(Ch 5)

# Lecture 8: Mass & Differentiation

Let's assume that we wish to solve the PDE:

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad \text{with } f = qu, \; u = \text{const} > 0 \;\&$$

periodic BCs.

The integral form of this problem is:

$$\int_{\Omega_e} \psi_i \left( \frac{\partial q_N}{\partial t} + \frac{\partial f_N}{\partial x} \right) d\Omega_e = 0$$

when $q_N(x,t) = \sum_{j=0}^{N} \psi_j(x)\, q_j(t)$ & $f_N(x,t) = \sum_{j=0}^{N} \psi_j(x) f_j(t)$

**CG Method** The CG formulation reads: find $q_N \in H^1$ s.t.

(8.1) $$\int_{\Omega_e} \psi_i \frac{\partial q_N}{\partial t}\, d\Omega_e + \int_{\Omega_e} \psi_i \frac{\partial f_N}{\partial x}\, d\Omega_e = 0 \quad \forall \psi \in H^1$$

w/ $q_N, \psi \in C^0$

To simplify the discussion, let us derive the CG formulation for $N=1$ → Linear basis functions.

∴ the approximations become:

(8.2) $$q_N(x,t) = \sum_{j=0}^{1} \psi_j(x)\, q_j(t) \quad \& \quad f_N(x,t) = \sum_{j=0}^{1} \psi_j(x) f_j(t)$$

From (8.2) we can write:

$$\text{(8.3)} \quad \frac{\partial f_N}{\partial t} = \sum_{j=0}^{1} \psi_j(x) \frac{d f_j}{dt}(t) \quad \& \quad \frac{\partial f_N}{\partial x} = \sum_{j=0}^{1} \frac{d\psi_j}{dx}(x) f_j(t)$$

which can be substituted into (8.1) to yield:

$$\text{(8.4)} \quad \int_{\Omega_e} \psi_i \sum_{j=0}^{1} \psi_j(x) \frac{d f_j}{dt}(t) \, d\Omega_e$$

$$+ \int_{\Omega_e} \psi_i \sum_{j=0}^{1} \frac{d\psi_j(x)}{dx} f_j(t) \, d\Omega_e = 0 \quad \forall \, i = 0, 1$$

Because $\frac{d f_j}{dt}(t)$ & $f_j(t)$ are independent of space, we can factor them from the integrals to yield:

$$\int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \frac{d f_j}{dt}(t) + \int_{\Omega_e} \psi_i \frac{d\psi_j}{dx} \, d\Omega_e \, f_j(t) = 0 \quad \forall \, i, j = 0, 1$$

The matrix-vector form of this problem then becomes:

$$\text{(8.5)} \quad M_{ij}^{(e)} \frac{d f_j^{(e)}}{dt} + D_{ij}^{(e)} f_j^{(e)} = 0$$

where the superscript $(e)$ has been added to remind the reader that the approximation is currently only defined within the element $\Omega_e$. We will discuss how to construct the global solution later in this lecture.

In (8.5), the mass & differentiation matrices are defined as:

$$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i(x)\,\psi_j(x)\, d\Omega_e$$

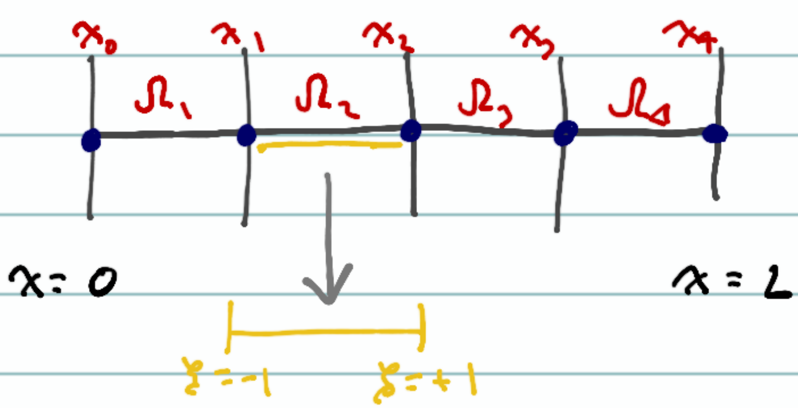
$$D_{ij}^{(e)} = \int_{\Omega_e} \psi_i(x)\, \frac{\partial \psi_j}{\partial x}(x)\, d\Omega_e \qquad i,j = 0,1$$

