MATH-H401 Numerical methods

Project 6: Modeling a Capacitor

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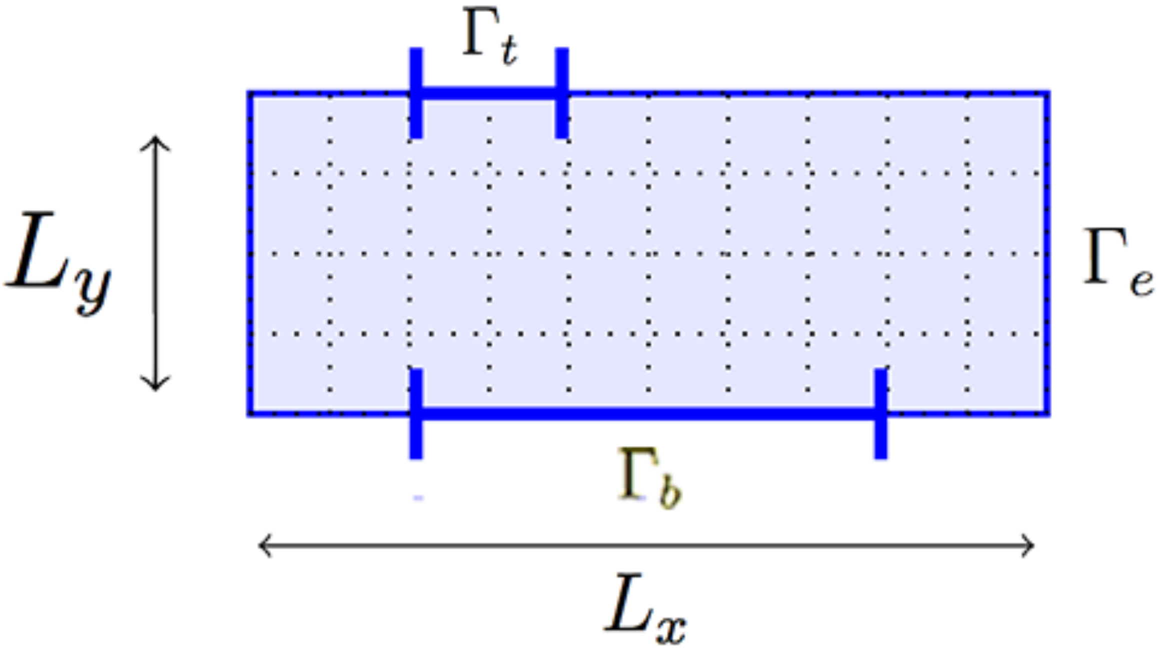
1. Introduction

