CSE1729: Introduction to Programming

Functional Programming in SCHEME: Substitution and Environment Semantics

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Recall: SUBSTITUTION SEMANTICS for function application

- Consider the function definition `(define (f x) <body>)`.
- In the future, if the evaluator encounters `(f <arg>)`, it will:
  - **Evaluate** `<arg>` (as usual), resulting in a value v.
  - Apply f to the value v. This is accomplished in two steps:
    - **Substitute** occurrences of `x` in `<body>` with v, and
    - **Evaluate** `<body>` (after substitution) and return the result.

Eval-Apply diagrams: An example

- Consider the definitions

```
> (define (square x) (* x x))
> (define (fourth x) (square (square x)))
```

- Then...let's explore this via an "eval-apply" diagram.

A red box indicates that an application/substitution is pending	A blue box indicates that an evaluation is pending

- In particular...

The rules for Eval/Apply, in diagrams

- Recall the standard evaluation rule. In our Eval/Apply diagram, it asserts that:

$$(\langle E_1\rangle\ \ldots\ \langle E_n\rangle) \rightarrow (\langle E_1\rangle\ \ldots\ \langle E_n\rangle)$$

- Recall the substitution semantics rule for function application. It asserts that with the function:

```
(define (f x) <body>)
```

we have:

```
(f v)  →  [x/v]<body>
```

Making **eval** and **apply**... **tick** and **tock**

`(fourth 5)`

Making **eval** and **apply**… **tick** and **tock**

```
(fourth 5)

(fourth 5)
```

Making **eval** and **apply**... **tick** and **tock**

```
(fourth 5)
```

```
(fourth 5)
```

Making **eval** and **apply**... **tick** and **tock**

```
(fourth 5)
```

```
(define (fourth x) (square (square x)))
```

```
(fourth 5)
```

```
[x/5](square (square x))
```

Making **eval** and **apply**...
tick and **tock**

```
(fourth 5)

(fourth 5)

[x/5](square (square x))

(square (square 5))
```

Making **eval** and **apply**... **tick** and **tock**

```
(fourth 5)
```

```
(fourth 5)
```

```
[x/5](square (square x))
```

```
(square (square 5))
```

Making **eval** and **apply**… **tick** and **tock**

```
(fourth 5)
```

```
(fourth 5)
```

```
[x/5](square (square x))
```

```
(square (square 5))
```

```
(square (square 5))
```

Making **eval** and **apply**... **tick** and **tock**

```
(fourth 5)
(fourth 5)
[x/5](square (square x))
(square (square 5))
(square (square 5))
(square (square 5))
```

Making **eval** and **apply**… **tick** and **tock**

```
(fourth 5)
(fourth 5)
[x/5](square (square x))
(square (square 5))
(square (square 5))
(square (square 5))
(square (square 5))
```

Making **eval** and **apply**...
tick and **tock**

```
(fourth 5)

(fourth 5)

[x/5](square (square x))

(square (square 5))

(square (square 5))

(square (square 5))

(square (square 5))

(square ([x/5](* x x)))
```

```
(define (square x) (* x x))
```

625

Making **eval** and **apply**...
tick and **tock**

```
(square ([x/5](* x x)))
```

```
(square (* 5 5))
```

Making **eval** and **apply**... **tick** and **tock**

```
(square ([x/5](* x x)))
```

```
(square (* 5 5))
```

```
(square (* 5 5))
```

Making **eval** and **apply**... **tick** and **tock**

```
(square ([x/5](* x x)))
```

```
(square (* 5 5))
```

```
(square (* 5 5))
```

```
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```

Making **eval** and **apply**... **tick** and **tock**

```
(square ([x/5](* x x)))

(square (* 5 5))

(square (* 5 5))

(square (* 5 5))

(square (* 5 5))
```

Making **eval** and **apply**...
tick and **tock**

```
(square ([x/5](* x x)))

(square (* 5 5))

(square (* 5 5))

(square (* 5 5))

(square (* 5 5))

(square 25)
```

625

Making **eval** and **apply**... **tick** and **tock**

```
(square ([x/5](* x x)))

(square (* 5 5))

(square (* 5 5))

(square (* 5 5))

(square (* 5 5))

(square 25)

[x/25](* x x)
```

```
(define (square x) (* x x))
```

625

Making **eval** and **apply**… **tick** and **tock**

```
(* 25 25)
```

Making **eval** and **apply**… **tick** and **tock**

```
(* 25 25)
```

```
(* 25 25)
```

625

Making **eval** and **apply**... **tick** and **tock**

```
(* 25 25)
```

```
(* 25 25)
```

```
(* 25 25)
```

Making **eval** and **apply**... **tick** and **tock**

```
(* 25 25)
```

```
(* 25 25)
```

```
(* 25 25)
```

```
625
```

625

Why all the fuss about application semantics?

- *Recursion* will be our principal tool for program development; application semantics are critical for understanding how, precisely, this works.

- This reflects the fact that **recursion is the principle tool used to construct rich mathematical objects**.

The factorial function

- Recall the factorial function:

$$n! = n \cdot (n\text{-}1) \cdot \ldots \cdot 1$$

- Alternatively, we could write:

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

- **This meaningfully defines a function, even though it is recursive**:

3! = 3 * 2! = 3 * 2 * 1! = 3 * 2 * 1 * 0! = 3 * 2 * 1 * 1 = 6

The factorial function

Factorial is usually written in postfix notation (gasp!)

- Recall the factorial function:

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$$3! = 3 * 2! = 3 * 2 * 1! = 3 * 2 * 1 * 0! = 3 * 2 * 1 * 1 = 6$$

Recursion in SCHEME

- We can represent such a definition in SCHEME:

```
> (define (factorial n)
    (if (= n 0)
        1
        (* n (factorial (- n 1)))
    )
  )
```

- Note, in particular, that `factorial` appears in the definition of `factorial`.
- This is a well-defined function; its meaning is determined by substitution semantics.

An Eval-Apply diagram for `factorial`

- We've introduced two colored boxes, which stand for "pending evaluation" and "pending application".
- Let us introduce a box for the `if` special form:

Behavior:

```
(if #t <then> <else>)  →  <then>

(if #f <then> <else>)  →  <else>
```

- With this in place, we can define:

```
(if <pred> <then> <else>)  →  (if <pred> <then> <else>)
```

Evaluation of `factorial`

```
(factorial 2)

(if (= 2 0) 1 (* 2 (factorial (- 2 1))))

(if (= 2 0) 1 (* 2 (factorial (- 2 1))))

(if #f 1 (* 2 (factorial (- 2 1))))

(* 2 (factorial (- 2 1)))

(* 2 (factorial (- 2 1)))

(* 2 (factorial (- 2 1)))

(* 2 (factorial 1))
```

and...hence...

```
(factorial 3)
(* 3 (factorial 2))
(* 3 (* 2 (factorial 1)))
(* 3 (* 2 (* 1 (factorial 0))))
```

Now what?

```
(factorial 0)
```

(substitution)

```
(if (= 0 0) 1 (* 0 (factorial (- 0 1))))
(if #t 1 (* 0 (factorial (- 0 1))))
```

(`if` special form)

```
1
```

! Green form does ***not*** evaluate all arguments !

Putting it all together…

```
(factorial 3)
(* 3 (factorial 2))
(* 3 (* 2 (factorial 1)))
(* 3 (* 2 (* 1 (factorial 0))))
(* 3 (* 2 (* 1 1)))
(* 3 (* 2 1))
(* 3 2)
6
```

Play it again...substitution semantics for `factorial`

Play it again...substitution semantics for `factorial`

```
(factorial 2)
```

Play it again...substitution semantics for `factorial`

```
(factorial 2)
```

```
(factorial 2)
```

Play it again...substitution semantics for `factorial`

```
(factorial 2)
```

```
(factorial 2)
```

```
(factorial 2)
```

Play it again...substitution semantics for `factorial`

```
(factorial 2)
```

```
(factorial 2)
```

```
(factorial 2)
```

```
[x/2](if (= x 0) 1 (* 2 (factorial (- x 1))))
```

Play it again…substitution semantics for `factorial`

```
(factorial 2)
```

```
(factorial 2)
```

`(factorial 2)`

```
[x/2](if (= x 0) 1 (* 2 (factorial (- x 1))))
```

Which is…

```
(if (= 2 0) 1 (* 2 (factorial (- 2 1))))
```

…

Some conclusions about Scheme from substitution semantics

- The **name** of "local variables" does not matter. Why? They are just placeholders for substitution!

- As far as Scheme is concerned

```
(define (double x) (* x 2))
```

and

```
(define (double y) (* y 2))
```

are identical!

- Why? For any value v,

```
[x/v](* x 2))
```

and

```
[y/v](* y 2))
```

are identical!

Variable shadows

- Substitution semantics explain what happens when a local variable has the same name as a variable in the enclosing environment.
- Question: How does the following code snippet behave?

```
> (define x 100)
> (define y 200)
> (define (add-to-y x) (+ x y))
> (add-to-y 2)
```

Variable shadows

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$x \longmapsto$

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```

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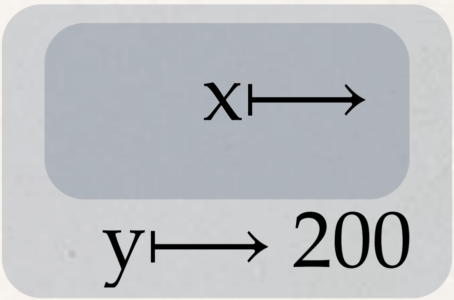

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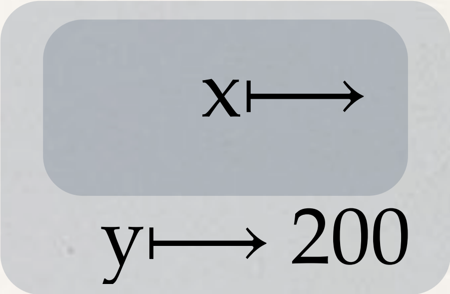


```
[x/2](+ x y)        (+ 2 y)
```

Variable shadows

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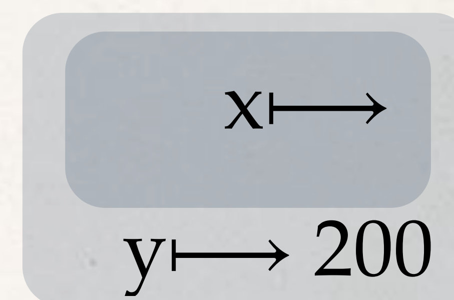



```
> (define x 100)
> (define y 200)
> (define (add-to-y x) (+ x y))
> (add-to-y 2)
```

```
[x/2](+ x y)  ⇒  (+ 2 y)
```

Note: x was never “looked up in this environment!”

Variable shadows

- With substitution semantics, variables are given values by two different processes:
  - Looking up in an environment, and
  - Substitution during function application.
- We can unify (and simplify) our understanding of Scheme by with *environment semantics*, which we will discuss again in more detail.

Recall the behavior of `factorial`

```
> (define (factorial n)
    (if (= n 0)
        1
        (* n (factorial (- n 1)))
    )
  )
```

- Let's focus on the behavior of factorial when called with 0.

`if`...it's just *got* to be special!

- The special evaluation rule for `if` is CRITICAL for this to work.
- Suppose that `(if <pred> <exp1> <exp2>)` evaluated all of its arguments (as per usual evaluation). Then...

```
(factorial 0)
```

expands to

```
(if (= 0 0) 1 (* 0 (factorial (- 0 1))))
```

which would require evaluation of...

`(= 0 0)` and `1` and `(factorial -1)`

This will never terminate...

…which would "unwind" eternally

```
(if #t
   1
   (* 0 (if #f
            1
            (* -1 (if #f
                      1
                      (* -2 (if #f
                                1
                                (* -3 (if #f
                                          1
                                          (* -4 (if ...
```

Remark: A similarly nonterminating computation would ensue if we called `(factorial -1)` or `(factorial (/ 1 2))`; why?

“Special” treatment of other primitive functions

- Thus, special “incomplete” evaluation is essential for meaningful recursive programming. For this reason, other primitive functions whose values can be determined by “incomplete” evaluation are also treated as special forms:

- $\texttt{(and <x}_1\texttt{> <x}_2\texttt{> ... <x}_n\texttt{>)}$ uses “short-circuited” evaluation. The expressions $\texttt{<x}_1\texttt{>}$, ... are evaluated one at a time, left to right. If any evaluate to `#f`, evaluation stops (and `#f` is returned). Otherwise, `#t` is returned.

- $\texttt{(or <x}_1\texttt{> <x}_2\texttt{> ... <x}_n\texttt{>)}$ uses “short-circuited” evaluation. The expressions $\texttt{<x}_1\texttt{>}$, ... are evaluated one at a time, left to right. If any evaluate to `#t`, evaluation stops (and `#t` is returned).

Another example: The Fibonacci numbers

- The *Fibonacci numbers* are defined by the rule:

$$F_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

- Note, then, that the sequence F_0, F_1, F_2, ... is

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

each is the sum of the previous two.

The Fibonacci numbers in SCHEME

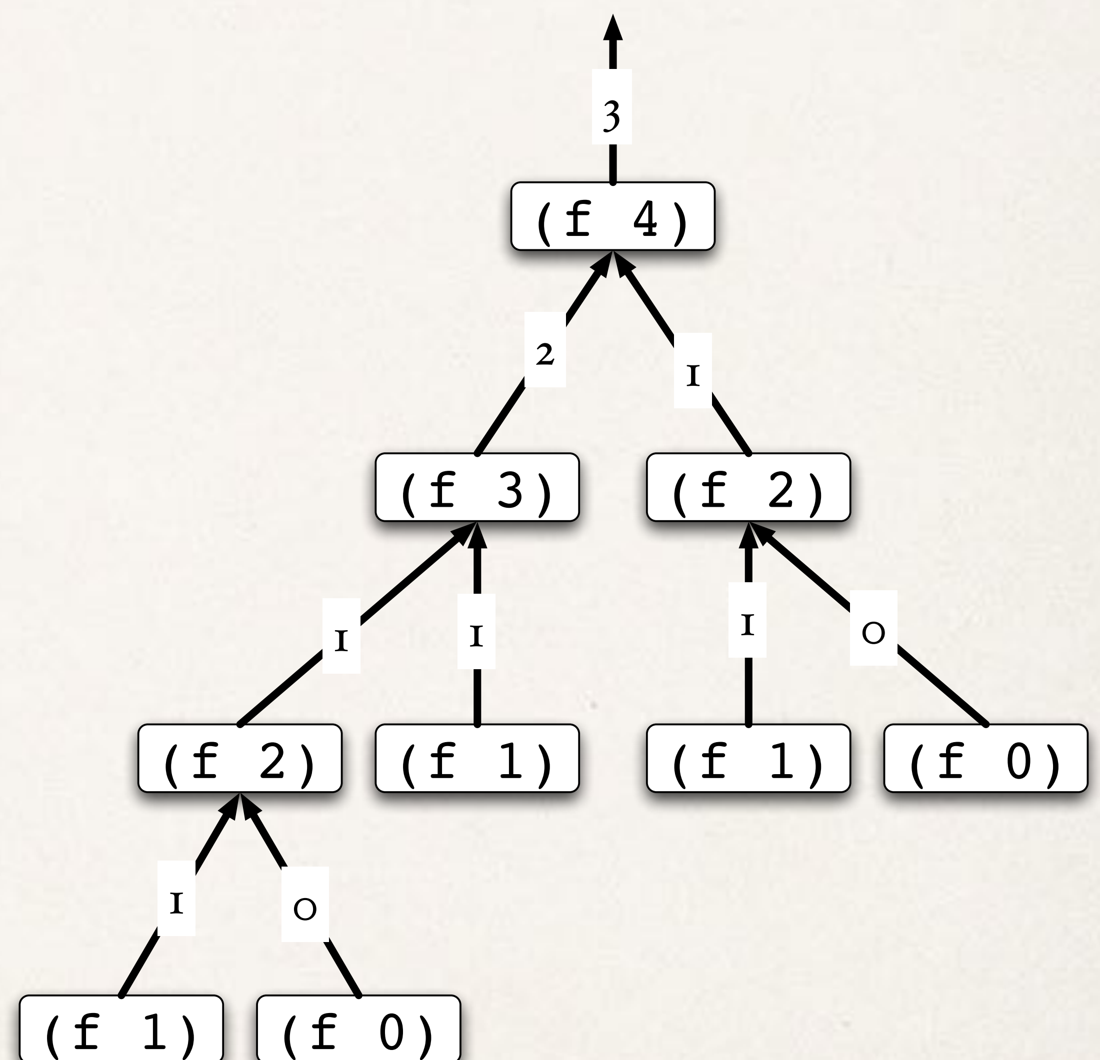
- As with the factorial function, we can naturally capture this definition in SCHEME.

```
(define (fib n)
        (cond ((= n 0) 0)
              ((= n 1) 1)
              ((> n 1) (+ (fib (- n 1))
                          (fib (- n 2))))
        )
)
```

- Notice, as with `factorial`, how closely the SCHEME definition can mirror the mathematical definition.

The Fibonacci *evaluation tree*

- The Fibonacci function gives rise to an "**evaluation tree**" as shown. Here each node returns the sum of the value of its children.
- Note that some "sub"-problems are evaluated many times.
- Question: How many times is (f 1) evaluated, in total?



Recursion is delicate business

```
> (define (recurse x) (recurse x))
> (recurse 1)
>
> (if #t 1 (recurse 1))
> 1
> (if #f 1 (recurse 1))
>
```

Recursion is delicate business

```
> (define (recurse x) (recurse x))
> (recurse 1)
>
> (if #t 1 (recurse 1))
> 1
> (if #f 1 (recurse 1))
>
```

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> (recurse 1)
>
> (if #t 1 (recurse 1))
> 1
> (if #f 1 (recurse 1))
>
```

“Iterative” constructs in SCHEME

- Consider computing the sum of the first n numbers in SCHEME.
- Note that $$\underbrace{(1 + \ldots + n)}_{\sum_{i=1}^{n} i} = n + \underbrace{(1 + \ldots + (n-1))}_{\sum_{i=1}^{n-1} i}$$
- And thus:

```
> (define (number-sum n)
    (if (= n 0)
        0
        (+ n (number-sum (- n 1)))))
> (number-sum 10)
55
```

The evaluation tree for `number-sum`

- `(number-sum 4)` generates a call to `(number-sum 3)`; it will add 4 to the result and return the value.

- `(number-sum 3)` generates a call to `(number-sum 2)`; it will add 3 to the result on return the value

…

- `(number-sum 0)` is called: returning `0`.

