

An Introduction to Element-based Galerkin
Methods on Tensor-Product Bases: Analysis,
Algorithms, and Applications

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