Project 6



Dimensions

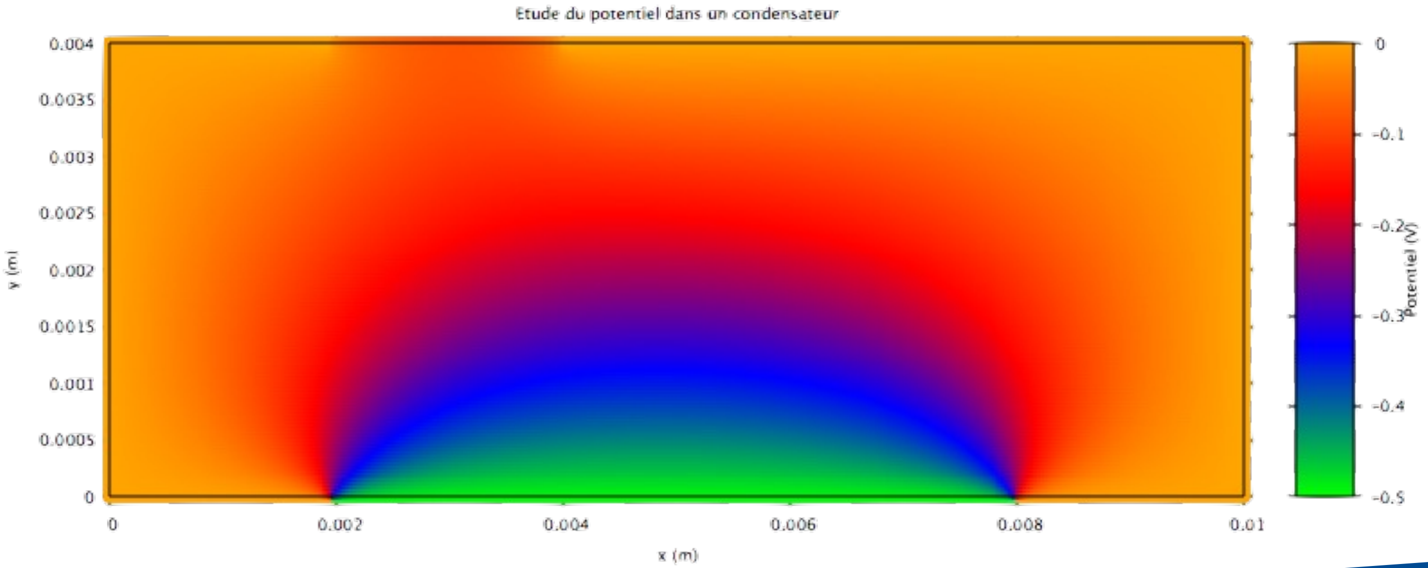
- Lx = 10 mm
- Ly = 4 mm

CB

- $\frac{du}{dn} = 0$ on Γ_t
- $u = -0.5$ on Γ_b
- $u = 0$ on Γ_e

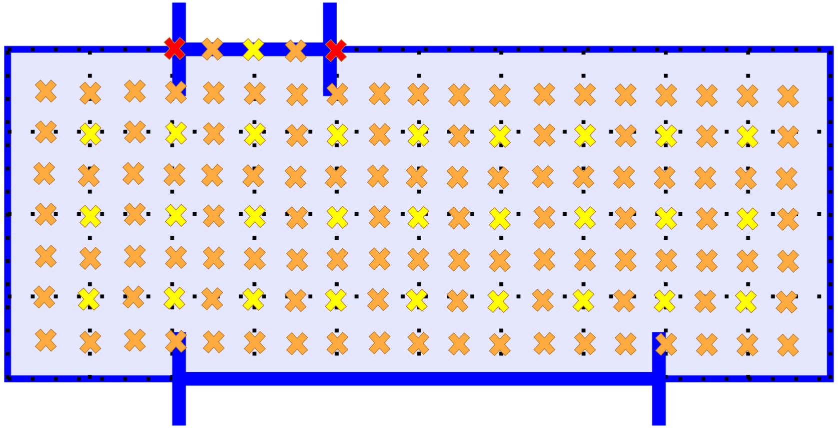
2. Neumann's CB

- Using finite differences
- Implemented in prob.c
- More unknowns

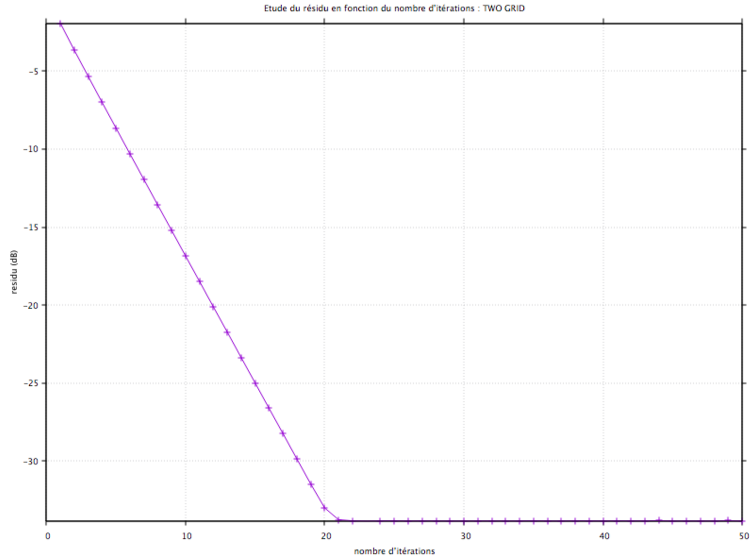


3.Two Grid

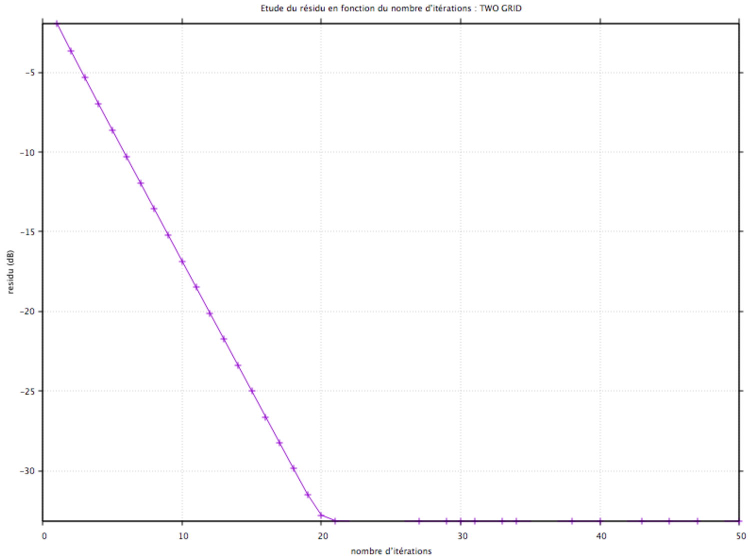
- Creation of symmetrized Gauss-Seidel
- Creation of the restriction
- Solve with UMFPACK $Au_c = r_c$
- Creation of the extension



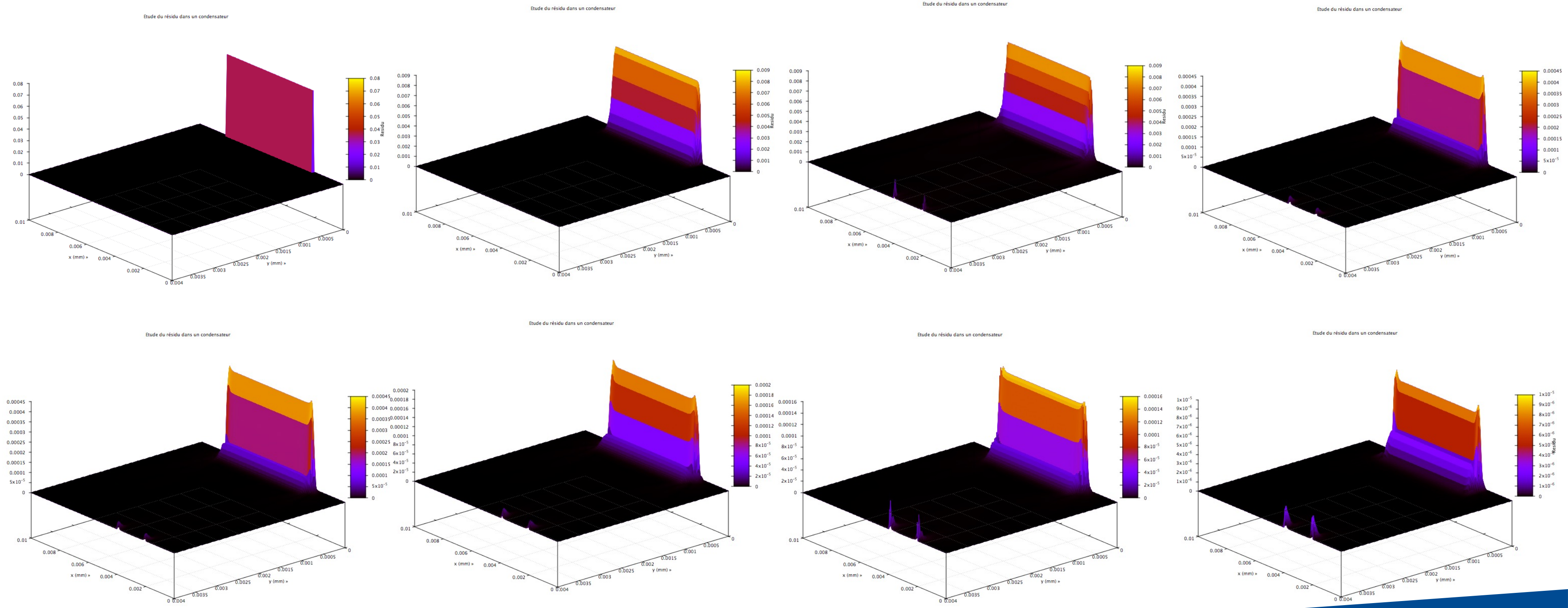
Result for a step of **h = 3.125 x 10 $^{-5}$ m**



