High-gain extended Kalman filter for continuous-discrete systems with asynchronous measurements

Aïda Feddaoui, Nicolas Boizot, Eric Busvelle and Vincent Hugel

May 4, 2018

Abstract

This paper investigates an adaptation of the high-gain Kalman filter for nonlinear continuous-discrete system with multirate sampled outputs under an observability normal form. The contribution of this article is twofold. First, we prove the global exponential convergence of this observer through the existence of bounds for the Riccati matrix. Second, we show that, under certain conditions on the sampling procedure, the observer's asynchronous continuous-discrete Riccati equation is stable and also, that its solution is bounded from above and below. An example, inspired by mobile robotics, with three outputs available is given for illustration purposes.

1 Introduction

The present paper deals with the design of observers for nonlinear multirate sampled-data systems under asynchronous sampling —i.e. control systems having continuous state dynamics and a discrete measurement procedure. This situation arises when the output vector of a control system is obtained through several sensors that do not have the same (possibly non-uniform) sampling rate. Such systems are often met in practice, for instance in global positioning problems, as in [33], or in the field of drone control [10]. Likewise, one can be confronted with such asynchronous systems in the fields of submarine robotics, as can be seen from [3, 31, 6], chemical engineering [41], or cultivation engineering [4].

As it is emphasised in [43], this state estimation problem can be tackled by considering one of the three following options. First, model the state dynamics as discrete and apply a known estimator for discrete state systems —see for example [4] in the linear setting. Second, lift the measurements into the space of continuous functions, e.g. with the help of a polynomial fit as it is done in [41] in the nonlinear setting. Third, directly consider the continuous model for the state dynamics and the discrete model for the measurements. This latter option is the one retained in the present paper, for nonlinear systems, in the framework of high-gain observers [18].

Considering the design of observers, or estimators, for linear multirate stochastic systems, [43] pose the problem in terms of Itô-Volterra equations associated to discrete measurements, which allows them to derive a very general optimal filter in this framework. Using the theory of vibrosolutions of integral equations with discontinuous measures, the authors provide an explicit solution in the form of a Kalman-like estimator. More recently, in [34], the authors model each sensor as a sample-and-hold device and perform a stability analysis based on Lyapunov-Krasovskii functionals. They also consider the problem of determining the maximum time interval between consecutive measurements that guarantees exponential stability. It is addressed under the guise of an optimisation problem in terms of linear matrix inequalities (LMI). In [28], the authors build upon the ideas of [25, 1] where an already designed continuous-time Luenberger-like observer is coupled with asynchronous inter-samples predictors. Finally, the problem under consideration is also addressed by using multirate versions of the Kalman filter, see for instance [27, 4, 24, 17]. In particular, in [17], the authors study the exponential convergence of the proposed observer and the preservation of observability for multirate systems. The present article extends this latter approach to nonlinear systems within the framework of high-gain observers.

In the nonlinear framework, there are many paths one can follow in order to perform data fusion for multirate systems, as it can be seen from [22]. In [41], the authors rely on a fully continuous, Luenberger-type, design where the missing measurements are predicted with the help of a polynomial interpolation method. More recently, [29] uses an already designed continuous-time, Luenberger-like, observer coupled with asynchronous inter-samples predictors. Also relying on a fixed correction gain, [40] propose a continuous observer for multirate systems where the measurements are updated whenever available, the sensors being seen as sample-and-hold devices. In this latter paper, the global exponential stability of the observer is proven assuming that the system under consideration is under an observability normal form distinct from the one used in the present work —see [19] for details.

Let us mention two more contributions based on Luenberger-like designs. In [42], the authors address the problem of robust multirate estimation in the sense that measurements are available in two time scales: *fast* and *slow*. Here, the *slow* measurements are shown to enhance the robustness of the estimation procedure with respect to modelling errors. For this purpose, the state variables need to be (locally) *integral detectable* from the *slow* measurements. Finally, in [11], the authors propose a discrete-time state estimation based on the Taylor series expansion of the system's dynamics. The analysis of the proposed observer follows the ideas of [18] regarding systems that are *observable for any inputs* but without using an explicit high-gain parameter.

A multirate moving horizon estimator is detailed in [30], and relies on a binary switching sequence in order to model the multirate sampling and predictions of the missing measurements.

Finally, the extended Kalman filter design has also been considered for multirate estimation, as it can be seen from [13, 14, 21, 35] where systems having two time scales are considered. In [10] a multirate extended Kalman filter is considered to perform data fusion onboard a small-scale helicopter.

The present paper details the design of a high-gain extended Kalman filter for the state estimation of multirate nonlinear systems. Following the ideas of [23, 12, 9, 17] the proposed observer consists of two steps: (i) an open-loop prediction when no measurements are available, and (ii) an impulsive correction each time a new measurement is available. This second step is performed accordingly to the actually measured outputs which may consist of a subset of the system's output vector only. The global exponential convergence is proven under the hypothesis that the system is under an observability normal form —see e.g. [19, 2, 15]. The main difficulties are, on the one hand, to deal with several non-uniform subdivisions of time in order to represent the asynchronous outputs, and on the other hand, proving that the observer's Riccati equation is bounded over time. This latter issue is handled by following the ideas developed in [7], where only the synchronous setting is considered.

The remainder of the article is as follows. In Section 2, the system under consideration is introduced. In particular, it introduces the notion of *virtual sensor* in order to take into account measurements that are always available at the same time steps. The observer proposed for this class of systems is defined in Section 3. Section 4 deals with the proof of the global exponential convergence of this observer. The demonstration heavily relies on the existence of bounds for the solution to the observer's Riccati equation. For the sake of clarity in the exposure, the proof of this result is given in appendix A. It basically follows the ideas developed in [7], with an increased complexity coming from the asynchronicity of the measurements that makes this exposure necessary. Section 5 is dedicated to an example coming from mobile robotics. Finally, Section 6 concludes the article.

Notations

- A **time subdivision** $\{t_k\}_{k\in\mathbb{N}}$ is meant as a strictly increasing sequence of real numbers with $t_0 = 0$ and $t_k \to \infty$ when $k \to \infty$.
- Id is the identity matrix with appropriate dimensions, $diag[v]$ denotes a diagonal matrix whose elements are the elements of v. Throughout the paper, v can either be a vector or a set of matrices. In this latter case, $diag[v]$ is to be understood as a block-diagonal matrix.
- For a square matrix M, $Tr(M)$ denotes the trace.
- If Ω is a set, we denote by $|\Omega|$ the cardinal of this set.
- *w.r.t.* is used as the short form of *with respect to*, and *s.p.d.* stands for *symmetric positive definite*. Oftentimes, time dependencies are omitted to make the notation less cluttered.

2 System under consideration

Let (Σ_c) be a nonlinear, observable, continuous system under the following observability normal form —see also [16, 19, 39]:

$$\begin{cases} \dot{x}(\tau) = & A(u(\tau))x(\tau) + b(x(\tau), u(\tau)), \quad x(0) = x_0 \\ y(\tau) = & C(\tau)x(\tau) \end{cases} \tag{Σ_c}$$

The state variable $x(\tau)$ lies in a compact subset χ of $\mathbb{R}^n$, the output $y(\tau)$ is in $\mathbb{R}^{n_y}$ and the input vector $u(\tau)$, which belongs to $\mathcal{U}_{adm} \subset R^{n_u}$, is bounded for all times. The state variable is decomposed into n_y subvectors as follows:

$$x(\tau) = (x_1(\tau), x_2(\tau), ..., x_{n_y}(\tau))'$$

where, for all $i \in \{1, ..., n_y\}$, $x_i(\tau)$ is in the compact subset $\chi_i \subset \mathbb{R}^{n_i}$ (and $\sum_{i=1}^{n_y} n_i = n$). Each subvector $x_i(\tau)$ is associated to the i^{th} output $y^i(\tau)$ and is written

$$x_i(\tau) = (x_i^1(\tau), x_i^2(\tau), ..., x_i^{n_i}(\tau))'$$

The dynamics of $x_i(\tau)$ are described by:

$$\begin{cases} \dot{x}_i(\tau) & = & A_i(u(\tau))x_i(\tau) + b_i(x(\tau), u(\tau)) \\ y^i(\tau) & = & C^i(u(\tau))x_i(\tau) \end{cases}$$

- $A_i(u)$ and $C^i(u)$ are, respectively, (n_i, n_i) and $(1 \times n_i)$ matrices of the form

$$A_i(u) = \begin{pmatrix} 0 & a_i^2(u) & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_i^{n_i}(u) \\ 0 & \dots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C^i(u) = \begin{pmatrix} a_i^1(u) & 0 & \cdots & 0 \end{pmatrix}$$

  where, for all $i \in \{1, ..., n_y\}$, $j \in \{1, ..., n_i\}$, $u \in \mathcal{U}_{adm}$, $0 < a_m < |a_i^j(u)| < a_M$. Moreover, we suppose that the elements of the C^i matrices are derivable at least once and have their derivative bounded over time[1].

- $b_i(x, u)$ is a $\mathcal{C}^1$ triangular, compactly supported, vector field whose last component is allowed to depend on the full state of (Σ_c):

$$b_i(x, u) = \begin{pmatrix} b_i^1(x_i^1, u) \\ b_i^2(x_i^1, x_i^2, u) \\ \vdots \\ b_i^{n_i-1}(x_i^1, ..., x_i^{n_i-1}, u) \\ b_i^{n_i}(x, u) \end{pmatrix}$$

[1] Although restrictive, this condition is necessary in order to apply Lemma (12) to a time varying matrix C.

We assume that, for all i, the Jacobian matrix $D_x b_i(x,u)$ of $b_i(x,u)$, computed *w.r.t.* x, is upper bounded by $L_b > 0$. Therefore, $b_i(x,u)$ has the Lipschitz property *w.r.t.* x (uniformly *w.r.t.* u): $\|b_i(x_i(t),u) - b_i(\tilde{x}_i(t),u)\| \leq L_b \|x_i(t) - \tilde{x}_i(t)\|$.

Finally, the full dynamics of system (Σ_c) are given by

$$A(u) = diag\left[A_1, \ldots, A_{n_y}\right], \quad b(x,u) = \begin{pmatrix} b_1(x,u) \\ \vdots \\ b_{n_y}(x,u) \end{pmatrix} \text{ and } C(u) = diag\left[C^1, \ldots, C^{n_y}\right]$$

To this plant, we associate the following continuous-discrete system with asynchronous, or multirate, measurements

$$\begin{cases} \dot{x}(\tau) = & A(u(\tau))x(\tau) + b(x(\tau), u(\tau)), \qquad x(0) = x_0 \\ y(\tau_k) = & C_{\sigma_k} x(\tau_k) \end{cases} \tag{Σ_{cda}}$$

Contrary to what was proposed in a previous work [17], the asynchronous measurement procedure is not modelled with respect to each output. Instead, we lump together outputs always available at the same time.

1. Let a **sensor** be a non-empty subset $\mathfrak{s}_i \subset \{1, \ldots, n_y\}$. It is associated to a vector $y^{(\mathfrak{s}_i)}(\tau) = \left\{y^j(\tau) \ : \ j \in \mathfrak{s}_i\right\}$. There are n_s sensors, with $0 < n_s \leq n_y$. In this work, we assume that the set of all sensors of (Σ_{cda}) is a partition of the set $\{1, \ldots, n_y\}$. Henceforth, it is assumed that a sensor is made of consecutive indexes. Indeed, this can always be achieved via a simple re-ordering of the output and state variables.

   The $(|\mathfrak{s}_i| \times n)$ output matrix corresponding to a sensor $\mathfrak{s}_i$ is denoted $C^{(\mathfrak{s}_i)}$ and is such that $y^{(\mathfrak{s}_i)}(\tau) = C^{(\mathfrak{s}_i)} x(\tau)$.

2. A subdivision of time $\left\{s_k^{(i)}\right\}_{k \in \mathbb{N}}$ is associated to each sensor $\mathfrak{s}_i$, and the global time subdivision $\{\tau_k\}_{k \in \mathbb{N}}$ is defined as:

   $$\{\tau_k\}_{k \in \mathbb{N}} := \bigcup_{i=1}^{n_s} \left\{s_l^{(i)}\right\}_{l \in \mathbb{N}}$$

   where elements belonging to several subdivisions $\left\{s_l^{(i)}\right\}$ are considered only once.

