

# <sup>1</sup> INS\_IEP: A MATLAB package for fitting Inelastic Neutron Scattering data.

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## <sup>6</sup> Summary

<sup>7</sup> Inelastic neutron scattering (INS) is a spectroscopic technique that can be used to measure  
<sup>8</sup> the magnetic excitations in materials with interacting electron spins. For samples composed of  
<sup>9</sup> finite-size clusters of magnetic moment-carrying atoms, such as single ions or molecular-based  
<sup>10</sup> magnets, INS experiments yield inelastic excitation with energies that correspond to the  
<sup>11</sup> eigenvalues of the spin Hamiltonian of the material being studied. Fitting a model spin system  
<sup>12</sup> to these experimental eigenvalues can be formulated as an inverse eigenvalue problem (IEP),  
<sup>13</sup> where the matrix formed is the spin Hamiltonian operator of the sample molecule. Solving  
<sup>14</sup> this IEP is less computationally expensive than fitting the full INS data, so can be used as an  
<sup>15</sup> initial proxy for the full fitting problem. INS\_IEP is a MATLAB package that uses deflated  
<sup>16</sup> numerical optimisation methods to systematically find multiple solutions to this IEP. The  
<sup>17</sup> package requires and is fully compatible with easyspin ([Stoll & Schweiger, 2006](#)), a package  
<sup>18</sup> for solving fitting problems in electron paramagnetic resonance (EPR).

## <sup>19</sup> Statement of need

<sup>20</sup> Neutrons are an excellent bulk probe of material properties since they carry no charge and  
<sup>21</sup> therefore penetrate deeply into matter. Neutrons also carry a quantum spin of a half, making  
<sup>22</sup> them a sensitive probe of magnetism ([Squires, 2012](#)). Reactors and spallation sources with neutron  
<sup>23</sup> scattering experiment capabilities serve the international research community with neutron  
<sup>24</sup> scattering experiment capabilities for material research. Inelastic neutron scattering (INS) is  
<sup>25</sup> one such experimental technique that can be used to study magnetism. In an INS experiment,  
<sup>26</sup> a sample under investigation is irradiated with a beam of neutrons and the scattered neutron  
<sup>27</sup> energy and momentum transfer are detected. For samples composed of finite-size clusters of  
<sup>28</sup> magnetic moment-carrying atoms, such as single ions or molecular-based magnets, the detected  
<sup>29</sup> neutron energy transfer gives direct access to the quantum spin excitations ([M. Baker & Mutka,  
2012; Furrer & Waldmann, 2013; "Spectroscopy Methods for Molecular Nanomagnets," 2014](#)).

<sup>30</sup> The energy of such excitations relates to the energy difference between eigenvalues of the  
<sup>31</sup> Hamiltonian matrix that describes the quantum spin dynamics of the compound in question.  
<sup>32</sup> Single-ion and molecular-based magnets are studied as prototype components (quantum bits,  
<sup>33</sup> sensors) for quantum technologies. INS can therefore provide crucial information concerning  
<sup>34</sup> the precise quantum properties of such systems. However, to relate the INS experimental  
<sup>35</sup> results to the Hamiltonian that describes quantum spin dynamics requires parameterisation of  
<sup>36</sup> matrix elements such that a set of eigenvalues and eigenstates matching the experiment are  
<sup>37</sup> determined. This situation is known as the inverse eigenvalue problem.

<sup>38</sup> To date, this problem is addressed in an iterative process where parameters of the Hamiltonian  
<sup>39</sup> are varied manually, often one at a time, and the resultant eigenvalues compared to the  
<sup>40</sup> experimental values - each such iteration requires an eigendecomposition of the Hamiltonian  
<sup>41</sup>

42 matrix. INS\_IEP presents an elegant solution to solving this problem, using algorithms to  
 43 calculate multiple parameter sets that minimise the difference in eigenvalues, reducing the  
 44 number of Hamiltonian matrix diagonalisations, and providing a more robust method to reliably  
 45 extract an accurate spin Hamiltonian model from INS experimental data.

