2 < 1$$

- $\phi_2 < 0$

$$\implies 2\phi_2 > -2 \implies \phi_2 > -1$$
$$\implies \boxed{|\phi_2| < 1}$$

For X_t to be causal, we need all of the roots of the characteristic polynomial to be outside of the unit circle. This is true when $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$. Therefore X_t is causal if and only if all the three conditions are satisfied.