3. For each τ_k there exist at least one sensor $\mathfrak{s}_i$ such that $s_l^{(i)} = \tau_k$ for some index l. Let σ_k denote the set of such sensors:

   $$\sigma_k = \left\{i \in \{1, ..., n_s\} | \exists \ l \in \mathbb{N} \text{ such that } s_l^{(i)} = \tau_k\right\}.$$

The above mentioned l index, probably differs from k, and is not the same from sensor to sensor. As such, for all $i \in \sigma_k$, $l_k^{(i)}$ denotes the index $l \in \mathbb{N}$ such that $s_l^{(i)} = \tau_k$.

The matrix C_{σ_k} associated to the set σ_k is the $\left(\sum_{i\in\sigma_k} |\mathfrak{s}_i| \times n\right)$ matrix made of the $C^{(\mathfrak{s}_i)}$ matrices that correspond to the output actually available at time τ_k:

$$C_{\sigma_k} = \begin{pmatrix} \vdots \\ C^{(\mathfrak{s}_i)} \\ \vdots \end{pmatrix}_{i\in\sigma_k} \qquad \text{and thus} \quad y_k = y(\tau_k) = C_{\sigma_k} x(\tau_k).$$

3 Definition of the multirate high-gain Kalman filter

The continuous-discrete asynchronous high-gain Kalman filter is defined in two parts:

1. two prediction equations when $\tau \in [\tau_{k-1}, \tau_k[$, $k \in \mathbb{N}^*$, with initial values $z_{k-1}^{(+)}$ and $S_{k-1}^{(+)}$;
2. two correction equations at time τ_k.

Notations:

- $z(\tau)$ is the estimated state for all $\tau \in [\tau_{k-1}, \tau_k[$;
- $z_k^{(-)}$ is the estimated state at time τ_k, at the end of a prediction step and before a correction step;
- $z_k^{(+)}$ is the estimated state at time τ_k, after a correction step. Therefore, $z_k^{(+)}$ is the initial estimated state of the new prediction interval $[\tau_k, \tau_{k+1}[$.

Prediction equations

$$\begin{cases} \dot{z}(\tau) = & A(u)z(\tau) + b(z,u) \\ \dot{S}(\tau) = & -\left(A(u) + D_x b(z,u)\right)' S(\tau) - S(\tau)\left(A(u) + D_x b(z,u)\right) - (SQ_\theta S)(\tau) \end{cases} \tag{$\mathcal{O}_1$}$$

Correction equations

$$\begin{cases} z_k^{(+)} = & z_k^{(-)} - {S_k^{(+)}}^{-1} \displaystyle\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)\prime} \left(R_\theta^{(\mathfrak{s}_i)}\right)^{-1} \left(C^{(\mathfrak{s}_i)} z_k^{(-)} - y_k^{(\mathfrak{s}_i)}\right) \left(s_{l_k^{(i)}}^{(i)} - s_{l_k^{(i)}-1}^{(i)}\right) \\ S_k^{(+)} = & S_k^{(-)} + \displaystyle\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)\prime} \left(R_\theta^{(\mathfrak{s}_i)}\right)^{-1} C^{(\mathfrak{s}_i)} \left(s_{l_k^{(i)}}^{(i)} - s_{l_k^{(i)}-1}^{(i)}\right) \end{cases} \tag{$\mathcal{O}_2$}$$

In other words, the correction at a time τ_k is made with respect to each measure $y_k^{(\mathfrak{s}_i)}$ that is actually available and involves a weighting factor equal to the time elapsed since the last time this specific output was measured.

The matrices Q_θ and $R_\theta^{(\mathfrak{s}_i)}$ —which can be time dependent[2] provided the constraints (1)-(2) below are met— are of the form

$$Q_\theta = \theta \Delta^{-1} Q \Delta^{-1} \quad \text{and} \quad R_\theta^{(\mathfrak{s}_i)} = \frac{1}{\theta} \delta^{(\mathfrak{s}_i)} R^{(\mathfrak{s}_i)} \delta^{(\mathfrak{s}_i)} \quad \text{where:}$$

- Q and $R^{(\mathfrak{s}_i)}$ are *s.p.d.* matrices, of dimensions $(n \times n)$ and $(|\mathfrak{s}_i| \times |\mathfrak{s}_i|)$ respectively which must lie in compact subsets such that:

$$\underline{q} \text{ Id } \leq Q \leq \overline{q} \text{ Id} \qquad \text{with } 0 < \underline{q} < \overline{q} \tag{1}$$

$$\underline{r}_i \text{ Id } \leq R^{(\mathfrak{s}_i)} \leq \overline{r}_i \text{ Id} \qquad \text{with } 0 < \underline{r}_i < \overline{r}_i \tag{2}$$

- $\delta^{(\mathfrak{s}_i)}$ and Δ are both diagonal matrices which construction relies on the quantity $n^* = \max(n_1, n_2, \ldots, n_{n_y})$ and on a fixed scalar $\theta \geq 1$:

$$\Delta = diag\left[\Delta_1, \ldots, \Delta_{n_y}\right] \quad \text{where} \quad \Delta_i = diag\left[\frac{1}{\theta^{n^*-n_i}}, \ldots, \frac{1}{\theta^{n^*-1}}\right]$$

$$\text{and} \quad \delta^{(\mathfrak{s}_i)} = diag\left[\left\{\theta^{n^*-n_j} \ : \ j \in (\mathfrak{s}_i)\right\}\right].$$

  Finally, $R = diag\left[R^{(\mathfrak{s}_1)}, \ldots, R^{(\mathfrak{s}_{n_s})}\right]$ and $R_\theta = diag\left[R_\theta^{(\mathfrak{s}_1)}, \ldots, R_\theta^{(\mathfrak{s}_{n_s})}\right]$, or equivalently, if one defines $\delta = diag\left[\delta^{(\mathfrak{s}_1)}, \ldots, \delta^{(\mathfrak{s}_{n_s})}\right]$: $R_\theta = \frac{1}{\theta}\delta R \delta$.

- The initial datum of the observer is made of the initial estimated state $z(0) \in \chi \subset \mathbb{R}^n$ and of $S(0)$, a *s.p.d.* matrix.

Remark 1. *1. The two matrices Q_θ and R_θ, built according to the normal form of an observable system constitute the high-gain formalism. The fixed parameter θ is the so-called high-gain parameter. When $\theta = 1$, the proposed observer is a simple extended Kalman filter for which the normal form allows to prove local convergence only —see e.g. [8].*

Although out of scope of the present work, a worth mentioning issue is the study of methods that allow to define and run the observer in the original coordinates of the system instead of the normal coordinates. Interested readers can refer to, e.g., [5, 26, 36] and references herein.

2. Although the definitions of Δ and δ may appear uselessly intricate, they are necessary in order to simplify forthcoming computations, in particular by preserving the Lipschitz constant of vector field $b(x, u)$ despite the change of variables performed at the beginning of the proof of convergence (cf. Sec. 4).

[2] This time dependency is not explicitly written in the observer's equations to make the notations less cluttered.

3. *According to Equation ($\mathcal{O}_2$), the matrix R cannot be taken s.p.d. Indeed, the elements of R that relates outputs that are not available at the same sampling time are not considered in ($\mathcal{O}_2$). In [17], R was a diagonal matrix. Here, our definition of sensors allows us to consider correlations between outputs that are always available at the same time —and usually given by the same physical sensor.*

4 Proof of convergence

This section is dedicated to the proof of convergence of observer ($\mathcal{O}_1$)-($\mathcal{O}_2$). It relies on the analysis of the dynamics of the estimation error: $\varepsilon(\tau) = z(\tau) - x(\tau)$, and is divided into two parts: the *preparation for the proof*, and the *exponential convergence*.

Preparation for the proof

Let us first consider the change of variables $\tilde{x} = \Delta x$, $\tilde{z} = \Delta z$ and $\tilde{\varepsilon} = \Delta\varepsilon$. We also denote $\tilde{b}(.,u) = \Delta b(\Delta^{-1}.,u)$, $D_x\tilde{b}(.,u) = \Delta D_x b(\Delta^{-1}.,u)\Delta^{-1}$ and $\tilde{S} = \Delta^{-1}S\Delta^{-1}$.

Lemma 1. [19]

1. *The vector field $\tilde{b}(\tilde{x},u)$ has the same Lipschitz constant as $b(x,u)$.*
2. *The Jacobian $D_x\tilde{b}(\tilde{x},u)$ has the same bound as $D_x b(x,u)$.*
3. *We also have the following relations:*
   - $\Delta A = \theta A\Delta$, *and* $A\Delta^{-1} = \theta\Delta^{-1}A$;
   - $\delta^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)}\Delta^{-1} = C^{(\mathfrak{s}_i)}$;
   - $\Delta^{-1}C^{(\mathfrak{s}_i)'} R_\theta^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)'}\Delta^{-1} = \theta C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)}$.

This change of variables allows us to remove the θ-dependance of the matrices $R_\theta^{(\mathfrak{s}_i)}$ and Q_θ. With the help of the relations given in Lemma 1, the observer's equations ($\mathcal{O}_1$)-($\mathcal{O}_2$) become:

$$\begin{cases} \dot{\tilde{z}}(\tau) = & \theta A(u)\tilde{z}(\tau) + \tilde{b}(\tilde{z},u) \\ \dot{\tilde{S}}(\tau) = & -\left(\theta A(u) + D_x\tilde{b}(\tilde{z},u)\right)'\tilde{S} - \tilde{S}\left(\theta A(u) + D_x\tilde{b}(\tilde{z},u)\right) - \theta\tilde{S}Q\tilde{S} \end{cases} \qquad (\tilde{\mathcal{O}}_1)$$

$$\begin{cases} \tilde{z}_k^{(+)} = & \tilde{z}_k^{(-)} \\ & -\theta\tilde{S}_k^{(+)^{-1}}\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\tilde{z}_k^{(-)} - \tilde{x}(\tau_k)\right)\left(s^{(i)}_{l_k^{(i)}} - s^{(i)}_{l_k^{(i)}-1}\right) \\ \tilde{S}_k^{(+)} = & \tilde{S}_k^{(-)} + \theta\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(s^{(i)}_{l_k^{(i)}} - s^{(i)}_{l_k^{(i)}-1}\right) \end{cases} \qquad (\tilde{\mathcal{O}}_2)$$

In order to proceed with the proof, we want to be able to bound all the elements of $\theta A(u) + D_x\tilde{b}(\tilde{z}, u)$, independently from θ. This is true for the lower bound since $\theta \geq 1$, but not for the upper bound. This issue is resolved with the help of a time reparametrization.

Let $\bar{\tau}$ be such that $\bar{\tau} = \theta\tau$. This infers a change on the subdivisions $\{\tau_k\}_{k\in\mathbb{N}}$ and $\left\{s_k^{(i)}\right\}_{k\in\mathbb{N}}$ for all $i \in \{1, ..., n_s\}$, as follows: $\bar{\tau}_k = \theta\tau_k$, and $\bar{s}_k^{(i)} = \theta s_k^{(i)}$ for all $k \in \mathbb{N}$. Moreover, we use the notation $\bar{z}(\bar{\tau}) = \tilde{z}(\tau)$ in the new time frame. The observer is now given by the set of equations ($\bar{\mathcal{O}}_1$)-($\bar{\mathcal{O}}_2$):

$$\begin{cases} \dfrac{d\bar{z}(\bar{\tau})}{d\bar{\tau}} &= A(\bar{u})\bar{z}(\bar{\tau}) + \frac{1}{\theta}\tilde{b}(\bar{z}, \bar{u}) \\ \dfrac{d\bar{S}(\bar{\tau})}{d\bar{\tau}} &= -\left(A(\bar{u}) + \frac{1}{\theta}D_x\tilde{b}(\bar{z}, \bar{u})\right)'\bar{S} - \bar{S}\left(A(\bar{u}) + \frac{1}{\theta}D_x\tilde{b}(\bar{z}, \bar{u})\right) - \bar{S}Q\bar{S} \end{cases} \tag{$\bar{\mathcal{O}}_1$}$$

$$\begin{cases} \bar{z}_k^{(+)} = & \bar{z}_k^{(-)} - \bar{S}_k^{(+)^{-1}} \displaystyle\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{z}_k^{(-)} - \bar{x}(\bar{\tau}_k)\right)\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) \\ \bar{S}_k^{(+)} = & \bar{S}_k^{(-)} + \displaystyle\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) \end{cases} \tag{$\bar{\mathcal{O}}_2$}$$

Exponential convergence

The rest of the proof is based on a Lyapunov function argument, the candidate function being $V(\bar{\varepsilon}) = (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})(\bar{\tau})$. Provided that $\bar{S}(\bar{\tau})$ remains *s.p.d.* then, $V(\bar{\varepsilon}) > 0$ for all $\varepsilon \neq 0_{\mathbb{R}^n}$. In the sequel, after stating a theorem that ensures the stability of the matrix $\bar{S}$, we compute the time derivative $V(\bar{\varepsilon})$ in order to display the exponential convergence of the proposed observer.

