(Ch 2)

Lecture 4: Overview of Methods

Finite Difference

Recall that for solving the 1d wave equation:

(4.1) $\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = 0$ we said that, using Taylor Series we could discretize the eq. as follows

$$\frac{\partial q_i}{\partial t} + u \sum_{k=-N}^{+N} \frac{\beta_k q_{i+k}^n}{\Delta x} = 0$$

where we analyzed:

Case 1: $\frac{dq_i}{dt} + u \frac{q_i^n - q_{i-1}^n}{\Delta x} = 0$

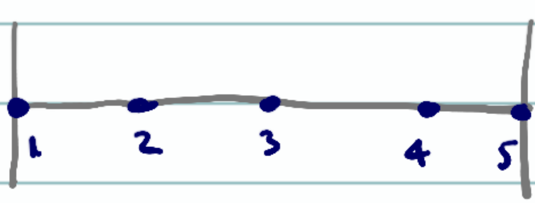
Case 2: $\frac{dq_i}{dt} + u \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x} = 0$

Case 3: $\frac{dq_i}{dt} + u \frac{q_{i+1}^n - q_i^n}{\Delta x} = 0$

Semi-discrete form, in space only. Time is left alone.

which we can write in a unified manner as follows:

(4.2) $\frac{dq_i}{dt} + u D_{ij} q_j = 0$ where D is the differentiation matrix. To see what this matrix looks like let's assume the following grid w/ Periodic BCs:



For this set-up we would get:

$\boxed{\text{Case 1}}$

$$D q = \frac{1}{\Delta x}\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}$$

$\boxed{\text{Case 2}}$

$$D q = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}$$

$\boxed{\text{HW 4.2}}$ Write D for Case 3.

In Summary, although typically, FDM methods are not written in this matrix form, we can nonetheless recast them as such to better see the connection b/w all of these methods. For now, let us rewrite our PDE as follows: (4.3) $\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0$ where $f = qu$

& so we can write the general high-order FDM discretization as follows:

(4.4) $M_{ij}\dfrac{dq_j}{dt} + D_{ij} f_j = 0$

where for FDM $M_{ij} = I_N$. We will now introduce Galerkin methods & revisit this form.

Galerkin Methods Instead of tackling the original eq. in differential form: $\dfrac{\partial q}{\partial t} + \dfrac{\partial f}{\partial x} = 0$, we will now seek solutions in the integral form of the equations.

Approximation First we introduce our approximation:

let $q_N(x,t) = \displaystyle\sum_{j=0}^{N} \psi_j(x)\,\tilde{q}_j(t)$ where $\psi(x)$ are interpolation functions & $\tilde{q}(t)$ are expansion coefficients. For now, let us assume that $f_N = f(q_N)$ which is unique since f is a linear function in q & we assume that u is known.

The discrete form of (4.3) is:

(4.5) $\dfrac{\partial q_N}{\partial t} + \dfrac{\partial f_N}{\partial x} = \epsilon$ where $\epsilon \neq 0$ because q_N is only a finite-dimensional expansion. If $N \to \infty$ then we would assume that $\epsilon \to 0$ (convergence)

Next, multiply (4.5) by ψ_i & integrate within the domain Ω_e :

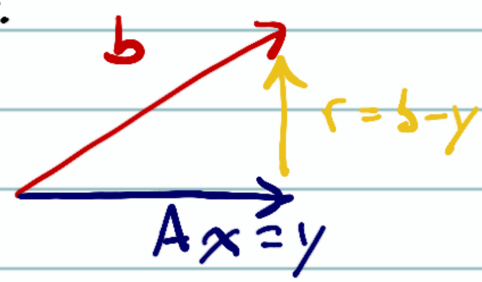
$$(4.6) \quad \int_{\Omega_e} \psi_i \frac{\partial f_N}{\partial t} \, d\Omega_e + \int_{\Omega_e} \psi_i \frac{\partial f_N}{\partial x} \, d\Omega_e = \int_{\Omega_e} \psi_i \epsilon \, d\Omega_e$$

s.t. $f_N, \psi \in V_N$ → V_N is a finite-dimensional vector space described later.

Next, we seek solutions, f, s.t. the space spanned by ψ is o.g. to the error. For this to be true requires that ψ_i & ϵ are o.g. A discrete inner product is defined as follows:

if $\int_{\Omega} u \, v \, d\Omega = 0$ then u & v are o.g in Ω.

