(Ch. 1)

# Lecture 3: Convergence & Norms

**Convergence** Lax Equivalence Theorem states that a numerical method must be consistent with the underlying continuous equations, i.e., take

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0 \xrightarrow[\text{upwind}]{1^{st}\text{ order}} \frac{q_i^{n+1} - q_i^n}{\Delta t} + u\frac{q_i^n - q_{i-1}^n}{\Delta x} + O(\Delta t, \Delta x) = 0$$

The discretized form recovers the continuous form as $(\Delta t, \Delta x) \to 0$ at the rate of $O(\Delta t, \Delta x)$.
Furthermore, if consistency is met, then stability (i.e., the numerical sol. is bounded) is the necessary & sufficient condition for convergence.

$\boxed{\text{Def}}$ Convergence means that, if the continuous eq. has a unique analytic solution, then given enough resolution our consistent & stable numerical method will converge to the "truth" at the rate of $O(\Delta t^K, \Delta x^N)$ when $K$ & $N$ are the orders of accuracy of our method in time & space, respectively.

However, if we wish to compare our numerical sol. w/ the truth then we need a metric to measure how close we are.

## Error Norms

We use norms to perform this task.

Let's assume that we live in a vector space, $L^p$.

Then let $g^{num}$ & $g^{exact}$ be the numerical & exact solutions. Next let us define the $L^p$ error as:

$$\text{(3.1)} \qquad \| g^{num} - g^{exact} \|_{L^p} = \left[ \sum_{i=1}^{N} \left| g_i^{num} - g_i^{exact} \right|^p \right]^{1/p}$$

where $\dim(g) = N$

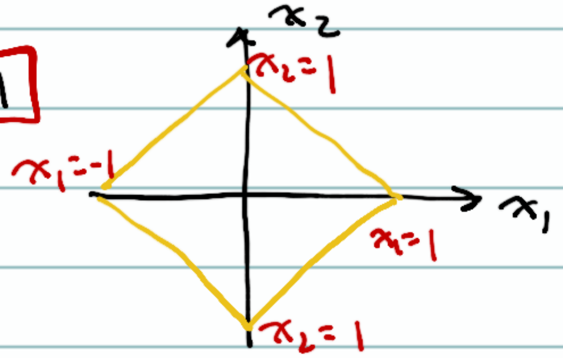
The most common error norms that one encounters are:

$$\text{(3.2.1)} \qquad L^1 : \| g^{num} - g^{exact} \|_{L^1} = \sum_{i=1}^{N} \left| g_i^{num} - g_i^{exact} \right|$$

$$\text{(3.2.2)} \qquad L^2 : \| g^{num} - g^{exact} \|_{L^2} = \sqrt{ \sum_{i=1}^{N} \left| g_i^{num} - g_i^{exact} \right|^2 }$$

$$\text{(3.2.3)} \qquad L^\infty : \| g^{num} - g^{exact} \|_{L^\infty} = \max_{\forall i=1,\dots,N} \left( \left| g_i^{num} - g_i^{exact} \right| \right)$$

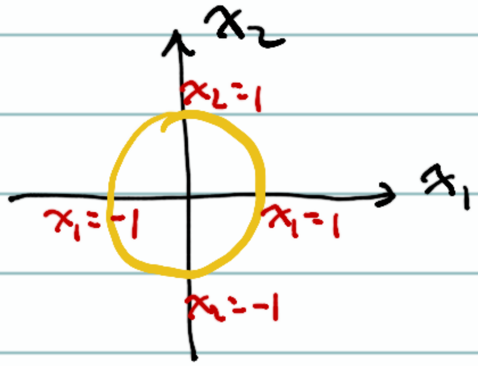
$\boxed{\text{EX } 3.1}$




$$|\bar{x}|_{L^1} = \sum_{i=1}^{N} |x_i| = 1$$

$$= |x_1| + |x_2| = 1 \rightarrow$$ unit ball under the $L^1$ norm or 1-norm

**Ex 3.2**




$$\|\bar{x}\|_{\ell^2} = \left[\sum_{i=1}^{N} |x_i|^2\right]^{1/2} = 1$$

$$= \underbrace{\sqrt{x_1^2 + x_2^2} = 1}_{\text{unit ball under the 2-norm}}$$

**Ex 3.3**


$$\|\bar{x}\|_{\ell^\infty} = \max_{1\le i\le N} \left(|x_i|\right) = 1$$

$$= \underbrace{\max\left(|x_1|, |x_2|\right)}_{\text{unit ball under } \infty\text{-norm}}$$