Theorem 2.

Let us consider the asynchronous, continue-discrete, Riccati equation of observer ($\bar{\mathcal{O}}_1$)*-*($\bar{\mathcal{O}}_2$)*, that is to say, with* $\mathcal{A} = A(\bar{u}) + \frac{1}{\theta}D_x\tilde{b}(\bar{z}, \bar{u})$*:*

$$\begin{cases} \dfrac{d\bar{S}(\bar{\tau})}{d\bar{\tau}} = & -\bar{\mathcal{A}}'\bar{S} - \bar{S}\bar{\mathcal{A}} - \bar{S}Q\bar{S} \\ \bar{S}_k^{(+)} = & \bar{S}_k^{(-)} + \displaystyle\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) \end{cases} \tag{3}$$

Here, $(\mathcal{A}, C)$ *is a time-dependent observable pair (in the classical sense) having elements belonging to the set*

$$\mathcal{A}_B = \left\{a = (a_{i,j}) \in L^\infty\left([0,T], \mathbb{R}^\eta\right) : \sup_{i,j}|a_{i,j}|_\infty \leq B \quad \textit{with} \quad B > 0\right\}.$$

Moreover, all the elements of C *are derivable at least once and have their derivative bounded over time.*

Then, $\bar{S}(\bar{\tau})$ is well defined and is s.p.d. for all times. Moreover, for all $\bar{T} > 0$, there exists constants $\mu_i > 0$, $i \in \{1, ..., n_s\}$, and $0 < \alpha < \beta$, such that, for all subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$, $\{\bar{\tau}_k\}_{k\in\mathbb{N}}$ with $\left(\bar{s}_k^{(i)} - \bar{s}_{k-1}^{(i)}\right) \leq \mu_i$, we have:

$$\alpha Id \leq \bar{S}(\bar{\tau}) \leq \beta Id \qquad \text{for all } \bar{\tau} \geq \bar{T}.$$

The constants α and β are independent from θ and the shape of the subdivisions.

Proof. The proof is detailed in Appendix A. ☐

Let us now resume the convergence study with the computation of $\frac{d}{d\bar{\tau}}V(\bar{\varepsilon})$:

$$\frac{d\bar{\varepsilon}}{d\bar{\tau}}(\bar{\tau}) = \frac{d}{d\bar{\tau}}(\bar{z} - \bar{x})(\bar{\tau}) = A(\bar{u})\bar{\varepsilon} + \frac{1}{\theta}\left(b(\bar{z}, \bar{u}) - b(\bar{x}, \bar{u})\right) \tag{4}$$

$$\begin{aligned} \frac{dV}{d\bar{\tau}}(\bar{\varepsilon}) &= \frac{d(\bar{\varepsilon}'\bar{S}\bar{\varepsilon})}{d\bar{\tau}}(\bar{\tau}) \\ &= \frac{2}{\theta}\left(\bar{\varepsilon}'\bar{S}\right)\left(\tilde{b}(\bar{z}, \bar{u}) - \tilde{b}(\bar{x}, \bar{u}) - D\tilde{b}(\bar{z}, \bar{u})\bar{\varepsilon}\right) - \left(\bar{\varepsilon}'\bar{S}Q\bar{S}\bar{\varepsilon}\right). \end{aligned} \tag{5}$$

Next, we determine the expression of $V_k^{(+)}$ —i.e. $V(\bar{\tau}_k)$ after a prediction step:

$$\begin{aligned} \bar{\varepsilon}_k^{(+)} &= \bar{z}_k^{(+)} - \bar{x}(\bar{\tau}_k) \\ &= \left[Id - \bar{S}_k^{(+)^{-1}} \sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)\right] \bar{\varepsilon}_k^{(-)} \end{aligned}$$

On the other hand, in $\bar{\tau}_k$, we have:

$$\begin{aligned} V_k^{(+)} &= \left(\bar{\varepsilon}'\bar{S}\bar{\varepsilon}\right)_k^{(+)} \\ &= \bar{\varepsilon}_k^{(-)'}\left[\bar{S}_k^{(+)} - 2\mathcal{M} + \mathcal{M}\bar{S}_k^{(+)^{-1}}\mathcal{M}\right]\bar{\varepsilon}_k^{(-)} \end{aligned} \tag{6}$$

where

$$\mathcal{M} = \sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right) = \bar{S}_k^{(+)} - \bar{S}_k^{(-)} \tag{7}$$

In (6), the matrix $\mathcal{M}$ is replaced by the right-hand side of (7). Simplifications lead to:

$$V_k^{(+)} = \bar{\varepsilon}_k^{(-)'}\left[\bar{S}_k^{(-)^{-1}}\bar{S}_k^{(+)}\bar{S}_k^{(-)^{-1}}\right]^{-1}\bar{\varepsilon}_k^{(-)}$$

Using ($\mathcal{O}_2$) again allows us to write:

$$\begin{aligned} \bar{S}_k^{(-)^{-1}}\bar{S}_k^{(+)}\bar{S}_k^{(-)^{-1}} &= \bar{S}_k^{(-)^{-1}}\left[\bar{S}_k^{(-)} + \sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)\right]\bar{S}_k^{(-)^{-1}} \\ &= \bar{S}_k^{(-)^{-1}} + \bar{S}_k^{(-)^{-1}} \underbrace{\sum_{i\in\sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)\bar{S}_k^{(-)^{-1}}}_{(\star)} \end{aligned}$$

Before going any further, let us remind the matrix inversion lemma.

Lemma 3 (Matrix inversion lemma)**.**

Let M be a s.p.d. matrix and R an invertible matrix. then

$$(M + MC'R^{-1}CM)^{-1} = M^{-1} - C'(R + CMC')^{-1}C.$$

In order to use this lemma, it is necessary to express the sum of matrices that appear in expression $(\star)$ as a product of matrices. First, let us denote:

$$R_{\sigma_k} = diag\left[\left\{R^{(\mathfrak{s}_i)} : i \in \sigma_k\right\}\right] \text{ and } I_{\sigma_k} = diag\left[\left\{\left(\bar{s}^{(i)}_{l^{(i)}_k} - \bar{s}^{(i)}_{l^{(i)}_k - 1}\right) Id : i \in \sigma_k\right\}\right].$$

Then the sum in $(\star)$ can be written:

$$\sum_{i \in \sigma_k} C^{(\mathfrak{s}_i)'} R^{(\mathfrak{s}_i)^{-1}} C^{(\mathfrak{s}_i)} \left(\bar{s}^{(i)}_{l^{(i)}_k} - \bar{s}^{(i)}_{l^{(i)}_k - 1}\right) = C'_{\sigma_k} R^{-1}_{\sigma_k} I_{\sigma_k} C_{\sigma_k}$$

By definition, I_{σ_k} is invertible and using[3] Lemma 3:

$$\begin{aligned}
\left[\bar{S}_k^{(-)^{-1}} \bar{S}_k^{(+)} \bar{S}_k^{(-)^{-1}}\right]^{-1} &= \left[\bar{S}_k^{(-)^{-1}} + \bar{S}_k^{(-)^{-1}} C'_{\sigma_k} R^{-1}_{\sigma_k} I_{\sigma_k} C_{\sigma_k} \bar{S}_k^{(-)^{-1}}\right]^{-1} \\
&= \bar{S}_k^{(-)} - C'_{\sigma_k} \left(R_{\sigma_k} I^{-1}_{\sigma_k} + C_{\sigma_k} \bar{S}_k^{(-)^{-1}} C'_{\sigma_k}\right)^{-1} C_{\sigma_k}
\end{aligned}$$

Therefore, we obtain the following system, for all $k \in \mathbb{N}$:

$$\begin{cases}
\dfrac{dV}{d\bar{\tau}}(\bar{\varepsilon}) = & \frac{2}{\theta}(\bar{\varepsilon}'\bar{S})[\tilde{b}(\bar{z},\bar{u}) - \tilde{b}(\bar{x},\bar{u}) - D_x\tilde{b}(\bar{z},\bar{u})\bar{\varepsilon}] - (\bar{\varepsilon}'\bar{S}Q\bar{S}\bar{\varepsilon}) \qquad \text{for } \bar{\tau} \in [\bar{\tau}_{k-1}, \bar{\tau}_k[\\
V_k^{(+)} = & (\bar{\varepsilon}'\bar{S}\bar{\varepsilon})_k^{(-)} - \bar{\varepsilon}_k^{(-)'} C'_{\sigma_k} \left(R_{\sigma_k} I^{-1}_{\sigma_k} + C_{\sigma_k} \bar{S}_k^{(-)^{-1}} C'_{\sigma_k}\right)^{-1} C_{\sigma_k} \bar{\varepsilon}_k^{(-)} \qquad \text{for } \bar{\tau} = \bar{\tau}_k
\end{cases}$$

Since $\tilde{b}(.,\bar{u})$ is Lipschitz in its first argument (uniformly *w.r.t* $\bar{u}$) and $D_x\tilde{b}(.,\bar{u})$ is upper bounded:

$$\begin{aligned}
\|\tilde{b}(\bar{z},\bar{u}) - \tilde{b}(\bar{x},\bar{u}) - D_x\tilde{b}(\bar{z},\bar{u})\bar{\varepsilon}\| &\leq L_b\|\bar{z} - \bar{x}\| + L_b\|\bar{\varepsilon}\| \\
&= 2L_b\|\bar{\varepsilon}\|
\end{aligned}$$

Theorem 2 provides bounds for $\bar{S}$ for times greater than a fixed $\bar{T} > 0$, and the constraints on Q are given in (1), thus leading to:

$$\frac{dV}{d\bar{\tau}}(\bar{\varepsilon}) \leq \left(\frac{4}{\theta}\frac{\beta}{\alpha}L_b - \alpha\underline{q}\right)(\bar{\varepsilon}'\bar{S}\bar{\varepsilon})$$

Since $\bar{S}(\bar{\tau})$ is positive definite, the derivative of $V(\bar{\varepsilon})$ is negative for θ chosen such that $\frac{4}{\theta}\frac{\beta}{\alpha}L_b - \alpha\underline{q} < 0$. Furthemore, we easily show that $V_k^{(+)} \leq V_k^{(-)}$ for all $k \in \mathbb{N}$. Indeed:

$$V_k^{(+)} = V_k^{(-)} - \bar{\varepsilon}_k^{(-)'} \underbrace{C_{\sigma_k}{}' \left(R_{\sigma_k} I^{-1}_{\sigma_k} + C_{\sigma_k} \bar{S}_k^{(-)^{-1}} C_{\sigma_k}{}'\right)^{-1} C_{\sigma_k}}_{(\diamond)} \bar{\varepsilon}_k^{(-)}.$$

[3]Note that, matrices R_{σ_k} and $I^{-1}_{\sigma_k}$ do commute. Indeed, by definition, each blocks of R_{σ_k} correspond to a block of $I^{-1}_{\sigma_k}$ made of an identity matrix times some constant parameter.

