MA4245
Mathematical Foundations of Galerkin Methods

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Credit Hours: 4

Class Schedule: -- Monday (10am-11am), Wednesday (1pm-2pm), Thursday (9am-11am) in SP-257.

Office Hours: -- Anytime my door is open or by appointment (email is best to get a hold of me).

Course Objective:

The goal of the course is to provide mathematical and practical definitions of Galerkin methods. The simplest definition of a Galerkin method is a method whereby the solution vector is approximated by a polynomial expansion that is then used to solve a partial differential equation; in this course, we shall cover hyperbolic, elliptic, and parabolic equations. Hyperbolic and parabolic equations yield time-dependent solutions and a method of lines approach will be used to introduce time-integrators. While many of the concepts introduced can be explained using one-dimensional domains, real understanding of Galerkin methods can only be accomplished in the context of multi-dimensional spatial domains; we shall explore these in two-dimensional domains (and possibly three-dimensions). While time will not permit a discussion of these methods on distributed-memory computers, algorithmic advantages of these methods will be discussed when appropriate. Those that have already taken MA4261 are encouraged to use the tools learned there to make the Galerkin codes performant.

Course Description:

Galerkin methods including global and local methods as well as continuous and discontinuous methods (including the hybridizable DG method) will be covered in detail; finite differences will be covered for comparing and contrasting. One-dimensional partial differential equations such as the linear advection-diffusion equation and the Poisson equation will be used to demonstrate the implementation of the methods. One-dimensional systems of equations such as the shallow water equations and the Euler equations will be covered as well. The two-dimensional Laplace and advection-diffusion equations will also be discussed as well as extensions to systems of nonlinear equations.

Course Requirements:

It will be assumed that the student has had prior courses on: ordinary differential equations (MA2121), partial differential equations (MA3132), linear algebra

(MA3046), and numerical analysis (MA3232). Proficiency in Matlab is necessary but would prefer to use Julia.

Primary Textbooks:

1. F.X. Giraldo, An Introduction to Element-based Galerkin Methods on Tensor-Product Bases (2019 version, available in PDF).

Topics to be covered:

1. Introduction: The importance of numerical methods/modeling for solving real world problems. Galerkin methods and their role in scientific computing.
2. Classification of partial differential equations (PDEs). Definitions of convergence, consistency and stability. Conservation and accuracy. Lax equivalence theorem.
3. Differential versus integral forms of the equations: i.e., finite differences versus Galerkin methods.
4. Galerkin methods, including: spectral, finite elements, spectral elements, finite volumes, and discontinuous Galerkin methods.
5. Interpolation and approximation theory (Lobatto, Legendre, and Chebyshev polynomials) and numerical quadrature (Legendre-Gauss, Lobatto-Gauss).
6. Explicit time-integration methods (Leapfrog, backward difference formulas, Runge-Kutta, Adams-Bashforth) and their corresponding stability regions.
7. 1D advection-diffusion and elliptic equations. Initial and boundary conditions. CFL condition and stability.
8. 1D systems of equations: shallow water and Euler.
9. Upwinding, numerical flux functions (for first and second order differential operators), numerical filters, numerical diffusion, and energy/entropy stable methods.
10. 2D linear scalar equations: Poisson and advection-diffusion equations; boundary conditions; the importance of grid generation.
11. Adaptive mesh refinement for both continuous and discontinuous Galerkin methods.
12. Hybridizable discontinuous Galerkin and stabilization methods.

Grading:

-Homework Projects (4): Each project is worth 25%

Each homework project must be submitted both electronically (with code ready to run) along with a PDF describing your methodology and results.

Assignments/Exams:

Four homework projects will be assigned.
The projects must be worked on individually but discussions among the students is not only allowed *but strongly encouraged*. No final exam will be given.

Week	(Lectures Hours) , Topic, Readings, Assignments
1 April 1-5	(1) Motivation and Background; (2) Consistency, Stability; (3) Convergence; (4) Classification of PDES Reading: Chapters 1 and 2
2 April 8-12	(1)-(4) Overview of Methods Reading: Chapter 3
3 April 15-19	(1)-(2) Galerkin Methods; (3) 1D Interpolation; (4) 1D Integration Reading: Chapters 3, 4, and 5
4 April 22-26	(1)-(4) Continuous Galerkin Methods for 1D Hyperbolic Equations Reading: Chapter 6 Project 1 Due: Interpolation and Integration (April 26 at 12pm)
5 April 29-May 3	(1)-(2) Discontinuous Galerkin Methods and (3)-(4) Unified CG/DG Methods for 1D Hyperbolic Equations Reading: Chapters 7 and 8
6 May 6-10	(1)-(2) CG for 1D Elliptic Equations; (3)-(4) DG for 1D Elliptic Equations Reading: Chapters 9 and 10 Project 2 Due: 1D Wave Equation for CG/ DG (May 10 at 12pm)

Week	Lectures Hours , Topic, Readings, and Assignments
7 **May 13-17**	**(1)-2) 2D Interpolation; (3)-(4) 2D Integration** **Reading: Chapters 11 and 12**
8 **May 20-24**	**(1)-(3) CG for 2D Elliptic Equations; (4) DG for 2D Elliptic Equations** **Reading: Chapters 13 and 14** **Project 3 Due: 1D Shallow Water and/or Euler for CG/DG (May 14 at 12pm)**
9 **May 27-31**	**(2)-(4) CG and DG for 2D Hyperbolic-Elliptic Equations** **Reading: Chapters 15, 16, and 17** **May 27 (Monday) Memorial Day: No Class**
10 **June 3-7**	**(1)-(4) Riemann Solvers, Filtering, and Stabilization** **Reading: Chapter 18**
11 **June 10-14**	**(1)-(2) Advanced Topics (Adaptive Mesh Refinement, HDG, and Stabilization)** **Reading: Chapter 2** **Project 4 Due: 2D Poisson Equation for CG and 2D Wave Equation for CG/DG (June 10 at 12pm)**