Lecture 14: 2D Interpolation & Integration

Introduction

So far, we have only considered 1D problems whereby we learned to define basis functions as follows:

$$(14.1) \qquad g_N^{(e)}(\xi) = \sum_{j=0}^{N} \psi_j(\xi,\eta)\, g_j^{(e)}$$

where ψ can be either modal or nodal functions.

We also learned that we can use Gauss points to integrate a function as follows:

$$(14.2) \qquad \int_{\hat{\Omega}} g_N^{(e)}(\xi)\, d\xi = \sum_{k=0}^{Q} \omega_k\, g_N^{(e)}(\xi)$$

$$= \sum_{k=0}^{Q} \omega_k \sum_{j=0}^{N} \psi_j(\xi)\, g_j^{(e)}$$

Extension to 2D

Extending these ideas to 2D is simple if we use tensor-products on quadrilaterals as follows:

$$\boxed{\text{Interpolation}} \qquad g_N^{(e)}(\xi,\eta) = \sum_{i=1}^{M_N} \psi_i(\xi,\eta)\, g_i^{(e)} \qquad (14.3)$$

where Eq. (14.3) is valid for any domain (quads, tris, etc) provided that $\psi(\xi,\eta)$ are defined. In this example $j=1,\ldots,M_N$ where $M_N = (N_\xi+1)(N_\eta+1)$ are the degrees of freedom w/in each element where N_ξ & N_η are the polynomial orders along the ξ & η directions.

Note that we can write the basis functions as follows:

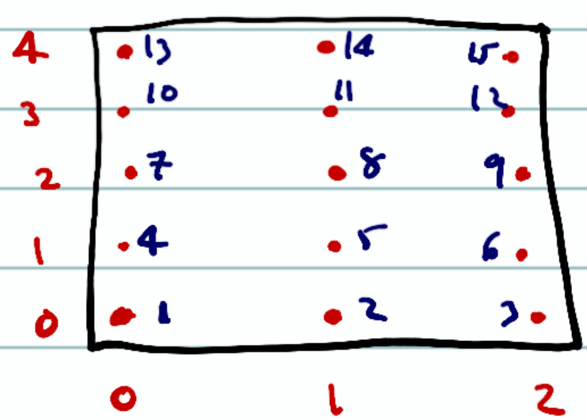
$$\psi_i(\xi,\eta) = h_j(\xi) \otimes h_k(\eta) \quad \text{where} \quad j=0,\ldots,N_\xi, \; k=0,\ldots,N_\eta$$

& $\quad i = j+1 + k(N_\xi+1)$

Of course, if we want to exploit the tensor-product nature of ψ then we would write (14.3) as follows:

$$(14.4) \qquad q_N^{(e)}(\xi,\eta) = \sum_{j=0}^{N_\xi} \sum_{k=0}^{N_\eta} h_j(\xi)\, h_k(\eta)\, q_{jk}^{(e)}$$

which we describe graphically in the figure below for $N_\xi = 2$, $N_\eta = 4$ & $M_N = (N_\xi+1)(N_\eta+1) = 15$



$\boxed{\text{Note}}$ that $h_j(\xi)$ & $h_k(\eta)$ are the 1D basis functions we have already defined previously.

Integration In a similar manner to extending interpolation to 2D, we can extend integration as follows:

From $$\int_{\hat{\Omega}} g_N^{(e)}(\xi)\, d\xi = \sum_{k=0}^{Q} w_k\, g_N^{(e)}(\xi_k)$$

we can write:

(14.5) $$\int_{\hat{\Omega}} g_N^{(e)}(\xi,\eta)\, d\xi\, d\eta = \sum_{k=1}^{M_Q} w_k\, g_N^{(e)}(\xi_k,\eta_k)$$

where $M_Q = (Q_\xi+1)(Q_\eta+1)$ & Q_ξ, Q_η are the order of integration along the ξ & η directions, respectively.

