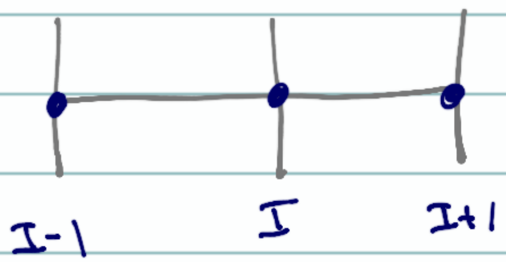
# Lecture 12: 1D CG Elliptic

Let us now consider the Poisson equation:

$$(12.1) \qquad \frac{d^2 q}{dx^2} = f(x) \quad w/ \quad q = q(x) \quad \& \quad q|_{\Gamma} = g(x).$$

## Finite Difference Method




Using FDM we can write:

$$q_{I+1} = q_I + \Delta x\, q_{x,I} + \frac{\Delta x^2}{2} q_{xx,I} + \frac{\Delta x^3}{6} q_{xxx,I} + O(\Delta x^4)$$

$$q_{I-1} = q_I - \Delta x\, q_{x,I} + \frac{\Delta x^2}{2} q_{xx,I} - \frac{\Delta x^3}{6} q_{xxx,I} + O(\Delta x^4)$$

which yields, after adding,

$$q_{I+1} + q_{I-1} = 2 q_I + \Delta x^2 q_{xx,I} + O(\Delta x^4)$$

rearranging:

$$q_{xx,I} = \frac{q_{I+1} - 2 q_I + q_{I-1}}{\Delta x^2} + O(\Delta x^2)$$

CG

To use EBG methods we begin w/ the approximation:

$$q_N^{(e)}(x) = \sum_{j=0}^{N} \psi_j(x)\, q_j^{(e)}$$ & substitute into (12.1)

which yields the Galerkin problem statement: find $q_N \in V$ s.t.

$$\int_{\Omega_e} \psi_i \frac{d^2 f_N^{(e)}}{dx} d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e \quad \forall\ \psi \in V. \tag{12.2}$$

Although not immediately obvious, the largest vector space V that can be used for CG is $H^1$. Let's see why this is so.

Let's take the following equivalence:

$$\frac{d}{dx}\left(\psi_i \frac{d f_N}{dx}\right) = \frac{d\psi_i}{dx}\frac{d f_N}{dx} + \psi_i \frac{d^2 f_N}{dx^2}$$

which allows us to write (12.2) as follows:

$$\int_{\Omega_e} \frac{d}{dx}\left(\psi_i \frac{d f_N^{(e)}}{dx}\right) d\Omega_e - \int_{\Omega_e} \frac{d\psi_i}{dx}\frac{d f_N^{(e)}}{dx} d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e \tag{12.3}$$

**Remark** We only need $H^1$ (& not $H^2$) because we have terms of the form $\int_{\Omega_e} \left(\vec{\nabla}\psi \cdot \vec{\nabla} f + \psi f\right) d\Omega_e$ & no higher order derivatives. This is $H^1$!

# CG 2nd Derivatives

To see what the 2nd derivative in CG looks like, let's begin w/ the identity

(12.4) $$\frac{d^2 f_N}{dx^2} = \frac{d^2 f_N}{dx^2}$$

where, on the left is what we want to approx. & on the right is how we will approx. it. I.e.

(12.5) $$\frac{d^2 f_N^{(e)}}{dx^2}(x) = \sum_{j=0}^{N} \psi_j(x)\, f_{xx,j}^{(e)} \qquad \text{s.t.} \qquad f_{xx,j}^{(e)} = \frac{d^2 f_N^{(e)}}{dx^2}(x_j)$$

Multiplying (12.4) by $\psi$ & integrating yields:

(12.6) $$\int_{\Omega_e} \psi_i \frac{d^2 f_N^{(e)}}{dx^2}\, d\Omega_e = \int_{\Omega_e} \frac{d}{dx}\left(\psi_i \frac{d f_N^{(e)}}{dx}\right) d\Omega_e$$

$$- \int_{\Omega_e} \frac{d\psi_i}{dx} \frac{d f_N^{(e)}}{dx}\, d\Omega_e$$

where we have used (12.5) for the RHS. Let us now introduce (12.5) into (12.6) along with:

(12.7) $$f_N^{(e)}(x) = \sum_{j=0}^{N} \psi_j(x)\, f_j^{(e)} \longrightarrow \frac{d f_N^{(e)}}{dx}(x) = \sum_{j=0}^{N} \frac{d\psi_j}{dx}(x)\, f_j^{(e)}$$

& invoke the FTC to arrive at:

(12.8) $$\int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, f^{(e)}_{xx,j} = \left[\psi_i \frac{d f_N^{(e)}}{dx}\right]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} d\Omega_e \, q_j^{(e)}$$

This can be written in the following matrix form

(12.9) $$M_{ij}^{(e)} \, f^{(e)}_{xx,j} = F_{ij}^{(e)} \frac{d q_j^{(e)}}{dx} - L_{ij}^{(e)} \, q_j^{(e)}$$

For simplicity, assume that $\left.\frac{dq}{dx}\right|_{\Gamma} = 0$ although for more general problems we would apply this boundary condition at the physical boundaries.

