Lecture 11: 1D CG/DG Hyperbolic

Introduction

For the 1D wave eq. $\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0$

we see that the CG representation is:

$$(11.1) \quad M_{ij}^{(e)} \frac{d q_j^{(e)}}{dt} + D_{ij}^{(e)} f_j^{(e)} = \square$$

where, upon DSS, $i,j = 0,\dots,N$, $e = 1,\dots,N_e$

we get:

$$(11.2) \quad M_{IJ} \frac{d q_J}{dt} + D_{IJ} f_J = \square \qquad I,J = 1, N_{point}$$

where $M_{IJ} = \bigwedge_{e=1}^{N_e} M_{ij}^{(e)}$

For DG, we obtained:

$$(11.3) \quad M_{ij}^{(e)} \frac{d q_j^{(e)}}{dt} + F_{ij}^{(e)} f_j^{(*)} - \tilde{D}_{ij}^{(e)} f_j^{(e)} = \square$$

$i,j = 0,\dots,N$

where $f^{(*)}$ is the numerical flux $\qquad e = 1,\dots,N_e$

Unified CG/DG

Looking at (11.1) & (11.3) we see little difference b/w CG & DG at least for the element eqs. The only difference is that for DG we used IBP & then added a numerical flux $f^{(*)}$. Let's do the same thing for CG which means that we arrive at:

$$(11.3) \quad M_{ij}^{(e)} \frac{df_j^{(e)}}{dt} + F_{ij}^{(e)} f_j^{(*)} - \tilde{D}_{ij}^{(e)} f_j^{(e)} = 0$$

Let us now apply DSS to (11.3).

DSS

Applying DSS to (11.3) yields:

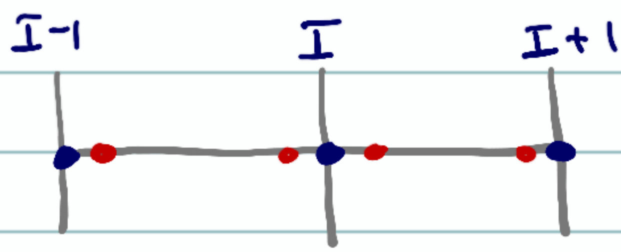
$$(11.4) \quad M_{IJ} \frac{df_J}{dt} + F_{IJ} f_J^{(*)} - \tilde{D}_{IJ} f_J = 0$$

$$I,J = 1,\dots,N_{point}$$

where we already know what M is & have a good idea what $\tilde{D}$ looks like since it is quite similar to D. In fact $\tilde{D} = D^T$. However, we don't quite know what F looks like.

DSS of F matrix

Let's look at the simple example for $N=1$ & $N_e=2$ as follows:



where we know that $F_{ij}^{(e)} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$

∴ we get

$$F_{IJ} = \bigwedge_{e=1}^{Ne} F_{ij}^{(e)} = \begin{pmatrix} -1 & 0 & \\ 0 & +1 \;\; -1 & 0 \\ & 0 & +1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

In general, we will find that

$$F_{IJ} = \begin{pmatrix} -1 & 0 & \cdots & & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & +1 \end{pmatrix}$$

when the only non-zero entries are at the left & right-most points (i.e., physical boundaries).

For periodic BCs $F_{IJ} = 0$ because $F_{1,1}$ cancels $F_{Npoin,Npoin}$

∴ we can safely use (11.3) for both CG & DG, & then applying DSS for both methods gives us the global matrix problem given by (11.4), i.e.,

$$M_{IJ} \frac{dq_I}{dt} + F_{IJ} f_J^{(*)} - \tilde{D}_{IJ} f_J = 0$$

System of Eqs.

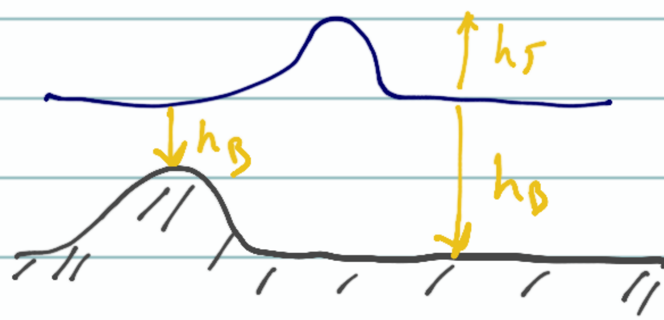
Using Eq. (11.4) we can now apply the unified CG/DG approach for systems of eqs.

