(Ch. 1)

Lecture 1: Background & Motivation

Galerkin methods are numerical methods for solving PDEs that rely on the integral form of the equations. E.g., for the conservation law:

(1.0) $$\frac{\partial q}{\partial t} + \bar{\nabla}\cdot\bar{F} = 0 \quad , \text{ let } \bar{F} = q\bar{u}$$

we solve it as follows:

(1.1) $$\int_{\Omega} \psi \left(\frac{\partial q}{\partial t} + \bar{\nabla}\cdot\bar{F} \right) d\Omega = 0$$

Why use Galerkin methods instead of other methods? There are numerous methods to choose from so to whittle them down, we should strive for the following traits (assuming the continuous problem is well-posed):

1. It must be consistent w/ the continuous problem, i.e., one is solving the same problem
2. It must be stable

absolutely necessary conditions

3. It must be as accurate as possible
4. It must be efficient

wishlist

5. It must be algorithmically & geometrically flexible.

Galerkin trait

The first 2 conditions are absolutely necessary & can be satisfied by standard methods. These two conditions are part of Hadamard well-posedness:

1. $\exists$ a sol.
2. it is unique
3. the sol. depends continuously on the initial data and/or boundary conditions.

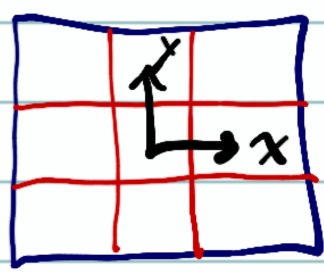
Condition 3 says that if $u(\bar{x},0) = U(\bar{x})$ is initial data then $\max|u-u'| \leq \epsilon$ for $\max|U(\bar{x}) - U'(x)| < \delta$ i.e., small changes in IC should only cause small changes in the solution.

The traits regarding accuracy, efficiency, & flexibility are strictly not necessary but they are desirable. It is certainly desirable to have methods that are both accurate & efficient.

Let us now consider algorithmic & geometric flexibility. Geometric flexibility means that the method is amenable to using all types of grids such as unstructured and adaptive. Algorithmic flexibility can mean many things but, here, let us define it to mean that code can be reused for other applications and that the method itself can be easily modified as required. It turns out that both geometric & algorithmic flexibility are, in fact, bound together. If a method is built upon a basic building block then it will be able to satisfy both types of flexibility.

Element-based Galerkin (EBG) methods are built directly on an element (e.g., quadrilateral or triangle in 2D). Therefore, for any type of problem or new eqs., once we know how to represent a function & its derivatives inside the element, then you can solve any problem w/ little effort.

Let's take a closer look at Galerkin methods.
Given the conservation law $\frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{F} = 0$ w/ appropriate ICs & BCs on the unit square



To solve this problem using Galerkin methods, we will represent $q(\bar{x},t)$ as follows: $q_N^{(e)}(\bar{x},t) = \sum_{j=1}^{M} \psi_j(\bar{x})\, q_j^{(e)}(t)$

inside each element $e=1,\ldots,N_e$ w/ an N^{th} order polynomial $\psi \in P_N$ having M gridpoints in each element.
We can also represent the flux function $\bar{F} = F_1\hat{i} + F_2\hat{j}$ as follows: $\bar{F}_N = \bar{F}(q_N^{(e)})$.

We can now seek solutions $q_N \in S$ s.t.

$$(1.2) \quad \int_\Omega \psi \left(\frac{\partial q_N^{(e)}}{\partial t} + \vec{\nabla} \cdot \vec{F}_N^{(e)} \right) d\Omega = 0 \quad \forall \ \psi \in S$$

There are a few things about Galerkin methods that become immediately obvious:

1. The success of the method depends on constructing N^{th} order functions ψ inside each element.
2. We need to be able to evaluate the integral in Eq. (1.2) accurately & efficiently.

Summary For this reason, learning about interpolation & integration (in 2d it is called quadrature & in 3d cubature) is important. This is the topic of Project 1.

Ex 1.0 Show AMR grid & movie (global tsunami)