## Basis Functions & Reference Element

We already learned that in 1D we can define the Lagrange basis functions as follows:

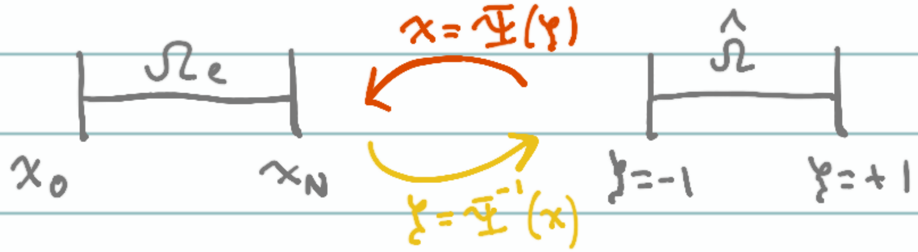
$$\psi_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{N} \left( \frac{x - x_j}{x_i - x_j} \right)$$

Next assume that we want to solve a PDE in the global domain:




We can choose to build $\psi_i(x)$ within each element $\Omega_e$ or we can build $\psi_i(\xi)$ in the reference element defined in the interval $\xi \in [-1, +1]$ $\rightarrow$ space for Legendre, Lobatto, etc

The idea is to map each element $\Omega_e$ as follows:




Upon performing this mapping, we can build the basis functions only in $\hat{\Omega}$ as such:

$$(8.6) \quad \psi_i(\xi) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \left( \frac{\xi - \xi_j}{\xi_i - \xi_j} \right); \quad \frac{d\psi_i}{d\xi}(\xi) = \sum_{\substack{n=0 \\ n \neq i}}^{N} \frac{1}{\xi_i - \xi_n} \prod_{\substack{j=0 \\ j \neq i \\ j \neq n}}^{N} \left( \frac{\xi - \xi_j}{\xi_i - \xi_j} \right)$$

$\boxed{\text{HW 2}}$ Derive $\psi_i$ & $\psi_i'$ for $N=1$ & $N=2$

<u>Map $x = \Phi(\xi)$</u> The map from the physical to the reference space is found by first expanding the physical coordinates using the basis functions (isoparametric map). I.e.,

$$(8.7) \quad x_N(\xi) = \sum_{j=0}^{N} \psi_j(\xi)\, x_j$$

& for $N=1$ we get

$$\begin{aligned} x_N(\xi) &= \psi_0(\xi)\, x_0 + \psi_1(\xi)\, x_1 \\ &= \tfrac{1}{2}(1-\xi)\, x_0 + \tfrac{1}{2}(1+\xi)\, x_1 \\ &= \frac{x_0 + x_1}{2} + \frac{\xi}{2}(x_1 - x_0) \end{aligned}$$

Let $\Delta x = x_1 - x_0$

$$= \frac{x_0 + (x_0 + \Delta x)}{2} + \frac{\xi}{2}\Delta x$$

(8.8.1) $$x_N(\xi) = x_0 + \frac{\Delta x}{2}(\xi + 1)$$

(8.8.2)
with the inverse map $\rightarrow$ $$\xi = 2\left(\frac{x(\xi) - x_0}{\Delta x}\right) - 1$$

$\Psi^{-1}: \xi \to x$

$\Psi: x \to \xi$

<u>Jacobian</u> From this map, we can compute the Jacobian of the mapping as follows:

$$\frac{d x_N(\xi)}{d\xi} \equiv \frac{dx}{d\xi} = \frac{\Delta x}{2} \quad \& \quad \frac{d\xi}{dx} = \frac{2}{\Delta x}$$

<u>Integration in the Reference element</u>

Now that we have the Jacobian of the transformation we can use it to perform integration inside the reference element as such:

$$M_{ij}^{(e)} \equiv \int_{\Omega_e} \psi_i(x)\psi_j(x)\, d\Omega_e = \int_{x_0}^{x_1} \psi_i(x)\,\psi_j(x)\, dx$$

$$= \int_{-1}^{+1} \psi_i(\xi)\,\psi_j(\xi)\,\frac{dx}{d\xi}\, d\xi \equiv \int_{-1}^{+1} \psi_i(\xi)\psi_j(\xi)\,\frac{\Delta x^{(e)}}{2} d\xi$$