- `(number-sum 4)`

The evaluation tree for `number-sum`

- `(number-sum 4)`

The evaluation tree for `number-sum`

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The evaluation tree for `number-sum`

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The evaluation tree for `number-sum`

- `(number-sum 4)`

`(number-sum 4)`

4 +

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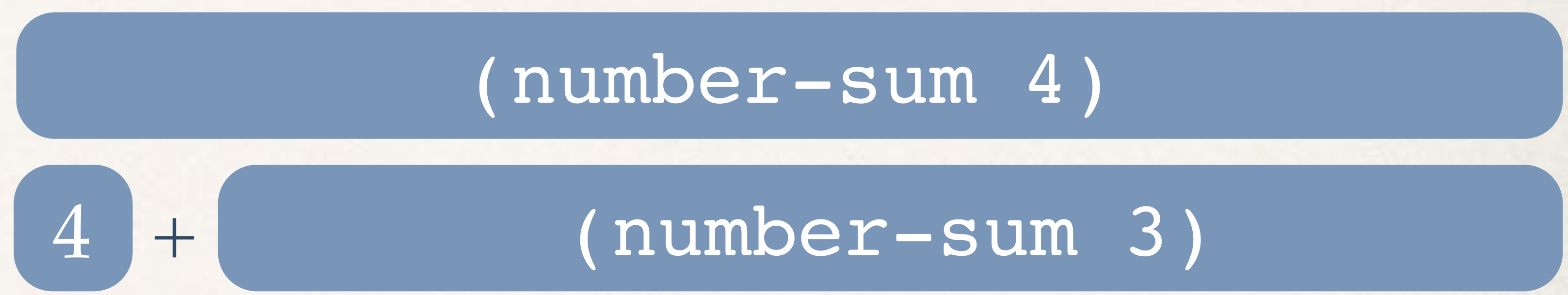


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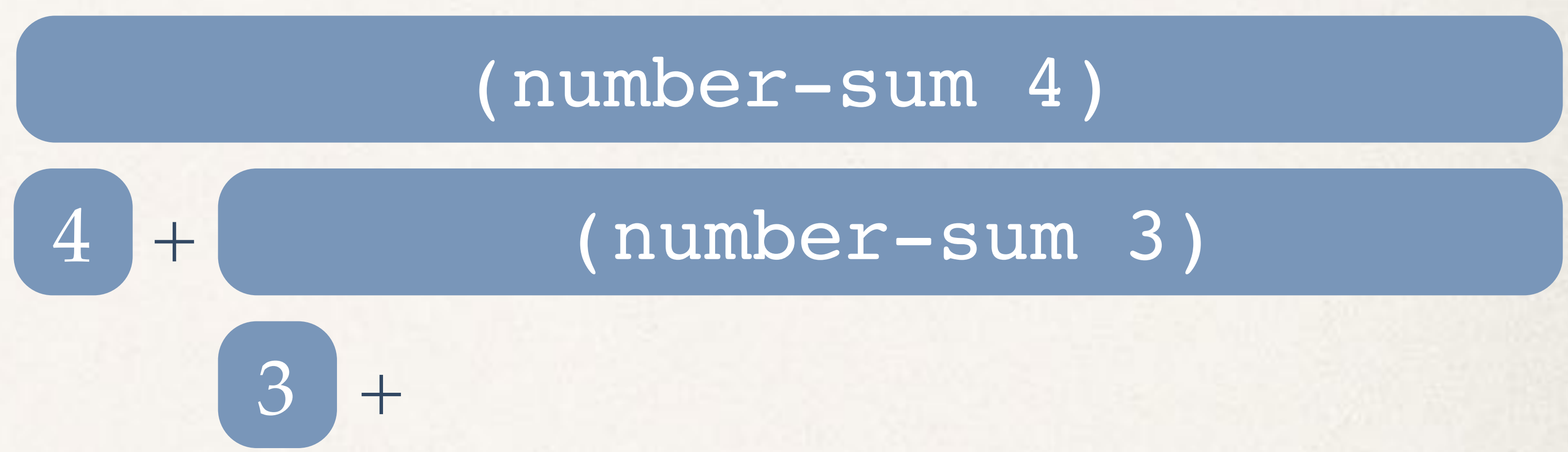


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55
```

The evaluation tree for `number-sum`

- `(number-sum 4)`

```
(number-sum 4)
4 + (number-sum 3)
    3 + (number-sum 2)
```

```
> (define (number-sum n)
    (if (= n 0)
        0
        (+ n (number-sum (- n 1)))))
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The evaluation tree for `number-sum`

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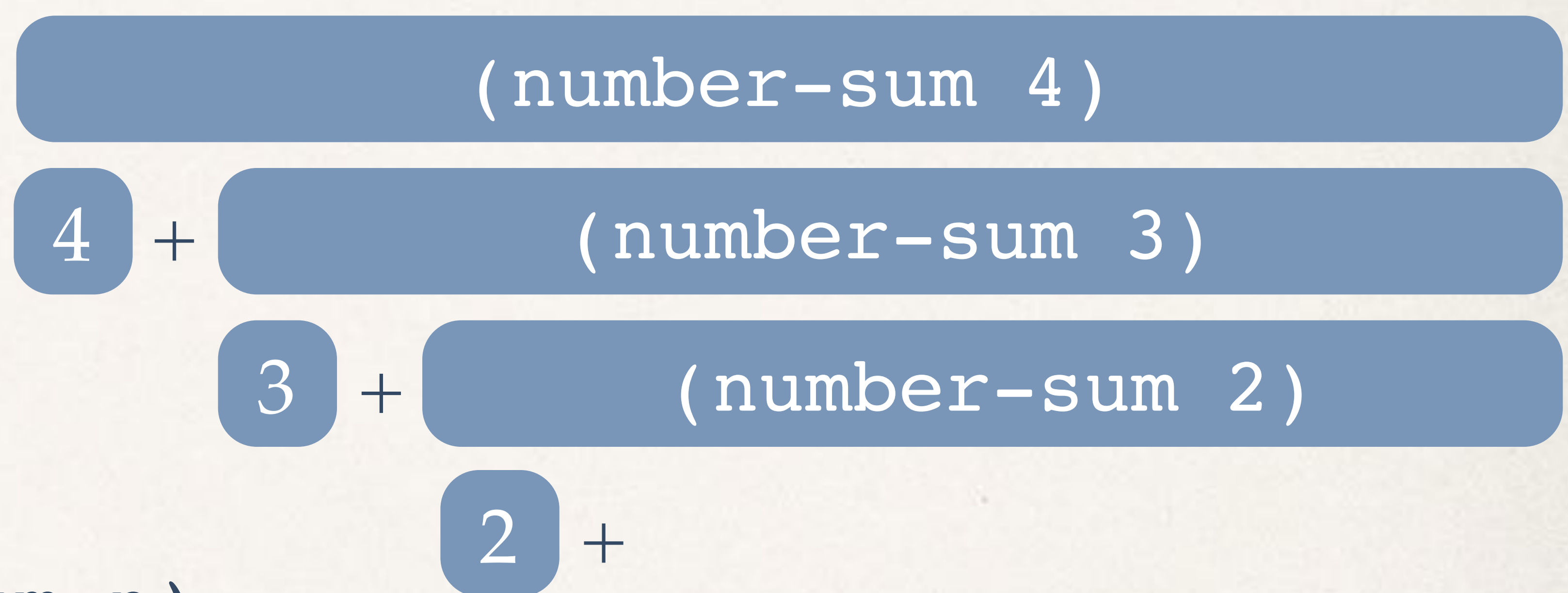


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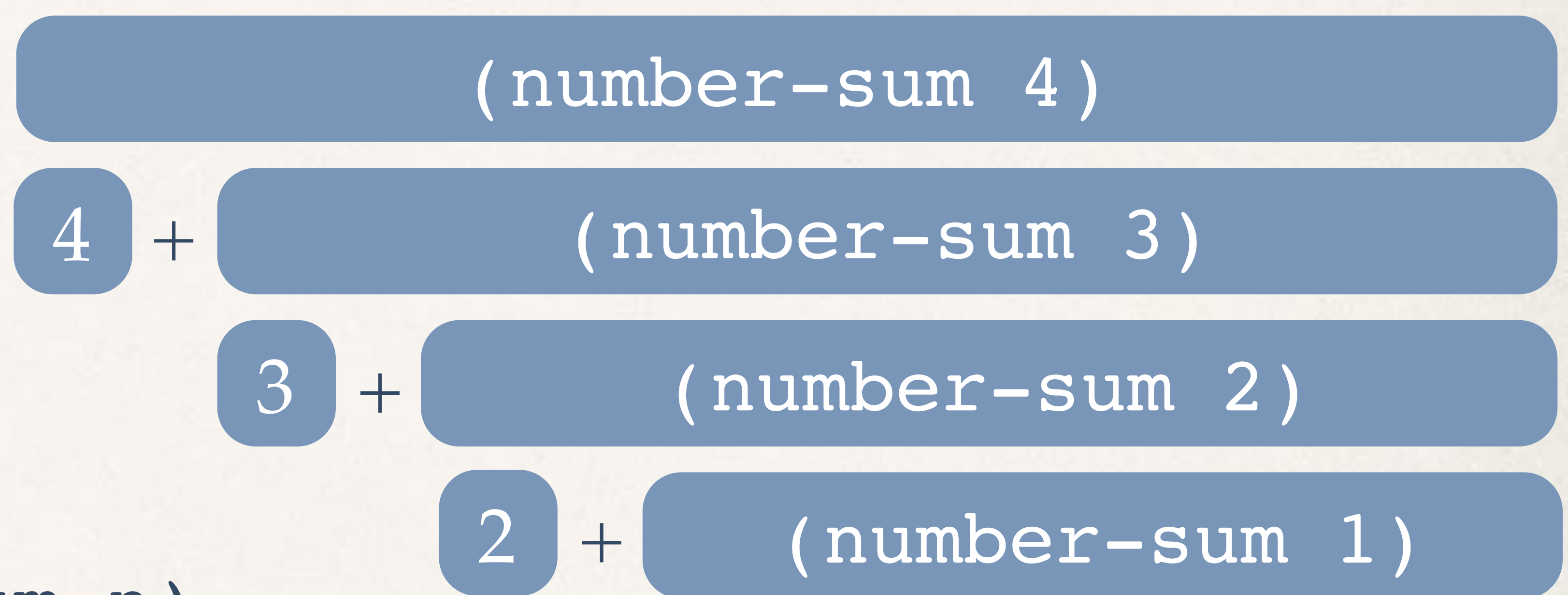


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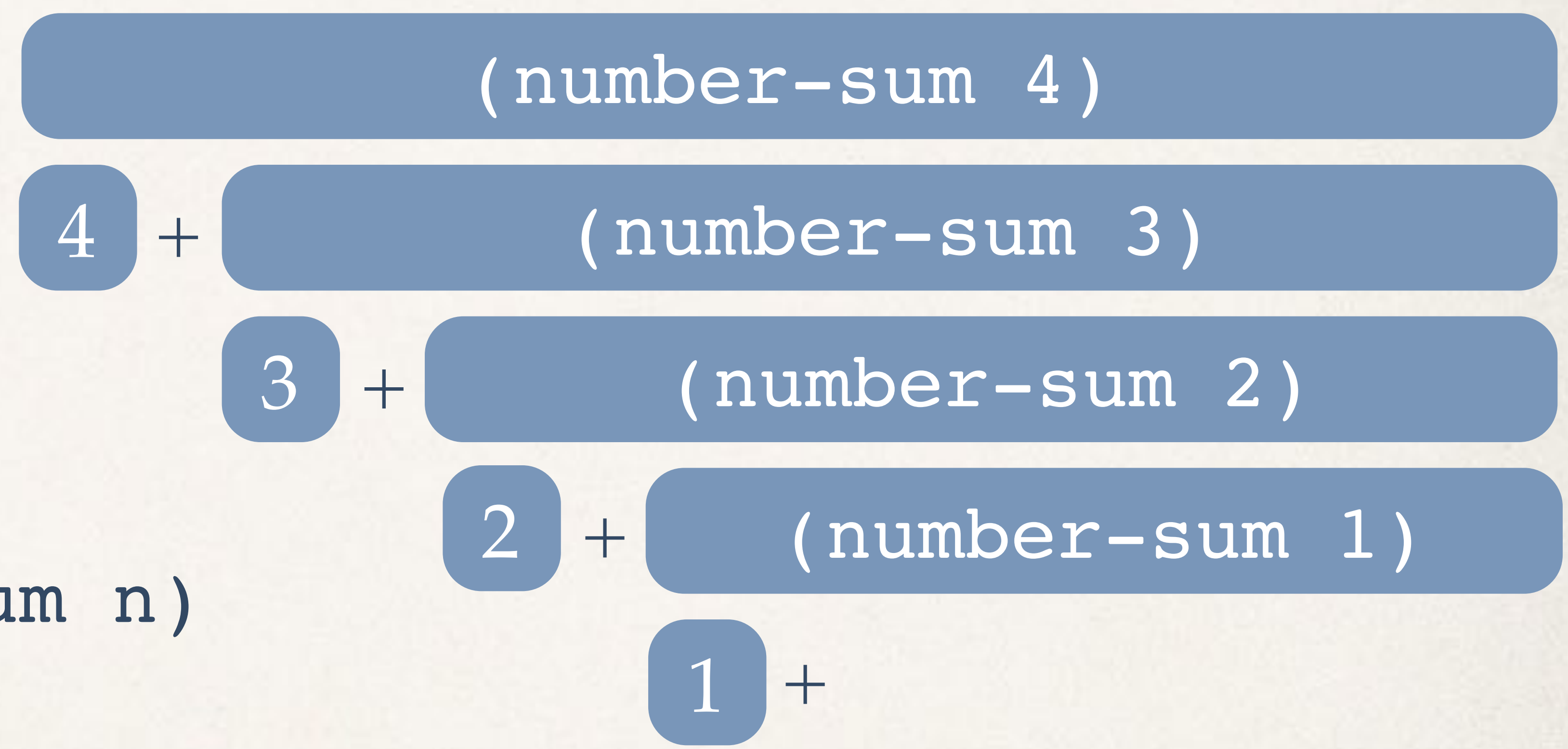


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```

The evaluation tree for `number-sum`

- `(number-sum 4)`

```
(number-sum 4)
4 + (number-sum 3)
    3 + (number-sum 2)
        2 + (number-sum 1)
            1 + (number-sum 0)
```

```
> (define (number-sum n)
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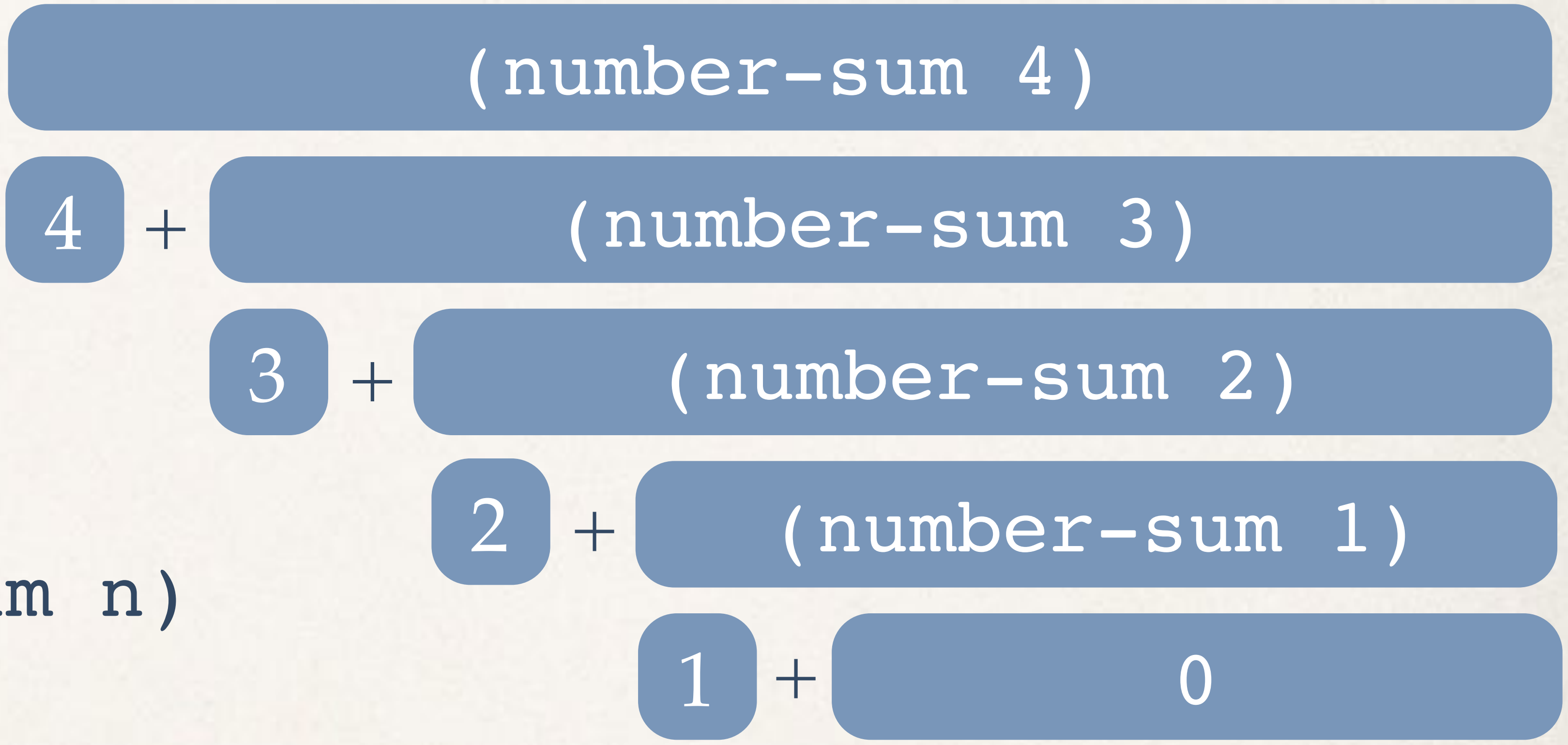
The evaluation tree for `number-sum`

- `(number-sum 4)`

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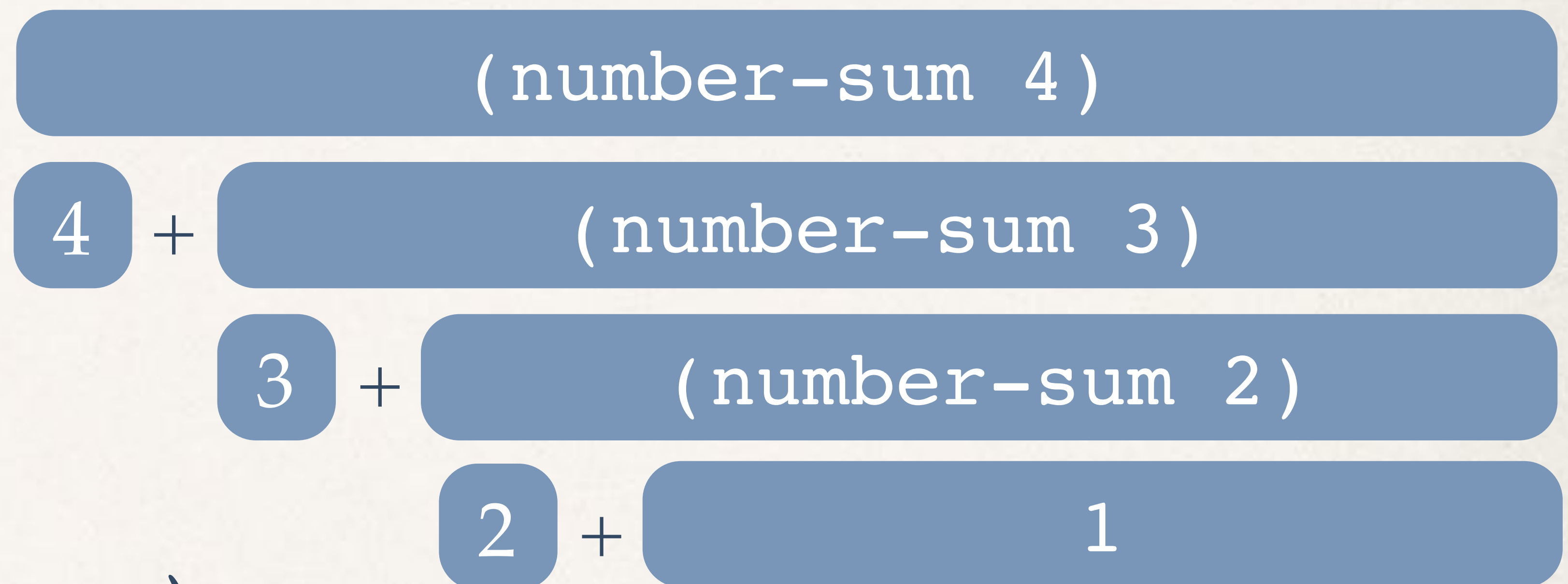
The evaluation tree for `number-sum`

- (number-sum 4)

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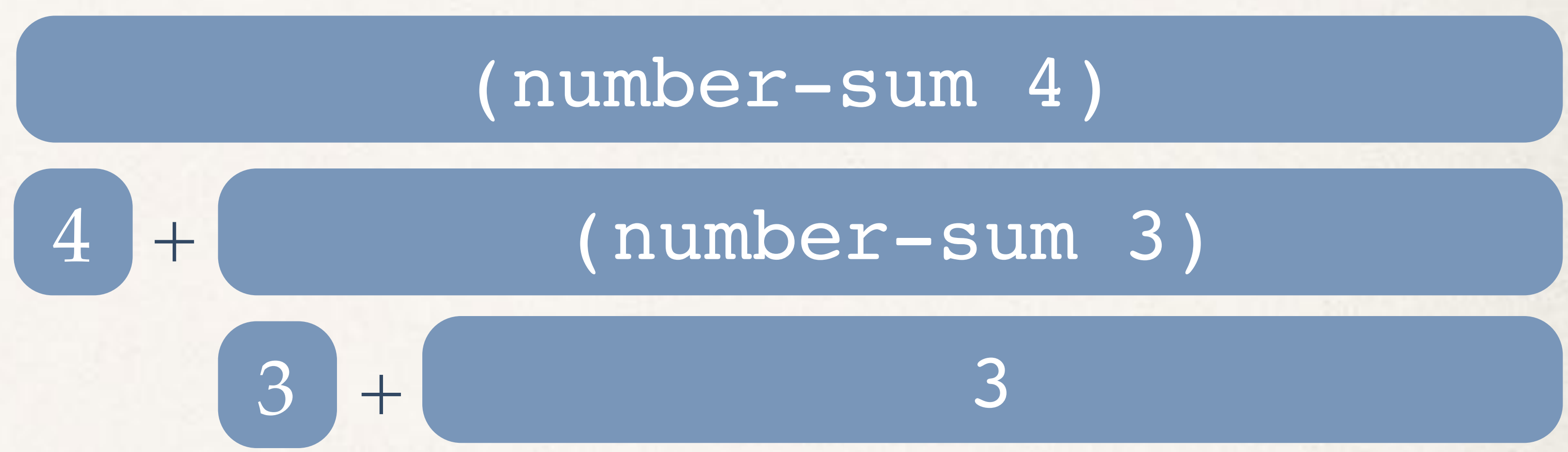
- (number-sum 4)

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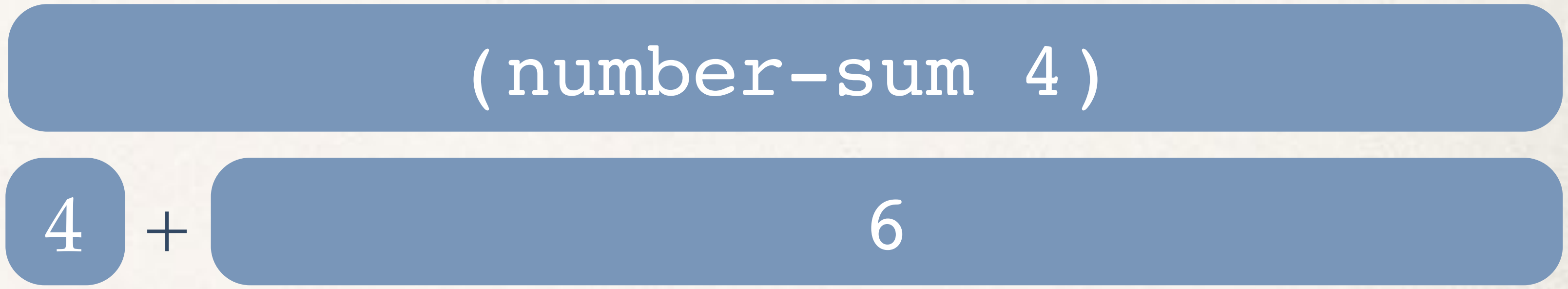
- `(number-sum 4)`

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The evaluation tree for `number-sum`

- `(number-sum 4)`

10

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        0
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> (number-sum 10)
55
```

Recursive decomposition requires love and understanding…

- Not all recursive decompositions of a problem are the same…
- There can be major conceptual and computational differences…

Example: Multiplication in terms of addition

- Consider the definition of multiplication as repeated addition:

$$a \times b = \underbrace{b + \cdots + b}_{a \text{ times}}$$

- We can express this in SCHEME:

```
(define (mult a b)
    (if (= a 0)
        0
        (+ b (mult (- a 1) b))))
```

Efficiency considerations

- How many recursive calls are generated by

```
(mult 200 2)⟵(mult 199 2)⟵(mult 198 2)⟵
```

- How about

```
(mult 2 200)⟵(mult 1 200)⟵(mult 0 200)
```

- We could write a more *efficient* program by "recursing on the smaller of a and b." Thus

A more efficient multiply...