We can write (14.5) using tensor-products as follows:

(14.6) $$\int_{\hat{\Omega}} g_N^{(e)}(\xi,\eta)\, d\hat{\Omega} = \int_{-1}^{+1}\int_{-1}^{+1} g_N^{(e)}(\xi,\eta)\, d\xi\, d\eta$$
$$= \sum_{k=0}^{Q_\xi} \sum_{l=1}^{Q_\eta} w_k^{(\xi)} w_l^{(\eta)}\, g_N^{(e)}(\xi_k,\eta_l)$$

Substituting (14.4) for $g_N^{(e)}(\xi_k,\eta_l)$ yields:

(14.7) $$\int_{\hat{\Omega}} g_N^{(e)}(\xi,\eta)\, d\xi\, d\eta = \sum_{k=0}^{Q_\xi} \sum_{l=0}^{Q_\eta} w_k^{(\xi)} w_l^{(\eta)} \sum_{i=0}^{N_\xi} \sum_{j=0}^{N_\eta} h_i(\xi_k) h_j(\eta_l)\, g_{ij}^{(e)}$$

where $h_i(\xi_k) = h^{(\xi)}_{i,k}$ & $h_j(\eta_l) = h^{(\eta)}_{j,l}$ & so we can write:

$$(14.8) \quad \int_{\hat{\Omega}} f_N^{(e)}(\xi,\eta)\, d\xi\, d\eta = \sum_{k=0}^{Q_\xi} \sum_{l=0}^{Q_\eta} \omega_k^{(\xi)} \omega_l^{(\eta)} \sum_{i=0}^{N_\xi} \sum_{j=0}^{N_\eta} h^{(\xi)}_{i,k} h^{(\eta)}_{j,l} f^{(e)}_{ij}$$

Extension to 3D

To go from $1D \to 2D \to 3D$ is again quite straightforward it: 1) we use the monolithic form $i=1,\ldots,M_N$ or 2) use tensor-products using 1D basis functions

Interpolation Note that we can write:

$$(14.9) \quad f_N^{(e)}(\xi,\eta,\zeta) = \sum_{l=1}^{M_N} \psi_l(\xi,\eta,\zeta)\, f_l^{(e)}$$

where $\psi_l(\xi,\eta,\zeta)$ are 3D interpolation functions & $M_N = (N_\xi+1)(N_\eta+1)(N_\zeta+1)$.

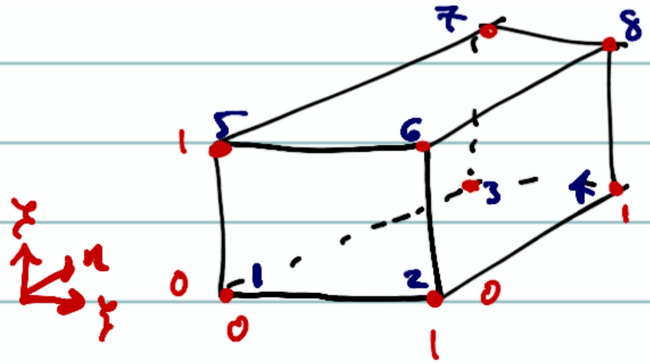
We can use tensor-products to define:

$$\psi_l(\xi,\eta,\zeta) = h_i(\xi) \otimes h_j(\eta) \otimes h_k(\zeta)$$

s.t.

$$l = i+1 + j(N_\xi+1) + k(N_\xi+1)(N_\eta+1)$$

which is shown in the figure below:



Using tensor-products we can write (14.9) as follows:

$$q_N^{(e)}(\xi,\eta,\zeta) = \sum_{i=0}^{N_\xi} \sum_{j=0}^{N_\eta} \sum_{k=0}^{N_\zeta} h_i(\xi)\, h_j(\eta)\, h_k(\zeta)\, q_{ijk}^{(e)} \tag{14.10}$$

where, once again, h are the 1D basis functions we have defined previously.

$\boxed{\text{Integration}}$ The 3D integral is defined as

$$\int_{\hat{\Omega}} q_N^{(e)}(\xi,\eta,\zeta)\, d\hat{\Omega} = \sum_{k=1}^{M_Q} \omega_k\, q_N^{(e)}(\xi_k,\eta_k,\zeta_k) \tag{14.11}$$

where $M_Q = (Q_\xi+1)(Q_\eta+1)(Q_\zeta+1)$ & ω_k $k=1,\ldots,M_Q$ are the 3D quadrature weights.

In terms of tensor-products we can write:

$$\int_{\hat{\Omega}} q_N^{(e)}(\xi,\eta,\zeta)\, d\hat{\Omega} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} q_N^{(e)}(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \tag{14.12}$$

$$= \sum_{i=0}^{Q_\xi} \sum_{j=0}^{Q_\eta} \sum_{k=0}^{Q_\zeta} \omega_i\, \omega_j\, \omega_k\, q_N^{(e)}(\xi_i,\eta_j,\zeta_k)$$

whe we use (14.10) to represent $\xi_N^{(e)}(\gamma_i, n_j, \xi_k)$