## Resulting Matrices for N = 1

We already know that $M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ &

we can construct $L^{(e)}$ as follows:

$$L_{ij}^{(e)} = \int_{x_0}^{x_N} \frac{d\psi_i}{dx}(x) \frac{d\psi_j}{dx}(x) \, dx = \int_{-1}^{+1} \frac{d\psi_i}{d\xi}(\xi)\frac{d\xi}{dx} \frac{d\psi_j}{d\xi}(\xi)\cancel{\frac{d\xi}{dx}} \cancel{\frac{dx}{d\xi}} d\xi$$

$$L_{ij}^{(e)} = \frac{2}{\Delta x^{(e)}}\int_{-1}^{+1} \frac{d\psi_i}{d\xi}(\xi) \frac{d\psi_j}{d\xi}(\xi) \, d\xi \qquad \text{where} \quad \psi_i = \tfrac{1}{2}(1+\xi_i \xi)$$

$$\& \quad \frac{d\psi_i}{d\xi} = \tfrac{1}{2}\xi_i \qquad \xi_i = -1, +1$$

$$L_{ij}^{(e)} = \frac{2}{\Delta x^{(e)}} \int_{-1}^{+1} \begin{pmatrix} \frac{d\psi_0}{d\xi}\frac{d\psi_0}{d\xi} & \frac{d\psi_0}{d\xi}\frac{d\psi_1}{d\xi} \\ \frac{d\psi_1}{d\xi}\frac{d\psi_0}{d\xi} & \frac{d\psi_1}{d\xi}\frac{d\psi_1}{d\xi} \end{pmatrix} d\xi = \frac{2}{\Delta x^{(e)}}\frac{1}{4}\int_{-1}^{+1} \begin{pmatrix} \xi_0^2 & \xi_0\xi_1 \\ \xi_1\xi_0 & \xi_1^2 \end{pmatrix}$$

$$L_{ij}^{(e)} = \frac{1}{2\Delta x^{(e)}} \begin{pmatrix} \xi & -\xi \\ -\xi & \xi \end{pmatrix}\Bigg|_{-1}^{+1} = \frac{1}{2\Delta x^{(e)}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$L_{ij}^{(e)} = \frac{1}{\Delta x^{(e)}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \longrightarrow$$

Note that $\sum_{i=0}^{N} L_{ij}^{(e)} = \sum_{j=0}^{N} L_{ij}^{(e)} = 0$

$$\underbrace{\& \quad \sum_{i=0}^{N} \sum_{j=0}^{N} L_{ij}^{(e)} = 0}_{}$$

This is true for all matrices representing a derivative

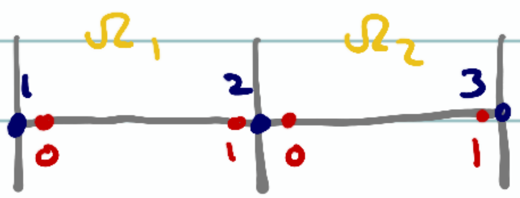
## Resulting Element Eqs.

On the reference element we arrive at the following eqs.

$$\frac{\Delta x^{(e)}}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} f_{xx,0}^{(e)} \\ f_{xx,1}^{(e)} \end{pmatrix} = -\frac{1}{\Delta x^{(e)}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f_0^{(e)} \\ f_1^{(e)} \end{pmatrix}$$


## Resulting Global Eqs.

For the grid configuration $N=1$ & $N_e = 2$:



We arrive at the following global matrix problem upon DSS: when we assume $\Delta x = \Delta x^{(1)} = \Delta x^{(2)}$

$$M = \frac{\Delta x}{6}\begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \\ & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \end{pmatrix} = \frac{\Delta x}{6}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$L = \frac{1}{\Delta x}\begin{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{pmatrix} = \frac{1}{\Delta x}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Diagonalizing the mass matrix yields:

$$\frac{\Delta x}{6}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} f_{xx,I-1} \\ f_{xx,I} \\ f_{xx,I+1} \end{pmatrix} = -\frac{1}{\Delta x}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} f_{I-1} \\ f_I \\ f_{I+1} \end{pmatrix}$$

which yields for the global gridpoint $I$:

$$\Delta x\, f_{xx,I} = -\frac{1}{\Delta x}\left( -f_{I-1} + 2f_I - f_{I+1} \right)$$

$$f_{xx,I} = \frac{f_{I-1} - 2f_I - f_{I+1}}{\Delta x^2}$$

which is identical to the centered finite difference representation that we know is $O(\Delta x^2)$.