We could write a new program to exploit this...

```
(define (fmult a b)
    (cond ((= a 0) 0)
          ((= b 0) 0)
          ((<= a b) (+ b (mult (- a 1) b)))
          ((>  a b) (+ a (mult a (- b 1))))))
```

Now it will only recurse min(a,b) times. Alternatively,

```
(define (fmult a b)
    (if (> a b) (mult b a) (mult a b)))
```

To be really fancy, we could reduce both a and b at the same time…

Remember that **ab = (a-1)(b-1) + a + b - 1**. Thus we could also express multiply as…

```
(define (fmult a b)
    (cond ((= a 0) 0)
          ((= b 0) 0)
          (else (+ -1
                   a
                   b
                   (fmult (- a 1) (- b 1))))))
```

This will also recurse min(a,b) times.

Actually, all three of these algorithms are terrible...why?

- With paper and pencil, how long would it take you to multiply two 16 digit numbers? Perhaps a few hours?
- With the program above, the computation

```
(fmult 1000000000000000 1000000000000000)
```

will never complete, even on a *very* fast computer. (Try it.) Why?

What does the evaluation tree look like?

- Well, 1000000000000000 will generate a call to
  - 999999999999999, and hence to
  - 999999999999998, and hence to
  - 999999999999997, and hence ....
- In total 10000000000000000 calls must be completed. If the computer could carry out a million calls per second, this would take 1000000000 seconds, a little over 30 years.

We can fix this by using more information about multiplication...

- On a computer dividing by 2 and multiplying by 2 can be done very quickly--we can improve our program:
- **Observation**: Suppose we wish to multiply x and y.
  - If we're lucky, x is even, and we have

$$x \times y = 2 \times \boxed{\left[\frac{x}{2} \times y\right]}$$

These operations can be done quickly

x has been reduced by **half** in this recursive call

Fast multiplication with division & multiplication by 2

- On a computer dividing by 2 and multiplying by 2 can be done very quickly--we can improve our program:
- **Idea**: To multiply x and y (positive whole numbers):
  - If x is odd, fix it! The answer is: y + 2 * [(x-1)/2 * y]

    (x-1 is even if x is odd, so (x-1)/2 is integer)

    Recursive calls

  - If x is even: the answer is: 2 * [x/2 * y]

Now, one of the numbers in the recursive call [...] has been significantly reduced--it's only half the previous size!

Capturing this idea in a Scheme program

- On a computer dividing by 2 and multiplying by 2 can be done very quickly--we can improve our program:

```
(define (even x)  (= (modulo x 2) 0))
(define (twice x) (* x 2))
(define (half x)  (/ x 2))

(define (rfmult a b)
    (cond ((= 0 a) 0)
          ((= 0 b) 0)
          ((even a) (twice (rfmult (half a) b)))
          (else     (+ b (twice (rfmult (half (- a 1))
                                     b)))
    )
)
```

How has the evaluation tree changed?

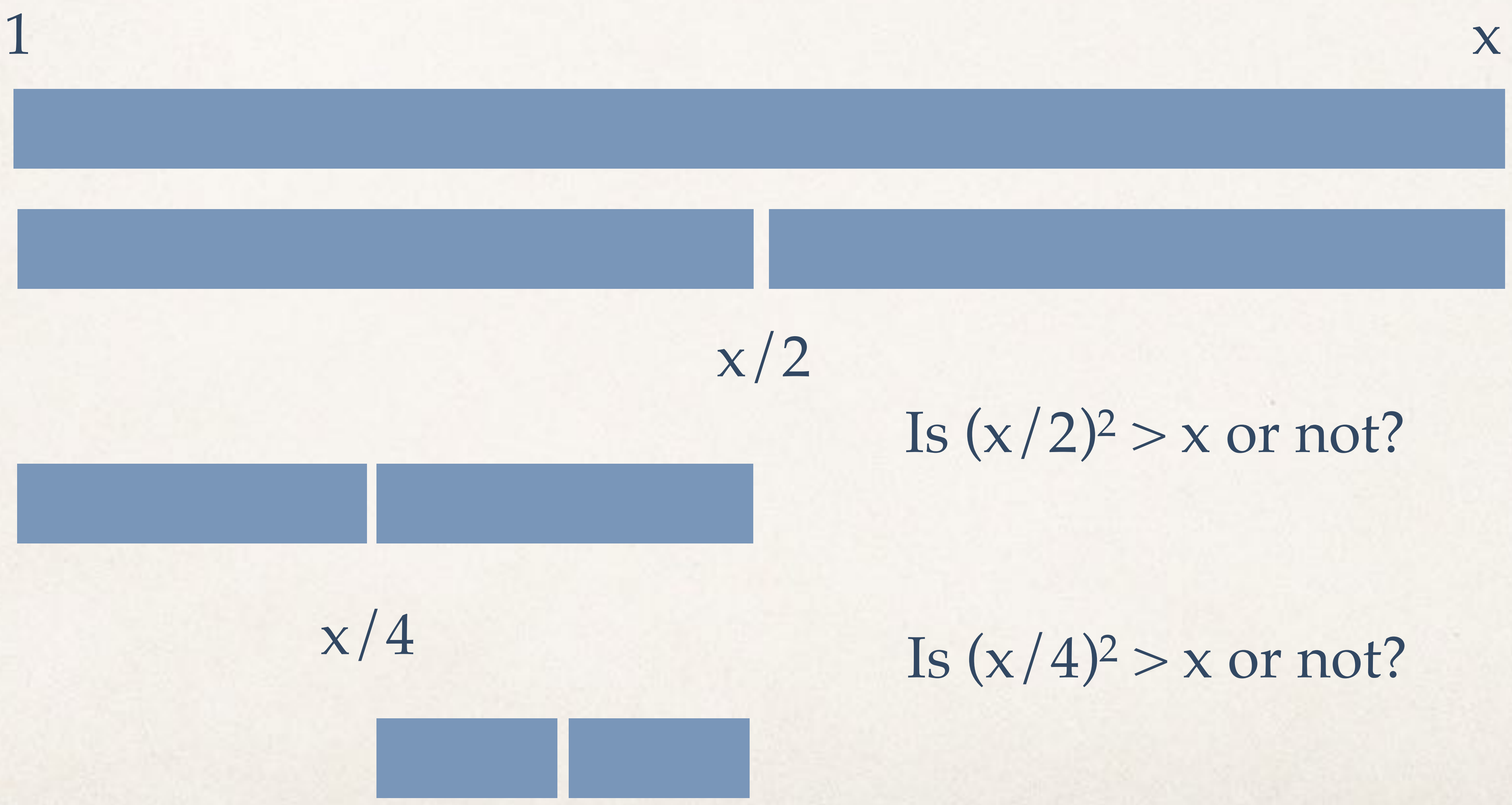
- Well, `(rfmult` 2^k `x)` will generate a call to
  - `(rfmult` 2^{k-1} `x)`, and hence to
  - `(rfmult` 2^{k-2} `x)`, and hence to
  - `(rfmult` 2^{k-3} `x), ...`
- In total, if called on `10000000000000000` $< 2^{54}$, only 54 calls must be completed. Your computer can do this in a fraction of a second. (Try it.)

Computing square roots by averaging

- One simple way to compute an approximation to the square root of a number x is to
  - Start with two guesses, a and b, with the property that
    $$a < \sqrt{x} < b$$
    (For example, if $x > 1$, we could start with $a = 1$, $b = x$.) Thus we know that the actual square root is between a and b.
  - If $(a + b)/2$ is larger than the square root (which we can check by comparing $[(a + b)/2]^2$ with x) we know the real square root lies between a and $(a + b)/2$.
  - Otherwise, the real square root lies between $(a + b)/2$ and b.

Square roots by “binary search”

Suppose $x > 1$. Then sqrt(x) is certainly between 1 and x.



Is $(x/2)^2 > x$ or not?

Is $(x/4)^2 > x$ or not?

…

For example...

- To compute the square root of 10:
  - start with the window: [1, 10] (we know the square root lies in this range).
  - Consider $(1 + 10)/2 = 5.5$. Since $5.5^2 > 10$, this is larger than sqrt(10).
  - Now we know the square root lies in [1, 5.5].
- Repeating this process, we find that it lies in [1, 3.25].
- Repeating again, we find that it lies in [2.125, 3.25]. ...

In SCHEME

```
(define (average a b) (/ (+ a b) 2))
(define (square a) (* a a))

(define (sqrt-converge x a b)
  (if (< (abs (- a b)) .000001)
      a
      (if (> (square (average a b)) x)
          (sqrt-converge x a (average a b))
          (sqrt-converge x (average a b) b))))
```

N.b.: `(average a b)` is referred to three times here.

Now, we might like to define a more attractive square root function that does not require choosing a and b:

```
(define (new-sqrt x) (sqrt-converge x 1 x))
```

Local variables

- `(average a b)` is referred to several times in `sqrt-converge`. Wouldn't it be nice if we could temporarily bind a "local" variable to this value?

- The `let` construct does exactly this:

```
(let ((x1 <expr1>)
      (x2 <expr2>)
        ...
      (xk <exprk>))
   <body-expr>)
```

- **Semantics**: Evaluate each `<expr`$_i$`>`, yielding a value v_i. Create a new environment by starting with the current one and binding each `x`$_i$ to v_i. Then return the value of `<body-expr>` in this environment.

`sqrt-converge` reloaded

```
(define (sqrt-converge x a b)
  (let ((avg (/ (+ a b) 2)))
     (if (< (abs (- a b)) .000001)
      a
      (if (> (square avg) x)
          (sqrt-converge x a avg)
          (sqrt-converge x avg b)))))
```

The `let` statement binds `avg` to (a+b)/2 for the shaded block of code

We say that the `let` statement constructs a new environment, just like the enclosing environment, but in which `abs` has been bound to the value of (a+b)/2.

Lets go crazy!

```
> (define a 3)
> a
3
> (let ((a 10)
        (b (+ a 1)))
    b)
4
> a
3
> (let ((a 10)
        (b (+ a 1)))
    a)
10
> a
3
```

Lets go crazy!

“Ambient” environment a:3

```
> (define a 3)
> a
3
> (let ((a 10)
        (b (+ a 1)))
    b)
4
> a
3
> (let ((a 10)
        (b (+ a 1)))
    a)
10
> a
3
```

Lets go crazy!

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> (define a 3)
> a
3
> (let ((a 10)
        (b (+ a 1)))
    b)
4
> a
3
> (let ((a 10)
        (b (+ a 1)))
    a)
10
> a
3
```
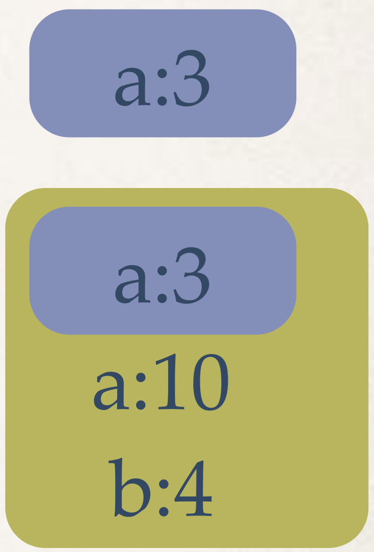

“Ambient” environment

Let environment



Lets go crazy!

```
> (define a 3)
> a
3
> (let ((a 10)
        (b (+ a 1)))
    b)
4
> a
3
> (let ((a 10)
        (b (+ a 1)))
    a)
10
> a
3
```

"Ambient" environment a:3

Let environment a:3 a:10 b:4

`a` *is unchanged in the enclosing environment!*

Lets go crazy!

```
> (define a 3)
> a
3
> (let ((a 10)
        (b (+ a 1)))
    b)
4
> a
3
> (let ((a 10)
        (b (+ a 1)))
    a)
10
> a
3
```

"Ambient" environment

a:3

`Let` environment

a:3
a:10
b:4

`a` *is unchanged in the enclosing environment!*

The original binding of
a is shadowed

a:3
a:10
b:4

Lets go crazy!

```
> (define a 3)
> a
3
> (let ((a 10)
        (b (+ a 1)))
    b)
4
> a
3
> (let ((a 10)
        (b (+ a 1)))
    a)
10
> a
3
```

"Ambient" environment — a:3

Let environment — a:3 / a:10 / b:4

`a` *is unchanged in the enclosing environment!*

The original binding of a is shadowed — a:3 / a:10 / b:4

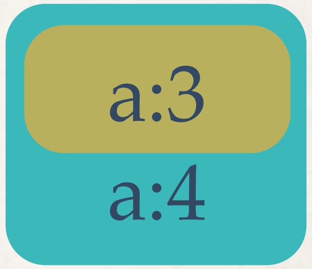
"Ambient" environment — a:3

Lets go even crazier!

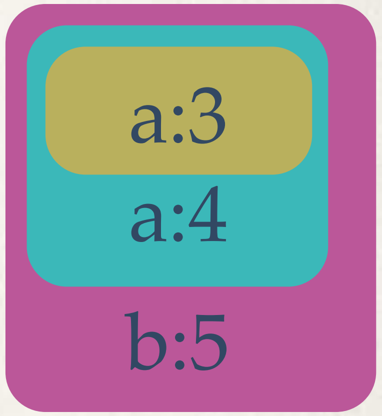
```
> (define a 3)
> (let ((a (+ a 1)))
    (let ((b (+ a 1)))
      b))
5
```

"Ambient" environment

Let environment



Let environment



Defining local *functions*…like local variables..but even more awesome!

- A number is **perfect** if it is the sum of its (proper) divisors; 6 = 3 + 2 + 1 is perfect; 8 is not 4 + 2 + 1 so it is not perfect.

A version using our tools so far…

```
(define (divides? a b) (= 0 (modulo b a)))

(define (divisor-sum n k)
  (cond ((= k 1) 1)
        ((divides? k n) (+ k (divisor-sum n (- k 1))))
        (else (divisor-sum n (- k 1)))))

(define (perfect? n) (= n (divisor-sum n (- n 1))))
```

Defining local *functions*…like local variables..but even more awesome!

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        (else (divisor-sum n (- k 1)))))

(define (perfect? n) (= n (divisor-sum n (- n 1))))
```

Divides only used here…it's local

Defining local *functions*…like local variables..but even more awesome!

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So: we may define `divides?` inside…

```
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```

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  (define (divides? a b) (= 0 (modulo b a)))
  (cond ((= k 1) 1)
        ((divides? k n) (+ k (divisor-sum n (- k 1))))
        (else (divisor-sum n (- k 1)))))

(define (perfect? n) (= n (divisor-sum n (- n 1))))
```

Local defines affect the local environment…

```
(define (f x)
  (define (average a b) (/ (+ a b) 2))
  (average 1 x))
```

- When f is called…
  - An environment is created (in which x is bound to the actual parameter)
  - The function `average` is defined, and added to the environment.
- …and…finally, the body `(average 1 x)` is evaluated.

Function application constructs new environments…

- Recall that `let` produces a (new) environment with new variable bindings. (Incidentally, you could also make this precise by means of substitution.)

- Recall, also, our variant of substitution semantics which we called "environment semantics"...

- This will alleviate the need to understand functions in terms of substitution: everything will be captured with environments.

- Also handles a difficulty: Function bodies that refer to variables that are not the formal parameter…

Variables in function bodies: What if you set them free?

- Consider the declaration `(define (f x) (+ a x))`.
- Question: To what does `a` refer?
- Remark: It's not so obvious what the answer should be!

```
> (define a 10)
> (define (f x) (+ x a))
> (f 100)
110
> (let ((a 20))
    (f 100))
```

Variables in function bodies: What if you set them free?

- Consider the declaration `(define (f x) (+ a x))`.
- Question: To what does `a` refer?
- Remark: It's not so obvious what the answer should be!

```
> (define a 10)
> (define (f x) (+ x a))
> (f 100)
110
> (let ((a 20))
    (f 100))
```

110

Oh no!
Substitution semantics is *WRONG*

```
> (define y 10)
> (define a 10)
> (define (f x) (+ x a))
> (f 100)
110
> (let ((a 20))
    (f 100))
```

- Substitution semantics is fine for simple function bodies.
- Function bodies with free variables require environment semantics… let's look in more detail…

Oh no!
Substitution semantics is *WRONG*

```
> (define y 10)
> (define a 10)
> (define (f x) (+ x a))
> (f 100)
110
> (let ((a 20))
    (f 100))
```

Substitution here would have given…120

- Substitution semantics is fine for simple function bodies.
- Function bodies with free variables require environment semantics… let's look in more detail…

Recall **Environment semantics:** A more sophisticated model of function application

- Consider `(define (f x) <body>)`.
- In the future, if the interpreter is called upon to evaluate f on the value v it will:
  - create a *new environment*, identical to the environment in which f was **defined**, but in which **`x`** has been bound to v. (This shadows any existing binding of **`x`** in def'n environment.)
  - evaluate the expression `<body>` in this new environment; the resulting value of `<body>` is the value this function returns.

Body evaluated in this environment

Lexical scope and variable clashes

- Scheme uses a precise set of rules to determine the binding of a variable. These conditions are known as *scoping rules* for the binding.
- SCHEME uses *lexical scope*.
- The other natural choice is *dynamic scope*.
- Example:

```
> (let ((x 10))
    (define (g y) (+ x y))
    (let ((x 100))
         (g 1000)))

1010
>
```

- Two potentially relevant environments:
  - The environment at the *time of definition* (in which x = 10).
  - The environment at time of *invocation* (in which x = 100).
- Lexical scoping rules (which SCHEME uses) always rely on the environment at *definition time*.