Let's take the 1D shallow water eqs:

(11.5) $$\frac{\partial h_S}{\partial t} + \frac{\partial U}{\partial x} = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left[\frac{U^2}{h} + \frac{1}{2} g h_S^2 + g h_S h_B \right] = g h_S \frac{d h_B}{d x}$$

where this means



with $h = h_S + h_B$ being the height of the water column & $U = hu$ is the momentum (or discharge in some terminology). We can write (11.5) as follows:

(11.6) $$\frac{\partial \bar{q}}{\partial t} + \frac{\partial \bar{f}}{\partial x} = \bar{S}$$ with:

$$\bar{q} = \begin{pmatrix} h_S \\ U \end{pmatrix}, \quad \bar{f} = \begin{pmatrix} U \\ \frac{U^2}{h} + \frac{1}{2} g h_S^2 + g h_S h_B \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} 0 \\ g h_S \frac{d h_B}{d x} \end{pmatrix}$$

We can discretize (11.6) in the usual way as follows:

Let $\bar{q}_N^{(e)}(x,t) = \sum_{j=0}^{N} \psi_j(x)\, \bar{q}_j^{(e)}(t)$ & $\bar{f}_N^{(e)}(x,t) = \bar{f}(q_N^{(e)})$

& $\bar{S}_N^{(e)} = \bar{S}(\bar{q}_N^{(e)})$

Plugging these approximations into (11.6) yields:

(11.7) $$\frac{\partial \bar{q}_N^{(e)}}{\partial t} + \frac{\partial \bar{f}_N^{(e)}}{\partial x} - \bar{S}_N^{(e)} = \epsilon$$ & multiplying by ψ & integrating yields:

(11.8) $$M_{ij}^{(e)} \frac{d\bar{q}_j^{(e)}}{dt} + F_{ij}^{(e)} \bar{f}_j^{(*)} - \tilde{D}_{ij}^{(e)} \bar{f}_j^{(e)} = \bar{S}_i^{(e)}$$

where we have seen all the matrices in this eq. The only term we have not seen before is the source vector $\bar{S}_i^{(e)}$.

Numerical Flux

Let us now define the numerical flux using Rusanov as follows:

$$\bar{f}^{(*)} = \frac{1}{2}\left[\bar{f}^{(e)} + f^{(k)} - |\lambda|\, \hat{n}^{(e,k)} \left(\bar{q}^{(k)} - \bar{q}^{(e)} \right) \right]$$

where $\bar{f} = \begin{pmatrix} h_s \\ U^2/h + \frac{1}{2} g h_s^2 + g h_s h_B \end{pmatrix}$ $\bar{q} = \begin{pmatrix} h \\ U \end{pmatrix}$

4 $\lambda = |u| + \sqrt{g h_s}$

Constructing Efficient CG/DG Algorithms

So far we have arrived at the following unified CG/DG global matrix problem (Alg. 6.7)

- Construct $M^{(e)}$, $\tilde{D}^{(e)}$, $F^{(e)}$
- Construct M, $\tilde{D}$, & F via DSS
- Compute $\hat{D} = M^{-1}\tilde{D}$ & $\hat{F} = M^{-1}F$
- for $n = 1 : N_{time}$ do

  for $I = 1 : N_{point}$

  $$R_I = \hat{D}_{IJ} f_J - \hat{F}_{IJ} f_J^{(*)}$$

  end

  $$\frac{d f_I}{dt} = R_I$$

  end

Although this algorithm is easy to understand, it is not the optimal way of constructing CG/DG algorithms.

Let us consider the following algorithm (Alg. 7.6):

- Construct $M^{(e)}$, $F^{(e)}$, & $\tilde{D}^{(e)}$
- Construct M via DSS
- For $n = 1 : N_{time}$ do

    for $e = 1 : N_e$ do

        for $i = 0 : N$ do

            $I = \text{intma}(i, e)$ → $I = \text{periodicity}(\text{intma}(i,e))$

            $R_I \mathrel{+}= \tilde{D}^{(e)}_{ij} \, f_j^{(e)}$ → implies a dot-product

        end

    end

    for $s = 1 : N_{faces}$ do

        $R_I \mathrel{-}= F^{(e)}_{ij} \, f_j^{(*)}$

    end

    $R_I = M^{-1}_{IJ} R_J$

    $\frac{dq_I}{dt} = R_I$ → it CG then $q_{N_{point}} = q_1$ for periodicity

end

Note that BCs have not been incorporated here for CG. We would still need to use Periodicity as before