Lecture 10: 1D DG Hyperbolic

Introduction

In lectures 8 & 9 we found that the element-wise CG matrix problem is written as:

$$(10.1) \qquad M_{ij}^{(e)} \frac{d q_j^{(e)}}{dt} + D_{ij}^{(e)} f_j^{(e)} = 0 \qquad i,j = 0,\ldots,N$$

$$\forall \left\{ \Omega_e \,\middle|\, \Omega = \bigcup_{e=1}^{N_e} \Omega_e \right\}$$

Using the map INTMA s.t. $(i,e) \rightarrow I$ we can use Eq. (10.1) to write the global matrix problem as follows:

$$(10.2) \qquad M_{IJ} \frac{d q_J}{dt} + D_{IJ} f_J = 0 \qquad I,J = 1,\ldots,N_p$$

where $M_{IJ} = \bigwedge_{e=1}^{N_e} M_{ij}^{(e)}$, etc.

DG Equations

To derive the DG discretization of $\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0$ we use the basis function expansion $q_N^{(e)} = \sum_{j=0}^{N} \psi_j(x)\, q_j^{(e)}(t)$ & $f_N^{(e)} = \sum_{j=0}^{N} \psi_j(x)\, f_j^{(e)}(t)$ & write: find $q_N^{(e)} \in L^2$ s.t.

$$(10.3) \qquad \int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t}\, d\Omega_e + \int_{\Omega_e} \psi_i \frac{\partial f_N^{(e)}}{\partial x}\, d\Omega_e = 0 \qquad \forall \psi \in L^2$$

which is exactly what we had for CG. However, since

$g_N^{(e)} \in L^2$ then Eq. (10.3) states that each element has a distinct solution & so the problem is uncoupled & thereby not well-posed. We can overcome this issue by the following two fixes: using integration by parts & defining a "numerical flux".

Integration by Parts (IBP)

The 2^{nd} term in (10.3) can be rewritten as:

$$\int_{\Omega_e} \frac{\partial}{\partial x}\left(\psi_i f_N^{(e)}\right) d\Omega_e = \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} f_N^{(e)} d\Omega_e + \int_{\Omega_e} \psi_i \frac{\partial f_N^{(e)}}{\partial x} d\Omega_e$$

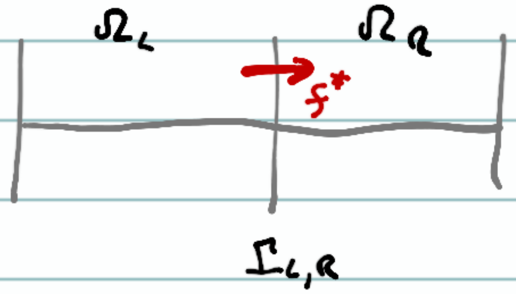
& integrating (using the FTC for the left term) yields:

$$\text{(10.4)} \quad \int_{\Omega_e} \psi_i \frac{\partial f_N^{(e)}}{\partial x} d\Omega_e = \left[\psi_i f_N^{(e)}\right]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} f_N^{(e)} d\Omega_e = 0$$

Subbing (10.4) into (10.3) yields:

$$\text{(10.5)} \quad \int_{\Omega_e} \psi_i \frac{\partial g_N^{(e)}}{\partial t} d\Omega_e + \underbrace{\left[\psi_i f_N^{(e)}\right]\Big|_{\Gamma_e}}_{= \hat{n}\, \psi_i f_N^{(e)}} - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} f_N^{(e)} d\Omega_e = 0$$

Numerical Flux Eq. (10.5) shows that we moved the derivative from $f_N^{(e)}$ to ψ. However, we still have the challenge that the boundary term $[\psi_i f_N^{(e)}]$ will yield a different value at Γ_e since $f_N^{(e)}$ is discontinuous across elements. Therefore we need to replace $[\psi_i f_N^{(e)}]$ by a continuous function s.t. what leaves one element enters the neighbor as such:



i.e., we need to replace $[\psi_i f_N^{(L)}]|_{\Gamma_L}$ by $[\psi_i f_N^{(*)}]|_{\Gamma_L}$ s.t.

$f_N^{(*)} = F(f_N^{(L)}, f_N^{(R)})$. The simplest such numerical flux is the mean value:

$$f_N^{(*)} \equiv \frac{1}{2}\left(f_N^{(L)} + f_N^{(R)}\right) := \{f_N\}$$

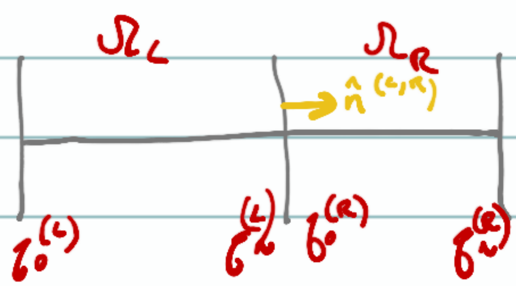
→ However, this choice is not ideal as we shall see.