Another example, pictorially

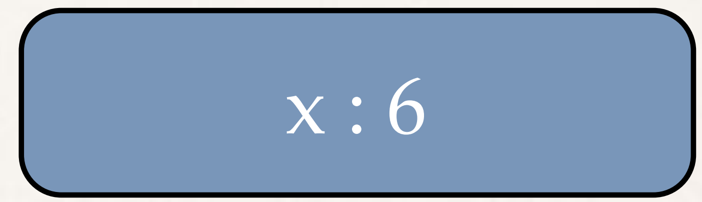
```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

Call to f

Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

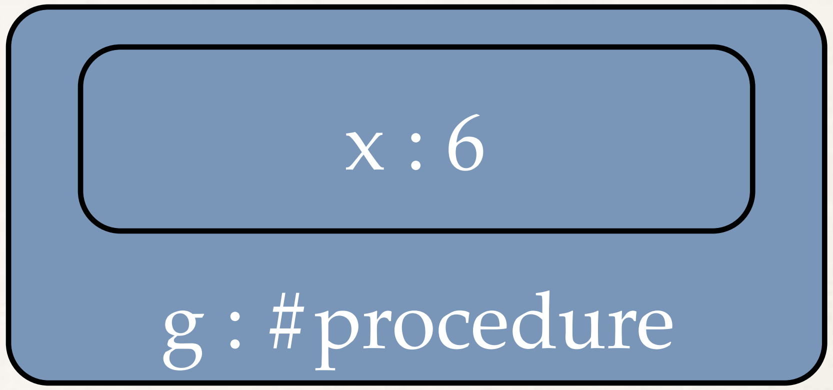
Binding for x



Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

define g

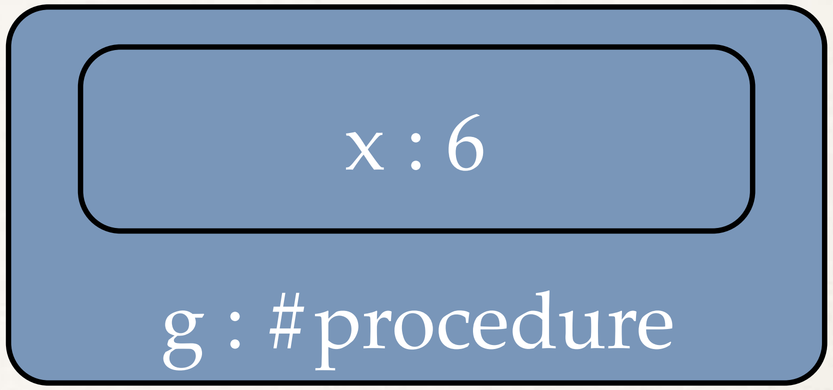


The definition of g *grabs the current environment and its binding of x. Any evaluation inside g that needs x will use this copy!*

Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

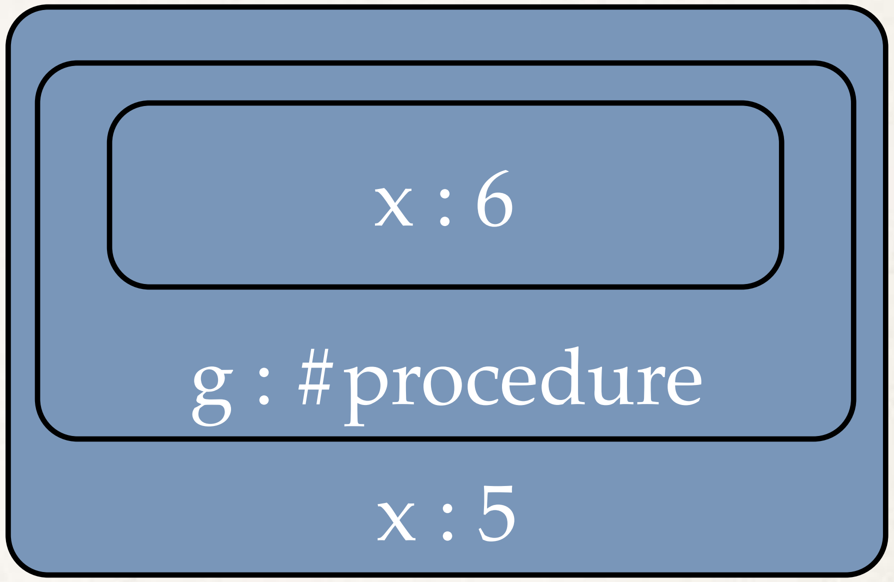
define g



The definition of g *grabs the current environment and its binding of x.*
Any evaluation inside g that needs x will use this copy!

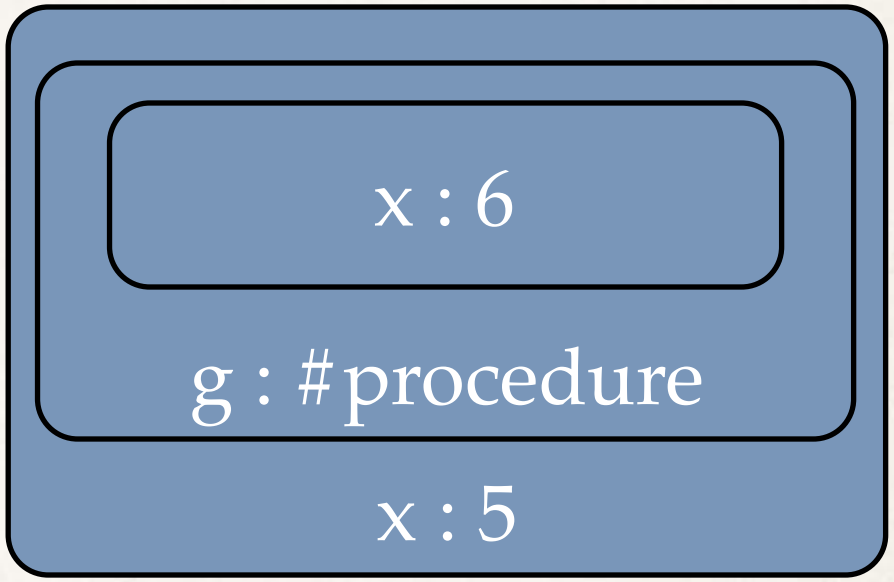
Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))    ← Bind x
    (g 11)))
> (f 6)
17
```



Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))      ← Call g
> (f 6)
17
```



Another example, pictorially

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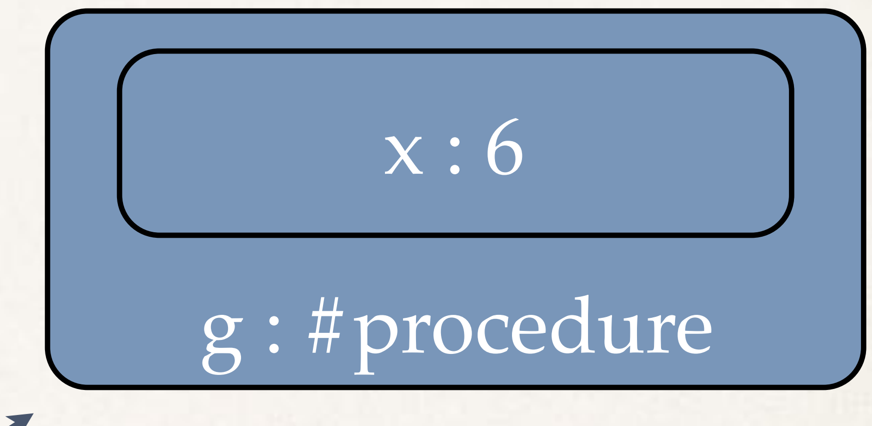
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

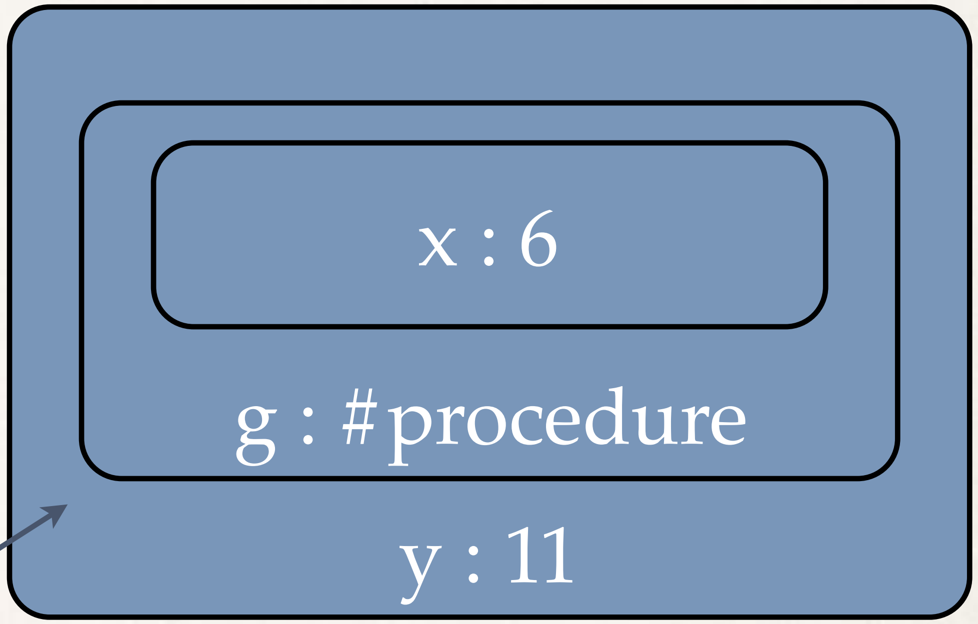


Starts with the definition environment!

Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

bind y

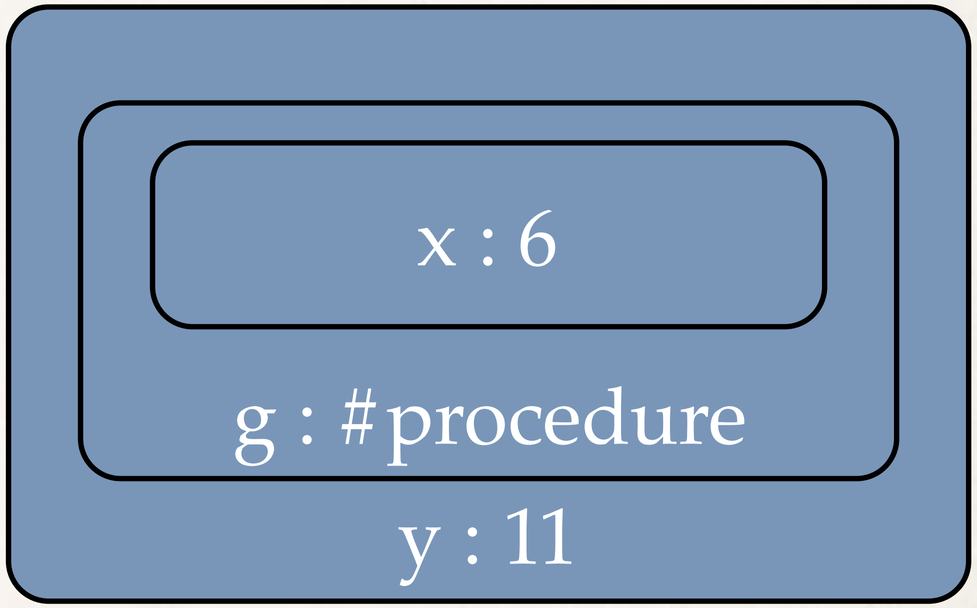


Starts with the definition environment!

Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

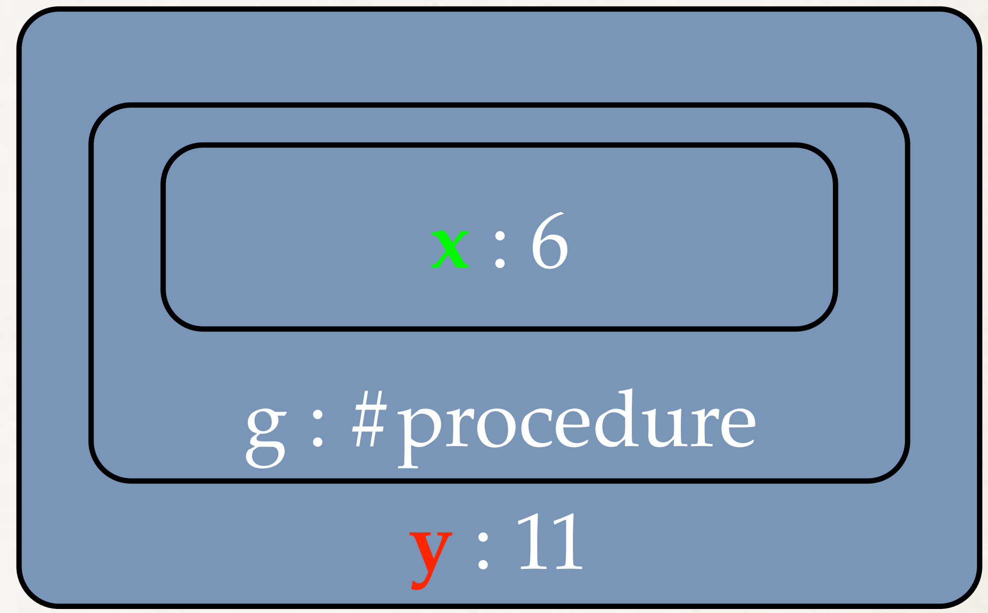
eval



Another example, pictorially

```
(define (f x)
  (define (g y)
    (+ x y))
  (let ((x 5))
    (g 11)))
> (f 6)
17
```

eval



Returns the value 17

Lexical scope and the life of a Scheme function

- "Free" variables in the body of a scheme function are assigned values from the environment in which the function was *defined*. This makes reasoning about their values easy, they are always drawn from the same environment!

f is defined

a:10

```
(define (f x)
  (+ x a))
```

f is applied

a:100

`(f 0)` = 10

a:1000

`(f 0)` = 10

…

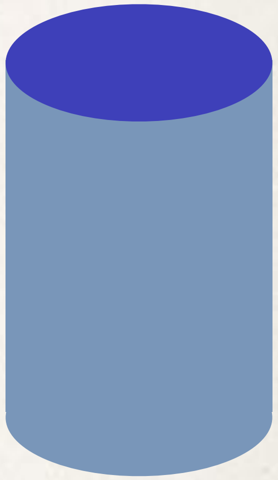
With lexical scope...

Functions behave like functions!

- "Free" variables in the body of a scheme function are assigned values from the environment in which the function was *defined*. This makes reasoning about their values easy, they are always drawn from the same environment!
- When definition environments never change, we are using **functional programming**.
- There are some cool things you can do by fiddling with definitional environments *after a function has been defined*. Part III of the course.

An example of environment semantics…

- Consider the following definition for computing the volume of a cylindrical solid of height h and radius r.



```
(define (volume h r) (* 3.1415 r r h))
```

- Evaluation can be understood in terms of an environment:

```
(volume 8 2)
```

`(* 3.1415 r r h)` in the environment `h:8` `r:2`

A more nuanced definition of `define`

- The `define` command forcibly adds a new binding in the current environment. This is one of the few ways that an environment can change in SCHEME.

- Thus

```
> (define a 6)
> (define a 60)
> (define (f x)
    (define x 5)
    (+ x a))
> (f a)
65
```

a:6

a:60

x:60

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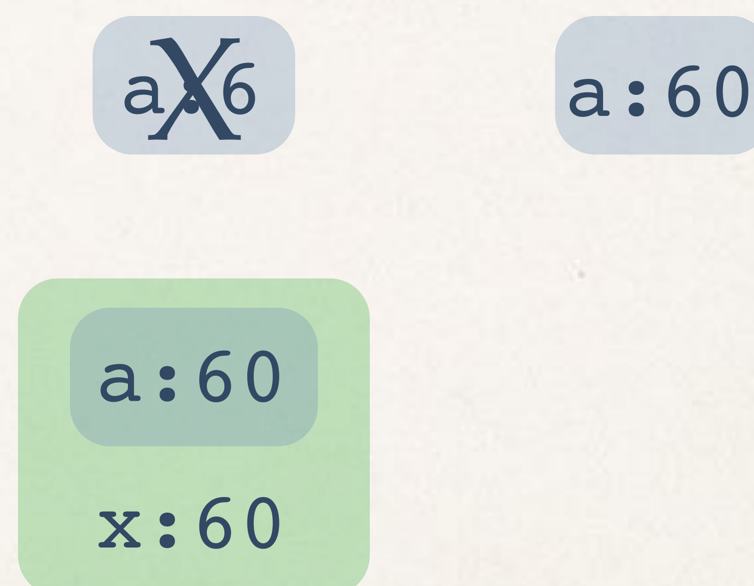
- Thus

```
> (define a 6)
> (define a 60)
> (define (f x)
    (define x 5)
    (+ x a))
> (f a)
65
```



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    (define x 5)
    (+ x a))
> (f a)
65
```

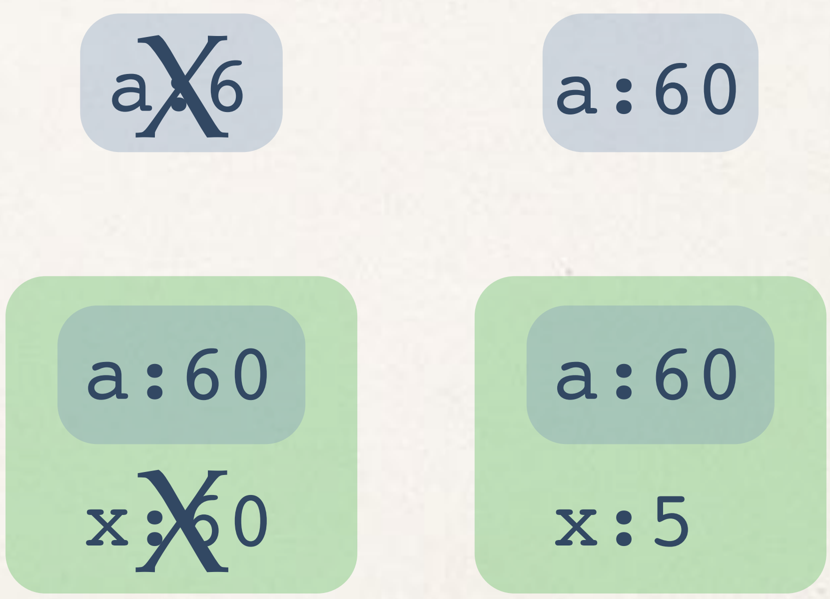

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- The `define` command forcibly adds a new binding in the current environment. This is one of the few ways that an environment can change in SCHEME.

- Thus

```
> (define a 6)
> (define (f x)
    (+ x a))
> (f 0)
6
> (define a 60)
> (f 1)
61
```

```
a:6
```

A more nuanced definition of `define`

- The `define` command forcibly adds a new binding in the current environment. This is one of the few ways that an environment can change in SCHEME.

- Thus

```
> (define a 6)
> (define (f x)
    (+ x a))
> (f 0)
6
> (define a 60)
> (f 1)
61
```

a:6

a:6

x:0

Environment created by (f 0)

A more nuanced definition of `define`

- The `define` command forcibly adds a new binding in the current environment. This is one of the few ways that an environment can change in SCHEME.

- Thus

```
> (define a 6)
> (define (f x)
    (+ x a))
> (f 0)
6
> (define a 60)
> (f 1)
61
```

~~a:6~~

a:60

a:6

x:0

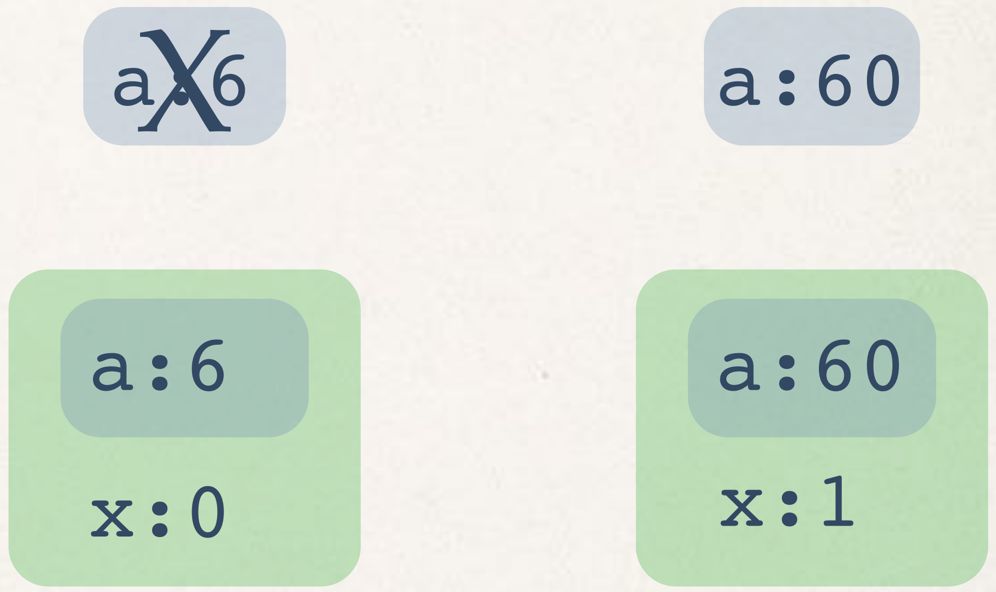
Environment created by (f 0)

A more nuanced definition of `define`

- The `define` command forcibly adds a new binding in the current environment. This is one of the few ways that an environment can change in SCHEME.

- Thus

```
> (define a 6)
> (define (f x)
    (+ x a))
> (f 0)
6
> (define a 60)
> (f 1)
61
```



Environment created by (f 0)

Environment created by (f 1)

Environment clutter and local functions

- Consider the definition

```
(define (square a)     (* a a))
(define (sqrt-converge x a b)
  (let ((avg (/ (+ a b) 2)))
    (if (< (abs (- a b)) .000001)
        a
        (if (> (square avg) x)
            (sqrt-converge x a avg)
            (sqrt-converge x avg b))))))
(define (new-sqrt x) (sqrt-converge x 1 x))
```

- We wished to define `new-sqrt`, but introduced many other functions into the environment. What if someone clobbers them or, in general, they clash with other functions?

Environmental protection…leave behind only what you intended

- Making internal structure (e.g., sqrt-converge) available to the user is dirty, provides opportunities for error.

- To avoid this, we can place the definitions inside `new-sqrt`:

```
(define (new-sqrt-i x)
  (define (square z) (* z z))
  (define (sqrt-converge t a b)
    (let ((avg (/ (+ a b) 2)))
      (if (< (abs (- a b)) .000001)
          a
          (if (> (square avg) t)
              (sqrt-converge t a avg)
              (sqrt-converge t avg b)))))
  (sqrt-converge x 1 x))
```

Environment

square: fn
sqrt-converge: fn
x: value

Scope

- Note, also, that `sqrt-converge` is called with `x`. If it is defined inside the environment of `new-sqrt`, `x` already appears in the environment! Thus we can further simplify the definition:

```
(define (new-sqrt-i x)
  (define (square z) (* z z))
  (define (sqrt-converge a b)
    (let ((avg (/ (+ a b) 2)))
      (if (< (abs (- a b)) .000001)
          a
          (if (> (square avg) x)
              (sqrt-converge a avg)
              (sqrt-converge avg b)))))
  (sqrt-converge 1 x))
```

How can we understand this? In terms of environment semantics

```
(define (new-sqrt-i x)
  ...
  (define (sqrt-converge a b)
    ...
     (if (> (square avg) x)
         ...))
  (sqrt-converge 1 x))
```

```
(new-sqrt-i 6)
```

Body:

```
(define (sqrt-converge a b)
  ...
   (if (> (square avg) x)
       ...))
(sqrt-converge 1 x))
```

- Consider the definition.
- Consider a call to `new-sqrt-i`.
- Creates an environment
  x:6
- The define is evaluated in this environment,
  x:6, `sqrt-converge`:fn

How can we understand this? In terms of environment semantics

```
(define (new-sqrt-i x)
  ...
  (define (sqrt-converge a b)
    ...
     (if (> (square avg) x)
           ...))
  (sqrt-converge 1 x))
```

```
(new-sqrt-i 6)
```

Body:

```
(define (sqrt-converge a b)
                              6
  ...
   (if (> (square avg) x)
         ...))
(sqrt-converge 1 x))
```

- Consider the definition.
- Consider a call to `new-sqrt-i`.
- Creates an environment

  x:6
- The define is evaluated in this environment,

  x:6, `sqrt-converge`:fn

How can we understand this? In terms of environment semantics

```
(define (new-sqrt-i x)
  ...
  (define (sqrt-converge a b)
    ...
     (if (> (square avg) x)
         ...))
  (sqrt-converge 1 x))
```

```
(new-sqrt-i 6)
```

Body:

```
(define (sqrt-converge a b)
                              6
  ...
   (if (> (square avg) x)
                   6
       ...))
(sqrt-converge 1 x))
```

- Consider the definition.
- Consider a call to `new-sqrt-i`.
- Creates an environment
  x:6
- The define is evaluated in this environment,
  x:6, `sqrt-converge`:fn

Example: Testing Primality

- Recall that a (whole, positive) number n is composite if it has a divisor other than 1 and n. Otherwise, it is *prime*.*

- What is a natural way to determine if a number p is composite? **Test to see if it has a divisor d, 1 < d < n.**

- Note: It's not obvious how to define `(composite p)` in terms of `(composite k)` for smaller k: we will need to introduce some other functions to help structure our computation.

