(Ch. 4)

Lecture 7: 1D Integration

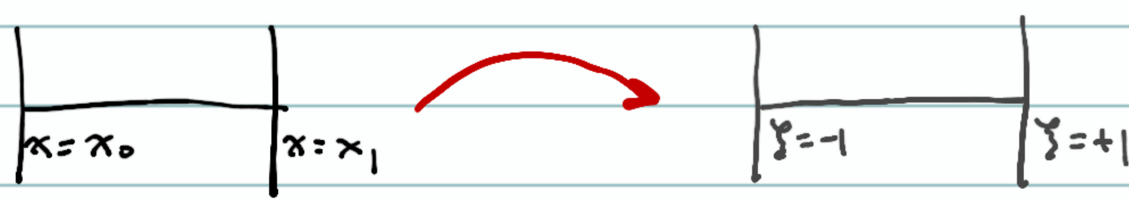
The goal of numerical integration is to approximate

$\int_{x_0}^{x_1} f(x)\,dx$ in some way.

Using the following change of coordinates: $\xi = 2\dfrac{(x - x_0)}{(x_1 - x_0)} - 1$ (7.1)

w/ inverse map $x = x_0 + \frac{1}{2}(\xi+1)\underbrace{(x_1 - x_0)}_{\Delta x}$ (7.2)

These maps yield the following:



where from (7.2) we get $\dfrac{dx}{d\xi} = \dfrac{\Delta x}{2}$

& from (7.1) $\dfrac{d\xi}{dx} = \dfrac{2}{\Delta x}$

$$\therefore \int_{x_0}^{x_1} f(x)\,dx = \int_{-1}^{+1} f(\xi)\,\frac{dx}{d\xi}\,d\xi = \int_{-1}^{+1} f(\xi)\,\frac{\Delta x}{2}\,d\xi$$

where now we approximate the last integral as follows:

$$\frac{\Delta x}{2}\int_{-1}^{+1} f(\xi)\, d\xi \simeq \frac{\Delta x}{2}\sum_{k=1}^{n} \omega_k f(\xi_k)$$

Choosing Quadrature Points

Assume that we can represent any function as follows:

$f(x) = p_n(x) + e_n(x)$ where p_n is the n^{th}-degree representation & e_n is the error.

Following the Newton's divided difference in Sec 4.2 we find that:

$$e_n = \underbrace{\varphi_n(x)\, f[x_0, x_1, \ldots, x_n, x]}$$

divided difference which is an M^{th}-degree polynomial s.t. $M > N$.

$\therefore$ if $f(x)$ is m^{th} degree & p is n^{th} degree then e_n is m^{th} degree where we have the following set-up:

(7.3)
$$\underbrace{f(x)}_{O(m)} = \underbrace{p_n(x)}_{O(n)} + \underbrace{\varphi_n(x)}_{O(n)}\, \underbrace{f[x_0, x_1, \ldots, x_n, x]}_{O(m-n)}$$

where $\varphi_n(x)$ is the O.G. generating polynomial (e.g., Legendre, Lobatto, etc.)

Integrating yields:

$$\int_{-1}^{+1} f(x)\,dx = \int_{-1}^{+1} p_n(x)\,dx + \int_{-1}^{+1} \varphi_n(x)\, f[x_0, x_1, \ldots, x_n, x]\,dx$$

when $\int_{-1}^{+1} \varphi_n(x)\, f[x_0, x_1, \ldots, x_n, x]\, dx = 0$

if $f[x_0, x_1, \ldots, x_n, x]$ is of $\leq O(n-1)$

Since φ_i $i = 0, \ldots, n$ form an O.G. basis in $\mathbb{R}^n$.

$\therefore$ we note that

$\varphi_n f[x_0, \ldots, x_n, x]$ will vanish for all polynomials of $O(2n-1) = O(n) + O(n-1)$.

Note that $n = N+1$ & so we get $O(2(N+1)-1) = O(2N+1)$

$\boxed{\text{Important Point}}$ Using the roots of N^{th} degree O.G. generating polynomials (φ_N) yields $O(2N+1)$ integration strength (accuracy) & we can write:

$$\int_{-1}^{+1} \varphi_N(x)\, f[x_0, x_1, \ldots, x_N, x]\, dx = \sum_{k=0}^{N} w_k\, \varphi_N(x_k)\, f[x_0, \ldots, x_N, x_k] = 0$$

will vanish when x_k are the roots (zeros) of φ_N.

Quadrature Weights Now that we know how to find the Quadrature Roots we next need to determine the weights.