## 46 Key Concepts

### 47 The Spin Hamiltonian

48 The Spin Hamiltonian,  $H$ , is an approximation of the Hamiltonian that uses spin coordinates  
 49 instead of orbital coordinates, and is widely used to model data arising from many spectroscopy  
 50 techniques (Launay & Verdaguer, 2014). It can be modeled as a linear combination of  
 51 interaction terms; in this package we will use the zero field interaction,  $H_{ZFI}$ , and the  
 52 electron-electron interaction,  $H_{EEI}$ :

$$H = H_{ZFI} + H_{EEI}.$$

53 Both of these terms can themselves be modelled as the linear sum of other basis matrices.  
 54 The zero field interaction can be written as:

$$H_{ZFI} = \sum_{-k \leq q \leq k} B_k^q O_k^q$$

55 where the  $O_k^q$  are Stevens Operators (Rudowicz & Chung, 2004), and  $B_k^q$  the associated  
 56 parameter. When there are multiple spin centres it is necessary to take Kronecker products of  
 57 the operator with identity matrices of the appropriate for each other spin centre.

58 When there are multiple spin centres it is also necessary to include an electron-electron  
 59 interaction term,  $H_{EEI}$ . This term will be the sum of interaction terms between each pair of  
 60 spin centres:

$$H_{EEI} = - \sum_{i \neq j} J_{ij} S_i \cdot S_j$$

61 where  $S_i$  is the vector of spin operators  $S_i = [S_x, S_y, S_z]$  for the  $i$ -th spin centre, and  $J_{ij}$  is  
 62 the parameter to be found that represents the strength of interaction between the two spin  
 63 centres. Note that in the isotropic case  $J$  can be thought of as a scalar value, but in the  
 64 anisotropic case will be a matrix where the off diagonals are skew symmetric (often the off  
 65 diagonals are assumed to be zero). While the summation is in theory over all spin centre  
 66 combinations, in practice many of these contributions will be negligible - often only the nearest  
 67 neighbour interactions are significant.

68 It is important to mention that these matrix operators can be very large. The size is defined  
 69 by the number of spin centres ( $n$ ) and the spin ( $S_i$ ) of each spin centre. The dimension of  
 70 matrices is given by:

$$\prod_i^n (2S_i + 1).$$

71 The operators are however highly sparse, this means that it is possible to use eigensolvers that  
 72 can take advantage of this sparsity.

### 73 Inverse Eigenvalue Problem

74 The INS experiments provide eigenvalues of the Spin Hamiltonian matrix of the sample, the  
 75 task of calculating the matrix from the eigenvalues is an inverse eigenvalue problem:

76 Let  $A(x)$  be the affine family of matrices,

$$A(x) = A_0 + \sum_{i=1}^{\ell} x_i A_i,$$

77 where  $x \in \mathbb{C}^\ell$  and  $A_0, \dots, A_\ell \in \mathbb{R}^{n \times n}$  are linearly independent Hermitian matrices, and denote  
 78 the ordered eigenvalues of  $A(x)$  as  $\lambda_1(x) \leq \dots \leq \lambda_n(x)$ . Then the least squares inverse  
 79 eigenvalue problem (LSIEP) is to find the parameters  $x \in \mathbb{R}^\ell$  that minimises

$$F(x) = \frac{1}{2} \|r(x)\|_2^2 = \frac{1}{2} \sum_{i=1}^m (\lambda_i(x) - \lambda_i^*)^2$$

80 where  $\lambda_1^* \leq \dots \leq \lambda_m^*$  are the experimental eigenvalues (Chu & Golub, 2005). In the case of INS  
 81 fitting the  $A_i$  basis matrices will be a combination of Stevens operators and electron-electron  
 82 exchange terms. The IEP described above is formulated as an least squares problem because  
 83 the number of eigenvalues that can be probed by INS experiments is often a small subset of  
 84 the full spectrum. Due to the low temperatures that these experiments are performed at (can  
 85 be as low as 1K) it is generally the smallest eigenvalues that are involved. Note also that since  
 86 it is the energy difference between the eigenvalues that is probed we actually have to modify  
 87 the IEP - either by adding an additional parameter (an identity matrix) that shifts the values  
 88 of the eigenvalues, or by changing the above formula for  $F$  to directly sum the difference in  
 89 eigenvalues thereby reducing the number of residual equations in  $r(x)$  by one.