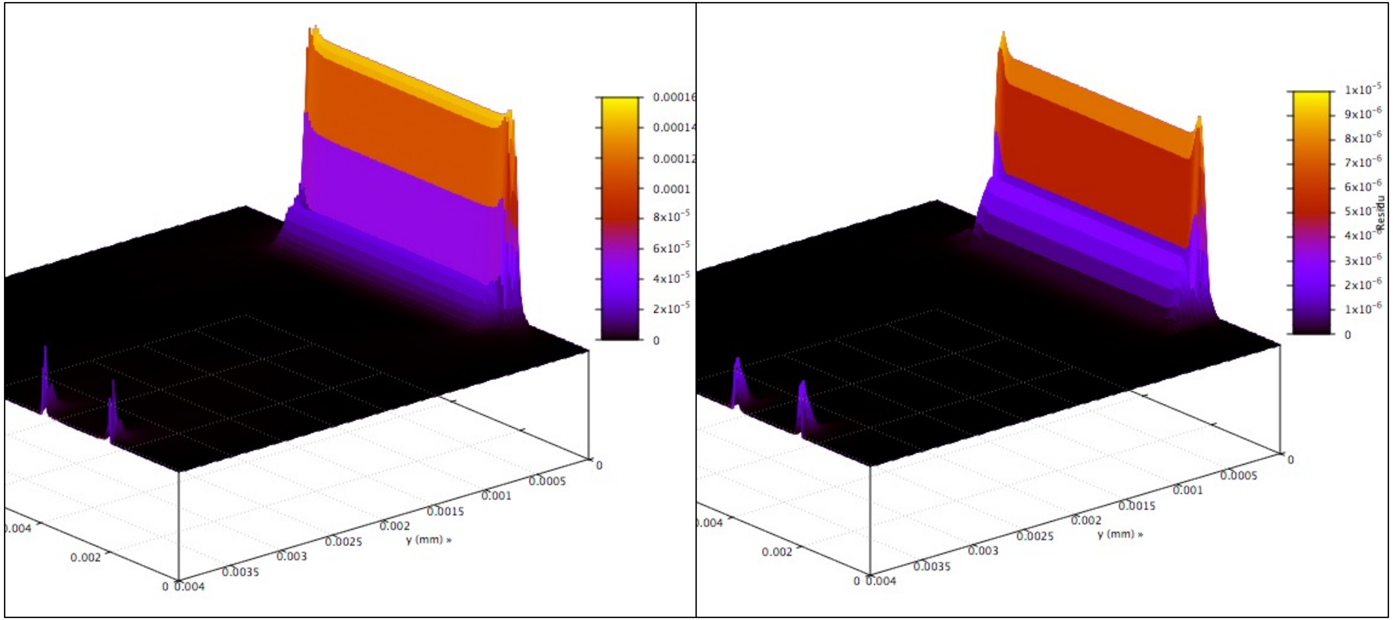
Result for a step of **h = 7.8125 x 10 $^{-6}$ m**