A better option is the upwind numerical flux:

$$f_N^{(*)} = \frac{1}{2}\left[f_N^{(L)} + f_N^{(R)} - \hat{n}\, u \left(g^{(R)} - g^{(L)}\right)\right]$$

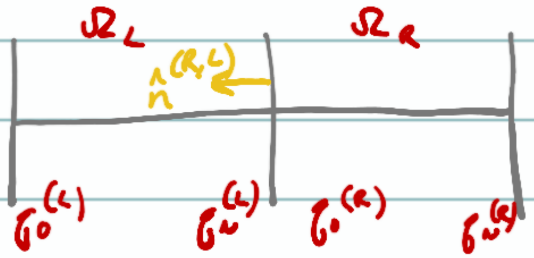
where $\hat{n}$ is the unit normal vector which points from $(L) \to (R)$.

For example: in the figure:



we get for Ω_L: $f^{(*,L)} = \frac{1}{2}\left[f_N^{(L)} + f_0^{(R)} - (1)\, u \left(g_0^{(R)} - g_N^{(L)}\right)\right]$

$$f^{(*,L)} = \frac{1}{2}\left[f_N^{(L)} + f_0^{(R)} - u\left(g_0^{(R)} - g_N^{(L)} \right) \right]$$



For Ω_R we get:

$$f^{(*,R)} = \frac{1}{2}\left[f_0^{(R)} + f_N^{(L)} - (-1)u\left(g_N^{(L)} - g_0^{(R)} \right) \right]$$

$$= \frac{1}{2}\left[f_0^{(R)} + f_N^{(L)} - u\left(g_0^{(R)} - g_N^{(L)} \right) \right]$$

which shows that $f^{(*,L)} = f^{(*,R)}$ & so $f^{(*)}$ is continuous. This means that since $[\psi_i f_N^{(*)}]|_{\Gamma_e} = \hat{n}\psi_i f_N^{(*)}$ then what flows out of Ω_L will flow into Ω_R which makes physical sense & so the problem is well-posed.

DG Reference Element Eqs. From Eq. (10.5) we can now write the local matrix problem as follows:

$$(10.6) \qquad M_{ij}^{(e)} \frac{dg_j^{(e)}}{dt} + F_{ij}^{(e)} f_j^{(*)} - \tilde{D}_{ij}^{(e)} f_j^{(e)} = 0$$

where $M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e$, $F_{ij}^{(e)} = [\psi_i \psi_j]|_{\Gamma_e}$
& $\tilde{D}_{ij}^{(e)} = \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} \psi_j \, d\Omega_e$

The only matrix that needs further discussion is the flux matrix : $F_{ij}^{(e)}$. Recall that this comes from

$$F_{ij}^{(e)} f_j^{(e)} \equiv \left[\psi_i f_N^{(*)}\right]\Big|_{\Gamma_e} = \left[\psi_i \sum_{j=0}^{N} \psi_j f_j^{(*)}\right]_{\Gamma_e} = \left[\psi_i \psi_j\right]\Big|_{\Gamma_e} f_j^{(*)}$$

For the special case where $N=1$ we get:

$\psi_0 = \frac{1}{2}(1-\xi)$, $\psi_1 = \frac{1}{2}(1+\xi)$

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

<u>Differentiation Matrix</u>

$$D_{ij}^{(e)} = \int_x \frac{d\psi_i}{dx}(x)\, \psi_j(x)\, dx = \int_{-1}^{+1} \frac{d\psi_i(\xi)}{d\xi} \cancel{\frac{d\xi}{dx}}\, \psi_j(\xi) \cancel{\frac{dx}{d\xi}}\, d\xi$$

$$= \int_{-1}^{+1} \frac{d\psi_i(\xi)}{d\xi} \psi_j(\xi)\, d\xi = \int_{-1}^{+1} \begin{pmatrix} \frac{d\psi_0}{d\xi}\psi_0 & \frac{d\psi_0}{d\xi}\psi_1 \\ \frac{d\psi_1}{d\xi}\psi_0 & \frac{d\psi_1}{d\xi}\psi_1 \end{pmatrix} d\xi$$

where:

$\psi_0 = \frac{1}{2}(1-\xi)$, $\frac{d\psi_0}{d\xi} = -\frac{1}{2}$

& so we get:

$\psi_1 = \frac{1}{2}(1+\xi)$, $\frac{d\psi_1}{d\xi} = \frac{1}{2}$

$$\tilde{D}_{ij}^{(e)} \equiv \frac{1}{4}\int_{-1}^{+1} \begin{pmatrix} -(1-\xi) & -(1+\xi) \\ (1-\xi) & (1+\xi) \end{pmatrix} d\xi$$

$$= \frac{1}{4}\begin{pmatrix} -\xi+\frac{1}{2}\xi^2 & -\xi-\frac{1}{2}\xi^2 \\ \xi-\frac{1}{2}\xi^2 & \xi+\frac{1}{2}\xi^2 \end{pmatrix}\Bigg|_{-1}^{+1} = \frac{1}{4}\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

Note that $\tilde{D} = D^T$ which is true in general for all polynomial orders. $\therefore \sum_{i=0}^{N} \tilde{D}_{ij} = 0 \quad \forall \; j = 0, \ldots, N$

Flux matrix The flux matrix is defined as follows:

$$F_{ij}^{(e)} = \left[\psi_i(x)\,\psi_j(x)\right]\Big|_{x_0}^{x_1} = \left[\psi_i(\xi)\,\psi_j(\xi)\right]\Big|_{-1}^{+1}$$

$$= \begin{pmatrix} \psi_0(\xi)\psi_0(\xi) & \psi_0(\xi)\psi_1(\xi) \\ \psi_1(\xi)\psi_0(\xi) & \psi_1(\xi)\psi_1(\xi) \end{pmatrix}\Bigg|_{-1}^{+1} = \frac{1}{4}\begin{pmatrix} (1-\xi)^2 & 1-\xi^2 \\ 1-\xi^2 & (1+\xi)^2 \end{pmatrix}\Bigg|_{-1}^{+1}$$

$$= \frac{1}{4}\begin{pmatrix} (1-1)^2-(1+1)^2 & (1-1)-(1-1) \\ (1-1)-(1-1) & (1+1)^2-(1-1)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

where $\sum_{i,j=0}^{N} F_{ij}^{(e)} = 0$ in general & $F_{ij}^{(e)} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & +1 \end{pmatrix}$

Element Eqs. With all of the element matrices defined we can now write (10.6) as follows:

(10.7) $$\frac{\Delta x^{(e)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \xi_0^{(e)} \\ \xi_1^{(e)} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} f_0^{(*)} \\ f_1^{(*)} \end{pmatrix}$$

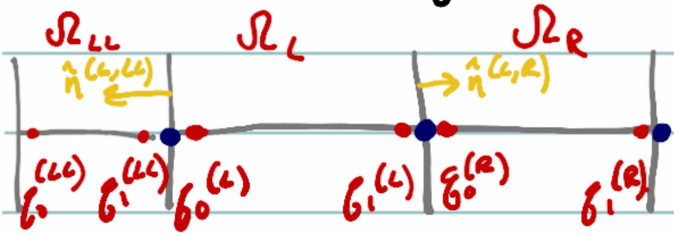
$$-\frac{1}{2}\begin{pmatrix}-1 & -1\\ +1 & +1\end{pmatrix}\begin{pmatrix} f_0^{(e)} \\ f_1^{(e)}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \equiv \square$$

Global Matrix Problem

Now that we know the local matrix problem given by Eq. (10.7) we can derive the global matrix problem.

Left Element Eqs.

From Eq. (10.7) we can write the left element eqs. for the following three element set-up:



(10.8) $$\frac{\Delta x^{(L)}}{6}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\frac{d}{dt}\begin{pmatrix} g_0^{(L)} \\ g_0^{(R)}\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} f_0^{(*;L,R)} \\ f_1^{(*;L,R)}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1 & -1\\ +1 & +1\end{pmatrix}\begin{pmatrix} f_0^{(L)} \\ f_1^{(L)}\end{pmatrix} = \square$$

where $f_i^{(L)} = u\, g_i^{(L)}$ &

$$f_1^{(*;L,R)} = \frac{1}{2}\left[f_1^{(L)} + f_0^{(R)} - \hat{n}^{(L,R)} u \left(g_0^{(R)} - g_1^{(L)}\right)\right]$$

$$= \frac{1}{2}\left[f_1^{(L)} + \cancel{f_0^{(R)}} - \cancel{u\, g_0^{(R)}} + u\, g_1^{(L)}\right]$$

Same

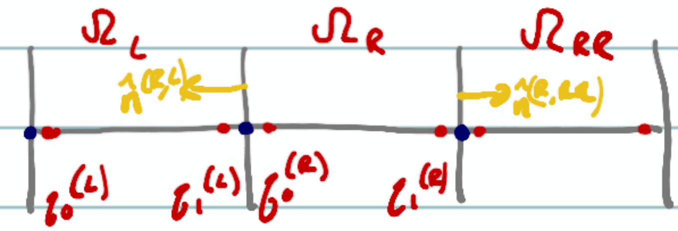
$$= f_1^{(L)}$$

$\therefore$ (10.8) becomes:

$$(10.9)\quad \frac{\Delta x^{(L)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} g_0^{(L)} \\ g_1^{(L)} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_1^{(LL)} \\ f_1^{(L)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_0^{(L)} \\ f_1^{(L)} \end{pmatrix} = \square$$

Right Element Eqs.