Since $\bar{S}(\bar{\tau})$ and $R_{\sigma_k} I_{\sigma_k}^{-1}$ are *s.p.d.* for all times, then the matrix $(\diamond)$ is at least positive semidefinite and:

$$V_k^{(+)} = (\bar{\varepsilon}' \bar{S} \bar{\varepsilon})_k^{(+)} \leq (\bar{\varepsilon}' \bar{S} \bar{\varepsilon})_k^{(-)} \quad \text{for all } k \in \mathbb{N}. \tag{8}$$

This shows the asymptotic convergence of the observer. Moreover, this convergence is exponential. Indeed, let $\bar{\tau}_k^\star = \min_{k\in\mathbb{N}} \left\{ \bar{\tau}_k : \bar{\tau}_k > \bar{T} \right\}$, then for $\bar{\tau} \in]\bar{T}, \bar{\tau}_k^\star[$:

$$\begin{aligned}
(\bar{\varepsilon}' \bar{S} \bar{\varepsilon})(\bar{\tau}) &= (\bar{\varepsilon}' \bar{S} \bar{\varepsilon})(\bar{T}) + \left(\frac{4}{\theta} \frac{\beta}{\alpha} L_b - \alpha \underline{q} \right) \int_{\bar{T}}^{\bar{\tau}} (\bar{\varepsilon}' \bar{S} \bar{\varepsilon})(v) dv \\
&\leq (\bar{\varepsilon}' \bar{S} \bar{\varepsilon})(\bar{T}) e^{(L_b \frac{4}{\theta} \frac{\beta}{\alpha} - \alpha \underline{q})(\bar{\tau} - \bar{T})}
\end{aligned} \tag{9}$$

where (9) has been obtained by using Grönwall's lemma.
For $\bar{\tau} \in [\bar{\tau}_k, \bar{\tau}_{k+1}[$, inequality (9) is true with $\bar{\tau}_k$ replacing $\bar{T}$. Then, using relation (8) we show by iteration that (9) is in fact true for all $\bar{\tau} > \bar{T}$, independently from the subdivions $\left\{ \bar{s}_k^{(i)} \right\}_{k\in\mathbb{N}}$ and $\left\{ \bar{\tau}_k^{(i)} \right\}_{k\in\mathbb{N}}$.

Since $\bar{\varepsilon}(\bar{\tau}) = \tilde{\varepsilon}(\tau)$, then inequality (9) becomes:

$$\|\tilde{\varepsilon}(\tau)\|^2 \leq \frac{\beta}{\alpha} \left\| \tilde{\varepsilon}\left(\frac{\bar{T}}{\theta} \right) \right\|^2 e^{\left(4 L_b \frac{\beta}{\alpha} - \theta \alpha \underline{q}\right)(\tau - \frac{\bar{T}}{\theta})} \quad \text{for all } \tau > \frac{\bar{T}}{\theta}.$$

Finally, following the definition of $\varepsilon(\tau) = \Delta^{-1} \tilde{\varepsilon}(\tau)$, since $\|\Delta^{-1}\| \leq \theta^{n^*-1}$ and $\|\Delta\| \leq 1$, we conclude that

$$\|\varepsilon(\tau)\|^2 \leq \theta^{2(n^*-1)} \frac{\beta}{\alpha} \left\| \varepsilon\left(\frac{\bar{T}}{\theta} \right) \right\|^2 e^{\left(4 L_b \frac{\beta}{\alpha} - \theta \alpha \underline{q}\right)(\tau - \frac{\bar{T}}{\theta})} \quad \text{for all } \tau > \frac{\bar{T}}{\theta}.$$

5 Illustrative example

Let us consider a boat[4] evolving in an area delimited by two beacons (denoted by A and B). As it is schematised in Figure 1a, $X = \binom{x}{y} \in \mathbb{R}^2$ is the position of the boat *w.r.t.* the reference frame attached to A, and $\theta \in \mathbb{R}$ is the orientation of the boat, that is the angle formed by axes $\mathcal{X}_r$ and $\mathcal{X}_b$ —this latter axis defining a reference frame attached to the boat.

An onboard rotational position sensor provides —when aligned with one of the two beacons— a measurement of the angle formed by axis $\mathcal{X}_b$ and the line that joins X to the concerned beacon. Furthermore, we assume that the signal received from A also provides the distance between A and the boat.

The dynamics are simply modelled by the following system:

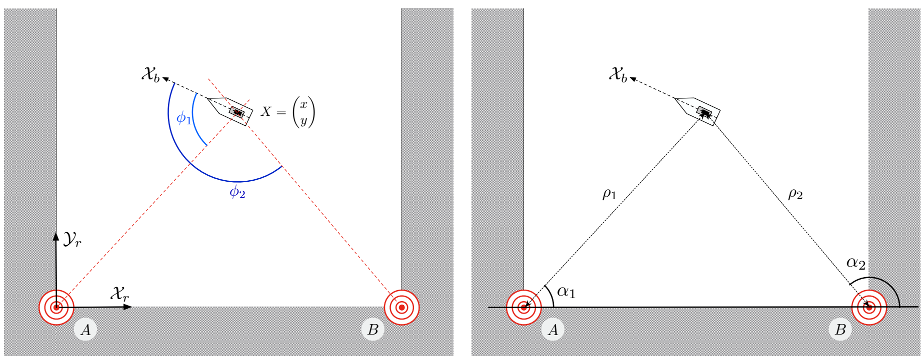
$$\begin{cases}
\dot{x}(\tau) = & v(\tau)\cos(\theta(\tau)) \\
\dot{y}(\tau) = & v(\tau)\sin(\theta(\tau)) \\
\theta(\tau) = & u(\tau)
\end{cases} \tag{Σ_{Boat}}$$

[4] Or a wheeled mobile robot.

where $u(\tau)$ and $v(\tau)$ are the controls.

In order to deal with simple equations for the output vector, the system is rewritten using polar coordinates *w.r.t.* both A and B —cf. Figure 1b. Let x_B be the abscissa of B in the $(\mathcal{X}_r, \mathcal{Y}_r)$ reference frame. Then, the angles α_1, α_2 and distances ρ_1, ρ_2 are such that:

- $x = \rho_1 \cos(\alpha_1)$ and $y = \rho_1 \sin(\alpha_1)$;
- $x = x_B + \rho_2 \cos(\alpha_2)$ and $y = \rho_2 \sin(\alpha_2)$.



(a) Notations (planar representation).

(b) Notations (polar representations).

Figure 1

In those new coordinates, the full dynamics are given by

$$\begin{cases} \dot{\alpha}_1 = & \frac{v}{\rho_1} \sin(\theta - \alpha_1) \\ \dot{\rho}_1 = & v \cos(\theta - \alpha_1) \\ \dot{\alpha}_2 = & \frac{v}{\rho_2} \sin(\theta - \alpha_2) \\ \dot{\rho}_2 = & v \cos(\theta - \alpha_2) \\ \dot{\theta} = & u \end{cases} \quad \text{and} \quad \xi(t) = \begin{pmatrix} \phi_1 \\ \rho_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \pi + \alpha_1 - \theta \\ \rho_1 \\ \pi + \alpha_2 - \theta \end{pmatrix} \tag{10}$$

This system is observable, see e.g. [38] and references herein. Actually, the exact position of the boat can be computed from the knowledge of $\phi_2 - \phi_1$ and ρ_1. The orientation of the boat can then be easily deduced. However, without both angle measurements observability is lost, and the rotational position sensor makes the angle measurements asynchronous which makes it an appropriate example.

System (10) is equipped with two sensors $\mathfrak{s}_1 = \{1, 2\}$ and $\mathfrak{s}_2 = \{3\}$. Note that with a quick enough rotational speed of the sensor, measurements take place alternatively. However, sampling times are non-uniform since the boat trajectory plays a role in determining the time when a measurement is available.

In order to apply the asynchonous high-gain Kalman filter under discussion, we put (10) under the normal observability form below by defining $z(\tau) = \xi(\tau)$.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \dfrac{v\sin(z_1)}{z_2} - u \\ -v\cos(z_1) \\ \dfrac{v\sin(z_3)}{\bar{\rho}(z_1, z_2, z_3)} - u \end{pmatrix} \tag{11}$$

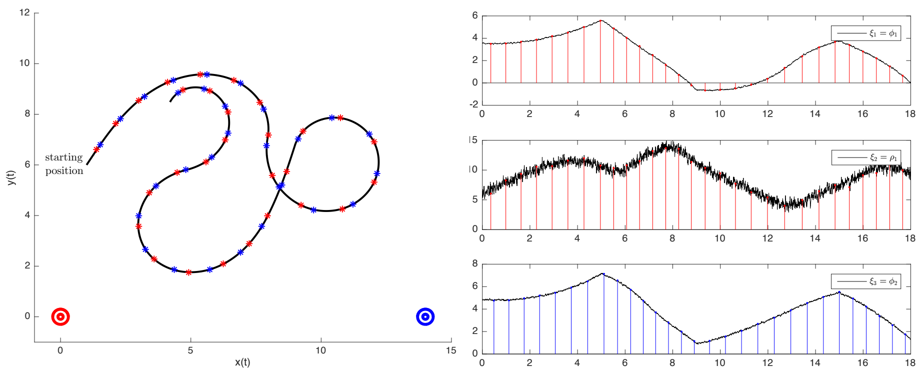
where $\bar{\rho}(z_1, z_2, z_3) = z_2\cos(z_3 - z_1) + \sqrt{x_B^2 - z_2^2}\sin^2(z_3 - z_1)$ is the expression of ρ_2 (as a function of $\phi_2 - \phi_1 = z_3 - z_1$ and $\rho_1 = z_2$) obtained through the law of cosines.

We considered the boat trajectory shown in Figure 2a. Here, the boat's speed (i.e. $v(\tau)$) is kept constant except for $\tau \in [5, 10]$ where it is momentarily raised to a higher constant value. The initial state of system is $x_0 = (1, 6, 1)$ and the initial state of the observer is directly set in the normal coordinates[5]. The Riccati equation's initial datum is set by solving an algebraic Riccati equation using the informations available at time $t = 0$.

The estimated trajectory, compared to the actual boat trajectory is shown in Figure 3. Figure 3a highlights the increased convergence speed due to a large high-gain parameter. Let us remark that when the high-gain parameter equals 1, the displayed observer fails to achieve convergence. Figure 3b shows the performance of the observer when additive noise is introduced in the output. We used a gaussian noise[6] colored through a first order discrete filter, as it is illustrated in Figure 2b. Because of the known sensibility of high-gain observers with respect to noise, only the lower value of the high-gain parameter was considered in the second experiment. This tradeoff between convergence efficiency and robustness *w.r.t.* measurement noise could be further investigated with the use of an adaptive scheme for the high-gain parameter in the spirit of [8, 16, 37].

[5] As a consequence, the initial guess lacks consistency *w.r.t.* the problem's physics which makes the task harder for the observer.

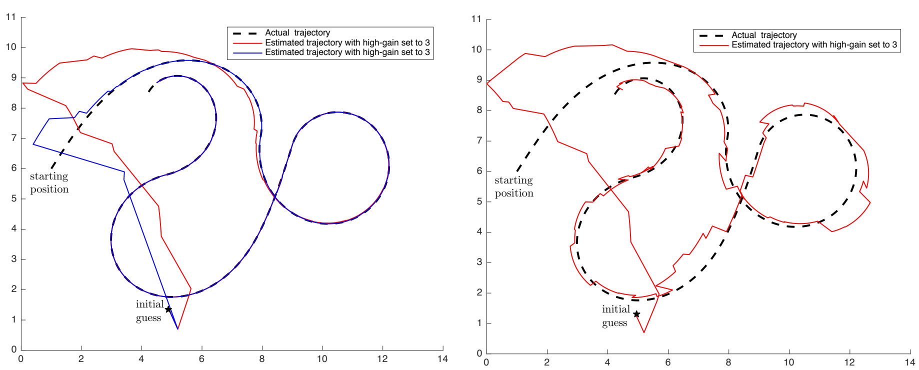
[6] Having its standard deviation equals to 0.1 for the angle measurements, and 1 for the distance measurements.



(a) Boat trajectory showing the position at sampling times.

(b) System's output signal with additive noise. The markers highlight sampling times.

Figure 2: Informations relative to the first (resp. second) beacon appear in red (resp. blue)



(a) Trajectory estimation without measurement noise.

(b) Trajectory estimation with additive noise on the measurement —cf. Figure 2b.