Evolution of the residue

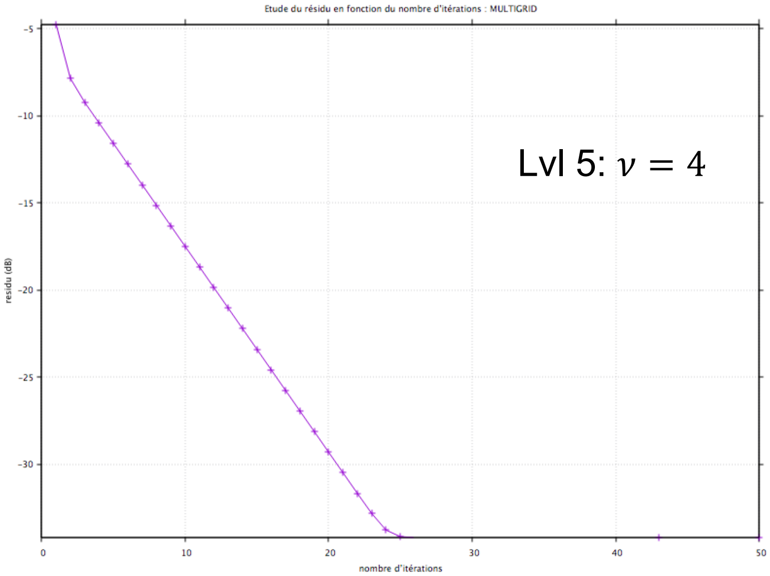


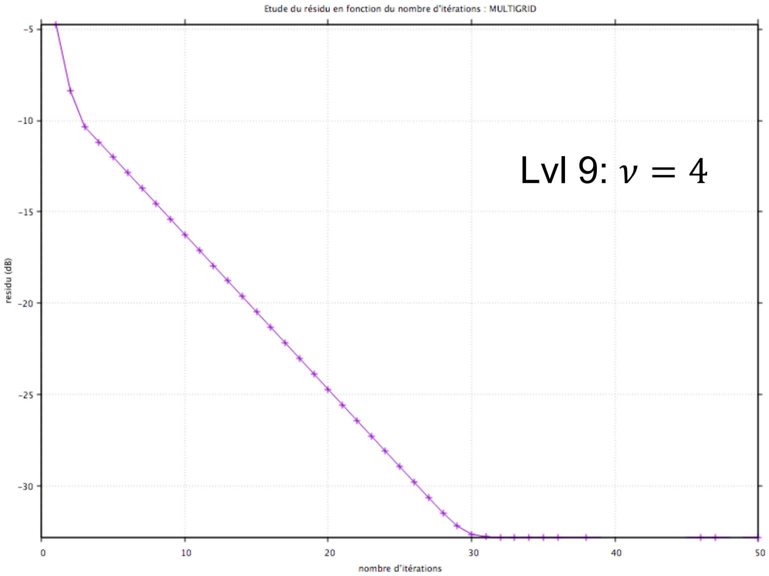
Before and after correction



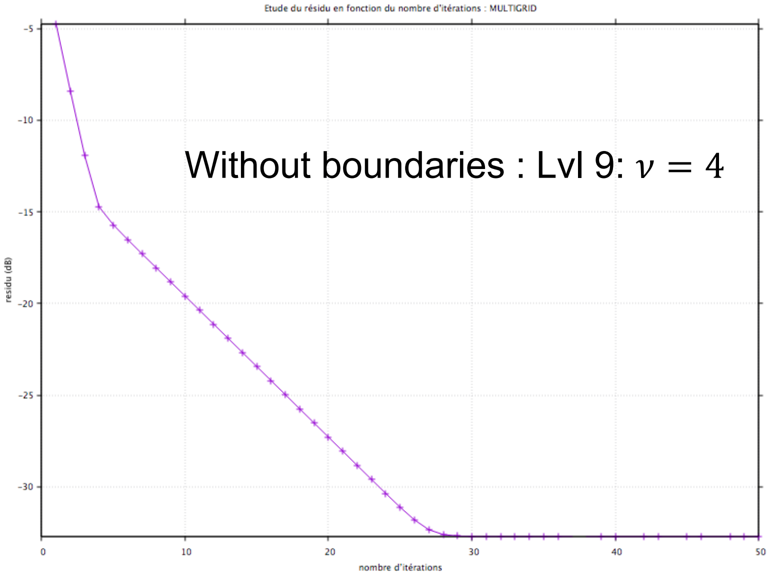
4. Multigrid

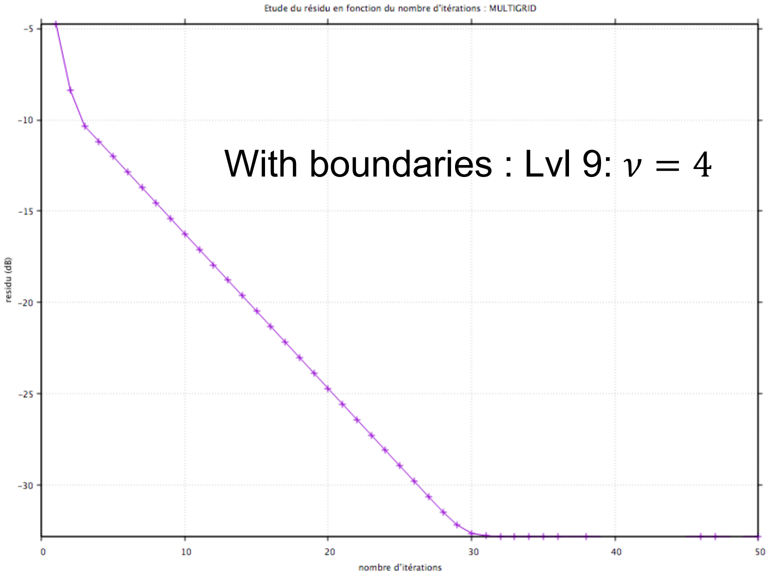
- Problem in the Multigrid : $\nu = 4$ instead of $\nu = 1 \rightarrow$ Visualization of the residue
- With Neumann boundaries $\rightarrow$ good convergence



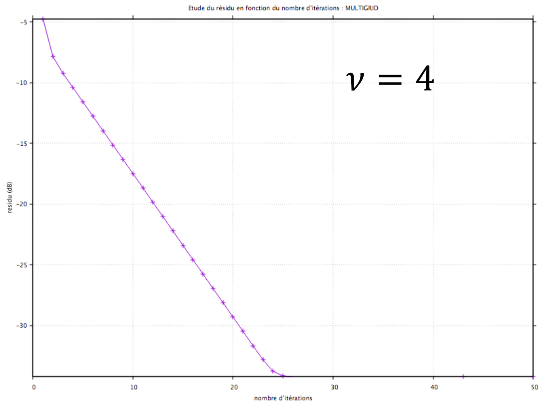


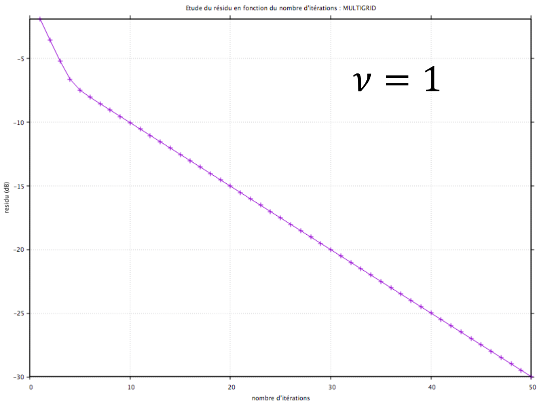
- Comparison WITH or WITHOUT Neumann boundaries
- No big ≠ + big break → Keeps Neumann boundaries

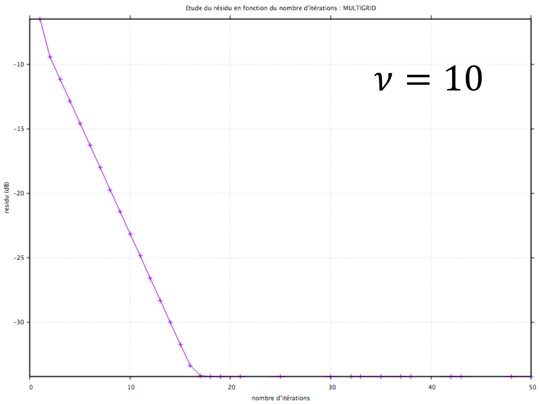




- For a level 5: passage of $\nu = 4$ instead of $\nu = 1$
- Remove Neumann Boundaries



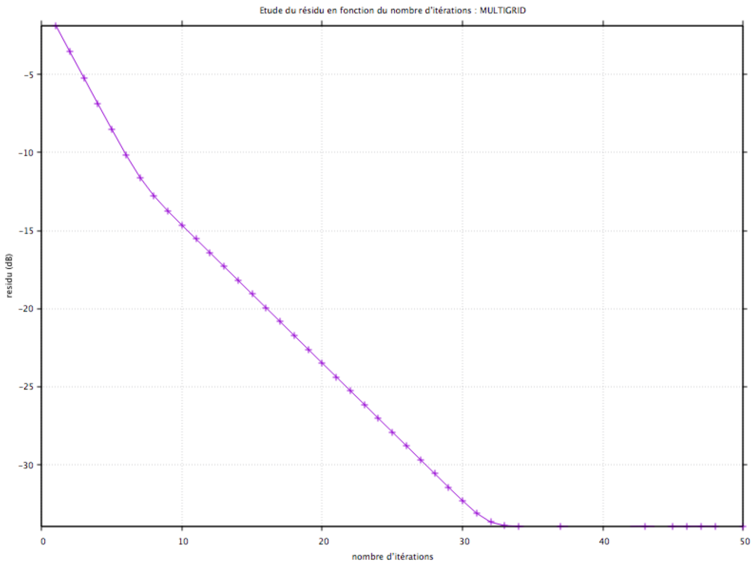




- Faster than Two Grid : for **h = 3.125 x 10 $^{-5}$ m** and **50** iterations

→ TG: 2.586298 s

→ MG: 0.224781s



<u>Level 5</u> : h = 3.125 x 10 $^{-5}$ m

- Faster than Two Grid : for **h = 3.125 x 10 $^{-5}$ m** and **50** iterations

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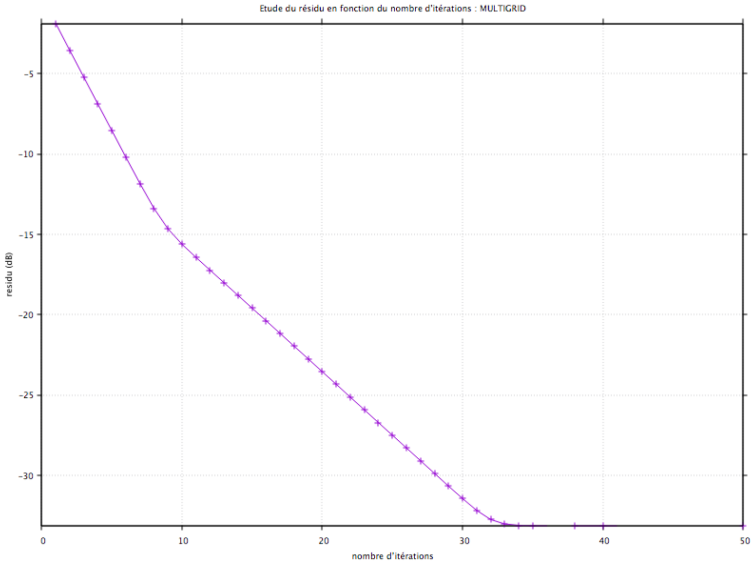


