
$Y = \sqrt{SNR}\cdot X + Z$

$Z \sim \mathcal{CN}(0,1)$

Rate $= R$

Code length $= n$

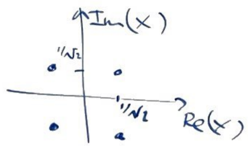
# of messages $= M = 2^{nR}$

Code $= \{x(1), x(2), \dots, x(M)\}$

$x(m) = (x_1(m), \dots, x_n(m))$

Random coding $x \in \mathcal{X}$ with probability $Q(x)$

Constellation $\mathcal{X}$, e.g.




Normalized to $\mathbb{E}[|X|^2] = 1$

where $\mathbb{E}[|X|^2] = \sum_{x \in \mathcal{X}} Q(x)|x|^2$

Channel transition probability $W(y|x)$, e.g.

$$W(y|x) = \frac{1}{\pi} e^{-|y - \sqrt{SNR}\cdot x|^2}$$

Channel output

$(y_1, \dots, y_n)$

Receiver/decoder estimates $m$ as $\hat{m} = g(y_1, \dots, y_n)$

We can show that we can transmit with error probability

$$P_e = 2^{-nE(R)}$$

where

$$E(R) = \max_{0 \le \rho \le 1} \{E_0(\rho) - \rho R\}$$

with

$$E_0(\rho) = -\log_2\left[\int_{y \in \mathbb{C}} dy \sum_{x \in \mathcal{X}} Q(x) W(y|x) \cdot \left(\frac{\sum_{\bar{x}} Q(\bar{x}) W(y|\bar{x})^{\frac{1}{1+\rho}}}{W(y|x)^{\frac{1}{1+\rho}}}\right)^{\rho}\right]$$

$$= -\log_2\left[\int_{y \in \mathbb{C}} dy \sum_{x \in \mathcal{X}} Q(x) G(y - \sqrt{SNR}\cdot x) \left(\frac{\sum_{\bar{x}} Q(\bar{x}) G(y - \sqrt{SNR}\cdot\bar{x})^{\frac{1}{1+\rho}}}{G(y - \sqrt{SNR}\cdot x)^{\frac{1}{1+\rho}}}\right)^{\rho}\right]$$

To simplify, we define $G(z) = \frac{1}{\pi} e^{-|z|^2}$ to write $W(y|x) = G(y - \sqrt{SNR}\cdot x)$ so that after the change of variable $z = y - \sqrt{SNR}\, x$,

$$E_0(\rho) = -\log_2\left[\int_{z \in \mathbb{C}} dz \sum_{x \in \mathcal{X}} Q(x) G(z) \underbrace{\left(\frac{\sum_{\bar{x}} Q(\bar{x}) G(z + \sqrt{SNR}\cdot x - \sqrt{SNR}\cdot\bar{x})^{\frac{1}{1+\rho}}}{G(z)^{\frac{1}{1+\rho}}}\right)}_{f_\rho(x,z)}{}^{\rho}\right]$$

$$= -\log_2\left[\sum_x Q(x) \int_{z \in \mathbb{C}} dz\, G(z) f_\rho(x,z)^\rho\right]$$

$$= -\log_2\left[\sum_x Q(x) \int_{z \in \mathbb{C}} dz\, \frac{1}{\pi} e^{-|z|^2} f_\rho(x,z)^\rho\right]$$

$$= -\log_2\left[\frac{1}{\pi}\sum_x Q(x) \int_{-\infty}^{\infty} dz_R \int_{-\infty}^{\infty} dz_I\, e^{-z_R^2} e^{-z_I^2} f_\rho(x, z_R + j z_I)^\rho\right]$$

$$\approx -\log_2\left[\frac{1}{\pi}\sum_x Q(x) \sum_{i=1}^{N} \sum_{k=1}^{N} w_i \cdot w_k\, f_\rho(x, z_i + j z_k)^\rho\right],$$

where $w_1, \dots, w_N$ are the quadrature weights and $z_1, \dots, z_N$ are the quadrature roots/nodes.