Figure 3

6 Conclusion

In this paper, a high-gain extended Kalman filter for nonlinear continuous-discrete systems with multirate sampled outputs has been presented and its global asymptotical convergence, proved. The proposed design consists of two steps: (i) an open loop prediction when no measurements are available, and (ii) an impulsive correction as soon as new measurements are available. To this end, each correction step involves a weighted sum of the output errors calculated on the basis of the measurements available at this sample time. In order to better handle possible cross-correlations between measurements alway available at the same time, *sensors* are defined as subsets of the output vector. Moreover, the Riccati matrix of the observer is shown to be bounded from above and below provided that (Σ_c), the underlying continuous system, is observable and for small enough sampling intervals.

Some improvements are left for the future. First of all, as it is illustrated in the example, the well known sensitivity of the high-gain design to measurement noise could be addressed with the help of an adaptive scheme in the spirit of [8, 16]. An approach taking into account several high-gain parameters instead of one only (i.e. one parameter per virtual sensor) in the spirit of [37] is another possible extension to the present work. The present study can also be conducted in the framework of hybrid systems, cf. [20], as is it done for synchronous hybrid systems in [32].

Finally, the presence of redundant sensors can lead to an improved version of the proposed design. Indeed, the maximum step size condition on the time subdivision of a given sensor could be relaxed provided there is an active redundant sensor —for example in submarine robotics the vehicle's speed available from a surface GPS is lost when the robot dives but can be obtained again (computed with respect to the ground) via a Doppler velocity log.

A Bounds for the solution of the Riccati equation

This section is dedicated to the proof of Theorem 2. It follows the structure of [7] where a similar result is proved for synchronous continuous-discrete systems. Although the present proof shares the same structure, differences specific to the asynchronous setting make this exposure necessary. However, only proofs having notable differences are detailed.

The complete argument of Theorem 2 is divided into two parts.

- In a first part, for a given $T^\star > 0$, we prove the existence of an upper bound for times greater than $T^\star$. Here, the argument mainly relies on the regularity of $\bar{S}$, and the bound depends on the maximum step size of the subdivision, $\{\bar{\tau}_k\}_{k\in\mathbb{N}}$ regardless of the underlying subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$.
- In a second time, we prove the existence of a lower bound for times greater

than $T_\star > T^\star$. In this second part, the result relies on the observability of the underlying continuous system (Σ_c), and requires small enough maximum time steps for each subdivision $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$.

The quantity $\bar{T}$ that appears in Theorem 2 is simply $\bar{T} = T_\star$.

In the following, we assume the existence of a positive definite solution $\bar{S}(\bar{\tau})$ on a small interval of time, which is ensured by the Sylvester criterion. We later on show that this interval of time is actually $\mathbb{R}^+$.

A.1 Upper bound

In order to prove that $\bar{S}_k^{(+)}$ is upper bounded for times greater than $T^\star$, we should remember that if $\bar{S}$ is a symmetric positive semidefinite matrix, then we have $\bar{S} \leq Tr(\bar{S})Id$.

Lemma 4. [7].
Let $\bar{S} : [0, T[\to S_n$ be a solution to $\frac{d\bar{S}}{d\bar{\tau}} = -\mathcal{A}'\bar{S} - \bar{S}\mathcal{A} - \bar{S}Q\bar{S}$, then for almost all $\bar{\tau} \in [0, T[$:

$$\frac{d}{d\bar{\tau}}Tr(\bar{S}) \leq -a(Tr(\bar{S}(\bar{\tau})))^2 + 2bTr(\bar{S}(\bar{\tau})) \quad where \left\{ \begin{array}{l} a = \frac{\lambda_{min}(Q)}{n} \\ b = \sup_{\bar{\tau}} Tr\left(\mathcal{A}'(\bar{\tau})\mathcal{A}(\bar{\tau})\right)^{\frac{1}{2}} \end{array} \right.$$

Lemma 5. [7].
Let a, b be two positive constants. Let $x : [0, T[\to \mathbb{R}^+$ (possibly $T = +\infty$) be an absolutely continuous function satisfying for almost all $0 < \tau < T$ the inequality:

$$\dot{x}(\tau) \leq -ax^2(\tau) + 2bx(\tau).$$

The roots of $-aX^2 + 2bX$ are $\frac{2b}{a}$ and 0. The solution $x(\tau)$ is such that:

$$x(\tau) \leq \max\left\{x(0), \tfrac{2b}{a}\right\} \text{ for all } \tau \in [0, T[.$$

In addition if $x(0) > \frac{2b}{a}$ then for all $\tau > 0 \in [0, T[$ we have the two inequalities:

$$x(\tau) \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\tau} - 1} \tag{12}$$

$$x(\tau) \leq \frac{2bx_0e^{2b\tau}}{ax_0\left(e^{2b\tau} - 1\right) + 2b} \tag{13}$$

Let us denote $r = \sup\limits_{i,k}\left(Tr\left(C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\right)\right)$. According to equation (3) and to the previous lemmas, upper bounding $\bar{S}$ turns into proving that $x_k^{(+)}$, solution of

$$\left\{ \begin{array}{rcl} \frac{dx}{d\tau} & = & -ax^2 + 2bx \\ x_k^{(+)} & = & x_k^{(-)} + r\sum\limits_{i\in\sigma_k}\left(\bar{s}_{l_1^{(i)}}^{(i)} - \bar{s}_{l_k^{(i)}-1}^{(i)}\right), \end{array} \right. \tag{14}$$

is bounded for all $\bar{\tau}_k > T^\star$, $k \in \mathbb{N}$, independently from the chosen subdivisions. It leads us to Lemma 6.

Lemma 6.
The solution of (14) *is such that:*

$$x(\bar{\tau}) \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}}-1} + n_s\, r\bar{\tau},$$

for any $\bar{\tau} > 0$*, before or after a discrete step.*

Proof. Bound (12) gives:

$$x_1^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + r\sum_{i\in\sigma_1}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right),$$

We denote $\chi_k = r\sum_{j=1}^{k}\sum_{i\in\sigma_k}\left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1}\right)$ and the previous inequality trivially becomes

$$x_1^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1. \tag{15}$$

We remark that (15) leads to the inequality below:

$$x_1^{(+)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + n_s\, r\,\bar{\tau}_1. \tag{16}$$

Let us now generalise this last inequality for all $k \in \mathbb{N}$. However, in order to do so, it is necessary to manipulate inequalities shaped as (15) instead of (16). Let us now rewrite bound (13) as follows:

$$x_2^{(-)} \leq \frac{2b}{a} + \frac{2b}{a}\frac{x_1^{(+)} - \frac{2b}{a}}{x_1^{(+)}\left(e^{2b(\bar{\tau}_2-\bar{\tau}_1)}-1\right) + \frac{2b}{a}}.$$

We want to replace $x_1^{(+)}$ by the upper bound found in (15).
Let us define the function

$$h(x) = \frac{2bxe^{2b\bar{\tau}}}{ax\left(e^{2b\bar{\tau}}-1\right)+2b}.$$

Its derivative *w.r.t.* x is

$$\begin{aligned} h^{'}(x) &= \frac{e^{2b\bar{\tau}}2b\left[ax(e^{2b\bar{\tau}}-1)+2b\right] - a(e^{2b\bar{\tau}}-1)xe^{2b\bar{\tau}}2b}{\left[ax(e^{2b\bar{\tau}}-1)+2b\right]^2} \\ &= \frac{e^{2b\bar{\tau}}(2b)^2}{\left[ax(e^{2b\bar{\tau}}-1)+2b\right]^2}. \end{aligned}$$

Since It is positive for all $\bar{\tau} > 0$, we can replace $x_1^{(+)}$ by its upper bound:

$$
\begin{aligned}
x_2^{(-)} &\leq \frac{2b}{a} + \frac{2b}{a} \frac{[\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1] - \frac{2b}{a}}{[\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + \frac{2b}{a}} \\
&\leq \frac{2b}{a} + \frac{2b}{a}\frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}\frac{1}{[\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + \frac{2b}{a}} \\
&\qquad + \frac{2b}{a}\frac{\chi_1}{\left[\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1\right](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + \frac{2b}{a}}.
\end{aligned}
$$

We lower bound the denominator of the last term with:

$$
[\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + \frac{2b}{a} \geq \frac{2b}{a},
$$

and the denominator of the second term with:

$$
[\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1} + \chi_1](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + \frac{2b}{a} \geq [\frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + \frac{2b}{a}.
$$

We also simplify $(2b/a)$ in those two terms:

$$
\begin{aligned}
x_2^{(-)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_1}-1}\frac{1}{([1 + \frac{1}{e^{2b\bar{\tau}_1}-1}](e^{2b(\bar{\tau}_2-\bar{\tau}_1)} - 1) + 1)} + \chi_1, \\
&\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2} - e^{2b\bar{\tau}_1} + e^{2b\bar{\tau}_1} - 1} + \chi_1.
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
x_2^{(+)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-1} + \chi_1 + r\sum_{i\in\sigma_2}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right) \\
&\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2}-1} + \chi_2
\end{aligned}
$$

$$
\text{with} \quad \chi_2 = r\left(\sum_{i\in\sigma_1}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right) + \sum_{i\in\sigma_2}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right)\right)
$$

Let us notice that for $i \in \sigma_1, \bar{s}^{(i)}_{l_1^{(i)}} = \bar{\tau}_1$, for $i \in \sigma_2, \bar{s}^{(i)}_{l_2^{(i)}} = \bar{\tau}_2$ —and so on and so forth for all $i \in \sigma_k, k \in \mathbb{N}, \bar{s}^{(i)}_{l_k^{(i)}} = \bar{\tau}_k$. At time $\bar{\tau}_2$, for a given sensor $\mathfrak{s}_i$, i belongs to one of the four following subsets:

1. $\sigma_1 \cap \sigma_2$
2. $\sigma_1 \setminus \sigma_2$
3. $\sigma_2 \setminus \sigma_1$
4. $\{1, ..., n_s\} \setminus \sigma_1 \setminus \sigma_2$

We consider first the case $i \in \sigma_1 \cap \sigma_2$, then:

- let $\lambda_2^{(i)}$ be defined as $\lambda_2^{(i)} = \max\{l \in \mathbb{N} \text{ such that } s_l^{(i)} \leq \tau_2\}$, thus $\lambda_2^{(i)} = 2$;
- recall that indexes $l_k^{(i)}$ are such that

$$\sigma_k = \left\{ i \in \{1, ..., n_s\} | \exists \; l_k^{(i)} \in \mathbb{N} \text{ such that } s^{(i)}_{l_k^{(i)}} = \tau_k \right\}$$

  then, $\bar{s}^{(i)}_{l_2^{(i)}} = \bar{s}^{(i)}_2 = \bar{\tau}_2$, $\bar{s}^{(i)}_{l_2^{(i)}-1} = \bar{s}^{(i)}_{l_1^{(i)}} = \bar{s}^{(i)}_1 = \bar{\tau}_1$ and $\bar{s}^{(i)}_{l_1^{(i)}-1} = \bar{s}^{(i)}_{l_0^{(i)}} = \bar{\tau}_0 = 0$

- The contribution of sensor $\mathfrak{s}_i$ to χ_2 is of the form:

$$r\left[\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right) + \left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right)\right] \quad = \quad r\sum_{j=1}^{\lambda_2^{(i)}} \left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right)$$

In the same way, and dealing with an $i \in \sigma_2 \setminus \sigma_1$, we find:

- $\lambda_2^{(i)} = 1$, $\bar{s}^{(i)}_{l_2^{(i)}} = \bar{s}^{(i)}_1 = \bar{\tau}_2$, and $\bar{s}^{(i)}_{l_2^{(i)}-1} = \bar{s}^{(i)}_0 = 0$;
- In this case, the contribution of sensor $\mathfrak{s}_i$ to χ_2 is of the form:

$$r\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right) \quad = \quad r\sum_{j=1}^{\lambda_2^{(i)}} \left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right)$$

By proceeding this way for all the other cases, we find that the contribution of sensor $\mathfrak{s}_i$ to χ_2 is always of the form:

$$\sum_{j=1}^{\lambda_2^{(i)}} \left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right) \tag{17}$$

and χ_2 also writes $\chi_2 = r\sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_2^{(i)}} \left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right) \leq r n_s \bar{\tau}_2$. Therefore,

$$\begin{aligned} x_2^{(+)} &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2} - 1} + r\sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_2^{(i)}} \left(\bar{s}^{(i)}_j - \bar{s}^{(i)}_{j-1}\right) \qquad (18) \\ &\leq \frac{2b}{a} + \frac{2b}{a}\frac{1}{e^{2b\bar{\tau}_2} - 1} + r n_s \bar{\tau}_2 \end{aligned}$$

We can generalise by induction (17) and (18) to any $k \in \mathbb{N} \setminus \{0\}$. To do so, we define $\lambda_k^{(i)} = \max\{l \in \mathbb{N} \text{ such that } s_l^{(i)} \leq \tau_k\}$ and, when $\lambda_k^{(i)} = 0$, we use the convention $\sum_{j=1}^{\lambda_k^{(i)}} (\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}) = 0$. This yields

$$\chi_k = r \sum_{j=1}^{k} \sum_{i \in \sigma_k} \left(\bar{s}^{(i)}_{l_k^{(i)}} - \bar{s}^{(i)}_{l_k^{(i)}-1} \right) = r \sum_{i=1}^{n_s} \sum_{j=1}^{\lambda_k^{(i)}} (\bar{s}_j^{(i)} - \bar{s}_{j-1}^{(i)}) \leq r n_s \bar{\tau}_k \tag{19}$$

and $x_k^{(+)} \leq \dfrac{2b}{a} + \dfrac{2b}{a} \dfrac{1}{e^{2b\bar{\tau}_i} - 1} + r\, n_s\, \bar{\tau}_k$.