90 As far as we are aware this is the first time that the fitting of INS data has been explicitly  
 91 formulated as an IEP. An advantage of this formulation is that there are explicit formulas for  
 92 the derivatives of  $r(x)$ . The first derivative (Jacobian) is:

$$J_r(x) = \begin{pmatrix} q_1(x)^T A_1 q_1(x) & \dots & q_1(x)^T A_\ell q_1(x) \\ \vdots & \ddots & \vdots \\ q_m(x)^T A_1 q_m(x) & \dots & q_m(x)^T A_\ell q_m(x) \end{pmatrix},$$

93 and the second derivative (Hessian) is:

$$(H_r)_{ij} = 2 \sum_{k=1}^m (\lambda_k - \lambda_k^*) \sum_{\substack{t=1 \\ \lambda_t \neq \lambda_k}}^m \frac{(q_t^T A_i q_k)(q_t^T A_j q_k)}{\lambda_k - \lambda_t}.$$

94 Another advantage is the number of constraints to fit is much smaller than fitting the spectrum  
 95 itself, as it corresponds to fitting only the locations of the peaks of the spectrum.

### 96 Methods

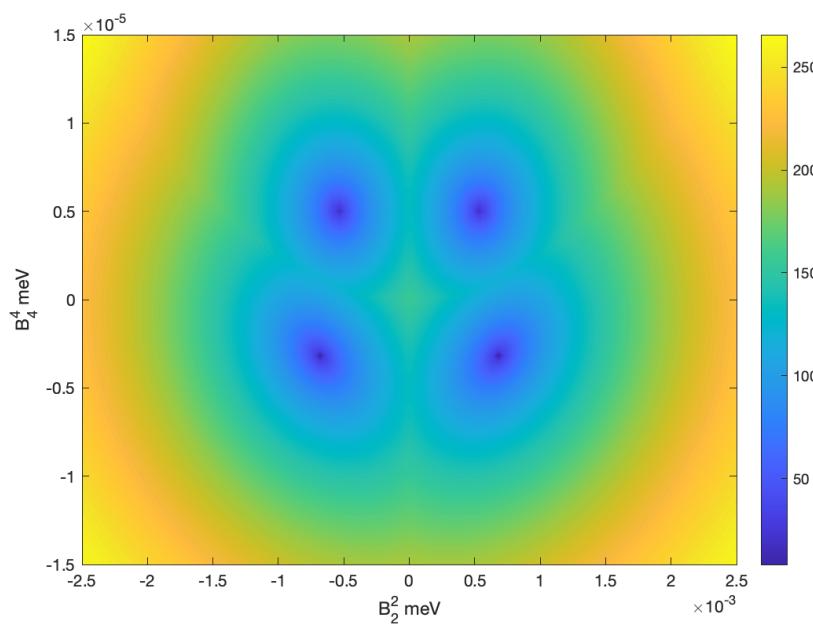
97 All of the methods used are iterative schemes of the form  $x^{k+1} = x^k + p^k$  where the step  $p^k$   
 98 uniquely defines each algorithm:

- 99     ▪ Newton's method:  $p^k = (J_r^T J_r + H_r r) - 1 J_r^T r$  (Nocedal & Wright, 2006)
- 100     ▪ Gauss-Newton method:  $p^k = (J_r^T J_r)^{-1} J_r^T r$  (Nocedal & Wright, 2006)
- 101     ▪ Lift and Projection Method:  $p^k = B^{-1} J_r^T r$  (Bloor Riley et al., 2025a)

