Lecture 9: Constructing CG Global Problem

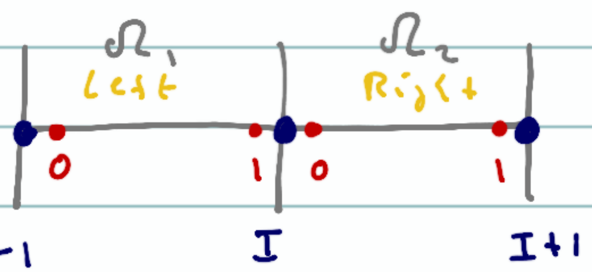
The 1D wave eq. on a reference element is written as:

$$(9.1)\quad M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + D_{ij}^{(e)} f_j^{(e)} = 0 \qquad \forall \quad i,j=0,\dots,N$$

Now we need to use this to build the global solution.

Global Matrix Problem

Let's assume the following simple example w/ only two elements $N_e=2$.



Let's consider the contribution of Eq. (9.1) to the global gridpoint I from both the left & right elements that claim it.

Left Element Using exact integration on N=1 Lobatto points we get:

$$(9.2)\quad \frac{\Delta x^{(L)}}{2}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\frac{dq_{I-1}}{dt}\\ \frac{dq_I}{dt}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}f_{I-1}\\ f_I\end{pmatrix} = 0$$

where the 1st row of (9.2) is the left element contrib to the global gridpoint $I-1$ & the 2nd row is the contrib to the global gridpoint I.

<u>Right Element</u> The right element eq. is:

$$(9.3) \qquad \frac{\Delta x^{(R)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{df_I}{dt} \\ \frac{df_{I+1}}{dt} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} f_I \\ f_{I+1} \end{pmatrix} = \square$$

where the 1st Row contribs. to I & 2nd row to $I+1$

<u>Total Contribution to I</u>

Summing the 2nd row of the left element & 1st row of right element yields: (assuming $\Delta x^{(L)} = \Delta x^{(R)} = \Delta x$

$$\frac{\Delta x}{6}\left(\frac{df_{I-1}}{dt} + 4\frac{df_I}{dt} + \frac{df_{I+1}}{dt}\right) + \frac{1}{2}\left(-f_{I-1} + f_I - f_I + f_{I+1}\right) = 0$$

$$\frac{\Delta x}{6}\left(\frac{df_{I-1}}{dt} + 4\frac{df_I}{dt} + \frac{df_{I+1}}{dt}\right) + \frac{1}{2}\left(f_{I+1} - f_{I-1}\right) = 0$$

Before analyzing the resulting numerical method, let us consider the general approach for building the global matrix problem

To go from the local (reference) element to the global problem requires a map $M : (i,e) \rightarrow I$ where (i,e) is the local element-wise index in Eq. (9.1) s.t. $i = 0, \dots, N$ are the DOF within all elements $e = 1, \dots, N_e$ & $I = 1, \dots, N_p$ denote the global DOF $= N_e N + 1$ in 1D.

The role of this map is to allow us to build the following:

$$(9.4)\quad M_{IJ} = \bigwedge_{e=1}^{N_e} M_{ij}^{(e)}$$ which does the following:

$$M_{IJ} = \begin{pmatrix} \frac{\Delta x^{(1)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \\ & \frac{\Delta x^{(2)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \end{pmatrix} \xrightarrow{\Delta x^{(1)} = \Delta x^{(2)} = \Delta x} = \frac{\Delta x}{6}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{matrix} \rightarrow I-1 \\ \rightarrow I \\ \rightarrow I+1 \end{matrix}$$

Similarly, we define the global differentiation matrix as follows:

$$(9.5)\quad D_{IJ} = \bigwedge_{e=1}^{N_e} D_{ij}^{(e)}$$ which does the following:

$$D_{IJ} = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} & \\ & \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{matrix} \rightarrow I-1 \\ \rightarrow I \\ \rightarrow I+1 \end{matrix}$$

The global matrix-vector problem becomes:

$$\frac{\Delta x}{6}\overbrace{\begin{pmatrix}2&1&0\\1&4&1\\0&1&2\end{pmatrix}}^{=M}\frac{d}{dt}\begin{pmatrix}f_{I-1}\\f_I\\f_{I+1}\end{pmatrix}+\frac{1}{2}\overbrace{\begin{pmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{pmatrix}}^{=D}\begin{pmatrix}f_{I-1}\\f_I\\f_{I+1}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Left-multiplying by

$$M^{-1}=\frac{1}{2\Delta x}\begin{pmatrix}7&-2&1\\-2&4&-2\\1&-2&7\end{pmatrix}$$ yields:

$$\frac{d}{dt}\begin{pmatrix}f_{I-1}\\f_I\\f_{I+1}\end{pmatrix}+\frac{1}{4\Delta x}\begin{pmatrix}-5&6&-1\\-2&0&2\\1&-6&5\end{pmatrix}\begin{pmatrix}f_{I-1}\\f_I\\f_{I+1}\end{pmatrix}=0$$

where the 2^{nd} row yields:

$$\frac{df_I}{dt}+\frac{f_{I+1}-f_{I-1}}{2\Delta x}=\emptyset+O(\Delta x^2)$$

$\therefore$ for CG w/ $N=1$ in 1D, we get the same accuracy as in a centered finite difference method.

DSS Operation To construct the global solution from the reference element representation (1.1) we use the Global Assembly (or Direct Stiffness Summation ≡ DSS) given by Eqs. (1.4) & (1.5). Algorithmically, we can write DSS as follows:

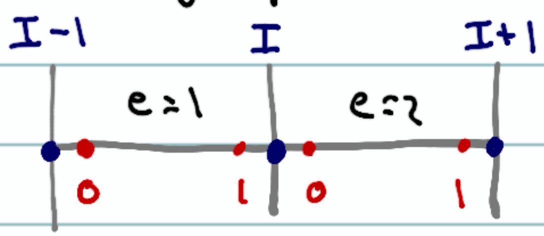
DSS Operation

```
M = 0
for e=1:Ne do
    for i=0:N do
        I = intma(i,e)
        for j=0:N do
            J = intma(j,e)
            M_IJ = M_IJ + M_ij^(e)
        end
    end
end
```

where this operation is the same for any element matrix $M_{ij}^{(e)}$ (such as $D_{ij}^{(e)}$ or any other matrix).

Note that the crux is now to define the map $\text{intma}(i,e) \to I$

Local to Global Map The map $(i,e) \rightarrow I$ allows us to go from the reference element numbering to the global gridpoint numbering as follows. For $N=1$, $N_e=2$:



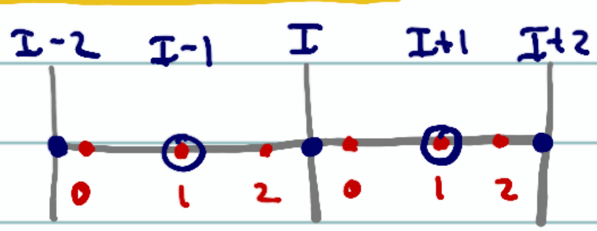
$\therefore$ let us define an integer array s.t.:

$$\text{intma}(0:1, e=1) = I-1,\ I \longrightarrow \text{element } \Omega_1$$

&

$$\text{intma}(0:1, e=2) = I,\ I+1 \longrightarrow \text{element } \Omega_2$$

For $N=2$, $N_e=2$ we have:



& so we get:

$$\text{intma}(0:2, e=1) = I-2,\ I-1,\ I \quad \&$$

$$\text{intma}(0:2, e=2) = I,\ I+1,\ I+2$$