Thus, n is composite if:
Some number between 2 and n-1 divides it evenly.

Idea: Let's say that a number is k-*smooth* if it has a divisor d so that $1 < d \leq k$.

Thus, n is composite if:
It is (n-1)-smooth.

A recursive expansion of... *being smooth*

- Note that n is k-*smooth* if k divides n or it is (k-1)-*smooth*. Why?
- Thus:

```
(define (divides a b) (= (modulo b a) 0))
(define (smooth n k)
  (and (>= k 2)
       (or (divides k n)
           (smooth n (- k 1)))))

(define (composite n) (smooth n (- n 1)))
```

Note: This uses **short-circuited evaluation** of `AND` and `OR`. How?

Without short-circuit ?

```
(define (divides a b) (= (modulo b a) 0))
(define (smooth n k)
  (if (< k 2)
      #f
      (if (divides k n)
          #t
          (smooth n (- k 1)))))

(define (composite n) (smooth n (- n 1)))
```

You can optimize this...

- With one possible exception, divisors come in pairs. If n has a divisor d for which 1 < d < n, then it has one
  - that is no more than n/2.
  - that is no more than sqrt(n).
- Why? Suppose that d * d′ = n. If both d and d′ were larger than sqrt(n), their product would be larger than n!
- Thus:

rounds down

```
(define (composite n) (smooth n (floor (sqrt n))))
```

Example: Testing primality

- As a number is prime when it is not composite,

```
(define (divides a b) (= (modulo b a) 0))
(define (smooth n k)
  (and (>= k 2)
       (or (divides k n)
           (smooth n (- k 1)))))
(define (isprime p)
  (not (smooth p (floor (sqrt p)))))
```

- Note: `(smooth k)` returns `#t` if there is a divisor of `n` between `2` and `k`.

- We can hide the definitions of `divides` and `smooth`...

Nesting the environments...

- As `divides` and `smooth` are only used inside `prime`:

```
(define (prime n)
  (define (divides a b) (= (modulo b a) 0))
  (define (smooth k)
    (and (>= k 2)
         (or (divides k n)
             (smooth (- k 1)))))
  (not (smooth (floor (sqrt n)))))
```

- INTERESTING SIMPLIFICATION: `(smooth k)` no longer needs to be passed n, it exists in the defining environment!

- Let's trace the environments...

The inner environment during a call to `(prime 6)`

If we make the invocation:

```
(prime 6)
```

the body is evaluated in an environment where n: 6.

```
(define (divides a b) (= (modulo b a) 0))
(define (smooth k)
  (and (>= k 2)
       (or (divides k n)
           (smooth (- k 1)))))
(not (smooth (floor (sqrt n)))))
```

Thus, `smooth` is defined in an environment where n = 6.

Notice that divides is always called with b = n. Further simplification...

One further simplification of the primality tester

```
(define (prime n)
  (define (divisor a) (= (modulo n a) 0))
  (define (smooth k)
    (and (>= k 2)
         (or (divisor k)
             (smooth (- k 1)))))
  (not (smooth (floor (sqrt n)))))
```

`divides` was always called with b = n.

Reading

- With some adaptations and omissions, the previous slides cover material from Section 1.2 of SICP.
- You might find it interesting to look over the Revised^5 Report on Scheme, a definition of the language. (Posted on the website.)

In SCHEME, functions are *first class objects*.

- Functions can be passed as arguments: Consider the following definition:

A function   A positive integer

```
(define (sum f n)
    (if (= n 0)
        (f 0)
        (+ (f n) (sum f (- n 1)))))
```

This computes the sum: f(0) + f(1) + ... + f(n)
Both f and n are passed as arguments.

Then…

- To compute the sum of the first n squares: $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

```
> (define (square x) (* x x))
> (sum square 5)
55
```

- To compute the sum of the first n cubes: $0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

```
> (define (third x) (* x x x))
> (sum third 5)
225
```

Another example. A tool for partial power series

- A power-series expander. `term` is a function that should return the coefficient of x^k.

```
(define (power-series x term k)
    (if (< k 0)
        0
        (+ (* (term k) (expt x k))
           (power-series x term (- k 1)))))
```

- Then `(power-series x term k)` should return:

$$\mathtt{term}(0) + \mathtt{term}(1)\, x + \mathtt{term}(2)\, x^2 + \ldots + \mathtt{term}(k)\, x^k .$$

Example power series: sine

$$\sin(x) \approx 1 \cdot x + 0 \cdot x^2 - \frac{1}{3!} \cdot x^3 + 0 \cdot x^4 + \frac{1}{5!} \cdot x^5 + 0 \cdot x^6 - \frac{1}{7!} \cdot x^7 + \cdots$$

- The *coefficients* of the even terms are always 0
- $$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

cos(x) has similar approximation
1 - x^2/2! + x^4/4! - x^6/6! + …

Generating partial power series

- Sin and Cos: Setting the stage

```
(define (fact n) (if (= n 0)
                     1
                     (* n (fact (- n 1)))))
```

- The term definitions:

```
(define (sin-term t)
  (if (odd? t)
      (/ (expt -1 (/ (- t 1) 2)) (fact t))
      0))
(define (cos-term t)
  (if (odd? t)
      0
      (/ (expt -1 (/ t 2)) (fact t))))
```

Understanding sin-term

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- The *coefficients* of the even terms are always 0

$$\sin(x) \approx 1 \cdot x + 0 \cdot x^2 - \frac{1}{3!} \cdot x^3 + 0 \cdot x^4 + \frac{1}{5!} \cdot x^5 + 0 \cdot x^6 - \frac{1}{7!} \cdot x^7 + \cdots$$

- term(0) = 0 even
- term(1) = 1
- term(2) = 0 even
- term(3) = - 1/3!
- term(4) = 0 even
- term(5) = + 1/5!
- term(6) = 0 even
- term(7) = - 1/7!

Understanding sin-term

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- The *coefficients* of the even terms are always 0

$$\sin(x) \approx 1 \cdot x + 0 \cdot x^2 - \frac{1}{3!} \cdot x^3 + 0 \cdot x^4 + \frac{1}{5!} \cdot x^5 + 0 \cdot x^6 - \frac{1}{7!} \cdot x^7 + \cdots$$

- term(0) = 0 even
- term(1) = 1
- term(2) = 0 even
- term(3) = - 1/3!
- term(4) = 0 even
- term(5) = + 1/5!
- term(6) = 0 even
- term(7) = - 1/7!

How to get the sign?

It *alternates!*

$$(-1)^{\frac{t-1}{2}}$$

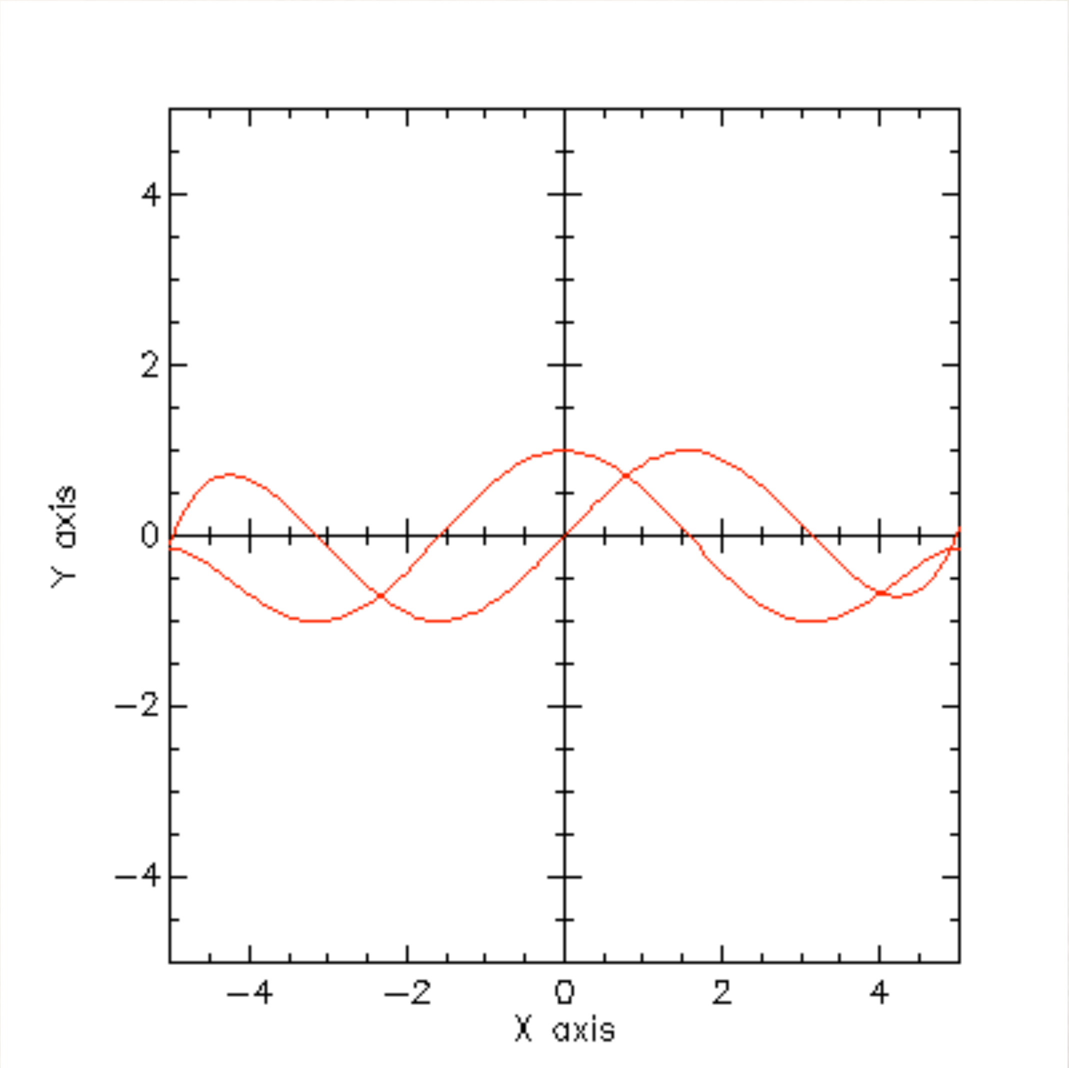
Now the power series are easy to generate

Now we can define functions from the first 10 terms of each power series:

```
(define (sin10 x) (power-series x sin-term 10))
(define (cos10 x) (power-series x cos-term 10))
```

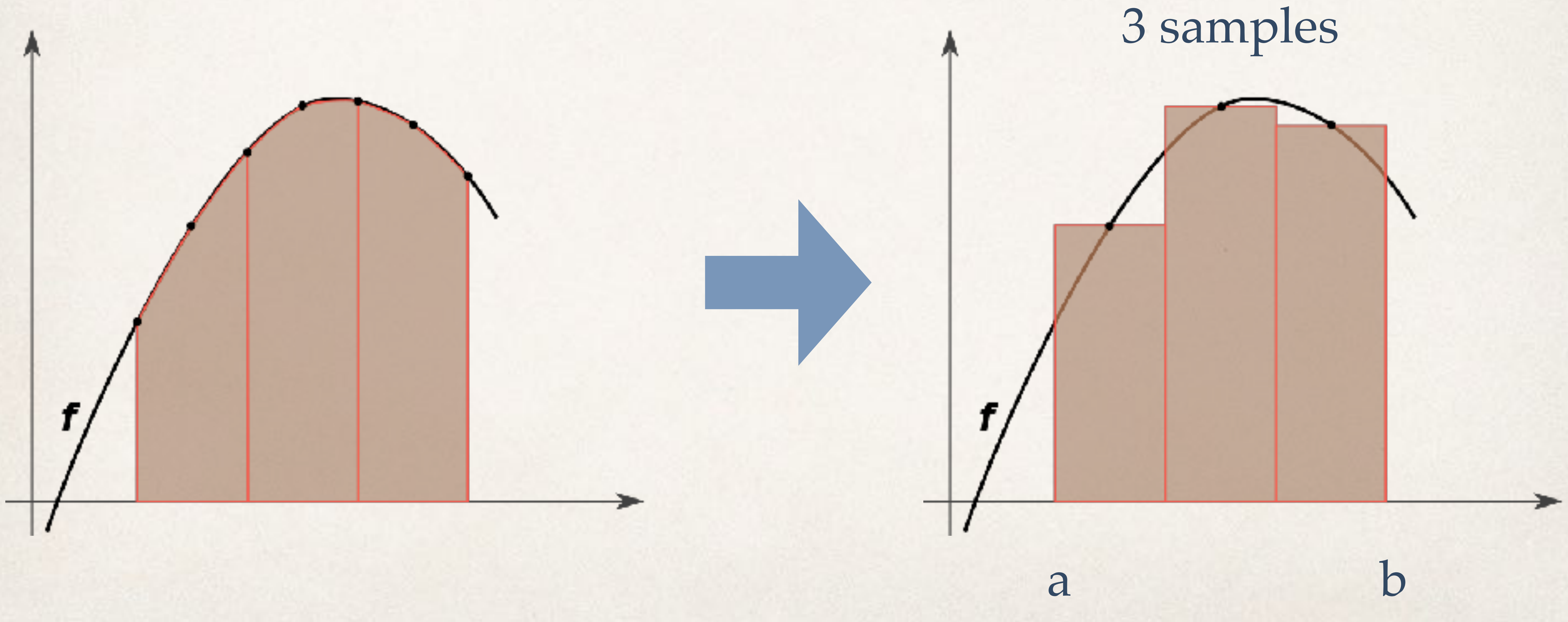
Then, for example,

```
(require plot)
(plot (mix (line sin10)
           (line cos10)))
```

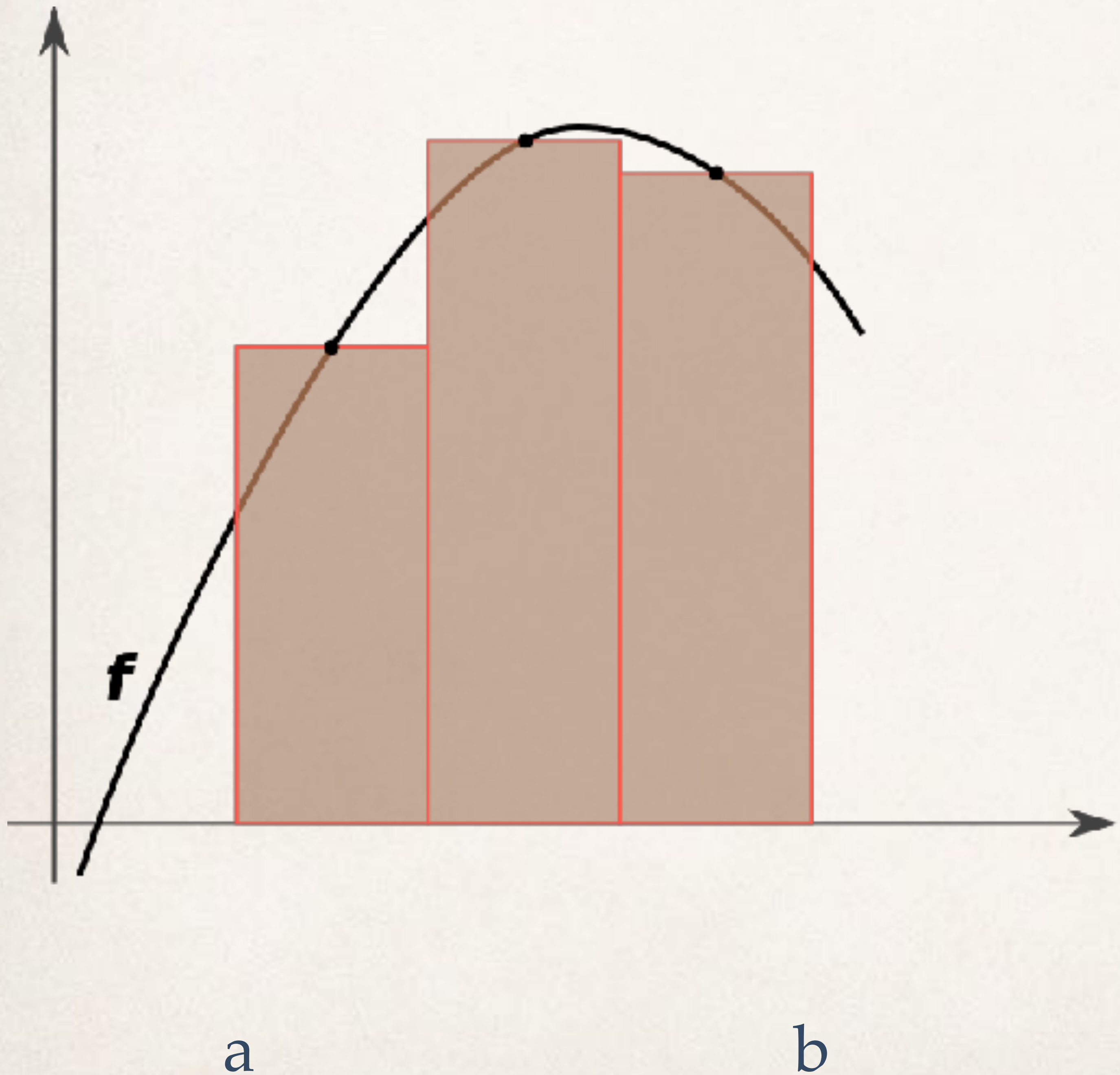


Numeric integration

- To approximate the area under a function, we approximate the area by a sequence of rectangles:



The technical details



- 3 samples, so...
- rectangle width = (b-a)/3 ,
- each sample position is

  (b-a)/6

  from the left side of its rectangle,
- area of each rectangle is

  f(sample) * (b-a)/3 .

Putting our `sum` function to work...a generic integrator

- We wish to approximate the area under f over the interval [a,b] by summing the areas of n equal width rectangles.
- `(sample k)` determines the height of the `k`th rectangle. Since our samples are the "midpoints" of the rectangle, you can check that they are:

```
(define (integrate f a b n)
  (define (sample k)
    (f (+ a (/ (- b a) (* 2 n))
          (* k (/ (- b a) n)))))
  (define (rectangle-area k)
    (* (sample k) (/ (- b a) n)))
  (sum rectangle-area (- n 1)))
```

This function is passed to `sum`

$$a + \frac{b-a}{2n} + \frac{k}{n}(b-a) \qquad k = 0, 1, \ldots, n-1$$

Unnamed functions

- SCHEME has a mechanism for defining functions without names:

  argument   body

  ```
  (lambda (x) (* x x))
  ```

  is the **function** that returns the square of its argument.

- If you wish to sum the values of the first n squares, instead of defining `square` first, you can directly pass the function:

```
> (sum (lambda (x) (* x x)) 10)
385
```

`Define` revisited

- If we enlarge our notion of value to include function values, we can simplify the definition of `define` as an operator that always binds a name to a value.

```
(define (square x) (* x x))
```

is the same as...

```
(define square (lambda (x) (* x x)))
```

`Let` revisited

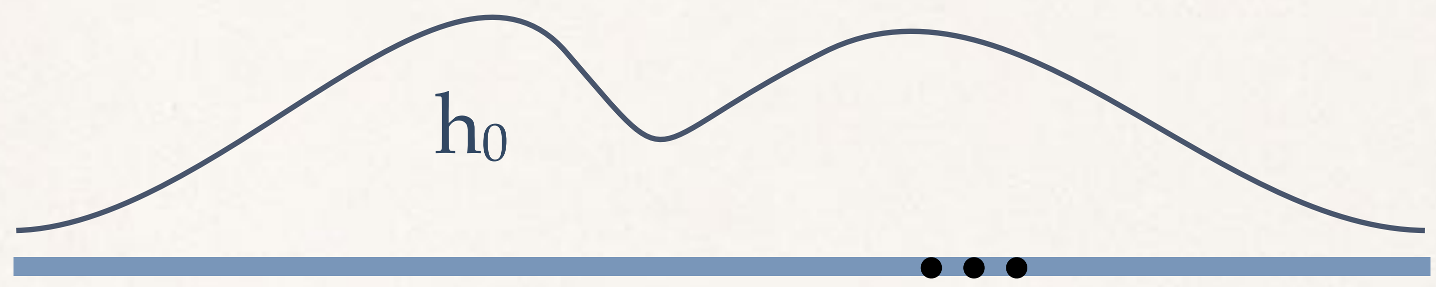
- We can express `let` using lambda and the standard application rule!

```
(let ((x1 <expr1>)
      ...
      (xk exprk>))
  <let-expr>)
```

...is the same as...

