MA4245 Mathematical Principles of Galerkin Methods
Project 2: 1D Wave Equation

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1 Continuous Problem

The governing partial differential equation (PDE) is

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0 \qquad \forall x \in [-1, +1]$$

where $f = qu$ and $u = 2$ is a constant. Thus, an initial wave $q(x, 0)$ will take exactly $t = 1$ time in order to complete one full revolution (loop) of the domain.

1.1 Initial Condition

Since the governing PDE is a hyperbolic system, then this problem represents an initial value problem (IVP or Cauchy Problem). We, therefore, need an initial condition. Let it be the following Gaussian

$$q(x, 0) = e^{-\left(\frac{x}{2\sigma}\right)^2}$$

where $\sigma = \frac{1}{8}$ and $x \in [-1, 1]$.

1.2 Boundary Condition

This problem also requires a boundary condition: let us impose periodic boundary conditions, meaning that the domain at $x = +1$ should wrap around and back to $x = -1$. Your solution variable q should have the same solution at $x = -1$ and $x = +1$.

2 Simulations

Write a code (or two) that uses both CG and DG. I strongly recommend that you code the CG version first. It is better to use the same code to do both CG and DG with a switch (if statement) to handle the communicator in both CG and DG. You need to show results for exact (let Q=N+1 be exact) AND inexact integration (Q=N) so write your codes in a general way.

2.1 Results You Need to Show

You must show results for linear elements $N = 1$ with increasing number of elements N_e and then show results for $N = 4$, $N = 8$, and $N = 16$ with increasing numbers of elements.

N=1 Simulations For linear elements, use $N_e = 16, 32$ and 64 elements. Plot the normalized L^2 error norm versus N_P (given below) for these 3 simulations on one plot.

N=4 Simulations For $N = 4$ use $N_e = 4, 8$ and 16 elements and plot the norms as above.

N=8 Simulations For $N = 8$ use $N_e = 2, 4$ and 8 elements and plot the norms as above.

N=16 Simulations For $N = 16$ use $N_e = 1, 2$ and 4 elements and plot the norms as above.

3 Helpful Relations

Error Norm The normalized L2 error norm that you should use is:

$$||error||_{L^2} = \sqrt{\frac{\sum_{k=1}^{N_P} \left(q^{numerical} - q^{exact}(x_k)\right)^2}{\sum_{k=1}^{N_P} q^{exact}(x_k)^2}} \tag{1}$$

where $k = 1, ..., N_P$ are $N_p = N_e N + 1$ global gridpoints and $q^{numerical}$ and q^{exact} are the numerical and exact solutions after one full revolution of the wave. Note that the wave should just stop where it began without changing shape (in a perfect world). Your solution will do that for lots of gridpoints (high resolution). At low resolution, you will see much error.

Time-Integrator To solve the time-dependent portion of the problem use the 2nd order RK method: for $\frac{\partial q}{\partial t} = R(q)$ let

$$q^{n+1/2} = q^n + \frac{\Delta t}{2} R(q^n)$$

$$q^{n+1} = q^n + \Delta t R(q^{n+1/2})$$

or a better time-integrator of your choice (DO NOT USE FORWARD EULER); feel free to use ODE45 in Matlab if you know how to use it or use the given time-integration functions in either Matlab or Julia given in the GitHub repository.

Make sure that your time-step Δt is small enough to ensure stability. Recall that the Courant number

$$C = u\frac{\Delta t}{\Delta x}$$

must be within a certain value for stability. For the 2nd order RK method I give you, it should be below $\frac{1}{4}$. For Δx take the difference of the first point in your domain $x_1 = -1$ and the next point x_2 since this will be the tightest clustering of points in your model. Another, more general, way is to take the minimum value of $x_{I+1} - x_I$ for all points $I = 1, ..., N_P - 1$.