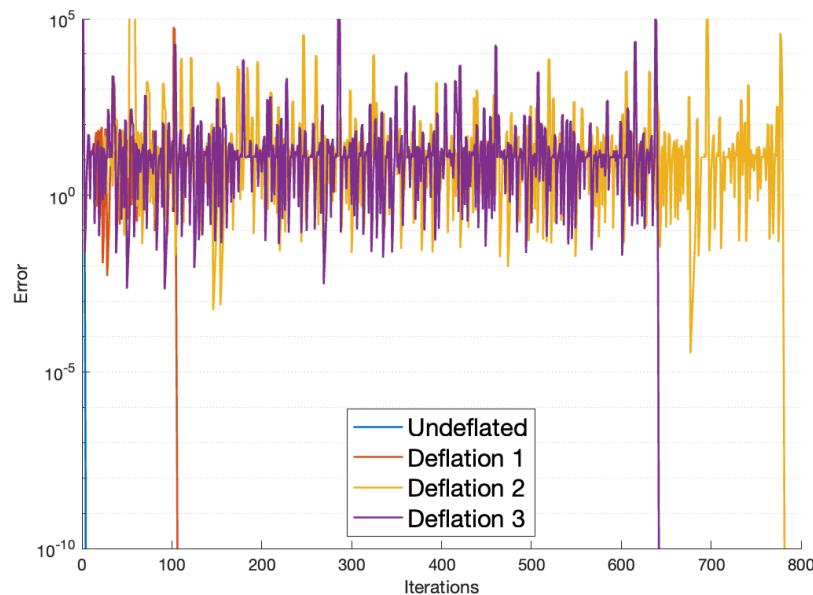
102 Where the matrix  $B$  is the Gram matrix formed from the frobenius inner products of the basis  
 103 matrices:  $B_{ij} = \langle A_i, A_j \rangle_F$ . The Lift and Projection method is a Riemannian Gradient descent  
 104 method (Bloor Riley et al., 2025a), inspired by the Lift and Projection method (Chu & Golub,  
 105 2005), specifically designed for solving IEPs. In (Bloor Riley et al., 2025a) it is proven that the  
 106 method is a strictly descending algorithm, that is it reduces the value of the objective function  
 107 every step. Both the deflated Gauss-Newton method and the Riemannian Gradient descent  
 108 Lift and Projection method are new methods designed for this package (Bloor Riley et al.,  
 109 2025a, 2025b).

## 110 Deflation

111 The number of eigenvalues that can be probed via INS experiments varies depending on the  
 112 equipment and sample in question, meaning that the fitting problem is often under or even  
 113 over determined. The IEP is also highly nonlinear and due to the experimental nature of the  
 114 data may be ill-posed. One consequence of this is that the solution space may be very ‘bumpy’,  
 115 that is there may exist many local minimisers to the problem. For example in Figure 1, there  
 116 are clearly 4 distinct solutions (for more details see Example 1 and the file Example1\_Mn12.m  
 117 in the examples folder). We seek to solve the problem of multiple local minima by the use  
 118 of Deflation, a numerical technique used to find multiple solutions to systems of equations  
 119 (Farrell et al., 2015). Fortunately it is cheap to apply deflation for the above methods, it is  
 120 simply a change to the length of the step - notably this means that the direction of each step  
 121 does not change. It is proven in (Bloor Riley et al., 2025b) that the deflated methods will not  
 122 converge to deflated points. The usual requirements still apply to the convergence of the new  
 123 methods - that the initial guess is close enough to the new minimum, and that the Jacobian is  
 124 full rank in a neighbourhood around that minimum. The rate of convergence of the deflated  
 125 methods is also more complicated, although the number of iterations required to converge can  
 126 go up with the number of deflations this is not a strict correlation, as can be seen in Figure 2.



**Figure 1:** Contour plot of how  $F$  varies with the two parameters  $B_2^2$  and  $B_4^4$  for the molecule Mn\_12 as described in Example 1. There are four locally minimising parameter pairs corresponding to the four blue regions



**Figure 2:** Comparison of the convergence behaviour for computing each solution in Example 1. The gradient of the lines as the method approaches the solution shows how the methods are quadratically convergent local to a minimum.

## 127 Examples

### 128 Example 1 - Mn12

129 The first example we will look at is Manganese-12-acetate. This is a well known example in the  
 130 INS and magnetism community, as one of the first molecules that behaves like a nano-sized  
 131 magnet with a molecular magnetic coercivity as well as its role in the research of quantum  
 132 tunnelling of magnetisation (Friedman et al., 1996; Sessoli et al., 1993).

