Lecture 13: 1D DG Elliptic

In this lecture we will describe how to solve the elliptic eq.

(13.1) $$\frac{d^2 u}{dx^2} = f(x) \qquad x \in [-L, +L]$$

using the two-step process:

(13.2a) $$Q = \frac{du}{dx}$$

(13.2b) $$\frac{dQ}{dx} = \frac{d^2 u}{dx^2} = f(x)$$

→ known as the flux formulation

We solve 2^{nd} order operators in this way because DG was initially designed to solve 1^{st} order operators.

1^{st} Order Operator w/ DG

Let's discuss how to construct a first derivative. Start w/ the equality:

(13.3) $$\frac{du}{dx} = \frac{du}{dx}$$

Next let $$\frac{du_N^{(e)}}{dx} = \sum_{j=0}^{N} \frac{d\psi_j(x)}{dx} q_j^{(e)} = \sum_{j=0}^{N} \psi_j(x)\, q_{x,j}^{(e)}$$

The middle term is the usual approx. of a derivative using the basis function derivative & the far right term is how we could represent the derivative if we knew the derivative at specific gridpoints x_j.

Multiplying (13.3) by ψ & integrating yields:

(13.4) $$\int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \; q_{x,j} = \int_{\Omega_e} \psi_i \frac{d q_N^{(e)}}{dx} \, d\Omega_e .$$

using the product rule:

$$\frac{d}{dx}\left(\psi_i q_N^{(e)}\right) = \frac{d\psi_i}{dx} q_N^{(e)} + \psi_i \frac{d q_N^{(e)}}{dx}$$

& subbing for the right in (13.4) yields:

(13.5) $$\int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \; q_{x,j}^{(e)} = \left[\psi_i q_N^{(e)}\right]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} q_N^{(e)} \, d\Omega_e$$

where we have used FTC for the 1st term on the right.

In matrix form, we now write:

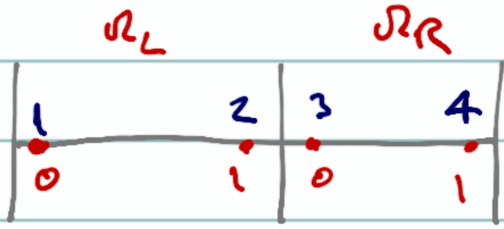
(13.6) $$M_{ij}^{(e)} q_{x,j}^{(e)} = F_{ij}^{(e)} q_j^{(e)} - \tilde{D}_{ij}^{(e)} q_j^{(e)}$$

where we now have to introduce a numerical flux.

Since there is no preferred direction of propagation in a derivative, then we will use the mean/averaged flux:
$$g^{(*)} = \{ g^{(e)} \} \equiv \tfrac{1}{2}\left(g^{(e)} + g^{(n)} \right)$$

Element Eqs.

For the $N=1$ & $N_e=2$ grid configuration shown below



we get the following element eqs:

Left Element

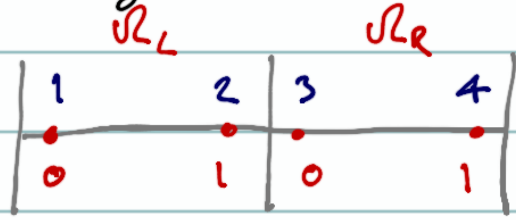
$$\frac{\Delta x^{(L)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} g_{x,1} \\ g_{x,2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g_1^{(*)} \\ g_2^{(*)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

Right Element

$$\frac{\Delta x^{(R)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} g_{x,3} \\ g_{x,4} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g_3^{(*)} \\ g_4^{(*)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} g_3 \\ g_4 \end{pmatrix}$$

Global Matrix

Applying DSS to the $N_e = 2$ $N = 1$ grid configuration



yields: where we assume $\Delta x = \Delta x^{(L)} = \Delta x^{(R)}$ for simplicity

$$\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 & & \\ 1 & 2 & & \\ & & 2 & 1 \\ & & 1 & 2 \end{pmatrix}\begin{pmatrix} f_{x,1} \\ f_{x,2} \\ f_{x,3} \\ f_{x,4} \end{pmatrix} = \begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1^* \\ f_2^* \\ f_3^* \\ f_4^* \end{pmatrix}$$

$$- \frac{1}{2}\begin{pmatrix} -1 & -1 & & \\ 1 & 1 & & \\ & & -1 & -1 \\ & & 1 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

Diagonalizing the mass matrix yields:

$$\frac{\Delta x}{2}\begin{pmatrix} f_{x,1} \\ f_{x,2} \\ f_{x,3} \\ f_{x,4} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1^{(*)} \\ f_2^{(*)} \\ f_3^{(*)} \\ f_4^{(*)} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

Note that the derivatives at q_2 & q_3 in fact may represent the same physical location.