Let's consider the three element set-up:



We get the element eqs.

$$(10.10)\quad \frac{\Delta x^{(R)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} g_0^{(R)} \\ g_1^{(R)} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} f_0^{(*;R,L)} \\ f_1^{(*;R,L)} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} f_0^{(R)} \\ f_1^{(R)} \end{pmatrix} = \square$$

where $f_0^{(*;R,L)} = \frac{1}{2}\left[f_0^{(R)} + f_1^{(L)} - \hat{n}^{(R,L)} u \left(g_1^{(L)} - g_0^{(R)} \right) \right]$

$$= \frac{1}{2}\left[\cancel{f_0^{(R)}} + f_1^{(L)} + u\left(g_1^{(L)} - \cancel{g_0^{(R)}} \right) \right]$$

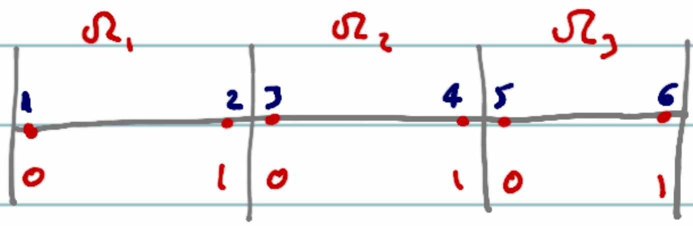
$$= f_1^{(L)} \rightarrow \text{Same as } f_1^{(*;L,R)}$$

$\therefore$ the Right element eqs. become:

$$(10.11)\quad \frac{\Delta x^{(R)}}{6}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\frac{d}{dt}\begin{pmatrix}g_0^{(R)}\\ g_1^{(R)}\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}\begin{pmatrix}f_1^{(L)}\\ f_1^{(R)}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1 & -1\\ +1 & +1\end{pmatrix}\begin{pmatrix}f_0^{(R)}\\ f_1^{(R)}\end{pmatrix} = 0$$

Constructing the Global Matrix Problem

Let's consider the following element configuration:



with 3 elements each w/ $N=1$ & a total of

$N_g = N_e(N+1) = 3(2) = 6$ DOF.

$\therefore$ the element eqs. are:

Ω_1

$$\frac{\Delta x^{(1)}}{6}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\frac{d}{dt}\begin{pmatrix}g_1\\ g_2\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}\begin{pmatrix}f_1^{(*)}\\ f_2^{(*)}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1 & -1\\ +1 & +1\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix} = 0$$

Ω_2

$$\frac{\Delta x^{(2)}}{6}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\frac{d}{dt}\begin{pmatrix}g_3\\ g_4\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}\begin{pmatrix}f_3^{(*)}\\ f_4^{(*)}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1 & -1\\ +1 & +1\end{pmatrix}\begin{pmatrix}f_3\\ f_4\end{pmatrix} = 0$$

Ω_3

$$\frac{\Delta x^{(3)}}{6}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\frac{d}{dt}\begin{pmatrix}g_5\\ g_6\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}\begin{pmatrix}f_5^{(*)}\\ f_6^{(*)}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1 & -1\\ +1 & +1\end{pmatrix}\begin{pmatrix}f_5\\ f_6\end{pmatrix} = 0$$

which in global form is: let $\Delta x = \Delta x^{(1)} = \Delta x^{(2)} = \Delta x^{(3)}$

$$
\frac{\Delta x}{6}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{pmatrix}
+ \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1^{(*)} \\ f_2^{(*)} \\ f_3^{(*)} \\ f_4^{(*)} \\ f_5^{(*)} \\ f_6^{(*)} \end{pmatrix}
- \frac{1}{2}\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ +1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & +1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & +1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix} = \square \tag{10.12}
$$

Note that (10.12) appears to be entirely block diagonal. However once we replace $f^{(*)}$ with its corresponding values, we will see that this matrix breaks the block diagonal form.

For the specific case of Rusanov numerical flux, this matrix becomes:

(10.13) $$F_{ij} f_j^{(*)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}$$

where the red font denote changes from the flux matrix given in (10.12). We can now see that F is not block diagonal since the yellow parentheses show those terms that would make the matrix block diagonal.