Lecture 15: 2D Elliptic CG

Introduction

So far we have learned how to solve the 1D Poisson problem: $\frac{d^2 g}{dx^2} = f(x)$ w/ appropriate BCs.

Let us now consider the multi-dimensional Poisson problem:

$$(15.1) \qquad \nabla^2 g(\bar{x}) = f(\bar{x}) \quad \text{w/} \quad \bar{x} = x_i \quad i=1,\dots,d$$

when $\nabla^2 = \bar{\nabla}\cdot\bar{\nabla}$ w/ BCs:

$$g\big|_{\Gamma_D} = g(\bar{x}) \quad \text{and/or} \quad \hat{n}_\Omega \cdot \bar{\nabla} g\big|_{\Gamma_N} = h(\bar{x})$$

Integral Form

To discretize (15.1) we first approximate $g_N^{(e)}(\bar{x})$ as follows:

$$(15.2) \qquad g_N^{(e)}(\bar{x}) = \sum_{i=1}^{M_N} \psi_i(x)\, g_i^{(e)} \quad \text{w/} \quad f_N^{(e)}(\bar{x}) = \sum_{i=1}^{M_N} \psi_i(\bar{x}) f_i^{(e)}$$

& subbing into (15.1) gives:

$$(15.3) \qquad \int_{\Omega_e} \psi_i \nabla^2 g_N^{(e)}\, d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)}\, d\Omega_e \quad \forall \; \psi \in H^1$$

where IBP gives: $\bar{\nabla}\cdot\left(\psi_i \bar{\nabla} g_N^{(e)}\right) = \bar{\nabla}\psi_i \cdot \bar{\nabla} g_N^{(e)} + \psi_i \nabla^2 g_N^{(e)}$

which gives in (15.3)

$$\int_{\Omega_e} \bar{\nabla}\cdot\left(\psi_i \bar{\nabla} g_N^{(e)}\right) d\Omega_e - \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{\nabla} g_N^{(e)} d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e \tag{15.4}$$

using the divergence theorem for the 1st term on the LHS yields:

$$\int_{\Gamma_e} \hat{n}\cdot\left(\psi_i \bar{\nabla} g_N^{(e)}\right) d\Gamma_e - \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{\nabla} g_N^{(e)} d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e \tag{15.5}$$

Subbing (15.2) into (15.5) yields:

$$\int_{\Gamma_e} \hat{n}\cdot\left(\psi_i \bar{\nabla} g_N^{(e)}\right) d\Gamma_e - \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \left(\sum_{j=1}^{M_N} \bar{\nabla}\psi_j g_j^{(e)}\right) d\Omega_e = \int_{\Omega_e} \psi_i \left(\sum_{j=1}^{M_N} \psi_j f_j^{(e)}\right) d\Omega_e \tag{15.6}$$

<u>Mapping $\bar{x} = \bar{\Psi}(\bar{\xi})$</u>

we now need to discuss the mapping from $\bar{x} \to \bar{\xi}$ & $\bar{\xi} \to \bar{x}$.

Recall from (15.2) that $f_N^{(e)}(\bar{x}) = \sum_{j=1}^{M_N} \psi_j(\bar{x}) f_j^{(e)}$
which means that we can write:

$$\bar{\nabla} f_N^{(e)}(\bar{x}) = \sum_{j=1}^{M_N} \bar{\nabla}\psi_j(\bar{x})\, f_j^{(e)}$$

where $\bar{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ in 2D where, e.g.,

$$\frac{\partial f_N^{(e)}}{\partial x}(\bar{x}) = \sum_{j=1}^{M_N} \frac{\partial \psi_j}{\partial x} f_j^{(e)}$$ & using the chain rule:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x}$$ where $\psi_i(\xi,\eta) = h_j(\xi) \otimes h_k(\eta)$

&

$$\frac{\partial \psi_i}{\partial \xi}(\xi,\eta) = \frac{\partial h_j(\xi)}{\partial \xi} \otimes h_k(\eta) \quad \& \quad \frac{\partial \psi_i}{\partial \eta}(\xi,\eta) = h_j(\xi) \otimes \frac{\partial h_k(\eta)}{\partial \eta}$$

where $h(\xi)$ & $h(\eta)$ are the basis functions defined in Ch. 3.

Metric Terms To go from $\bar{\xi} \to \bar{x}$ & $\bar{x} \to \bar{\xi}$
we need to define the mapping as follows, let:

$$d\bar{x} = \frac{\partial \bar{x}}{\partial \xi} d\xi + \frac{\partial \bar{x}}{\partial \eta} d\eta$$ as follows:

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

which we write in matrix form:

$$(15.7) \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}}_{= J \text{ is the Jacobian of the map from } \bar{\xi} \to \bar{x}} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

The determinant $\det(J) = |J| = x_\xi y_\eta - x_\eta y_\xi$.

The inverse map can be found as follows:

$$d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy$$

$$d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy$$

which in matrix form is:

$$(15.7) \quad \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}}_{= J^{-1} \text{ is the inverse Jacobian}} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Eq. (15.6) & (15.7) tell us that the map from $\bar{x} \to \bar{\xi}$ & $\bar{\xi} \to \bar{x}$ are given as follows:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = J \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} \quad \& \quad \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = J^{-1} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

where $J = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}$ & $J^{-1} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}$ (15.8)

where $J^{-1} = \frac{1}{|J|} \begin{pmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{pmatrix}$ (15.9)

Equating (15.8) & (15.9) shows that

$$\xi_x = \frac{1}{|J|} y_\eta, \quad \xi_y = -\frac{1}{|J|} x_\eta,$$

$$\eta_x = -\frac{1}{|J|} y_\xi, \quad \eta_y = \frac{1}{|J|} x_\xi$$

Next, we need to define $\bar{x}_\xi$ & $\bar{x}_\eta$. Using the basis function expansion, yields:

from $\bar{x}_N^{(e)}(\xi,\eta) = \sum_{j=1}^{M_N} \psi_j(\xi,\eta)\, \bar{x}_j^{(e)}$ we write.

$$\frac{\partial \bar{x}_N}{\partial \xi}(\xi,\eta) = \sum_{j=1}^{M_N} \frac{\partial \psi_j(\xi,\eta)}{\partial \xi}\, \bar{x}_j^{(e)}$$

which allows us to build all of the metric terms.
The alg. for doing this is described in Alg. 12.1.