Moreover, we can generalize this inequality to any $\bar{\tau} > 0$, before and after an update.

$$x(\bar{\tau}) \leq \frac{2b}{a} + \frac{2b}{a} \frac{1}{e^{2b\bar{\tau}} - 1} + r\, n_s\, \bar{\tau}. \qquad \square$$

Lemma 7. [7]

Let us define the functions

$$\begin{aligned} \phi(\bar{\tau}) &= \frac{2b}{a} + \frac{2b}{a} \frac{1}{e^{2b\bar{\tau}} - 1} + n_s\, r\bar{\tau}, \\ \psi_{x_0}(\bar{\tau}) &= \frac{2b x_0 e^{2b\bar{\tau}}}{a x_0 \left(e^{2b\bar{\tau}} - 1\right) + 2b} + n_s\, r\bar{\tau}. \end{aligned}$$

There exists $\mu_\phi > 0$, and $\mu_\psi(x_0) > 0$ such that $\phi(\bar{\tau})$, respectively $\psi_{x_0}(\bar{\tau})$, is a decreasing function for $\bar{\tau} \in]0, \mu_\phi]$, respectively for $\bar{\tau} \in [0, \mu_\psi(x_0)]$.
Moreover $\mu_\psi(x_0)$ is an increasing function of x_0.

Lemma 8. [7]

Consider the Riccati equation (3) and the assumptions of Theorem 2. Let $T^\star > 0$ be fixed. There exist two scalars $\beta_2 > 0$ and $\mu > 0$ such that

$$\bar{S}_k^{(+)} \leq \beta_2 Id,$$

for all $T^\star \leq \bar{\tau}_k$, $k \in \mathbb{N}$, for all subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k \in \mathbb{N}}$, $\{\bar{\tau}_k\}_{k \in \mathbb{N}}$ such that $\bar{\tau}_k - \bar{\tau}_{k-1} < \mu$. This bound is also valid during prediction intervals.

A.2 Lower bound

We now prove that $\bar{S}(\bar{\tau})$ is also lower bounded for times greater than a fixed $T_\star > T^\star$.

Lemma 9. [7, 19]

For any $\lambda \in \mathbb{R}^*$, any solution $\bar{S} : [0, T[\to S_m$ (Possibly, $T = +\infty$) of

$$\frac{d\bar{S}}{d\bar{\tau}} = -\mathcal{A}^{'}(\bar{\tau})\bar{S} - \bar{S}\mathcal{A}(\bar{\tau}) - \bar{S}Q\bar{S},$$

we have for all $\bar{\tau} \in [0, T[$:

$$\bar{S}(\bar{\tau}) = e^{-\lambda\bar{\tau}}\varphi_a(\bar{\tau},0)\bar{S}_0\varphi_a^{'}(\bar{\tau},0) \\ +\lambda\int_0^{\bar{\tau}} e^{-\lambda(\bar{\tau}-v)}\varphi_a(\bar{\tau},v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau},v)dv \tag{20}$$

where $\varphi_a(\bar{\tau},s)$ is such that: $\begin{cases} \dfrac{d\varphi_a(\bar{\tau},s)}{d\bar{\tau}} &= -\mathcal{A}^{'}(\bar{\tau})\varphi_a(\bar{\tau},s), \\ \varphi_a(s,s) &= Id. \end{cases}$

Lemma 10. **[7, 19]**

Let $\bar{S} : [0; e(\bar{S})[\to S_m$ be a maximal positive semi definite solution of

$$\frac{d}{d\bar{\tau}}\bar{S} = -\mathcal{A}^{'}\bar{S} - \bar{S}\mathcal{A} - \bar{S}Q\bar{S}.$$

If $\bar{S}(0) = \bar{S}_0$ is positive definite then

$$e(\bar{S}) = +\infty \text{ and } \bar{S}(\bar{\tau}) \text{ is positive definite for all } \bar{\tau} \geq 0.$$

Thus, for any arbitrary time subdivision $\{\bar{\tau}_k\}_{k\in\mathbb{N}^*}$, the solution $\bar{S}$ to the asynchronous continuous discrete Riccati equation (3) is positive definite for all times provided that $\bar{S}_0$ is positive definite.

Following Lemma 9, and for a fixed $\lambda > 0$, $\bar{S}_1^{(+)}$ is written:

$$\bar{S}_1^{(+)} = e^{-\lambda\bar{\tau}_1}\varphi_a(\bar{\tau}_1,0)\bar{S}_0\varphi_a^{'}(\bar{\tau}_1,0) \\ +\lambda\int_0^{\bar{\tau}_1} e^{-\lambda(\bar{\tau}_1-v)}\varphi_a(\bar{\tau}_1,v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_1,v)dv \\ +\sum_{i\in\sigma_1} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right) \tag{21}$$

At time $\bar{\tau}_2$, the formula yields:

$$\bar{S}_2^{(+)} = e^{-\lambda(\bar{\tau}_2-\bar{\tau}_1)}\varphi_a(\bar{\tau}_2,\bar{\tau}_1)\bar{S}_1^{(+)}\varphi_a^{'}(\bar{\tau}_2,\bar{\tau}_1) \\ +\lambda\int_{\bar{\tau}_1}^{\bar{\tau}_2} e^{-\lambda(\bar{\tau}_2-v)}\varphi_a(\bar{\tau}_2,v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_2,v)dv \\ +\sum_{i\in\sigma_2} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right)$$

Replacing $\bar{S}_1^{(+)}$ by the expression obtained in (21) leads to:

$$\bar{S}_2^{(+)} = e^{-\lambda\bar{\tau}_2}\varphi_a(\bar{\tau}_2,0)\bar{S}_0\varphi_a^{'}(\bar{\tau}_2,0) \\ +\lambda\int_0^{\bar{\tau}_2} e^{-\lambda(\bar{\tau}_2-v)}\varphi_a(\bar{\tau}_2,v)\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_2,v)dv \\ +e^{-\lambda(\bar{\tau}_2-\bar{\tau}_1)}\varphi_a(\bar{\tau}_2,\bar{\tau}_1)\sum_{i\in\sigma_1} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_1^{(i)}} - \bar{s}^{(i)}_{l_1^{(i)}-1}\right)\varphi_a^{'}(\bar{\tau}_2,\bar{\tau}_1) \\ +\sum_{i\in\sigma_2} C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\left(\bar{s}^{(i)}_{l_2^{(i)}} - \bar{s}^{(i)}_{l_2^{(i)}-1}\right)$$

We iterate this procedure in order to compute $S_k(+)$ for any k:

$$\begin{aligned}\bar{S}_k^{(+)} &= e^{-\lambda\bar{\tau}_k}\varphi_a(\bar{\tau}_k,0)\bar{S}_0\varphi_a^{'}(\bar{\tau}_k,0)\\ &\quad+\lambda\int_0^{\bar{\tau}_k} e^{-\lambda(\bar{\tau}_k-v)}\varphi_a(\bar{\tau}_k,v)\left(\bar{S}(v)-\frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)\varphi_a^{'}(\bar{\tau}_k,v)dv\\ &\qquad+\sum_{j=1}^{k}\sum_{i\in\sigma_j} e^{-\lambda(\bar{\tau}_k-\bar{\tau}_j)}\varphi_a(\bar{\tau}_k,\bar{\tau}_j)C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\varphi_a^{'}(\bar{\tau}_k,\bar{\tau}_j)\left(\bar{s}^{(i)}_{l_j^{(i)}}-\bar{s}^{(i)}_{l_j^{(i)}-1}\right)\end{aligned} \tag{22}$$

This last equation is of the form $S_k(+) = (I) + (II) + (III)$, in this order.

- (I) Since $\bar{S}_0$ is positive definite, (I) is at least positive semi-definite.
- (II) Let us pick $\lambda > \beta\bar{q}$, then $\left(\bar{S}(v) - \frac{\bar{S}(v)Q\bar{S}(v)}{\lambda}\right)$ is positive definite, and (II) is at least positive semi-definite.

We now concentrate our efforts on (III) since it is the quantity that is actually bounded from below for all $\bar{\tau} > \bar{T}^\star$.

Let us define[7]:

1. the time $0 < \rho < \bar{\tau}_k$ such that $\bar{\tau}_k - \rho = T_\star$;
2. the index $\lambda_\rho^{(i)}$ as $\lambda_\rho^{(i)} = \max\{l \in \mathbb{N} : \bar{s}_l^{(i)} \leq \rho\}$ which always exists as soon as $\bar{\tau}_k > T_\star$.

Then, we use relation (19) that appears in the proof of lemma 6 to rewrite (III) as

$$(III) = \sum_{i=1}^{n_s}\sum_{j=1}^{\lambda_k^{(i)}} e^{-\lambda\left(\bar{\tau}_k-\bar{s}_j^{(i)}\right)}\varphi_a\left(\bar{\tau}_k,\bar{s}_j^{(i)}\right)C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\varphi_a^{'}\left(\bar{\tau}_k,\bar{s}_j^{(i)}\right)\left(\bar{s}_j^{(i)}-\bar{s}_{j-1}^{(i)}\right)$$

Since all the terms of the sum (III) are symmetric positive semidefinite matrices:

$$(III) \geq \sum_{i=1}^{n_s}\sum_{j=\lambda_\rho^{(i)}+1}^{\lambda_k^{(i)}} e^{-\lambda\left(\bar{\tau}_k-\bar{s}_j^{(i)}\right)}\varphi_a\left(\bar{\tau}_k,\bar{s}_j^{(i)}\right)C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\varphi_a^{'}\left(\bar{\tau}_k,\bar{s}_j^{(i)}\right)\left(\bar{s}_j^{(i)}-\bar{s}_{j-1}^{(i)}\right)$$

From the properties of the resolvent φ_a, the above inequality can be rewritten, with $\bar{a}(\bar{\tau}) = a(\bar{\tau}+\rho)$:

$$(III) \geq \sum_{i=1}^{n_s}\sum_{j=\lambda_\rho^{(i)}+1}^{\lambda_k^{(i)}} e^{-\lambda\left(\bar{\tau}_k-\bar{s}_j^{(i)}\right)}\varphi_{\bar{a}}\left(\bar{\tau}_k-\rho,\bar{s}_j^{(i)}-\rho\right)C^{(\mathfrak{s}_i)'}R^{(\mathfrak{s}_i)^{-1}}C^{(\mathfrak{s}_i)}\varphi_{\bar{a}}^{'}\left(\bar{\tau}_k-\rho,\bar{s}_j^{(i)}-\rho\right)\left(\bar{s}_j^{(i)}-\bar{s}_{j-1}^{(i)}\right)$$

[7] ρ is defined *w.r.t.* k —since we need our relations to remain valid for any $\bar{\tau}_k$ large enough— and should be understood as ρ_k. This latter notation is however not used for readability reasons.