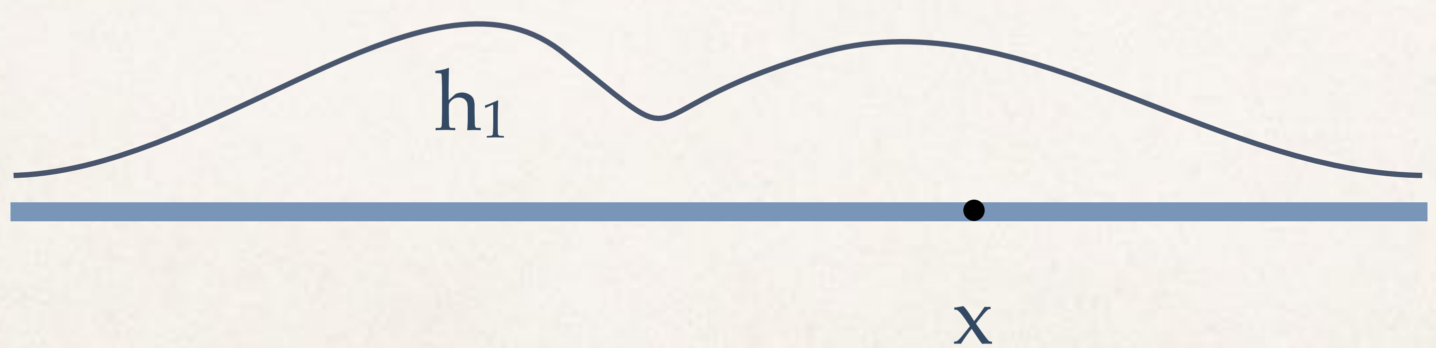
```
((lambda (x1 ... xk) <let-expr>)
 <expr1> ... <exprk>)
```

The heat flow equation



$$h_1(x) = \frac{h_0(x - dx) + h_0(x + dx)}{2}$$

The average of two close points



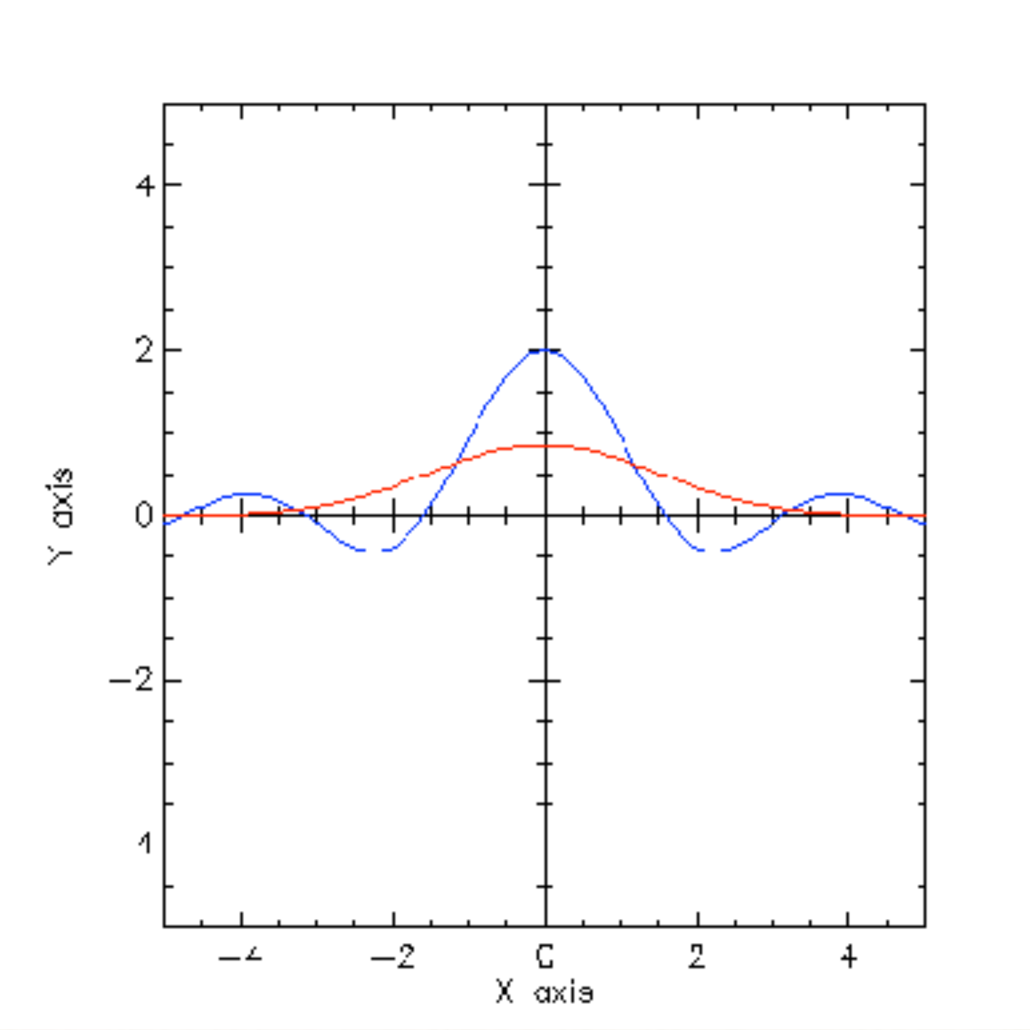
Returning functions as “values”

- The `lambda` form provides an easy way to return a function as a value.

```
(define (heat-flow f dx)
  (lambda (x) (/ (+ (f (+ x dx))
                    (f (- x dx)))
                 2)))

(define (heat-flow-evolve f dx t)
  (if (= t 0)
      f
      (heat-flow (heat-flow-evolve f dx (- t 1)) dx)))
```

Heat flow evolution

```
(define (curve x)
  (/ (sin (* 2 x)) x))

(plot (mix (line curve #:color 'blue)
           (line (heat-flow-evolve
                    curve
                    .5
                    8)
                 #:color 'red)))
```



Reminder about the life and times of an environment…

- Environments contain bindings of variables to values.
- The `define` command destructively adds a binding to an environment.
- There are two ways that new environments are created:
  - During function evaluation.
  - During `let` evaluation.
  - These new environments **always** inherit all bindings from the environment from which they were created: however, the bindings of arguments shadow existing bindings.

(Actually, these are the same internal process)

An example

```
(define (f x)
  (define (g y) (+ x y))
  (define (h x) (+ x (g x)))
  (h (+ x 10)))

(f 100)
```

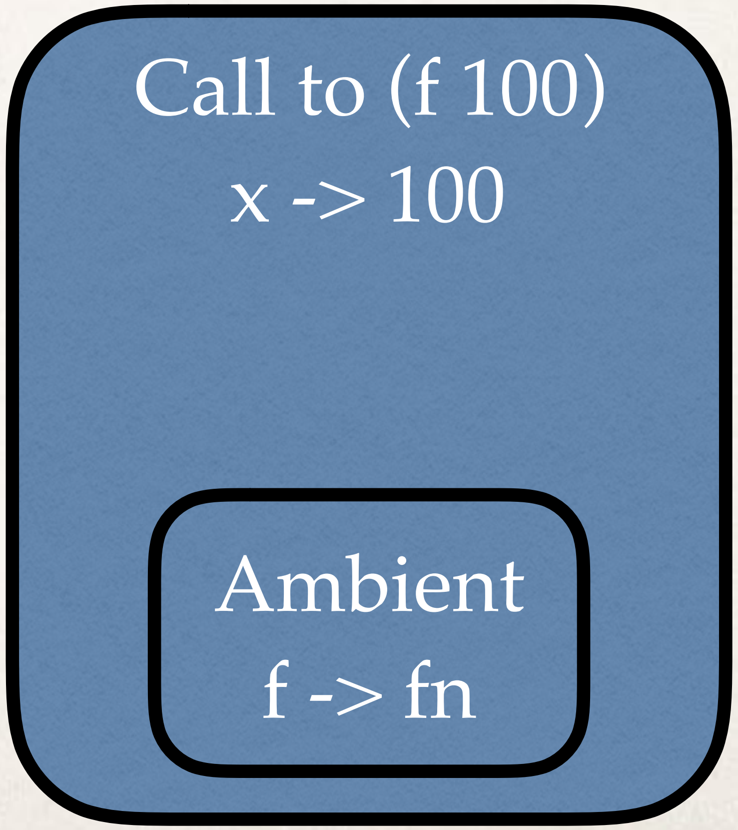
An example

```
(define (f x)
  (define (g y) (+ x y))
  (define (h x) (+ x (g x)))
  (h (+ x 10)))

(f 100)
```

Ambient
f -> fn

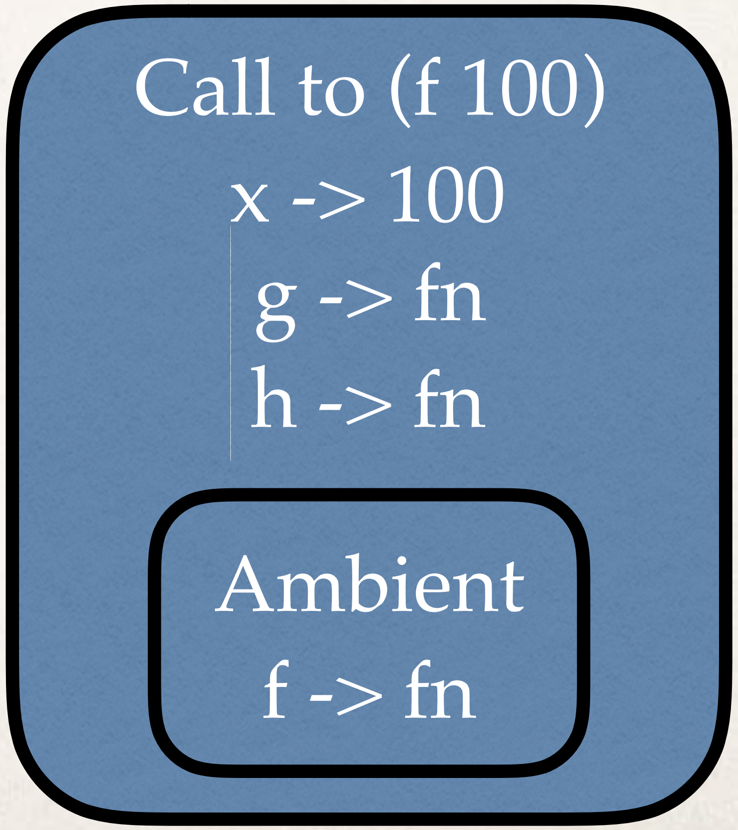
An example

```
(define (f x)
  (define (g y) (+ x y))
  (define (h x) (+ x (g x)))
  (h (+ x 10)))

(f 100)
```



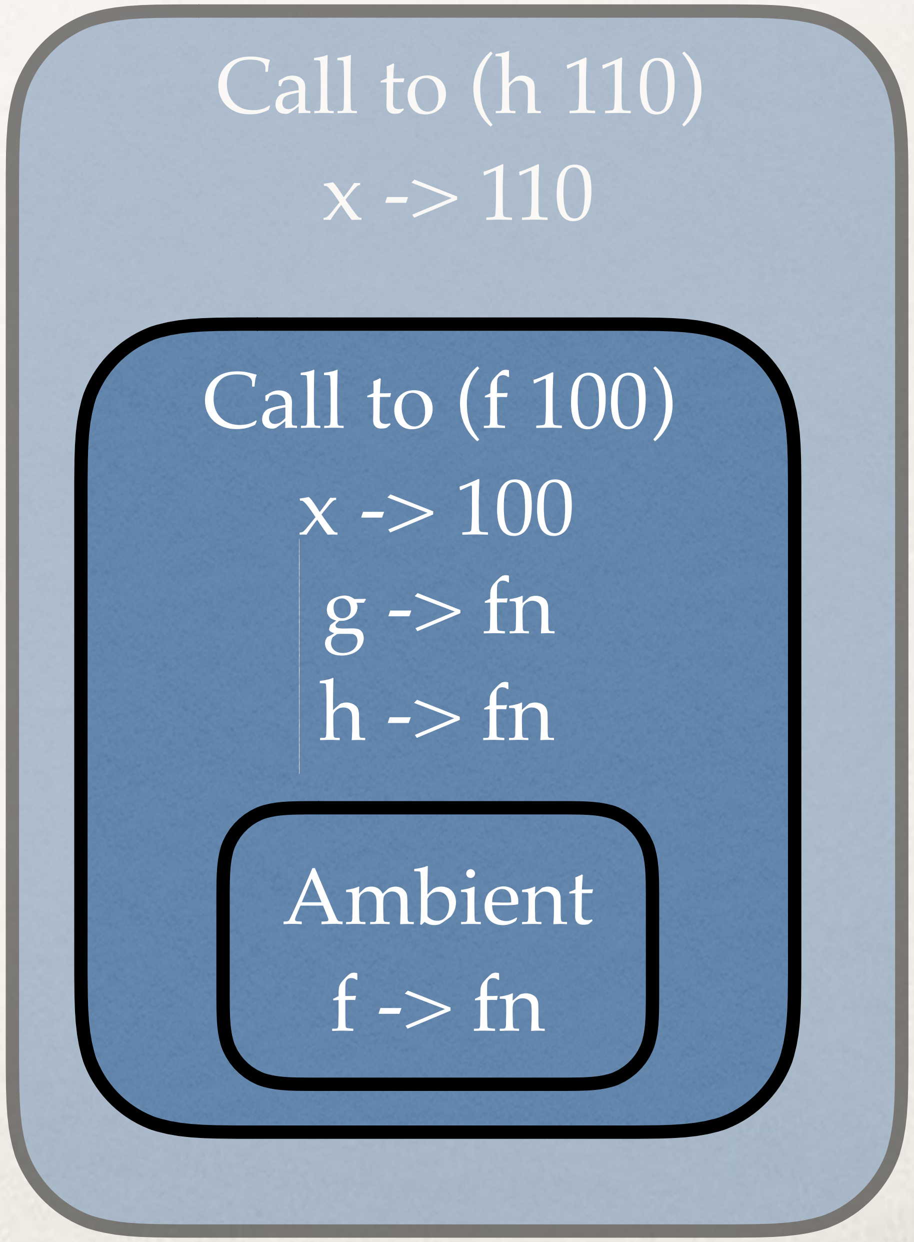
An example

```
(define (f x)
  (define (g y) (+ x y))
  (define (h x) (+ x (g x)))
  (h (+ x 10)))

(f 100)
```



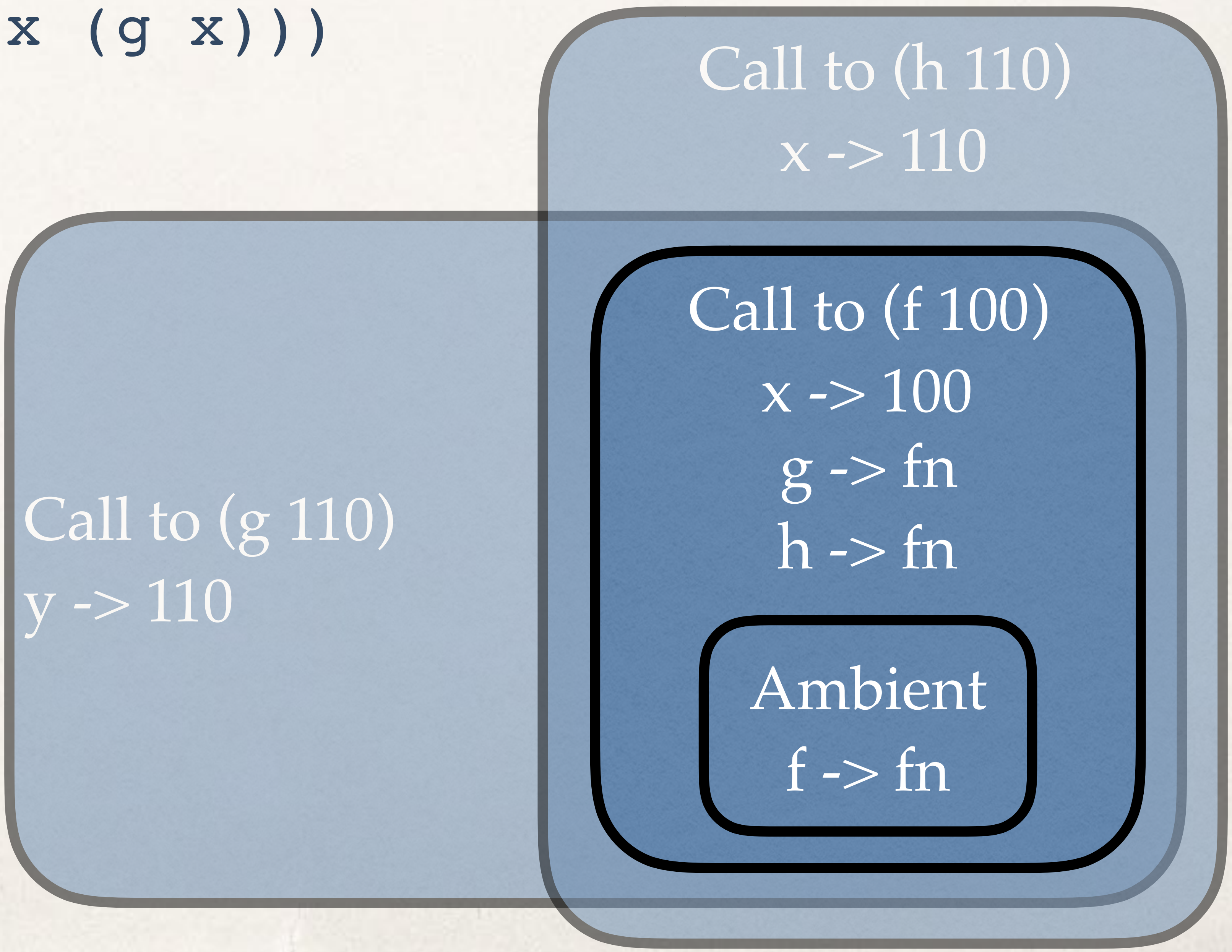
An example

```
(define (f x)
  (define (g y) (+ x y))
  (define (h x) (+ x (g x)))
  (h (+ x 10)))

(f 100)
```



An example

```
(define (f x)
  (define (g y) (+ x y))
  (define (h x) (+ x (g x)))
  (h (+ x 10)))

(f 100)
```

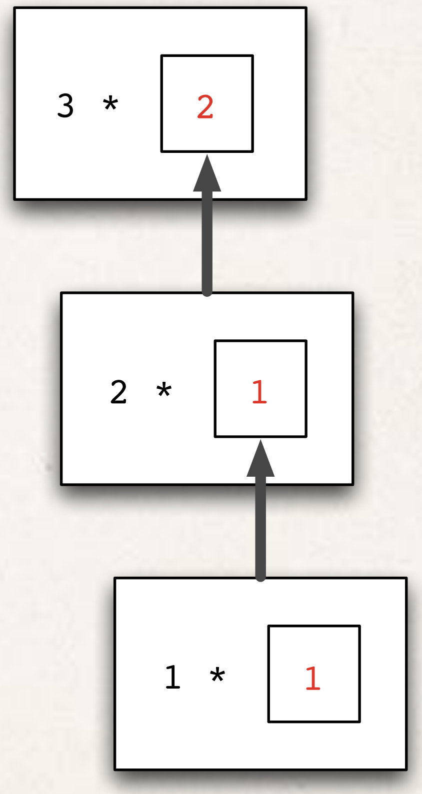


Recursion vs. iteration: "Recursion"

- Consider the familiar factorial function:

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (- n 1)))))
```

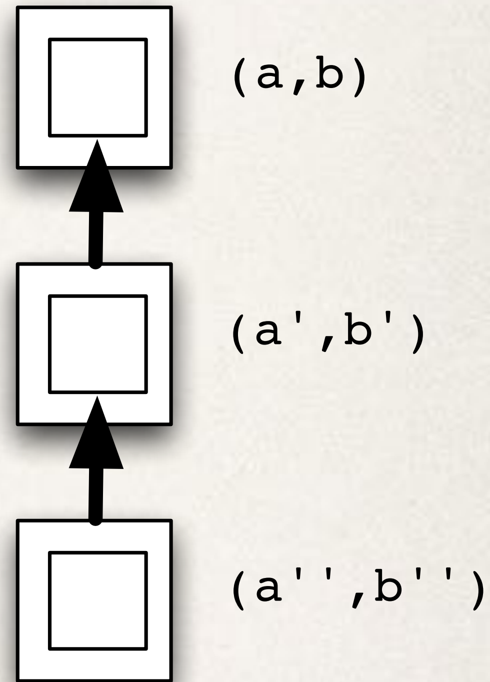
- Let's trace the evaluation of `(fact 3)`. Note how the multiplications
  `(* 3 □)`, `(* 2 □)`, ...
  are *pending* while the recursive calls complete.



Recursion vs. iteration: "Iteration"

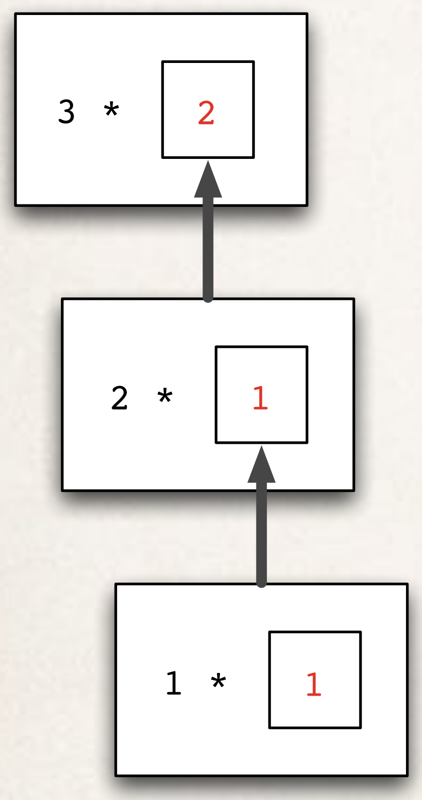
- Consider the `sqrt-converge` function we defined for extracting square roots:

```
(define (sqrt-converge a b)
  (let ((avg (/ (+ a b) 2)))
    (if (< (abs (- a b)) .000001) a
        (if (> (square avg) x)
            (sqrt-converge a avg)
            (sqrt-converge avg b)))))
```



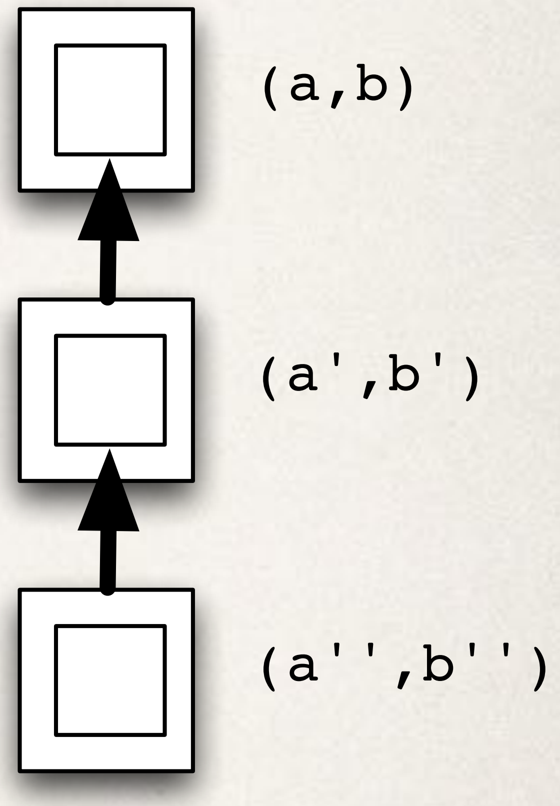
- Note that a call to `(sqrt-converge a b)` typically generates a call to `(sqrt-converge a′ b′)`. In fact, the *result* of `(sqrt-converge a b)` is simply the the *result* of `(sqrt-converge a′ b′)` *without further processing or pending operations*. This is called tail recursion.

Tail recursion requires no memory of pending operations



- In general, *recursion requires memory of the local state* of the calling procedure (including local variables and pending operations) in order to compute a final value.

- Tail recursion (or iteration) *does not require any such memory*. The value of the calling process is simply the value of the subprocess. The caller's environment can be discarded.



Conversion to tail recursion typically requires passing state

- Converting a function definition to a tail recursive call can significantly speed-up computation.
- Recall the original version of factorial:

Pending

```
(define (fact n)
  (if (= n 0) 1
      (* n
         (fact (- n 1)))))
```

- Idea: *Let's send the pending operation along to the subprocess*. The subprocess is responsible for: computing (n-1) factorial *and* multiplying the result by n.

This results in...

- New definition: function that computes a factorial *and* multiplies by a second “accumulator” argument.

