(Ch 2)

Lecture 5: Introduction to Galerkin Methods

So far we have described certain numerical methods for solving the 1D PDE $\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$. However, to be more general we need to abstract out the specific governing eqs. by defining a differential operator as follows: $L(u) = 0$ where, in the 1D PDE case we have $L = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = \frac{d}{dt}$ → Lagrangian derivative

No matter what "L" is, we still need to approximate our solution vector q as follows:

$$q_N(x,t) = \sum_{i=0}^{N} \psi_i(x)\, \tilde{q}_i(t)$$

Then the Galerkin problem becomes: find $\tilde{q} \in V_N$ s.t.

$$\int_\Omega \psi\, L(q_N)\, d\Omega = 0 \quad \forall\ \psi \in V_N$$ where

V_N is the vector space where the sol. $\tilde{q}$ & basis functions ψ live.

CG: V_N is H^1 (Sobolev space) } defined shortly.
DG: V_N is L^2 (Hilbert space)

Hilbert Space Is an inner product space with IP defined as $(u,v) = \int_\Omega uv \, d\Omega$ for functions u & v

that are complete. I.e., $\exists$ a set of linearly independent (LI) vectors which can be used to represent that function. A complete set of LI vectors forms a basis. The integral $\int_\Omega d\Omega$ is defined in the L^2 sense.

L^2 Space Is defined as the space w/ square-integrable functions

$$\int_\Omega |u|^2 \, d\Omega < \infty \rightarrow \text{bounded integral}$$

Sobolev Space Is defined from a hierarchy of Hilbert spaces as such:

$$(u,v)_{H^k} = \int_\Omega \sum_{i=0}^{k} u^{(i)} v^{(i)} \, d\Omega$$

where $u^{(i)}$ denotes the i^{th} derivative of u.

For 2^{nd} order PDEs, we only need H^1 which is written as follows:

$$(u,v)_{H^1} = \int_\Omega \left(\bar{\nabla} u \cdot \bar{\nabla} v + uv \right) d\Omega \rightarrow \text{CG space}$$

Galerkin Vector Spaces

We can now define the following vector spaces:

$$V_N^{CG} = \{ \psi \in H^1(\Omega) : \psi \in P_N \} \quad \text{and}$$

$$V_N^{DG} = \{ \psi \in L^2(\Omega_e) : \psi \in P_N \ \& \ \Omega = \bigcup_{e=1}^{N_e} \Omega_e \}$$

where we now understand what we mean by H^1 & L^2.
But how is ψ defined?

Derivation of Basis Functions (SLO)

The Sturm-Liouville operator:

$$\frac{d}{dx}\left(p(x) \frac{d\varphi}{dx}(x) \right) + \lambda \, \omega(x) \varphi(x) = 0$$

describes a general 2^{nd} order differential operator for specific choices of $p(x)$ & $\omega(x)$ & $x \in [a,b]$

$\boxed{\text{EX 5.1}}$ Let $a=0$, $b=2\pi$ w/ periodic BCs &
$p(x) = \omega(x) = 1$ yields:

(5.1) $$\frac{d^2 \varphi(x)}{dx^2} + \lambda \varphi(x) = 0$$

which is satisfied by $\psi(x) = e^{i\sqrt{\lambda}\,x} \equiv \sum_{n=0}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$

which is the Fourier Series

$\boxed{\text{Ex } 5.2}$ Let $a=-1$, $b=+1$ w/o periodicity &
let $p(x) = 1-x^2$ & $w(x) = 1$ gives:

$$(5.2) \quad \frac{d}{dx}\left[(1-x^2)\frac{d\psi}{dx}\right] + \lambda\psi = 0$$

& expanding, yields:

$$(1-x^2)\frac{d^2\psi}{dx^2} - 2x\frac{d\psi}{dx} + \lambda\psi = 0$$

which is satisfied by the power series $\psi(x) = \sum_{n=0}^{\infty} a_n x^n$
s.t. $P_n = a_n x^n$ & give:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$\vdots$$

$$P_{n+1}(x) = \frac{2n+1}{n+1} P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

or $P_n(x) = \frac{2n-1}{n} P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x)$

which are the Legendre Polynomials

$\boxed{\text{Ex } 5.3}$ Let $a=-1$, $b=+1$ w/o periodicity &
$p(x) = \sqrt{1-x^2}$ & $w(x) = \frac{1}{\sqrt{1-x^2}}$ which yields:

$$(5.3) \quad \frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{d\varphi}{dx}\right] + \frac{\lambda}{\sqrt{1-x^2}}\,\varphi = 0$$

or

$$\sqrt{1-x^2}\,\frac{d^2\varphi}{dx^2} - \frac{x}{\sqrt{1-x^2}}\,\frac{d\varphi}{dx} + \frac{\lambda}{\sqrt{1-x^2}}\,\varphi = 0 \longrightarrow \text{Simplifying}$$

$$(1-x^2)\,\frac{d^2\varphi}{dx^2} - x\,\frac{d\varphi}{dx} + \lambda\varphi = 0$$

which also has a power series solution: $\varphi(x) = \sum_{n=0}^{\infty} a_n x^n$

when we let $T_n = a_n x^n$.