If we denote by μ_i the maximum time step of a subdivision $\left\{s_k^i\right\}_{k\in\mathbb{N}}$, we notice that $e^{-\lambda\left(\bar{\tau}_k-\bar{s}_j^{(i)}\right)} \geq e^{-\lambda(T_\star+\mu_i)}$. Since $\bar{R}_{\sigma_k}^{-1}$ is defined in a compact subset, therefore, we need to find a lower bound for the following expression:

$$\sum_{i=1}^{n_s}\sum_{j=\gamma_k^{(i)}+1}^{\lambda_k^{(i)}} \varphi_{\bar{a}}\left(\bar{\tau}_k-\bar{s}^{(i)}_{\gamma_k^{(i)}}, \bar{s}_j^{(i)}-s^{(i)}_{\gamma_k^{(i)}}\right) C^{(\mathfrak{s}_i)\prime}C^{(\mathfrak{s}_i)}\varphi_{\bar{a}}^{\prime}\left(\bar{\tau}_k-\bar{s}^{(i)}_{\gamma_k^{(i)}}, \bar{s}_j^{(i)}-\bar{s}^{(i)}_{\gamma_k^{(i)}}\right)\left(\bar{s}_j^{(i)}-\bar{s}_{j-1}^{(i)}\right)$$

Let us first redefine the subdivisions as follows:

- we denote $\hat{s}_j^{(i)} = \bar{s}^{(i)}_{j+\lambda_\rho^{(i)}}-\rho$, with $\hat{s}_0^{(i)}=0$ for all $i\in\{1,...,n_s\}$;
- each new subdivision $\{\hat{s}_j^{(i)}\}$ has $k_*^{(i)}+1$ elements, with $k_*^{(i)}=\lambda_k^{(i)}-\lambda_\rho^{(i)}$. Hence, $\hat{s}^{(i)}_{k_*^{(i)}} = \bar{s}^{(i)}_{\lambda_k^{(i)}}-\rho$ for all $i\in\{1,...,n_s\}$;
- we denote the subdivision $\{\hat{\tau}_j\}$ by $\{\hat{\tau}_j\} = \bigcup_i \left\{\hat{s}_j^{(i)}\right\}$, where elements belonging to several subdivisions are considered only once.

Thus, we can show that (III) has a lower bound if we can prove that

$$\sum_{i=1}^{n_s}\sum_{j=1}^{k_*^{(i)}} \varphi_{\bar{a}}\left(T_\star,\hat{s}_j^{(i)}\right)C^{(\mathfrak{s}_i)\prime}C^{(\mathfrak{s}_i)}\varphi_{\bar{a}}^{\prime}\left(T_\star,\hat{s}_j^{(i)}\right)\left(\hat{s}_j^{(i)}-\hat{s}_{j-1}^{(i)}\right) \tag{23}$$

has a lower bound for all subdivisions $\{\hat{\tau}_j\}$ and $\left\{\hat{s}_j^{(i)}\right\}$, $i\in\{1,...,n_s\}$, having a maximum time step size denoted by μ_i.

Let us now define $\psi_{\bar{a}}(\bar{\tau},s) = \left(\varphi_{\bar{a}}^{-1}(\bar{\tau},s)\right)^{\prime}$, which is in fact the resolvant of system $\dot{x}=\mathcal{A}(\bar{\tau})x(\bar{\tau})$. Since $\psi_{\bar{a}}(\bar{\tau},s)=\psi_{\bar{a}}^{-1}(s,\bar{\tau})$, we can rewrite (23) as follows:

$$G_{cda}(T_\star) = \sum_{i=1}^{n_s}\sum_{j=1}^{k_*^{(i)}} \psi_{\bar{a}}^{\prime}\left(\hat{s}_j^{(i)},T_\star\right)C^{(\mathfrak{s}_i)\prime}C^{(\mathfrak{s}_i)}\psi_{\bar{a}}\left(\hat{s}_j^{(i)},T_\star\right)\left(\hat{s}_j^{(i)}-\hat{s}_{j-1}^{(i)}\right) \tag{24}$$

We call this latter quantity the *asynchronous continuous-discrete Gram observability matrix* associated to a time $T^\star>0$. It is actually the key object that allows us to lower bound the Riccati matrix $\bar{S}_k$. In the following we show that, provided the time steps are small enough, $G_{cda}(T_\star)$ is as close as needed to the continuous time Gram observability matrix. To do so, we need the two following extra lemmas.

Lemma 11. e.g. [19]
Let $\Psi_a(\tau,s)$ denote the resolvant of the following time-dependent, observable, system:

$$\begin{aligned}\dot{x} &= \mathcal{A}(\tau)x(\tau)\\ y(\tau) &= Cx(\tau)\end{aligned}$$

where the elements of $\mathcal{A}$, denoted by $(a_{i,j})$, are functions living in a subset[8]

$$\mathcal{A}_B = \left\{ a = (a_{i,j}) \in L^\infty\left([0,T], \mathbb{R}^\eta\right), \ \sup_{i,j} |a_{i,j}|_\infty \leq B \quad with \quad B > 0 \right\}.$$

For a given $T > 0$, the (continuous) Gram observability matrix is defined as

$$G_c(T) = \int_0^T \psi_a^{'}(v,T) C^{'} C \psi_a(v,T) dv \tag{25}$$

Then, there exist positive scalars $0 < \mathfrak{a} < \mathfrak{b}$ depending on B and T only, such that

$$\mathfrak{a} Id \leq G_c(T) \leq \mathfrak{b} Id$$

Lemma 12.

Let $m(t)$, $t \in [0,T]$, be a $(n \times n)$ symmetric matrix, at least differentiable once.
Let μ be a positive constant, and $\{\bar{\tau}_j\}_{j \in \mathbb{N}}$ an arbitrary subdivision of $[0,T]$ such that $\bar{\tau}_j - \bar{\tau}_{j-1} \leq \mu$, for all $j \in \mathbb{N}$, with $\bar{\tau}_0 = 0$ and $\bar{\tau}_k$ the maximal element of the subdivision such that $T - \bar{\tau}_k \leq \mu$. We suppose that all the coefficients of m have their derivative bounded over time.
Then

$$\int_0^T m(v) dv - \sum_{j=1}^k m(\bar{\tau}_j)\left(\bar{\tau}_j - \bar{\tau}_{j-1}\right) \leq \mu \left(KT + L\right) Id,$$

where $L = \sup_{\bar{\tau}} \|m(\bar{\tau})\|_2$, with $\|.\|_2$ the matrix norm induced by the euclidean norm, $K = \frac{n}{2} \max_{k,l,\bar{\tau}} \left(\left| m^{'}_{k,l}(\bar{\tau}) \right| \right)$, with $m^{'}_{k,l}(\bar{\tau})$ the element of the k^{th} row and l^{th} column of the matrix $m^{'}(\bar{\tau})$.

Proof. The proof of this lemma is mainly based on that of Lemma 3.11 in [7], with small differences discussed in Remark 2 at the end of the present section.

Let $M(t)$ be a primitive matrix of $m(t)$, that is to say a matrix whose elements are the primitives of the elements of $m(t)$. We have the identity

$$\int_0^T m(v) dv = M(T) - M(0) = \sum_{j=1}^k \left[M(\bar{\tau}_j) - M(\bar{\tau}_{j-1}) \right] + \int_{\bar{\tau}_k}^T m(v) dv.$$

We can apply the Taylor-Lagrange expansion on each element M_{kl}:

$$M_{kl}(\bar{\tau}_{i-1}) = M_{kl}(\bar{\tau}_i) + \left(\bar{\tau}_{i-1} - \bar{\tau}_i\right) m_{kl}(\bar{\tau}_i) + \frac{\left(\bar{\tau}_{i-1} - \bar{\tau}_i\right)^2}{2} m^{'}_{kl}(\xi_{kl,i})$$

where $\xi_{kl,i} \in [\bar{\tau}_{i-1}, \bar{\tau}_i]$. We have thus, the relation

$$\sum_{i=1}^k M(\bar{\tau}_{i-1}) = \sum_{i=1}^k M(\bar{\tau}_i) + \sum_{i=1}^k m(\bar{\tau}_i)\left(\bar{\tau}_{i-1} - \bar{\tau}_i\right) + \sum_{i=1}^k \left(\frac{\left(\bar{\tau}_{i-1} - \bar{\tau}_i\right)^2}{2} \mathfrak{R}_i \right)$$

[8] The value of η depends on the problem under consideration. When x is of dimension n, then $\eta \leq n^2$, depending on the number of identically null or constants elements of $\mathcal{A}$.

where $(\mathfrak{R}_i)_{kl} = m'_{kl}(\xi_{kl,i})$. Therefore

$$\begin{aligned}
\int_0^T m(v)dv - \sum_{i=1}^{k} m(\bar{\tau}_i)(\bar{\tau}_i - \bar{\tau}_{i-1}) &= \sum_{i=1}^{k}[M(\bar{\tau}_i) - M(\bar{\tau}_{i-1})] - \sum_{i=1}^{k}(\bar{\tau}_i - \bar{\tau}_{i-1})m(\bar{\tau}_i) + \int_{\bar{\tau}_k}^{T} m(v)dv \\
&= \int_{\bar{\tau}_k}^{T} m(v)dv - \sum_{i=1}^{k}\left(\frac{(\bar{\tau}_{i-1} - \bar{\tau}_i)^2}{2}\mathfrak{R}_i\right) = \mathfrak{A} + \mathfrak{B}.
\end{aligned}$$

We now use the definition of the matrix inequality to upper bound matrix $\mathfrak{B}$. Let x be a non zero element of $\mathbb{R}^n$:

$$\begin{aligned}
x'\left[-\sum_{i=1}^{k}\left(\frac{(\bar{\tau}_{i-1} - \bar{\tau}_i)^2}{2}\mathfrak{R}_i\right)\right]x &= -\frac{1}{2}\sum_{i=1}^{k}\left((\bar{\tau}_{i-1} - \bar{\tau}_i)^2 x'\mathfrak{R}_i x\right) \\
&\leq \frac{1}{2}\sum_{i=1}^{k}\left((\bar{\tau}_{i-1} - \bar{\tau}_i)^2 \sum_{k,l}|x_k|\,|\mathfrak{R}_i|_{kl}\,|x_l|\right) \\
&\leq \frac{1}{2}\mu \max_{k,l,i}(|\mathfrak{R}_i|_{kl})\left(\sum_{i=1}^{k}\bar{\tau}_{i-1} - \bar{\tau}_i\right)\left(\sum_{k,l}|x_k|\,|x_l|\right) \\
&\leq \frac{1}{2}\mu \max_{k,l,i}(|\mathfrak{R}_i|_{kl})\left(\sum_{i=1}^{k}\bar{\tau}_{i-1} - \bar{\tau}_i\right)\frac{1}{2}\left(\sum_{k,l}|x_k|^2 + |x_l|^2\right) \\
&\leq \mu\frac{n}{2}\max_{k,l,i}(|\mathfrak{R}_i|_{kl})T\,\|x\|^2.
\end{aligned}$$

Let us now upper bound matrix $\mathfrak{A}$. Since $m(\bar{\tau})$ is symmetric, for a given $\bar{\tau} \in \mathbb{R}^+$, $m(\bar{\tau}) \leq \|m(\bar{\tau})\|_2 Id$ where $\|.\|_2$ is the matrix norm induced by the euclidian norm, i.e. $\|m(\bar{\tau})\|_2 = \sup_{\|x\|_2}\|m(\bar{\tau})x\|_2$. Thus $\int_{\bar{\tau}_k}^{T} m(\bar{\tau}) \leq \sup_{\bar{\tau}}\|m(\bar{\tau})\|_2\,\mu Id$.

Those two upper bounds give us the result. □

The two preceding lemmas allow us to conclude this section's proof.

Lemma 13.

Consider the Riccati equation (3), and the assumptions of Theorem 2. Let $T_\star > T^\star$ be fixed. Then, there exist constants $\mu_i > 0$, $i \in \{1, ..., n_s\}$, and $\alpha_2 > 0$ such that, for all subdivisions $\left\{\bar{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$, $\{\bar{\tau}_k\}_{k\in\mathbb{N}}$ with $\left(\bar{s}_k^{(i)} - \bar{s}_{k-1}^{(i)}\right) \leq \mu_i$,

$$\alpha_2 Id \leq \bar{S}_k^{(+)} \quad \textit{as soon as} \quad \bar{\tau}_k > T_\star.$$

Proof.