For the integral: $I = \int_{-1}^{+1} f(x)\, dx$ we can approx. $f(x)$ as:

$$f(x) = \sum_{j=0}^{N} \psi_j(x)\, f_j$$ where ψ are some selected basis functions (e.g. Lagrange)

& So we can write I as follows:

$$I = \int_{-1}^{+1} \left(\sum_{j=0}^{N} \psi_j(x) f_j \right) dx \approx \sum_{k=0}^{N} \omega_k \left(\sum_{j=0}^{N} \psi_j(x_k) f_j \right)$$

reordering:

$$I \approx \sum_{j=0}^{N} \left(\int_{-1}^{+1} \psi_j(x)\, dx \right) f_j = \sum_{j=0}^{N} \left(\sum_{k=0}^{N} \omega_k \psi_j(x_k) \right) f_j$$

i.e., we need:

$$\sum_{k=0}^{N} \omega_k \psi_j(x_k) = \int_{-1}^{+1} \psi_j(x)\, dx$$

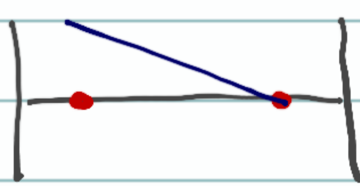
Using the cardinality property of Lagrange polynomials we find that since $\psi_j(x_k) = \begin{cases} 1 & \text{for } j=k \\ 0 & \text{for } j \neq k \end{cases}$

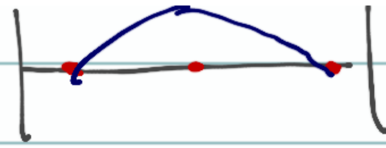
we get: $\omega_j = \int_{-1}^{+1} \psi_j(x)\, dx$.

Closed form expressions are given in Sec. 4.3.3

<u>Determining the Number of Quadrature Points</u>

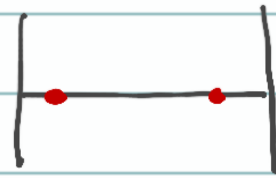
How many points are required to determine an N^{th} degree polynomial?

e.g., let $N=1$ $\rightarrow$  we need 2 points

For $N=2$ $\longrightarrow$  we need 3 points

& in general we need $N+1$ points to determine an N^{th} degree polynomial. Similarly, if we have $N+1$ DoF then we can find P_N, i.e., order = DoF − 1 in 1D.

DoF for Quadrature

Legendre/Chebyshev:  For $N+1$ points

we also have $N+1$ corresponding weights.

$\therefore$ the total DoF is $N+1$ roots + $N+1$ weights $= 2N+2$

& $\therefore$ $\boxed{\text{order} = \text{DoF} - 1 = 2N+1}$ is the maximum polynomial order that we can integrate exactly w/ the $N+1$ roots of an N^{th}-degree polynomial.

Lobatto: For $N+1$ points with Lobatto, we only actually have $N-1$ DoFs for the roots since we fixed $x=\pm 1$, but still have $N+1$ DoFs for the weights for a total of $N-1$ roots + $N+1$ weights

$= 2N$

& the order will be $\boxed{\text{order} = \text{DoF} - 1 = 2N-1}$

which is the maximum polynomial order that can be integrated exactly by the $N+1$ roots for an N^{th}-degree Lobatto polynomial.

Summary $\displaystyle\int_{-1}^{+1} f(x)\,dx = \int_{-1}^{+1} p_N(x)\,dx + \underbrace{\int_{-1}^{+1} \psi_N(x)\, f[x_0,\dots,x_N,x]\,dx}_{e_N(x)}$

where $\displaystyle\int_{-1}^{+1} e_N(x)\,dx \simeq \sum_{k=0}^{N} \omega_k\, e_N(x_k) \longrightarrow 0$ for $e_N \in P_{2N+1}$

if x_k $k=0,\dots,N$ are the roots of $\psi_N(x)$.

To derive an N^{th} degree Lagrange polynomial requires $N+1$ points which means that if we use the roots of ψ_N then we get integration accuracy $= \text{DOF} - 1$

$$= 2(N+1) - 1$$

$$= O(2N+1)$$

This is assuming that we are free to choose $N+1$ roots & $N+1$ weights $= 2N+2$ DoFs. $\longrightarrow$ this is the case for, e.g., Legendre & Chebyshev.

For Lobatto we only have the freedom to choose $N-1$ roots & $N+1$ weights $= 2N$ DoFs & so the max polynomial we can integrate exactly is $O(2N-1)$.