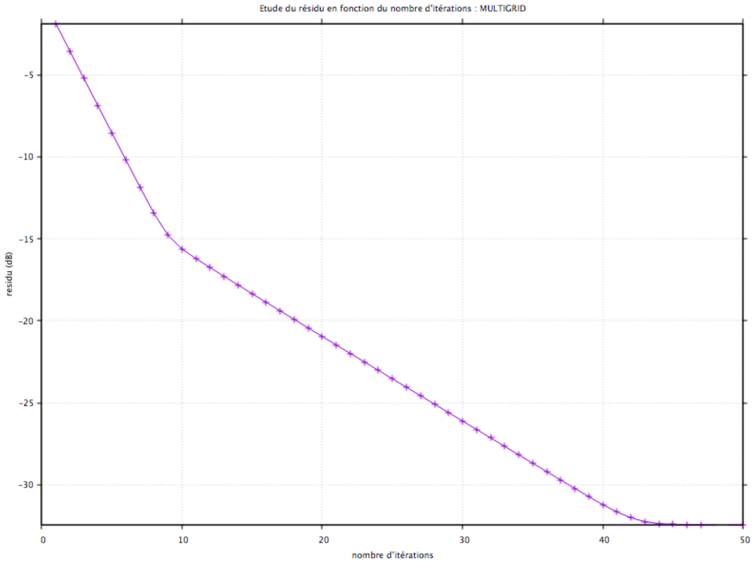
<u>Level 7</u> : h = 7.8125 x 10 $^{-6}$ m

ECOLE POLYTECHNIQUE DE BRUXELLES

- Faster than Two Grid : for **h = 3.125 x 10 $^{-5}$ m** and **50** iterations

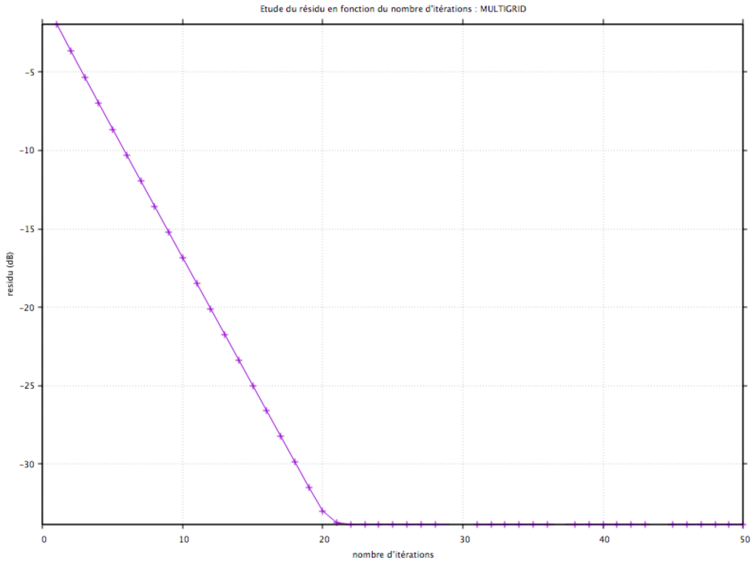
→ TG: 2.586298 s

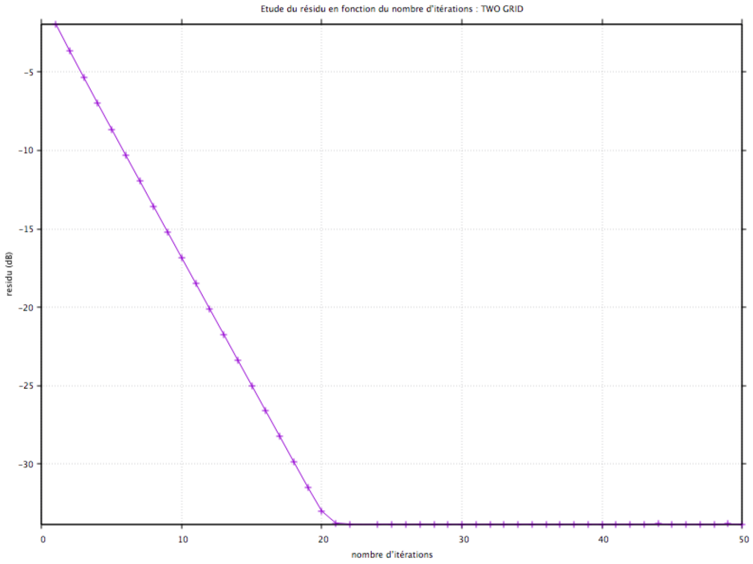
→ MG: 0.224781s



<u>Level 9</u> : h = 1.953125 x 10 $^{-6}$ m

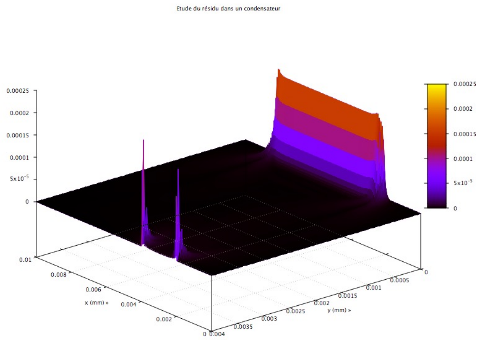
- MG can be used as a Two Grid : for **h = 3.125 x 10 $^{-5}$ m** (resolution at **h = 6.25 x 10 $^{-5}$ m**)