We start from Equation (24), the asynchronous continuous-discrete Gram observability matrix at time $T^\star > 0$:

$$G_{cda}(T_\star) = \sum_{i=1}^{n_s} G_{cda}^{(i)}(T_\star) = \sum_{i=1}^{n_s}\sum_{j=1}^{k_*^{(i)}} \psi'_{\bar{a}}\left(\hat{s}_j^{(i)}, T_\star\right) C^{(\mathfrak{s}_i)\prime} C^{(\mathfrak{s}_i)} \psi_{\bar{a}}\left(\hat{s}_j^{(i)}, T_\star\right)\left(\hat{s}_j^{(i)} - \hat{s}_{j-1}^{(i)}\right)$$

Let us consider the continuous Gram matrix $G_c(T^\star)$, defined in Lemma 12, which also writes:

$$G_c(T_\star) = \sum_{i=1}^{n_s} \psi_a^{'}(v, T_\star) C^{(\mathfrak{s}_i)'} C^{(\mathfrak{s}_i)} \psi_a(v, T_\star) = \sum_{i=1}^{n_s} G_c^{(i)}(T_\star)$$

By lemma 12, for all $i \in \{1, ..., n_s\}$, there are constants $L > 0$ and $K_i > 0$:

$$G_c^{(i)}(T_\star) - G_{cda}^{(i)}(T_\star) \leq \mu_i (K_i T_\star + L) Id \tag{26}$$

Let us apply Lemma 11 on $G_c(T_\star)$:

$$\begin{aligned}
\alpha Id &\leq G_c(T_\star) \\
&\leq \sum_{i=1}^{n_s} G_c^{(i)}(T_\star) - \sum_{i=1}^{n_s} G_{cda}^{(i)}(T_\star) + \sum_{i=1}^{n_s} G_{cda}^{(i)}(T_\star) \\
&\leq \sum_{i=1}^{n_s} G_{cda}^{(i)}(T_\star) + \sum_{i=1}^{n_s} \mu_i (K_i T_\star + L) Id
\end{aligned}$$

Therefore

$$\left[\alpha - \sum_{i=1}^{n_s} \mu_i (K_i T_\star + L)\right] Id \leq G_{cda}(T_\star)$$

As a consequence, if all the μ_i are such that $\left(\alpha - \sum_{i=1}^{n_y} \mu_i (K_i T_\star + L)\right) > 0$, then, independently from the shape of the subdivisions $\left\{\hat{s}_k^{(i)}\right\}_{k\in\mathbb{N}}$ and $\{\hat{\tau}_k\}_{k\in\mathbb{N}}$, there exist a positive α_2 such that:

$$\alpha_2 Id \leq \bar{S}_k^{(+)}$$

This bound is also valid during prediction intervals. □

Remark 2. Erratum to [7]. *The reason why we need Lemma 12 instead of simply re-using Lemma 3.11 of [7] is because it should be used there as well. Indeed, the following mistake —which doesn't invalidate the main result of the article and is corrected by Lemma 12— is done in [7].*

The very end of Proposition 3.12, which corresponds to Lemma 13 in the present paper relies on the relation:

$$\mathfrak{a} Id \leq G_c(T^\star) \leq G_c(T^\star + \varepsilon) \quad \textit{for} \quad \varepsilon > 0. \tag{27}$$

Going back to the definition of the Gram observability matrix (25), we see that the argument of $G_c(.)$ plays a part both as the integration upper limit, but also in the definition of the resolvent matrix ψ_a. As such, the integrands of $G_c(T^\star)$ and $G_c(T^\star + \varepsilon)$ are not the same functions, which implies that 27 is not always true.

However, this issue is resolved by following the procedure used in Equation (26) above.

References

[1] T. Ahmed-Ali, I. Karafyllis, and F. Lamnabhi-Lagarrigue. Global exponential sampled-data observers for nonlinear systems with delayed measurements. *Systems and control letters*, 62(7):539–549, 2013.

[2] Vincent Andrieu, Madiha Nadri, Ulysse Serres, and Jean-Claude Vivalda. Self-triggered continuous discrete observer with updated sampling period. *Automatica*, 62:106–113, 2015.

[3] C. Barbēlatē, V. De Carolis, M. W. Dunnigan, Y. Petillot, and D. Lane. An adaptive controller for autonomous underwater vehicles. In *Proceeding og IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, pages 1658 – 1663, 2015.

[4] O. Barrero, J. Petersen, and S. B. Jorgensen. Multirate data assimiliation in a cultivation process. In *IFAC proceedings volumes*, volume 41, pages 11433–11438, 2008.

[5] P. Bernard, V. Andrieu, and L. Praly. Nonlinear observer in the original coordinates with diffeomorphism extension and jacobian completion. *HAL Id : hal-01199791, version 2, ARXIV : 1509.04865*, 2018.

[6] M. Blain, S. Lemieux, and R. Houde. Implementation of a ROV navigation system using acoustic/doppler sensors and kalman filtering. In *Oceans 2003. Celebrating the Past ... Teaming Toward the Future (IEEE Cat. No.03CH37492)*. Institute of Electrical and Electronics Engineers (IEEE), 2003.

[7] N. Boizot and E. Busvelle. On the stability of a differential riccati equation for continuous-discrete observers. *International Journal of Control*, 89(2):322–336, 2016.

[8] N. Boizot, E. Busvelle, and J-P. Gauthier. An adaptive high-gain observer for nonlinear systems. *Automatica*, 46(9):1483–1488, 2010.

[9] Nicolas Boizot, Eric Busvelle, and Jean-Paul Gauthier. Adaptive-gain extended kalman filter: Extension to the continuous-discrete case. In *Proceedings of the European control conference*, 2009.

[10] P.-J. Bristeau, E. Dorveaux, D. Vissière, and N. Petit. Hardware and software architecture for state estimation on an experimental low-cost small-scaled helicopter. *Control Engineering Practice*, 18(7):733–746, 2010.

[11] W. Cao and Y. C. Soh. Nonlinear multi-rate current state estimation: convergence analysis and application to biological systems. *Computers and Chemical Engineering*, 28:1623–1633, 2004.

[12] F. Deza, E. Busvelle, J-P. Gauthier, and D. Rakotopara. High-gain estimation for nonlinear systems. *Systems and Control Letters*, 18(4):295–299, 1992.

[13] M. F. Ellis, T. W. Taylor, V. Gonzales, and K. F. Jensen. Estimation of the molecular weight distribution in batch polymerization. *AIChE Journal*, 34(8):1341–1353, 1988.

[14] M. F. Ellis, T. W. Taylor, and K. F. Jensen. On-line molecular weight distribution estimation and control in batch polymerization. *AIChE Journal*, 40(3):445–462, 1994.

[15] M. Farza, I. Bouraoui, T. Menard, R. B. Abdennour, and M. M'Saad. Adaptive observers for a class of uniformly observable systems with nonlinear parametrization and sampled outputs. *Automatica*, 50(11):2951–2960, 2014.

[16] M. Farzaa, M. Ouederab, R. B. Abdennourb, and M. M'Saada. High gain observer with updated gain for a class of mimo nonlinear systems. *International Journal of Control*, 84:270–280, 2011.

[17] Aïda Feddaoui, Nicolas Boizot, Eric Busvelle, and Vincent Hugel. A kalman filter for linear continuous-discrete systems with asynchronous measurements. In *Proceedings of the 56th IEEE Conference on Decision and Control (CDC)*, pages 2813–2818, 2017.

[18] J-P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems, applications to bioreactors. *IEEE Transactions on Automatic Control*, 37(6):875–880, 1992.

[19] J-P. Gauthier and I. Kupka. *Deterministic Observation Theory and Applications*. Cambridge University Press, 2001.

[20] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[21] R. D. Gudi, S. L. Shah, and M. R. Gray. Adaptive multirate state and parameter estimation strategies with applications to a bioreactor. *AIChE Journal*, 41(11):2451–2464, 1995.

[22] J. A. Isaza, H. A. Botero, and H. Alvarez. State estimation using non-uniform and delayed information: A review. *Internal Journal of Automation and Computing*, https://doi.org/10.1007/s11633-017-1106-7, 2018.

[23] A. H. Jazwinski. *Stochastic Processus and Filtered Theory*. Academic press, New-York,, 1970.

[24] L. Jiang, L. Yan, Y. Xia, Q. Guo, M. Fu, and K. Lu. Asynchronous multirate multisensor data fusion over unreliable measurements with correlated noise. *IEEE Transactions on Aerospace and Electronic Systems*, 53(5):2427–2437, 2017.

[25] I. Karafyllis and C. Kravaris. From continuous-time design to sampled-data design of observers. *IEEE Transactions on Automatic Control*, 54(9):2169–2174, 2009.

[26] F. Lafont, E. Busvelle, and J-P. Gauthier. An adaptive high-gain observer for wastewater treatment systems. *Journal of Process Control*, 21:893–900, 2011.

[27] W. Li, S. L. Shah, and D. Xiao. Kalman filters in non-uniformly sampled multirate systems: for fdi and beyond. *Automatica*, 44(1):199–208, 2008.

[28] C. Ling and C. Kravaris. Multi-rate observer design for process monitoring using asynchronous inter-sample output predictions. *AIChE Journal*, 63(8):3384–3397, 2017.

[29] C. Ling and C. Kravaris. Multi-rate sampled-data observers based on a continuous-time design. In *Proceedings of the 56th IEEE Conference on Decision and Control (CDC)*, pages 3664–3669, 2017.

[30] A. Liu, W. Zhang, and J. Chen. Moving horizon estimation for multi-rate systems. In *Proceedings of the 54th IEEE annual Conference on Decision and Control (CDC)*, pages 6850–6855, 2015.

[31] Divine Maalouf, Ahmed Chemori, and Vincent Creuze. $\mathcal{L}_1$ adaptive depth and pitch control of an underwater vehicle with real-time experiments. *Ocean Engineering, Elsevier*, 98:66–77, 2015.

[32] B. P. Malladi, R. G. Sanfelice, and E. Butcher. Robust hybrid kalman filter for a class of nonlinear systems. In *Proc. 2018 American Control Conference (ACC)*, volume To Appear, 2018.

[33] David Mayhew and Pushkin Kachroo. Multirate sensor fusion for gps using kalman filtering, fuzzy methods, and map matching. In *Proceedings of SPIE: Mobile Robots XIII and Intelligent Transportation Systems*, volume 3525(1), pages 440–449, 1998.

[34] M. Moarref and L. Rodrigues. Observer design for linear multi-rate sampled-data systems. In *Proc. American Control Conference (ACC)*, pages 5319–5324, 2014.

[35] M. A. Myers, S. Kang, and R. H. Luecke. State estimation and control for systems with delayed off-line measurements. *Computers and Chemical Engineering*, 20(5):585–588, 1996.

[36] A. Rapaport and A. Maloum. Design of exponential observers for nonlinear systems by embedding. *Int. J. Robust Nonlinear Control*, 14:273–288, 2004.

[37] M. Rashedi, J. Liu, and B. Huang. Distributed adaptive high-gain extended kalman filtering for nonlinear systems. *International Journal of Robust and Nonlinear Control*, 27:4873–4902, 2017.

[38] P. Rives and M. Devy. *La robotique mobile*, chapter Perception pour la robotique. Editions Hermès, 2001.

[39] K. D. Sebesta and N. Boizot. A real-time adaptive high-gain EKF, applied to a quadcopter inertial navigation system. *IEEE Transactions on Industrial Electronics*, 61:495–503, 2014.

[40] Y. Shen, D. Zhang, and X. Xia. Continuous observer design for a class of multi-output nonlinear systems with multi-rate sampled and delayed output measurements. *Automatica*, 75:127–132, 2017.

[41] S. Tatiraju, M. Soroush, and B. A. Ogunnaike. Multirate nonlinear state estimation with application to a polymerization reactor. *AIChE Journal*, 45(4), 1999.

[42] N. Zambare, M. Soroush, and B. A. Ogunnaike. A method of robust multirate state estimation. *Journal of Process Control*, 13:337–355, 2003.

[43] H. Zhang, M. V. Basin, and M. Skliar. Itô-volterra optimal state estimation with continuous, multirate, randomly sampled, and delayed measurements. *IEEE Transactions on Automatic Control*, 52(3):401–416, 2007.