```
(define (fact-accumulate n a)
  (if (= n 0) a
      (fact-accumulate (- n 1)
                       (* n a))))
```

Returns: (factorial of n) x (a)

Wrapping this to conceal the internal machinery

```
        (define (fact-tr n)
          (define (fact-accumulate m a)
            (if (= m 0) a
                (fact-accumulate (- m 1)
                                 (* m a))))
          (fact-accumulate n 1))
```

Nothing pending

- Now this is tail recursive.
- Why is **accumulate** an appropriate name for the second argument?

Another example

- Consider computing the sum of the first n numbers in SCHEME.
- Note that $\underbrace{(1 + \ldots + n)}_{\sum_{i=1}^{n} i} = n + \underbrace{(1 + \ldots + (n-1))}_{\sum_{i=1}^{n-1} i}$
- And thus:

```
> (define (number-sum n)
    (if (= n 0)
        0
        (+ n (number-sum (- n 1)))))
> (number-sum 10)
55
```

Recursive or iterative?

Recursive or iterative?

- Recursive, as it needs to remember pending operations.

Recursive or iterative?

- Recursive, as it needs to remember pending operations.

```
(define (number-sum n)
   (if (= n 0)
       0
       (+ n (number-sum (- n 1)))))
```

Can we make it iterative?

- What sort of state do we need to bring along?

Can we make it iterative?

- What sort of state do we need to bring along?

```
(define (sum-iter n a)
  (if (= n 0)
      a
      (sum-iter (- n 1)(+ a n))))
```

```
(sum-iter 10 0) => 55
```

Can we make it iterative?

- Package the iterative function to hide the extra parameter

Can we make it iterative?

- Package the iterative function to hide the extra parameter

```
(define (sum-i n)
  (define (sum-iter n a)
    (if (= n 0)
        a
        (sum-iter (- n 1)(+ a n))))
  (sum-iter n 0))
```

Another example: The Fibonacci numbers

- The *Fibonacci numbers* are defined by the rule:

$$F_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

- Note, then, that the sequence F_0, F_1, F_2, ... is

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

each is the sum of the previous two.

The Fibonacci numbers in SCHEME

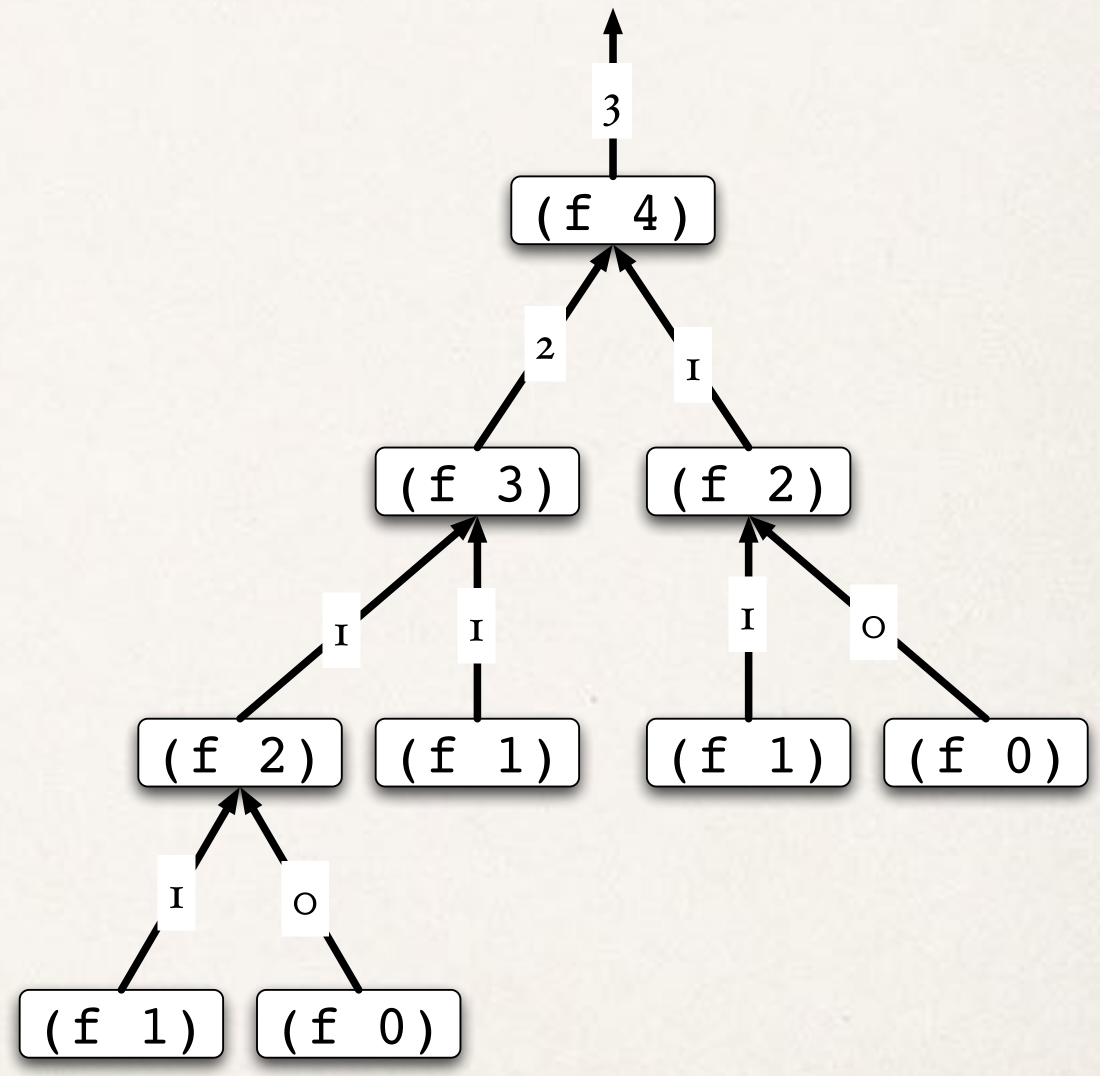
- As with the factorial function, we can naturally capture this definition in SCHEME.

```
(define (fib n)
        (cond ((= n 0) 0)
              ((= n 1) 1)
              ((> n 1) (+ (fib (- n 1))
                          (fib (- n 2))))
        )
)
```

- Notice, as with `factorial`, how closely the SCHEME definition can mirror the mathematical definition.

The Fibonacci *evaluation tree*

- The Fibonacci function gives rise to an "**evaluation tree**" as shown. Here each node returns the sum of the value of its children.
- Note that some "sub"-problems are evaluated many times.
- Question: How many times is (f 1) evaluated, in total?



Recursive or iterative?

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- Recursive, as it needs to remember pending operations.

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```
(define (fib n)
   (cond ((= n 0) 0)
         ((= n 1) 1)
         (else
          (+ (fib (- n 1))(fib (- n 2))))))
```

Can we make it into an iterative process?

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- If you were calculating Fibonacci numbers by hand, how would you do it?

What state do we need to compute fib(n)?

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- We need the previous 2 fibs.

Let's bring them along!

- Like with sum, we use parameters:

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- Like with sum, we use parameters:

```
(define (fib-iter n a b)
   (if (= n 0)
       a
       (fib-iter (- n 1) b (+ a b))))

(fib-iter n 0 1) => same as (fib n)
```

Tracing (fib-iter 5 0 1)

- Using this code:

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  (if (= n 0)
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(fib-iter n 0 1)
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(fib-iter 5 0 1)
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(fib-iter n 0 1)
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(fib-iter n 0 1)
```

```
(fib-iter 5 0 1)
```

```
(fib-iter 4 1 1)
```

Tracing (fib-iter 5 0 1)

- Using this code:

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(fib-iter 5 0 1)
```

```
(fib-iter 4 1 1)
```

```
(fib-iter 3 1 2)
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Tracing (fib-iter 5 0 1)

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```
(fib-iter 5 0 1)
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```
(fib-iter 4 1 1)
```

```
(fib-iter 3 1 2)
```

```
(fib-iter 2 2 3)
```

```
(fib-iter 1 3 5)
```

```
(fib-iter 0 5 8)
```

Tracing (fib-iter 5 0 1)

- Using this code:

```
(define (fib-iter n a b)
  (if (= n 0)
      a
      (fib-iter (- n 1)
                b
                (+ a b))))

(fib-iter n 0 1)
```

```
(fib-iter 5 0 1)
(fib-iter 4 1 1)
(fib-iter 3 1 2)
(fib-iter 2 2 3)
(fib-iter 1 3 5)
(fib-iter 0 5 8)
5
```

Package fib-iter:

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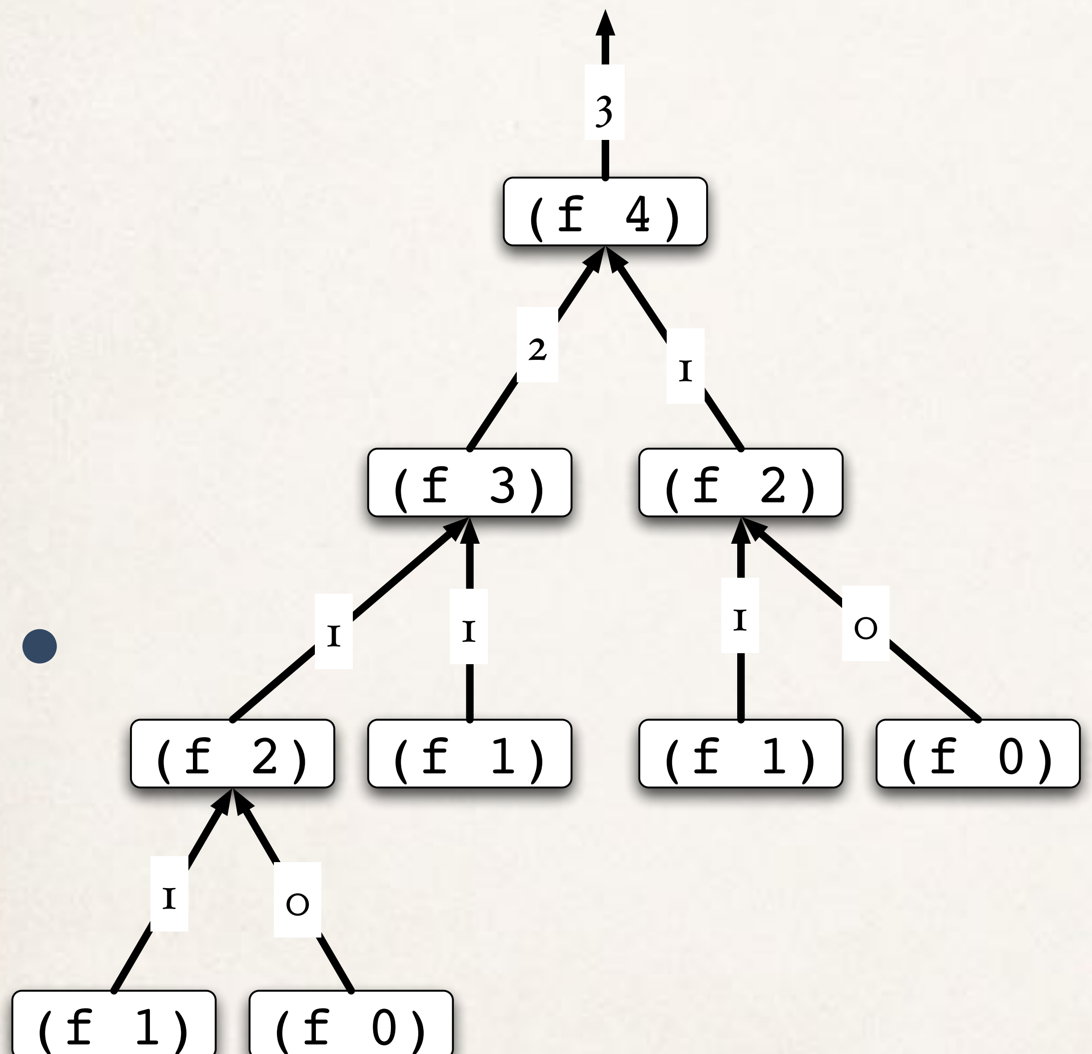
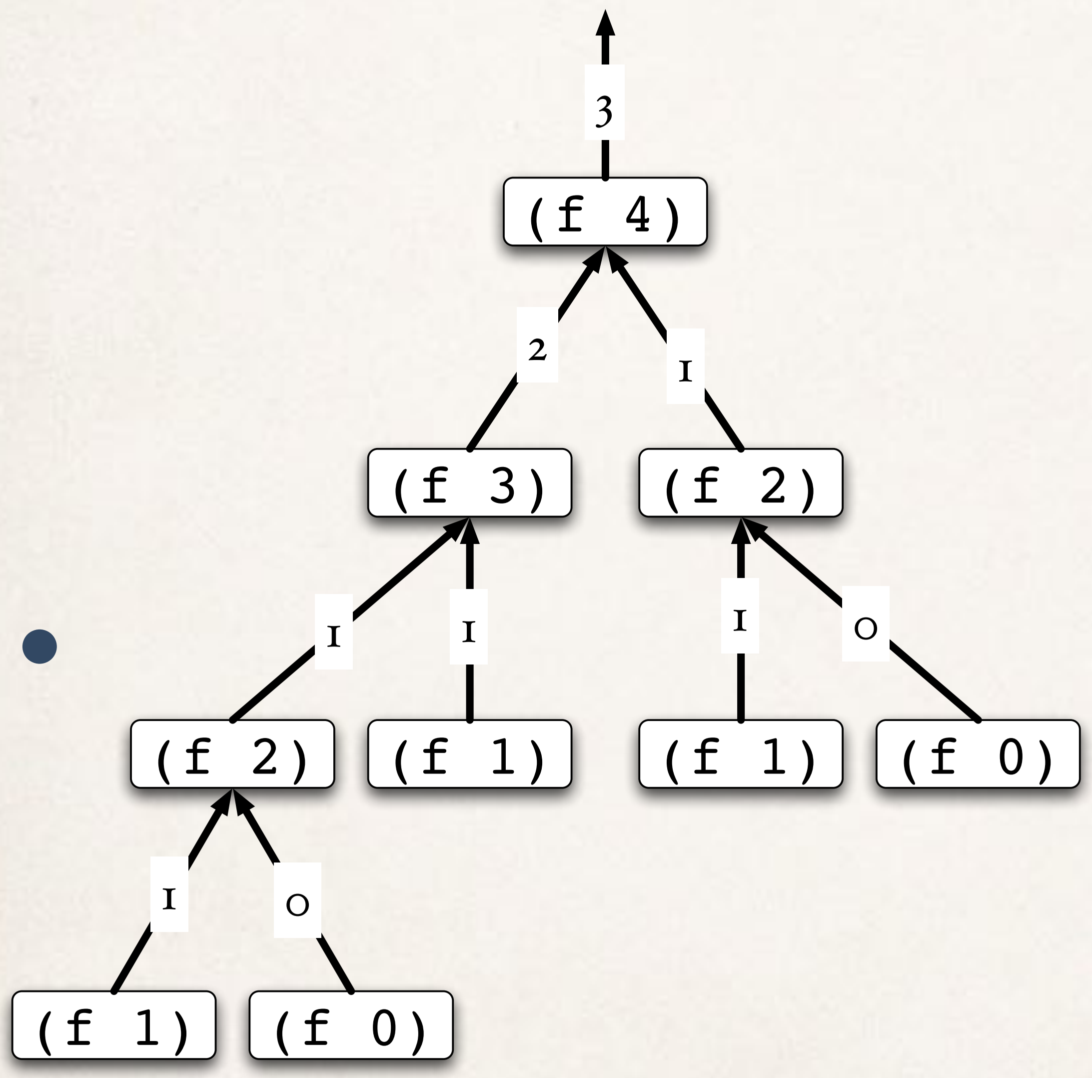
```
(define (fib-i n)
  (define (fib-iter n a b)
    (if (= n 0)
        a
        (fib-iter (- n 1) b (+ a b))))

  (fib-iter n 0 1))
```

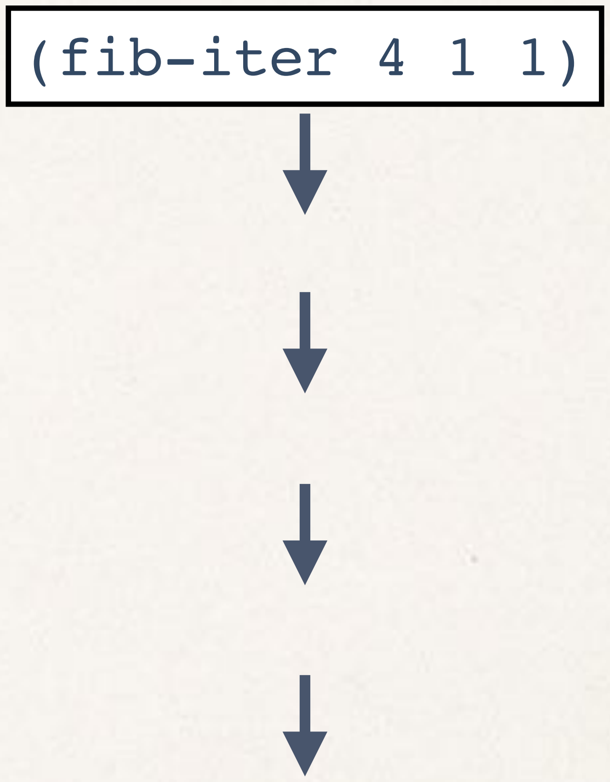
Time and space savings?



Time and space savings?

•





Time and space savings?