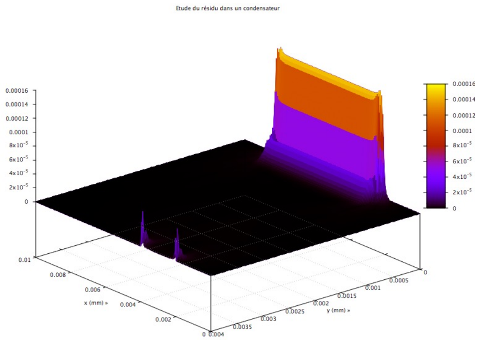


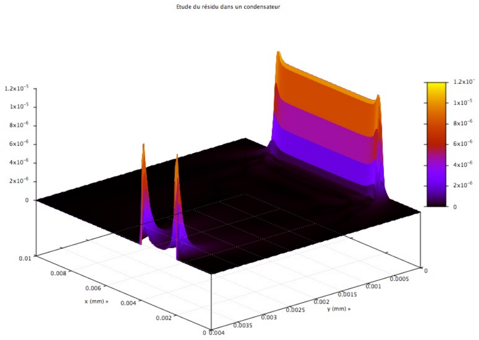


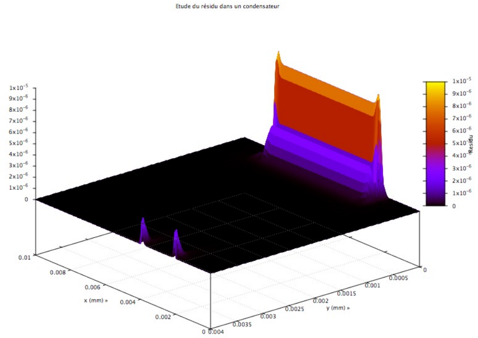
Multigrid

Two Grid









Multigrid convergence

ECOLE POLYTECHNIQUE DE BRUXELLES

Level 9 : h = 1.953125 x 10 $^{-6}$ m

Stability criterion

$\|r\| = \mathrm{C\ v}\ \|A\|\ \|\tilde{u}\|$

$\|r\| \approx 0{,}1201642$

$\|A\| = \lambda_{max}(A) \approx 2{,}097152.10^{12}$

$\|\tilde{u}\| = 639{,}904812$

$\rightarrow C \approx 0{,}81$

Calculation of ρ and τ

$\rho = \dfrac{r_n}{r_{n-1}} \approx 0{,}5786$

$\lambda_{max}(B^{-1}A) \approx 1$

$\rho(T) = \max(1 - \tau\lambda_{min}, \tau\lambda_{max} - 1)$

$\rho(T) = 1 - \tau\lambda_{min} \leftrightarrow \lambda_{min} = 0.4213$

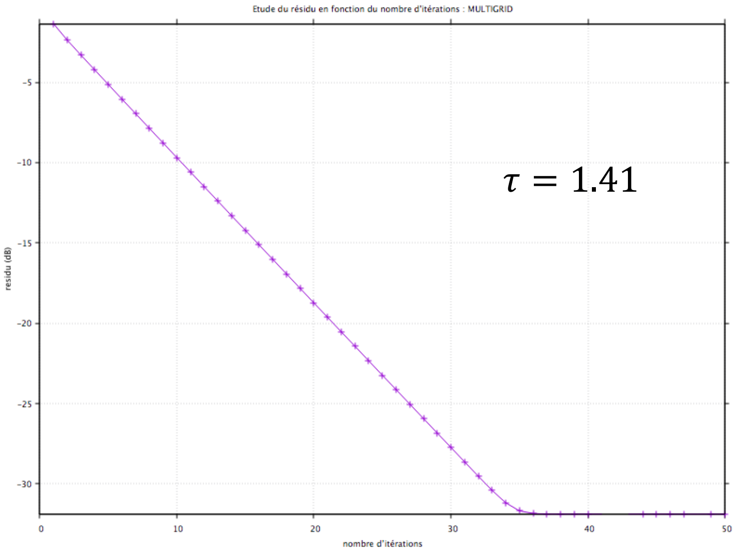
$\rightarrow \tau_{opti} = \dfrac{2}{\lambda_{max} + \lambda_{min}} \approx 1{,}41$

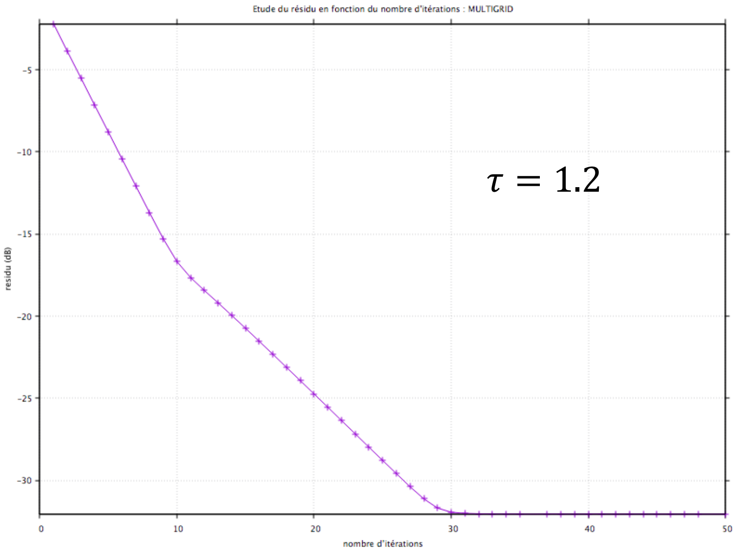
Conclusion

$\rho_{exp} = \dfrac{r_n}{r_{n-1}} \approx 0{,}4162$

$\rho_{th} = \dfrac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \approx 0.4072$

Results


Etude du résidu en fonction du nombre d'itérations : MULTIGRID
$\tau = 1.41$
residu (dB)
nombre d'itérations


Etude du résidu en fonction du nombre d'itérations : MULTIGRID
$\tau = 1.2$
residu (dB)
nombre d'itérations

Level 5 : h = 3.125 x 10 $^{-5}$ m

Stability criterion

$\|r\| = \mathrm{C}\,\mathrm{v}\,\|A\|\,\|\tilde{u}\|$

$\|r\| \approx 0.00003020493$

$\|A\| = \lambda_{max}(A) \approx 8192000000$

$\|\tilde{u}\| = 39.820483$

$\rightarrow C \approx 0.84$

Calculation of ρ and τ

$\rho = \frac{r_n}{r_{n-1}} \approx 0.4497$

$\rho(T) = 1 - \tau\lambda_{min} \leftrightarrow \lambda_{min} = 0.5503$

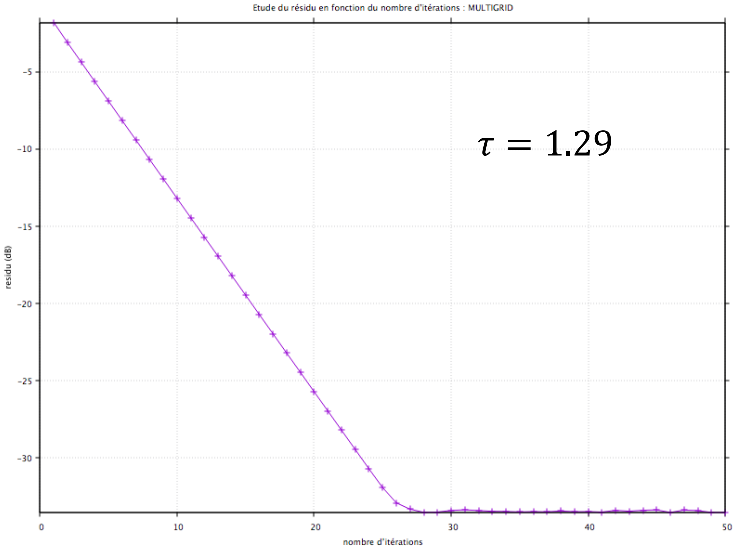
$\rightarrow \tau_{opti} = \frac{2}{\lambda_{max} + \lambda_{min}} \approx 1.29$