For q_2 we get:

$$\frac{\Delta x}{2} q_{x,2} = q_2^{(*)} - \tfrac{1}{2}(q_1 + q_2)$$

using the mean flux for $q_2^{(*)} = \tfrac{1}{2}(q_2 + q_3)$ yields:

$$\frac{\Delta x}{2} q_{x,2} = \tfrac{1}{2}\left[q_2 + q_3 - q_1 - q_2 \right]$$

$$q_{x,2} = \frac{q_3 - q_1}{\Delta x}$$ if $q \in C^0$ then $q_3 = q_2$

& we get:

$$q_{x,2} = \frac{q_2 - q_1}{\Delta x}$$

For q_3 we get:

$$\frac{\Delta x}{2} q_{x,3} = -q_3^{(*)} + \tfrac{1}{2}(q_3 + q_4)$$

using $q_3^{(*)} = \tfrac{1}{2}(q_2 + q_3)$ yields:

$$\frac{\Delta x}{2} q_{x,3} = \tfrac{1}{2}\left[-q_2 - q_3 + q_3 + q_4 \right]$$

$$f_{x,j} = \frac{f_4 - f_2}{\Delta x}$$ if $f \in C^0$ $f_2 = f_3$ & we get:

$$f_{x,j} = \frac{f_4 - f_3}{\Delta x}$$

Summary This approximation gives us $O(\Delta x)$ first derivatives.

Second Order Differential Operators

So far we have learned to represent a first derivative as follows:

$$(13.7)\quad M_{ij}^{(e)} f_{x,j}^{(e)} = F_{ij} f_j^{(e)} - \tilde{D}_{ij}^{(e)} f_j^{(e)} \longrightarrow \frac{df^{(e)}}{dx}$$

We can apply what we learned here to approximate $\frac{d^2 f}{dx^2}$ as follows:

1. Let $Q = \frac{df}{dx}$ & solve for Q using (13.7)
2. Write $\frac{d^2 f}{dx^2} = \frac{dQ}{dx}$ & solve for $\frac{d^2 f}{dx^2}$ using (13.7)

Auxiliary Variable Q To solve for Q we use the expansions: $Q_N^{(e)}(x) = \sum_{j=0}^{N} \psi_j(x)\, Q_j^{(e)}$ & $f_N^{(e)}(x) = \sum_{j=0}^{N} \psi_j(x) f_j^{(e)}$

multiply by ψ & integrate to get:

$$\int_{\Omega_e} \psi_i Q_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \frac{d\phi_N^{(e)}}{dx} \, d\Omega_e$$

using IBP: $\int_{\Omega_e} \psi_i \frac{d\phi_N^{(e)}}{dx} d\Omega_e = \left[\psi_i \phi_N^{(e)}\right]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} \phi_N^{(e)} d\Omega_e$

yields:

$$\int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, Q_j^{(e)} = \left[\psi_i \phi_N^{(e)}\right]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} \, d\Omega_e \, \phi_j^{(e)}$$

& in matrix form:

$$M_{ij}^{(e)} Q_j^{(e)} = F_{ij} \phi_j^{(e)} - \tilde{D}_{ij}^{(e)} \phi_j^{(e)}$$

$\frac{dQ}{dx}$

The eq. for $\frac{dQ}{dx}$ is as follows:

$\frac{d^2\phi}{dx^2} = \frac{dQ}{dx}$ which, after applying the same strategy as used for $Q = \frac{d\phi}{dx}$ yields:

$$M_{ij}^{(e)} \phi_{xx,j}^{(e)} = F_{ij}^{(e)} Q_j^{(e)} - \tilde{D}_{ij}^{(e)} Q_j^{(e)}$$

Solution of the 1D Elliptic

We can now use the previous machinery to solve the elliptic Poisson problem:

$$\frac{d^2 q}{dx^2} = f(x) \quad w/ \quad q \in [-L, +L] \quad \& \quad q\big|_{\Gamma_0} = g(x)$$

$$\& \quad \frac{dq}{dx}\bigg|_{\Gamma_N} = h(x)$$

We break the problem down into the following two-steps:

1. $Q = \frac{dq}{dx}$

2. $\frac{dQ}{dx} = f(x)$

which results in

$$(13.8) \qquad M_{ij}^{(e)} Q_j^{(e)} = F_{ij}^{(e)} q_j^{(*)} - \tilde{D}_{ij}^{(e)} q_j^{(e)}$$

$$\text{and} \qquad = F_{q\,ij}^{(*)} q_j - \tilde{D}_{ij}^{(e)} q_j^{(e)}$$

$$(13.9) \qquad F_{ij}^{(e)} Q_j^{(*)} - \tilde{D}_{ij}^{(e)} Q_j^{(e)} = M_{ij}^{(e)} f_j^{(e)}$$

$$= F_{Q\,ij}^{(*)} Q_j^{(e)} - \tilde{D}_{ij}^{(e)} Q_j^{(e)} = M_{ij}^{(e)} f_j^{(e)}$$

which can be solved in this two-step process or we can use (13.8) to replace Q in (13.9) as follows:

(13.10) $Q_i^{(e)} = M^{-1} D^{DG}_{ij} q_j$ where $D^{DG} = F^{(*)} - \tilde{D}^{(e)}$

& let $\hat{D}^{DG} = M^{-1} D^{DG}$

We can now sub (13.10) into (13.9) to get:

(13.11) $$D^{DG} M^{-1} D^{DG} Q^{(e)} = M^{(e)} f^{(e)}$$

$\therefore$ we note that the DG Laplacian operator is in fact: $L = D^{DG} M^{-1} D^{DG}$.