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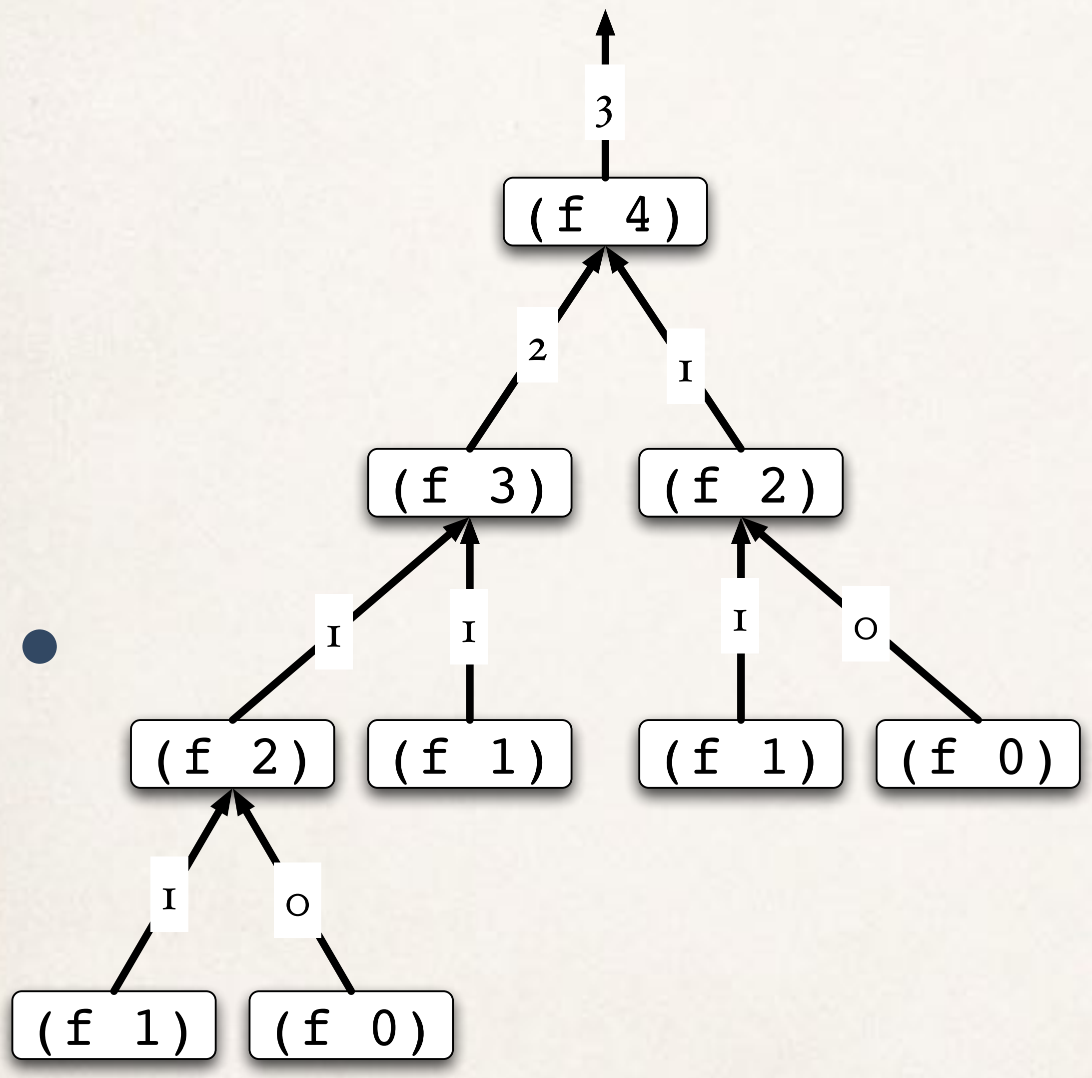
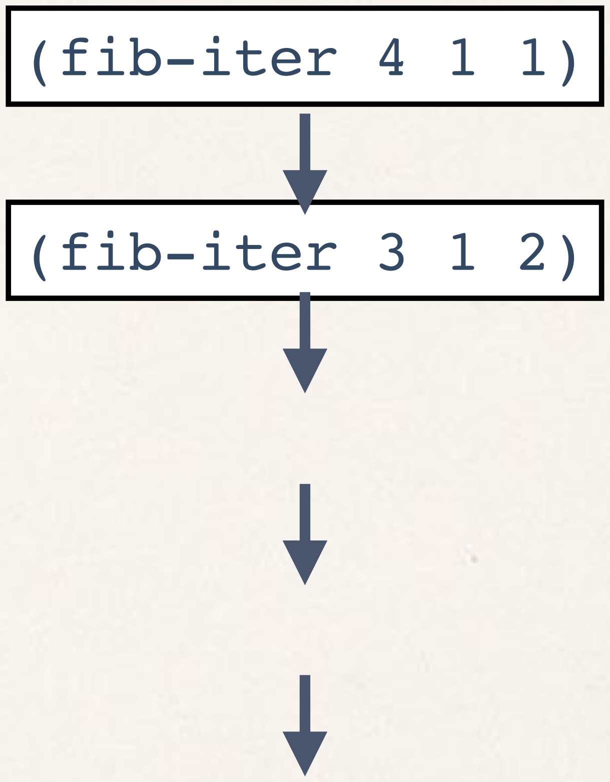
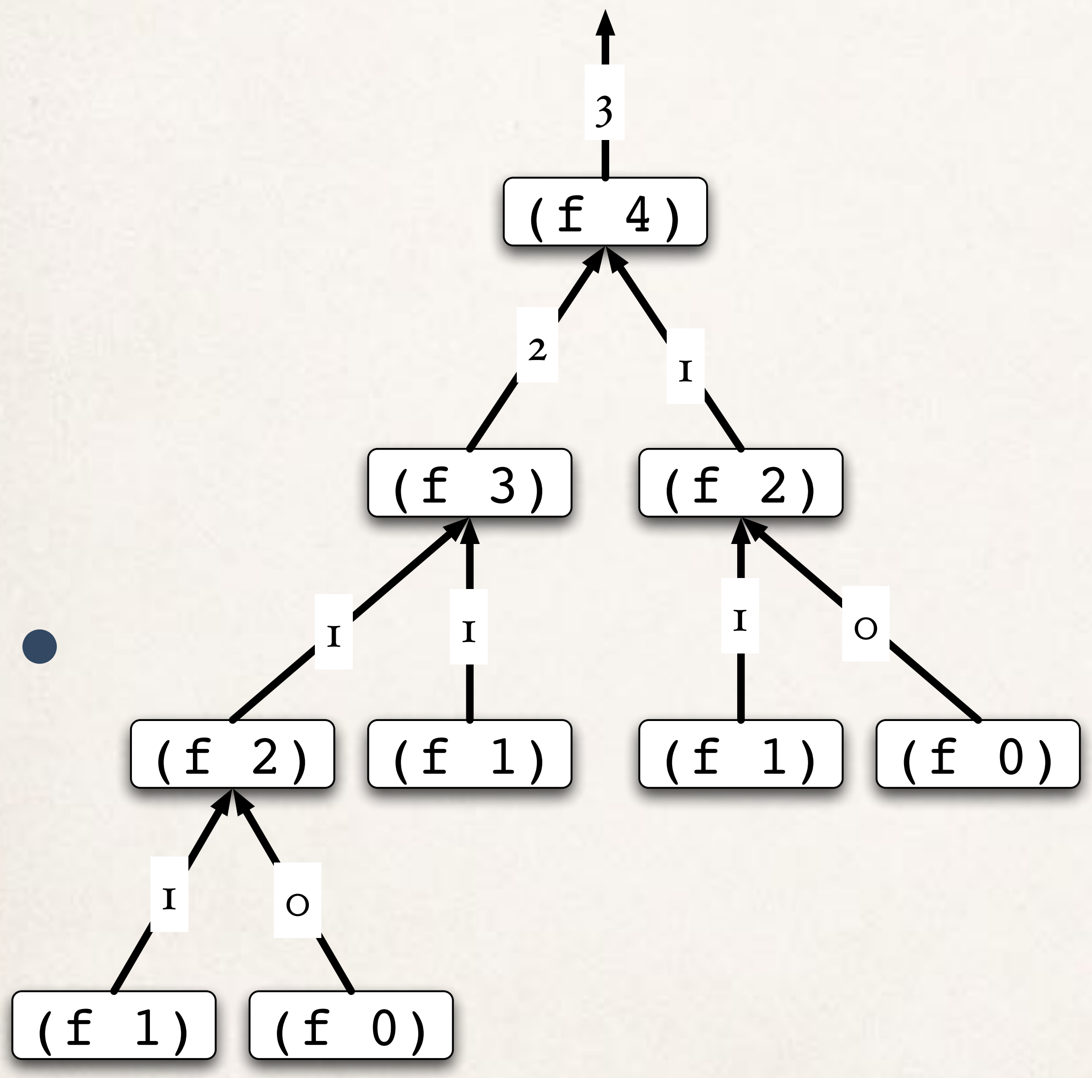
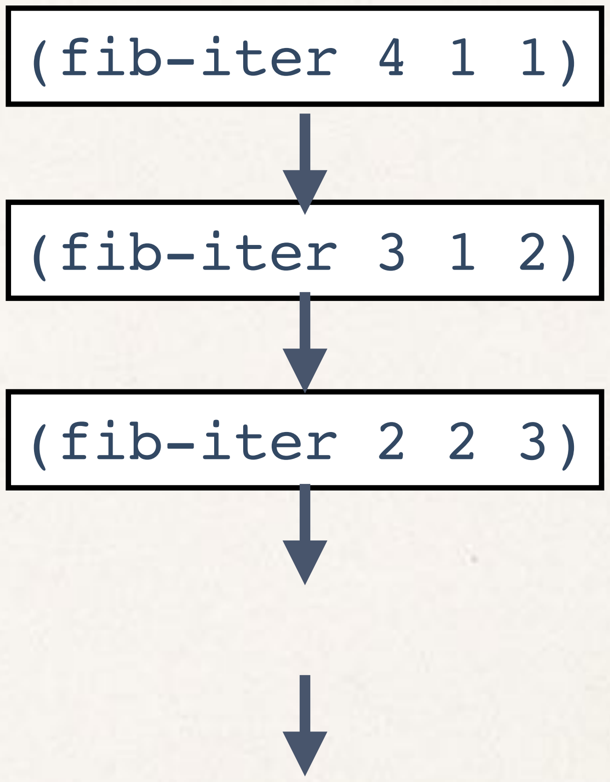




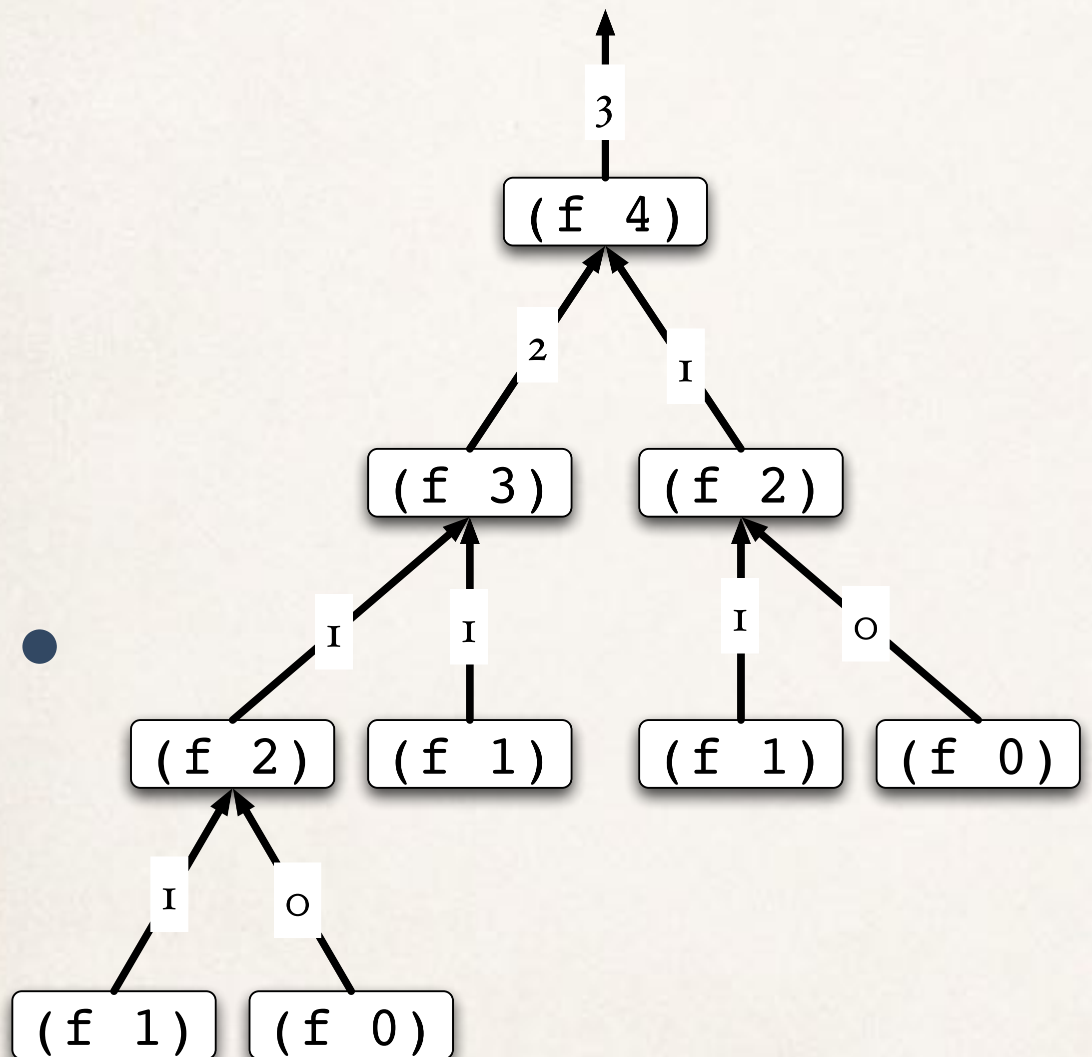
Time and space savings?

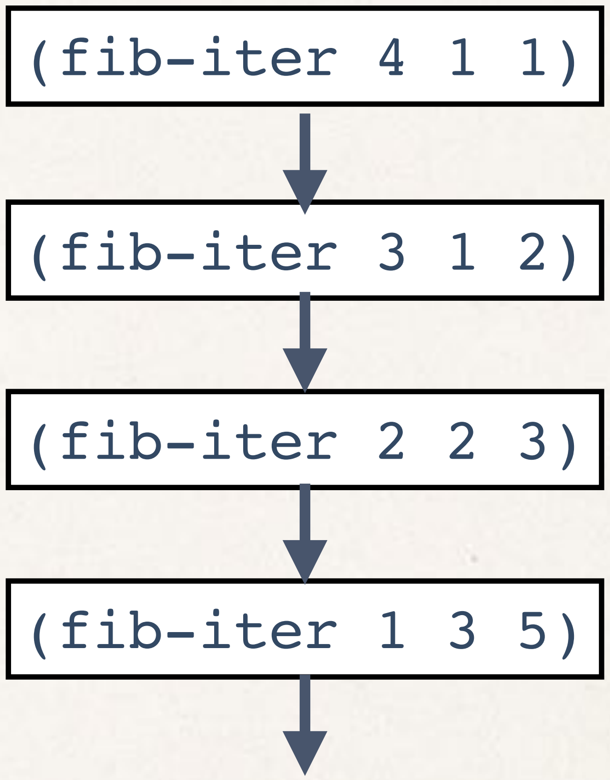
- 



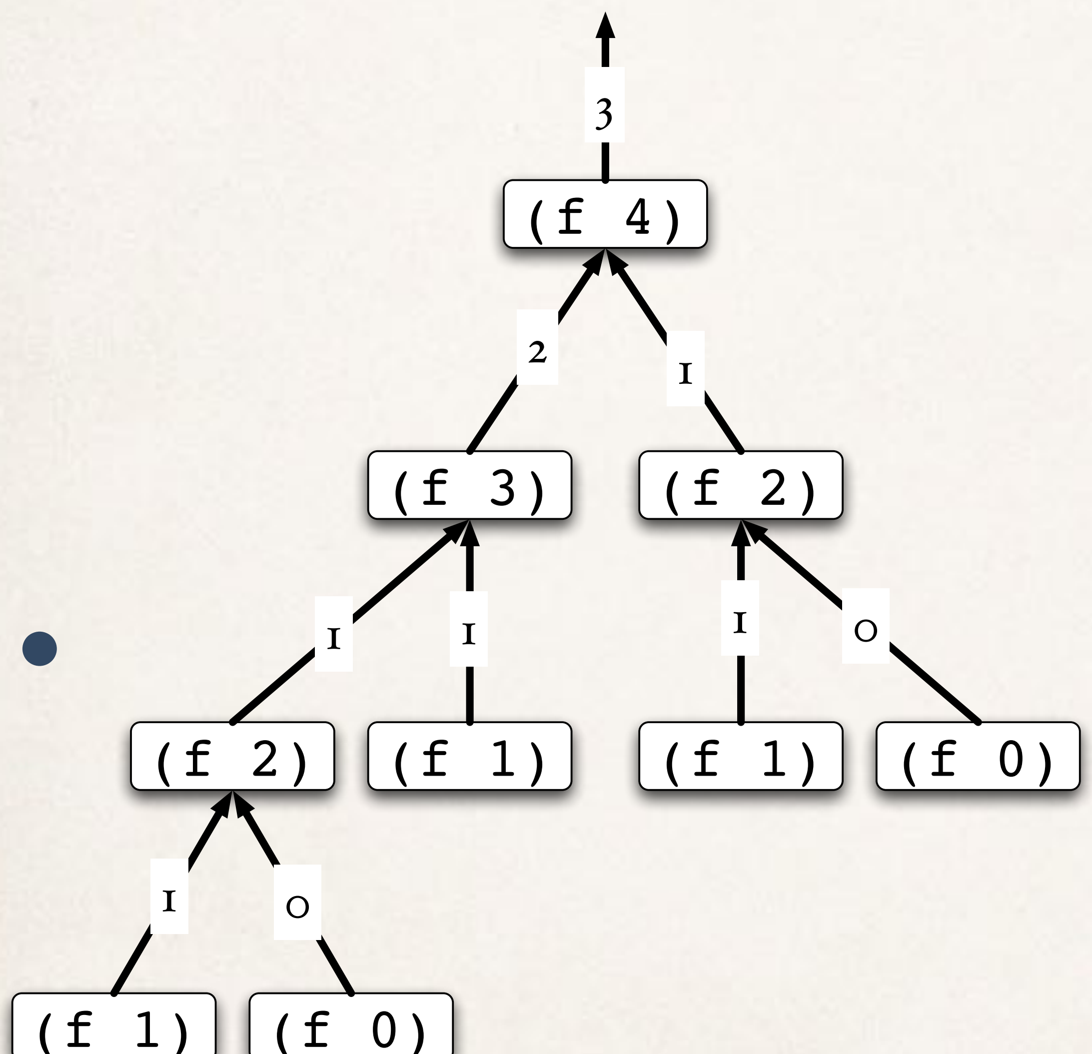
Time and space savings?

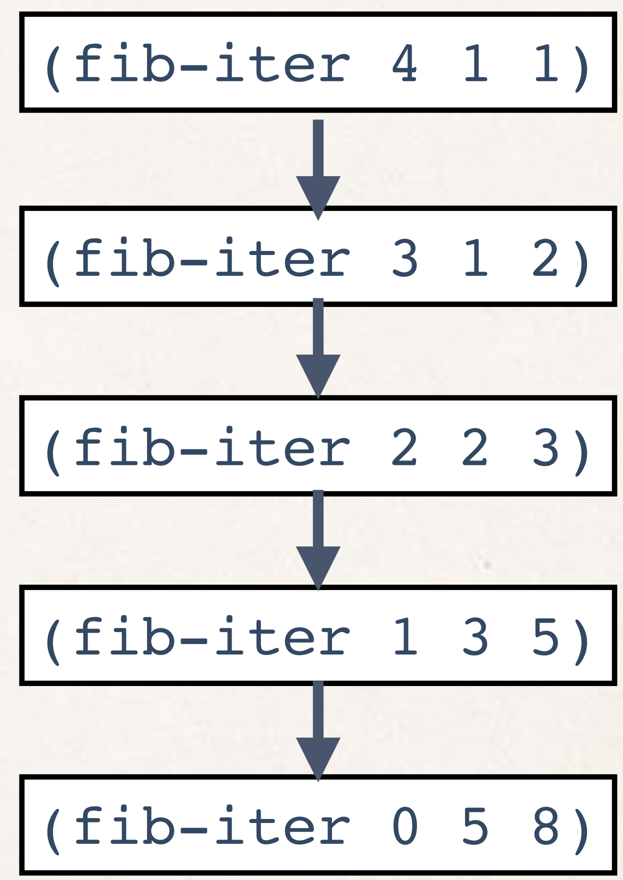
-





Time and space savings?





Continuation passing: a principled method to induce tail recursion

- Instead of: call **f** on **<arg>**, then apply **g** to the result:

```
(g (f <arg>))
```

e.g.

```
(* n (fact (- n 1))
```

- We call **f**^{c} with **<arg>** and ask it to apply **g** to the value when it is complete. In this case **g** is the *continuation*: what would have been done when **f** returned.

(f^c <arg> g)

e.g.

The continuation

```
(fact-c (- n 1) (lambda (k) (* k n)))
```

Factorial, continuation passing style

- Basic ingredient: function fact-c that computes the factorial of its first argument and then...*applies its second argument to the result*.

Compute factorial of `m`, then apply `c`

```
(define (fact-c m c)
  (if (= m 0) (c 1)
      (fact-c (- m 1)
              (lambda (x) (c (* m x))))))
```

Note! In order to be tail recursive, `fact-c` asks the recursively called `fact-c` to finish the computation, multiplying by `m`. (and, then, applying the continuation `c` fact-c was called with.).

The final product: `factorial` in continuation passing style

```
(define (fact-cps n)
  (define (fact-c m c)
    (if (= m 0) (c 1)
        (fact-c (- m 1)
                (lambda (x) (c (* m x))))))
  (fact-c n (lambda (x) x)))
```

- Note: It's tail recursive. Observe how the continuations "pile-up" in the recursive call. This second argument holds all of the pending operations.
- Note: initially, we simply call `fact-c` with the identity continuation.