$\boxed{\text{Ex 4.1}}$ The least-squares sol of $Ax = b$ is drawn as follows:



when $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ when $m > n$.

The residual (error) is minimized when y o.g. r.

In this example, $y = f_N, \psi$ & $b = f$, & $r = \epsilon$.

Therefore we seek f_N s.t. $\int_{\Omega_e} \psi \epsilon \, d\Omega_e = 0$.

This leads to the following integral form:

$$(4.7) \quad \int_{\Omega_e} \psi_i \frac{\partial \xi_N}{\partial t} \, d\Omega_e + \int_{\Omega_e} \psi_i \frac{\partial f_N}{\partial x} \, d\Omega_e = 0$$ (Strong form)

Integration by Parts (weak form)

The strong form means that we use the same operators as in the original differential equation. Once discretized we then have to apply BCs by altering the values of the solution vector ξ directly (strong form). There is another way to solve this problem & it requires integration-by-parts (IBP).

Note that in 1D we can write:

$$\frac{\partial}{\partial x}(\psi f_N) = \frac{\partial \psi}{\partial x} f_N + \psi \frac{\partial f_N}{\partial x}$$ & so (4.7) can be written as follows:

$$(4.8) \quad \int_{\Omega_e} \psi_i \frac{\partial \xi_N}{\partial t} \, d\Omega_e + \int_{\Omega_e} \frac{\partial}{\partial x}(\psi f_N) \, d\Omega_e - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} f_N \, d\Omega_e = 0$$

Using the Fundamental Theorem of Calculus for the 2^{nd} term allows us to write:

$$(4.9) \quad \int_{\Omega_e} \psi_i \frac{\partial \xi_N}{\partial t} \, d\Omega_e + \left[\psi f_N\right]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} f_N \, d\Omega_e = 0$$ (weak form)

where Γ_e is the boundary of Ω_e (sometimes denoted by $\partial\Omega_e$).

Strong vs. Weak Forms

Eqs. (4.7) & (4.9) define the strong & weak forms of all Galerkin methods. In the weak form, we now have a less intrusive way of enforcing BCs through the 2^{nd} term in (4.9). There is a way of doing the same with the strong form but it requires a 2^{nd} application of IBPs & will be discussed in later lectures.

We end this lecture by emphasizing the important connection b/w the strong & weak forms. Recall that to go from the strong to weak form we used the following identity:

$$\int_{\Omega_e} \psi_i \frac{\partial f_N}{\partial x}\, d\Omega_e = \int_{\Omega_e} \frac{\partial}{\partial x}(\psi_i f_N)\, d\Omega_e - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} f_N\, d\Omega_e$$

which must be satisfied at the discrete level in order to formally conserve f_N. What does this mean?

$\boxed{\text{EX 4.2}}$ Let $q = \rho$ & $f = \rho u$ be the variables in Eqs. (4.7) & (4.9) & so we have:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \longrightarrow \text{Conservation of Mass}$$

when we write the second term as follows:

$$\int_{\Omega_e} \psi_i \frac{\partial (\rho u)_N}{\partial x} d\Omega_e = [\psi_i \rho u]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} (\rho u)_N d\Omega_e$$

$\therefore$ the final form is written as:

$$\int_{\Omega_e} \psi_i \frac{\partial \rho_N}{\partial t} d\Omega_e + \int_{\Omega} \psi_i \frac{\partial}{\partial x} (\rho u)_N d\Omega_e = 0 \rightarrow \text{Strong}$$

or

$$\int_{\Omega_e} \psi_i \frac{\partial \rho_N}{\partial t} d\Omega_e + [\psi_i (\rho u)_N]\Big|_{\Gamma_e} - \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} (\rho u)_N d\Omega_e = 0 \rightarrow \text{weak}$$

if we let $\psi = \text{const}$ & assume periodic BCs then the weak form gives:

$$\int_{\Omega_e} \frac{\partial \rho_N}{\partial t} d\Omega_e = 0 \rightarrow \text{Mass is conserved}$$

For the strong form, we need to take special precautions to arrive at the same form because it is not immediately obvious that this will be true.

However, if it is true, then we can say that the method satisfies a discrete IBPs. Methods that satisfy this are known as Summation-by-Parts (SBP).