Working this out yields:

$T_0(x) = 1$

$T_1(x) = 1$

$\vdots$

$T_n(x) = 2x\,T_{n-1}(x) - T_{n-1}(x) \longrightarrow$ Chebyshev Polynomials

Orthogonality Property

Let's write the SLO as $L\varphi = 0$. For φ to be o.g. functions requires that L is self-adjoint:

$$(5.4) \quad (L\varphi, \psi) = (\varphi, L^*\psi) \quad \text{where} \quad L = L^*.$$

Following this argument requires that $(\varphi, \psi) = 0$ when $\varphi = \varphi_i$ & $\psi = \varphi_j$ $\forall\ i \neq j$. However, this will only be satisfied if the IP is defined as:

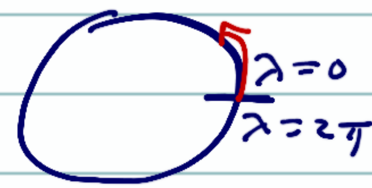
$$(u,v) = \int_a^b \varphi_i(x)\,\varphi_j(x)\,\omega(x)\,dx \quad \text{where:}$$

$$\omega(x) = \begin{cases} 1 \to \text{Fourier} \\ 1 \to \text{Legendre} \\ \frac{1}{\sqrt{1-x^2}} \to \text{Chebyshev} \end{cases}$$

For this reason, we tend to choose, e.g., Legendre over Chebyshev.

Constructing O.G. Polynomials in Multiple Dimensions

Summary In 1d w/ periodicity we obtain Fourier Series.



In 1D w/o periodicity we obtain Legendre Polynomials

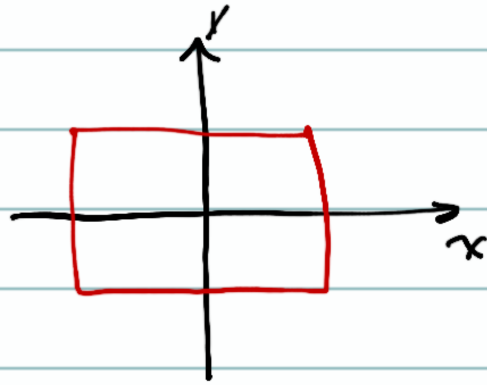
Spherical Domain

On Latitude circles we have periodicity so we use Fourier

On Longitude Semi-circles we don't have periodicity so we use Legendre polynomials.

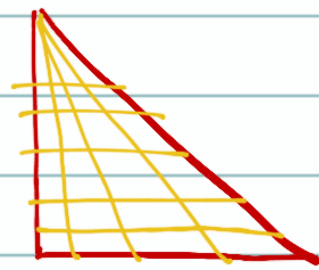
$\therefore$ we write $\varphi_{mn}(\lambda,\theta) = F_m(\lambda) \otimes P_n(\theta)$ →
tensor product of 1D functions

Square Domain



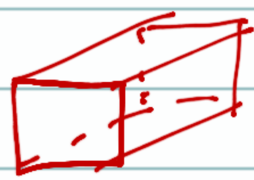
w/o Periodicity, we use: $\varphi_{mn}(x,y) = P_m(x) \otimes P_n(y)$

Triangular Domain



warped tensor products (PKD polynomials)

Hexahedron



Let $\varphi_{mno}(x,y,z) = P_m(x) \otimes P_n(y) \otimes P_o(z)$

Bottom Line Tensor Products are nice because they are easy & convenient. When we learn about integration we will see yet another major advantage.

Modal vs. Nodal Functions

Note that we can approximate a function in two different ways:

$\boxed{\text{Modal}}$: $g_N(x) = \sum_{i=0}^{N} \varphi_i(x)\, \tilde{g}_i$ where, e.g., $\varphi_i = P_i$

& $i = 0, \dots, N$ are the Legendre polynomials that are the natural basis on $x \in [-1, +1]$ for non-periodic domains, & $\tilde{g}_i$ are the amplitudes of the frequencies φ_i.

$\boxed{\text{Nodal}}$ Alternatively, we can write:

$$g_N(x) = \sum_{i=0}^{N} L_i(x)\, g_i$$ where L_i are

Lagrange polynomials & g_i are the sol. variables at the physical points defined by L_i, s.t.

$$L_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

$\boxed{\text{Modal} \rightarrow \text{Nodal}}$ Since they define the same function we can write:

$$g_N(x) \equiv \sum_{j=0}^{N} \varphi_j(x)\, \tilde{g}_j = \sum_{j=0}^{N} L_j(x)\, g_j$$

& $g_N(x_i) \equiv \underbrace{P_{ij}\, \tilde{g}_j = L_{ij}\, g_j}$

$\therefore \quad g_i = L^{-1}_{ik} P_{kj}\, \tilde{g}_j \quad$ & $\quad \tilde{g}_i = P^{-1}_{ik} L_{kj}\, g_j$

This Legendre transform is used quite often in Galerkin methods.