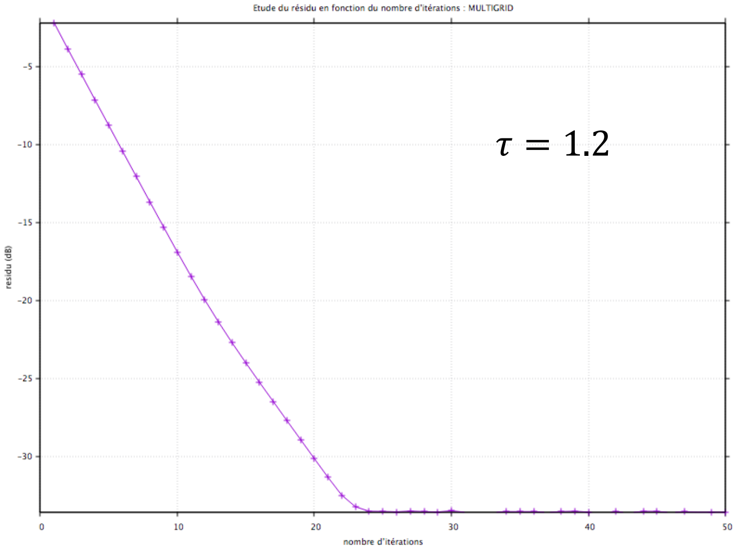
Conclusion

$\rho_{exp} = \frac{r_n}{r_{n-1}} \approx 0.3692$

$\rho_{th} = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \approx 0.29$

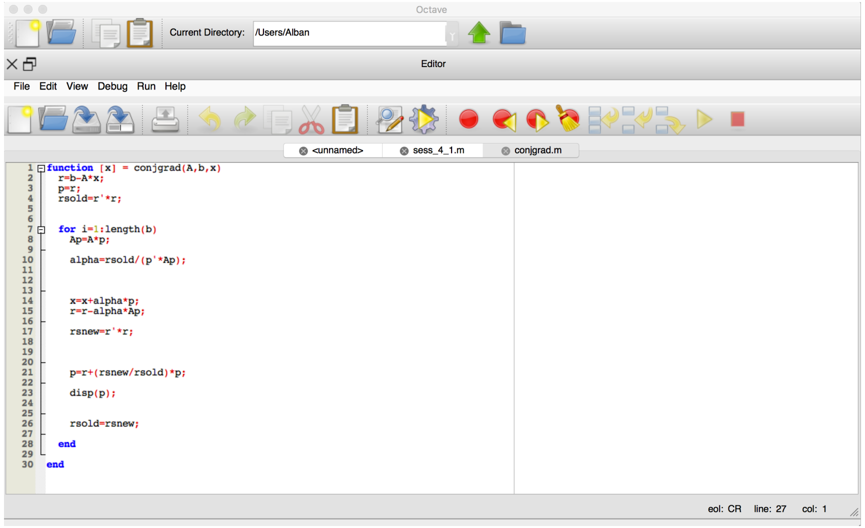
Results

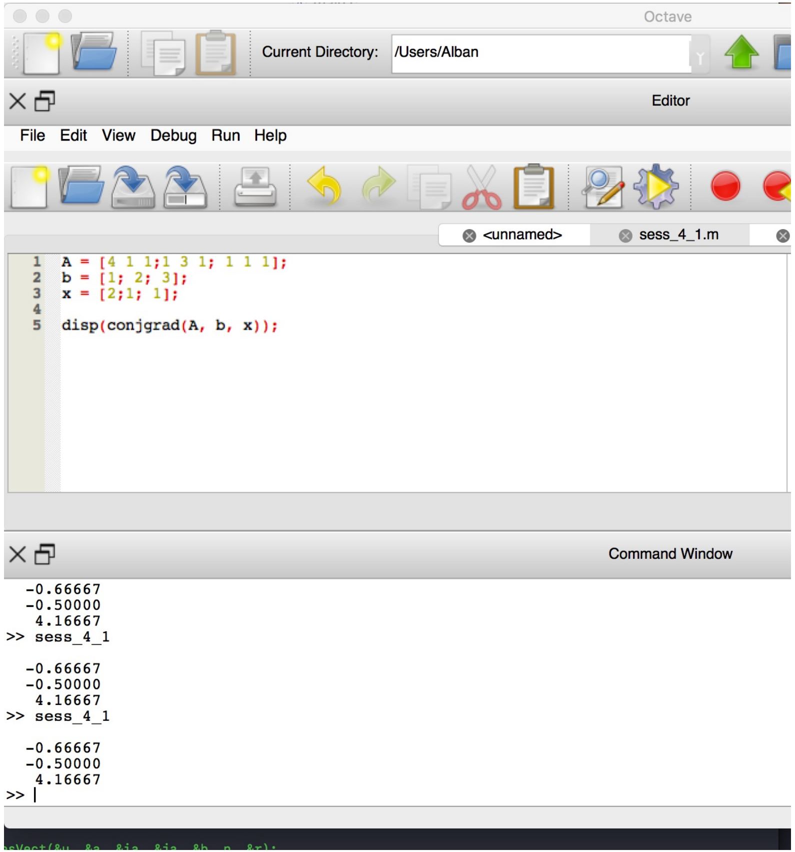

Etude du résidu en fonction du nombre d'itérations : MULTIGRID
residu (dB)
nombre d'itérations
$\tau = 1.29$


Etude du résidu en fonction du nombre d'itérations : MULTIGRID
residu (dB)
nombre d'itérations
$\tau = 1.2$

5. Conjugate gradient

- Conjugate gradient code converges to a residual $\propto 10^{-3}$ in 15 iterations
- Works with Octave $\rightarrow$ Gets GC without preconditioner

