MA4245 Mathematical Principles of Galerkin Methods
Project 1: Interpolation and Integration

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This project has three parts:

1. Interpolate a known function using specific sampled points,

2. Compute the derivative of the function using the basis functions, and

3. Integrate the function within the domain $x \in [-1, +1]$.

1 Interpolation

Let us use the function:

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad x \in [-1, +1]. \tag{1}$$

Using nodal basis functions $L_i(x)$ construct an interpolant of $f(x)$ as follows:

$$f_N(x_k) = \sum_{i=0}^{N} L_i(x_k)f_i \tag{2}$$

where x_k are $k = 1, ..., 50$ evenly spaced points and the Nth order interpolation points are:

1. equally-spaced points,

2. the roots of the Lobatto polynomials (these are called the Legendre-Gauss-Lobatto points),

3. the roots of the Legendre polynomials (these are called the Legendre-Gauss points).

M-files to compute LGL and LG roots are provided for your use (Legendre_Gauss_Lobatto.m and Legendre_Gauss.m, respectively, but both m-files require Legendre_Poly.m which is also provided).

Once you construct the Lagrange polynomials $L_i(x)$ check to see how accurate your interpolation becomes as you increase N. Run values of $N = 1, ..., 64$. To determine how well your interpolant is, use the normalized error norms defined at the end of this document. Plot the error as a function of order N.

2 Derivative

For the same analytic function

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad x \in [-1, +1]. \tag{3}$$

using the Lagrange polynomials evaluate the derivative of $f(x)$ and compare with the exact solution using error norms. Use all three types of points as above.

3 Integration

Using the interpolation functions from the previous part, sample the basis functions at LGL and LG integration points in order to perform the Gauss quadrature:

$$\int_{-1}^{+1} f_N(x)dx = \int_{-1}^{+1} \sum_{i=0}^{N} L_i(x)f_i dx = \sum_{k=0}^{N} w_k \left(\sum_{i=0}^{N} L_i(x_k)f_i\right) \tag{4}$$

where x_k are the Nth order Gauss quadrature points and compare against the exact integral of $f(x)$ for $x \in [-1, +1]$.

4 Write-Up

Turn in a few pages (enough to cover all the points) discussing what you see for each of the three interpolation points. Which points allow you to construct a good solution and which do not. Corroborate your arguments with error norm plots and show me a figure or two showing what happens when one of the interpolation point sets breaks-down. Please append your matlab code (m-files) at the end of your report.

Be sure to build a routine (I call it Lagrange_basis or something like that) that gives you a matrix L_{ij} of Lagrange polynomials for the interpolation points (i) evaluated at the integration or sampling points (j). This will make it easier to use for Project 2 where you will need all this machinery again.

5   Helpful Relations

The normalized L2 error norm that you should use is:

$$||error||_{L^1} = \frac{\sum_{k=1}^{N_s} |\, f_N(x_k) - f(x_k)\,|}{\sum_{k=1}^{N_s} |\, f(x_k)\,|}, \tag{5}$$

$$||error||_{L^2} = \sqrt{\frac{\sum_{k=1}^{N_s} (f_N(x_k) - f(x_k))^2}{\sum_{k=1}^{N_s} f(x_k)^2}}, \tag{6}$$

and

$$||error||_{L^\infty} = \frac{\max_{1 \leq k \leq N_s} |\, f_N(x_k) - f(x_k)\,|}{\max_{1 \leq k \leq N_s} |\, f(x_k)\,|} \tag{7}$$

where $k = 1, ..., N_s$ are the number of points for which you are computing the error norms (it differs whether you are doing the interpolation and derivatives or the integration). Note that you will not be able to compute an L^∞ error for the integration since the integration only gives you one value.

Recall that the equation for the Lagrange polynomials is

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{(x - x_j)}{(x_i - x_j)}. \tag{8}$$

Differentiating this expression yields

$$\frac{dL_i}{dx}(x) = \sum_{\substack{k=0 \\ k \neq i}}^{N} \left(\frac{1}{x_i - x_k} \right) \prod_{\substack{j=0 \\ j \neq i \\ j \neq k}}^{N} \frac{(x - x_j)}{(x_i - x_j)}. \tag{9}$$