133 The Spin Hamiltonian of this system, using the giant spin approximation, can be represented  
 134 as a  $21 \times 21$  matrix modelled using 4 Stevens operators (Bircher et al., 2004):

$$H = B_2^0 O_2^0 + B_4^0 O_4^0 + B_2^2 O_2^2 + B_4^4 O_4^4 \in \mathbb{R}^{21 \times 21}$$

135 We utilise the same spin system syntax as easyspin, so to set up the problem we first set up  
 136 the model, along with initial guesses for the parameters:

```
%One spin centre (because giant spin approximation)
Sys0.S=10;
%Four Stevens operators
Sys0.B2 = [-100,0,-1000,0,0];
Sys0.B4 = [-1,0,0,0,-1,0,0,0,0];
```

137 Then we input the experimental eigenvalues - these are typically shifted such that the smallest  
 138 eigenvalue is zero - and define which parameters to fit. Note that all values given must be in  
 139 gigahertz, so it may be useful to use conversions.

```
rcm = 29979.2458;    meV = rcm*8.065; %Conversions values
%Input calculated eigenvalues:
Exp.ev = [0,0,1.24,1.24,2.3,2.3,3.18,3.18,3.91,3.91,4.5,4.5,
4.97,4.97,5.32,5.32,5.54,5.59,5.69,5.75,5.78].*meV;
```

%Note that these eigenvalues are simulated from the parameters given in `[@bircher_trans]`

```
%Vary all non zero parameters (no Fixed parameters):
Vary = Sys0;
```

140 Then all that is required is to call `INS_IEP` with these three inputs:

```
SysOut = INS_IEP(Sys0,Vary,Exp);
```

141 If we wish to find all four solutions as shown in [Figure 1](#) then we use the additional option:

```
Opt.NDeflations = 4;
SysOut = INS_IEP(Sys0,Vary,Exp,Opt);
```

142 In this case `SysOut` will be an array of four spin structures each containing a distinct locally optimal solution. It is possible to access information about the convergence of each deflation by using `SysOut.Output`. For example by utilising the iterates recorded, stored in `SysOut.Output.Iterates` it is possible to plot a graph of convergence, as can be seen in [Figure 2](#). The `Output` structure also contains the value of  $F$  at the final point, as well as the number of iterations it took to get there.

148 A full list of options is provided in the help of `INS_IEP`.

## 149 Example 2 - Chromium(iii) Horseshoes

150 The second example concerns antiferro-magnetically coupled chain of six chromium(III) ions ([M. L. Baker et al., 2011](#)). Because there are multiple spin centres an electron-electron interaction term is required. The spin hamiltonian is a  $4096 \times 4096$  matrix composed of two Stevens operators and one interaction term, since it is known a priori that each spin centre will have the same value parameters we pin the parameters here, by setting the initial guess as the same value:

```
Sys0.S = [1.5 1.5 1.5 1.5 1.5 1.5];
Sys0.B2 = [1 0 -1 0 0;
           1 0 -1 0 0;
           1 0 -1 0 0;
           1 0 -1 0 0;
           1 0 -1 0 0;
           1 0 -1 0 0];
Sys0.J = [100,0,0,0,0,100,0,0,0,100,0,0,100,0,0,100];
Vary = Sys0;
Exp.ev = [0,0.355,0.457,0.497,1.576,1.577,1.592,1.629,1.632,
          2.97,2.98,3.002,3.004,3.01,3.038,3.821,3.824,3.827,
          3.837,3.856,3.879,3.888,3.895,3.903].*meV;
```

156 Note that only 24 eigenvalues were found experimentally, so this will form a partial LSIEP. To  
157 find the solution system is as simple as:

```
Opt.GradientTolerance = 1e-3; %Use additional stopping criterion.
SysOut= INS_IEP(Sys1,Vary1,Exp,Opt);
```

158 It is possible to find multiple minimising systems even if they do not make any sense physically,  
159 however due to the scaling of the problem a change in the default deflation parameters is  
160 necessary:

```
Opt = struct('NDeflations',5,'Sigma',1e-7,'StepTolerance',1e-3 );
SysOut= INS_IEP(Sys0,Vary,Exp,Opt);
```

161 The output contains five different spin systems that all have the same eigenvalues as input.  
162 Like in example one some of the minima found differ only by a sign change of the  $B_2^2$  parameter,  
163 but all minima have the same exchange term.

<sup>164</sup> Additional examples can be found in the Examples folder.

<sup>165</sup> Note also that using mint ([M. L. Baker, 2022](#)), which is fully compatible with INS\_IEP it is  
<sup>166</sup> possible to simulate the INS spectrum of any calculated system - which can then be compared  
<sup>167</sup> to the experimental spectrum. A description of how to do this can also be found in the  
<sup>168</sup> examples folder.

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