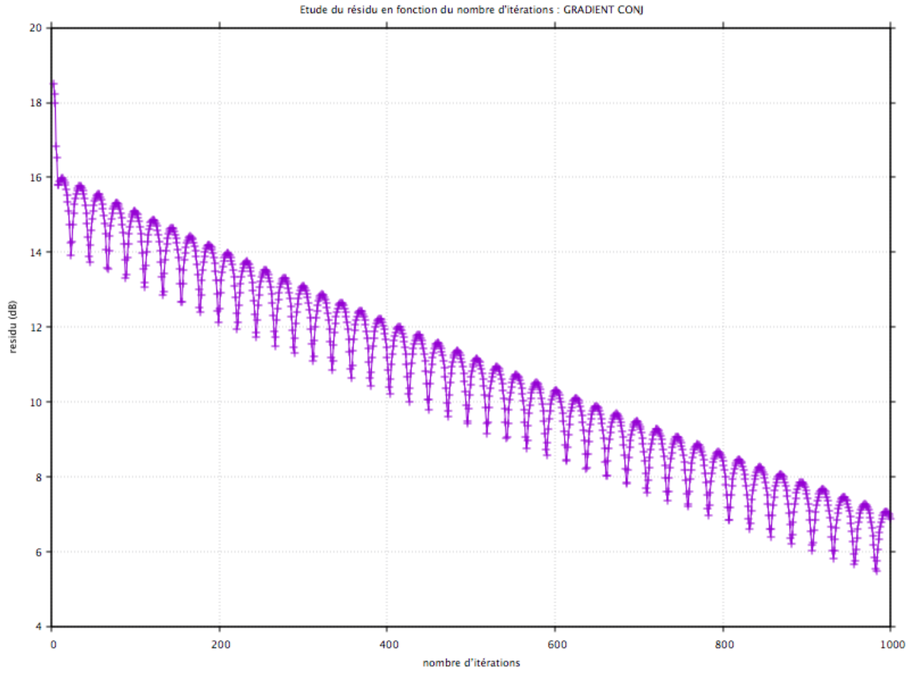





```c
/* DECLARATION DE VARIABLES */
double beta = 0, rsSold = 0, rsNew = 0, alpha = 0, numAlpha = 0, denAlpha = 0;
double *Ad;
printf("\nhello\n");
//Choix pour le problème du cours
n = 3;
Ad = malloc(n*sizeof(double));
a = malloc(n*n*sizeof(double));
u = malloc(n*sizeof(double));
r = malloc(n*sizeof(double));
d = malloc(n*sizeof(double));
b = malloc(n*sizeof(double));
ja = malloc(9*sizeof(int));
ia = malloc(4*sizeof(int));


a[0] = 4.0; a[1] = 1.0; a[2] = 1.0; a[3] = 1.0;a[4] = 3.0; a[5] = 1.0; a[6] = 1.0; a[7] = 1.0;a[8] =
    1.0;
ja[0] = 0; ja[1] = 1; ja[2] = 2;ja[3] = 0; ja[4] = 1; ja[5] = 2;ja[6] = 0; ja[7] = 1; ja[8] = 2;
ia[0] = 0; ia[1] = 3; ia[2] = 6; ia[3] = 9;
b[0] = 1; b[1] = 2; b[2] = 3;

CG(&ia, &ja, &a, &b, &u, n);

printf("\nSOLOO %f %f %f\n", u[0], u[1], u[2]);
```

```
|GENERATION DU PROBLEME|

PROBLEM: mx = 321 my = 129   n = 40576   nnz = 201986

SOLUTION -0.666667 -0.500000 4.166667
MacBook-Pro:CG Sans preconditionner Alban$
```

- With preconditioner : bad convergence → jumps



Appendices

ECOLE POLYTECHNIQUE DE BRUXELLES

Interlude : propriétés de la norme euclidienne

Propriété 1 : soient A une matrice symétrique et λ_i, $i = 1, ..., n$, ses valeurs propres. Alors

$$\|A\|_2 \;=\; \max_i |\lambda_i| \;=\; \max_{\mathbf{v}} \frac{|\mathbf{v}^T A \mathbf{v}|}{\mathbf{v}^T \mathbf{v}} \;.$$

Soit $\mathbf{p}_i$ le vecteur propre normalisé associé à λ_i, $i = 1, ..., n$. L'ensemble de ces vecteurs forment une base orthonormale et tout vecteur $\mathbf{v}$ a une représentation $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{p}_i$ dans cette base. On a alors

$$\|A\|_2^2 \;=\; \max_{\mathbf{v}} \frac{\|A\mathbf{v}\|_2^2}{\|\mathbf{v}\|_2^2} \;=\; \max_{\mathbf{v}} \frac{\mathbf{v}^T A^2 \mathbf{v}}{\mathbf{v}^T \mathbf{v}} \;=\; \max_{\alpha_1,\dots,\alpha_n} \frac{\sum_{i=1}^n \lambda_i^2 \alpha_i^2}{\sum_{i=1}^n \alpha_i^2} \;=\; \max_i |\lambda_i|^2 \,.$$

Par analogie avec les deux dernières égalités on a aussi

$$\max_{\mathbf{v}} \frac{|\mathbf{v}^T A \mathbf{v}|}{\mathbf{v}^T \mathbf{v}} \;=\; \max_{\alpha_1,\dots,\alpha_n} \frac{\left|\sum_{i=1}^n \lambda_i \alpha_i^2\right|}{\sum_{i=1}^n \alpha_i^2} \;=\; \max_i |\lambda_i| \,.$$

Propriété 2 : pour tout matrice A rectangulaire

$$\|A^T A\|_2 \;=\; \|A\|_2^2 \,.$$

$$\|A^T A\|_2 \;=\; \max_{\mathbf{v}} \frac{|\mathbf{v}^T A^T A \mathbf{v}|}{\mathbf{v}^T \mathbf{v}} \;=\; \max_{\mathbf{v}} \frac{\|A\mathbf{v}\|_2^2}{\|\mathbf{v}\|_2^2} \;.$$

3.3 Relaxation (5)

(p. 22)

ULB

ECOLE POLYTECHNIQUE DE BRUXELLES

Using this optimal value of τ , one obtains

$$\rho(T) \;=\; \frac{\lambda_{\max}(B^{-1}A)-\lambda_{\min}(B^{-1}A)}{\lambda_{\max}(B^{-1}A)+\lambda_{\min}(B^{-1}A)} \;=\; \frac{\kappa-1}{\kappa+1}\,,$$

where κ is the condition number of the preconditioned system:

$$\kappa \;=\; \kappa(B^{-1}A) \;=\; \frac{\lambda_{\max}(B^{-1}A)}{\lambda_{\min}(B^{-1}A)}\,.$$

The corresponding estimate of the number of iterations is (see above) the smallest integer satisfying

$$k_\varepsilon \;\geq\; \frac{\ln\frac{1}{\varepsilon}}{\ln\frac{1}{\rho(T)}} \;=\; \frac{\ln\frac{1}{\varepsilon}}{\ln\frac{\kappa+1}{\kappa-1}}\,. \qquad (2)$$

where

$$\ln\frac{\kappa+1}{\kappa-1} \;=\; \ln\frac{1+\kappa^{-1}}{1-\kappa^{-1}} \;\geq\; \frac{2}{\kappa}$$

(with $\mathcal{O}(\kappa^{-3})$ error).

- $k_\varepsilon \geq \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln\left(\frac{1}{\rho}\right)}$
- $k_\varepsilon \geq 15$(for $\varepsilon = 10^{-6}$ et $\rho = 0{,}4$)