where $\Delta x^{(e)}$ is the size of the element $\Omega_e$

Similarly, we write:

$$D_{ij}^{(e)} = \int_{x_0}^{x_1} \psi_i(x) \frac{d\psi_j(x)}{dx}\, dx$$

$$= \int_{-1}^{+1} \psi_i(\xi) \left(\frac{d\psi_j(\xi)}{d\xi} \frac{d\xi}{dx}\right) \left(\frac{dx}{d\xi}\, d\xi\right)$$

$$= \int_{-1}^{+1} \psi_i(\xi) \frac{d\psi_j(\xi)}{d\xi}\, d\xi$$

**More on Map $x = \Psi(\xi)$**

The map $\Psi$ is bijective. I.e., it is both injective & surjective as follows:

| Injective | Surjective | Bijective (affine) |
|---|---|---|
| $1 \to A$ | $1 \to A$ | $1 \to A$ |
| $2 \to B$ | $2 \to B$ | $2 \to B$ |
| $3 \to C$ | $3 \to C$ | $3 \to C$ |
| $D$ | $4 \to C$ | $4 \to D$ |
| one-to-one map<br>But output may not have an input | every output has an input | Perfect one-to-one map |

**Mass Matrix** The mass matrix is given by:

$$(8.9)\quad M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{2}\int_{-1}^{+1} \psi_i(\xi)\,\psi_j(\xi)\, d\xi \qquad i,j = 0,\dots,N$$

& for $N=1$ we have $\psi_0 = \frac{1}{2}(1-\xi)$, $\psi_1 = \frac{1}{2}(1+\xi)$ which gives for (8.9)

$$(8.10)\quad M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{2}\int_{-1}^{+1}\begin{pmatrix} \psi_0^2 & \psi_0\psi_1 \\ \psi_1\psi_0 & \psi_1^2 \end{pmatrix} d\xi$$

$$= \frac{\Delta x^{(e)}}{2}\int_{-1}^{+1} \frac{1}{4}\begin{pmatrix} (1-\xi)^2 & 1-\xi^2 \\ 1-\xi^2 & (1+\xi)^2 \end{pmatrix} d\xi$$

$$= \frac{\Delta x^{(e)}}{8}\int_{-1}^{+1}\begin{pmatrix} 1-2\xi+\xi^2 & 1-\xi^2 \\ 1-\xi^2 & 1+2\xi+\xi^2 \end{pmatrix} d\xi$$

$$= \frac{\Delta x^{(e)}}{8}\left.\begin{pmatrix} \xi-\xi^2+\frac{1}{3}\xi^3 & \xi-\frac{1}{3}\xi^3 \\ \xi-\frac{1}{3}\xi^3 & \xi+\xi^2+\frac{1}{3}\xi^3 \end{pmatrix}\right|_{-1}^{+1}$$

$$= \frac{\Delta x^{(e)}}{8}\begin{pmatrix} \underset{8/3}{2+\frac{2}{3}} & \underset{4/3}{2-\frac{2}{3}} \\ \underset{4/3}{2-\frac{2}{3}} & \underset{8/3}{2+\frac{2}{3}} \end{pmatrix} = \frac{\Delta x^{(e)}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

## Mass Matrix & Numerical Integration

Note that the product of basis functions in the mass matrix is a 2N polynomial. This means that if we use either N Legendre points or N+1 Lobatto points, that we can integrate them exactly.

**Exact Integration**

Let's use Q=2 Lobatto points: $\xi = [-1, 0, +1]$

$\& \ \omega = [\frac{1}{3}, \frac{4}{3}, \frac{1}{3}]$

& so we have:

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{2} \int_{-1}^{+1} \psi_i(\xi)\psi_j(\xi)\, d\xi = \frac{\Delta x^{(e)}}{2} \sum_{k=0}^{Q} \omega_k \psi_i(\xi_k)\psi_j(\xi_k)$$

$$= \frac{\Delta x^{(e)}}{8} \sum_{k=0}^{Q} \omega_k \begin{pmatrix} (1-\xi_k)^2 & 1-\xi_k^2 \\ 1-\xi_k^2 & (1+\xi_k)^2 \end{pmatrix}$$

e.g. $1-\xi_k^2$ term $= \omega_0(1-\xi_0)^2 + \omega_1(1-\xi_1)^2 + \omega_2(1-\xi_2)^2$

which yields $= \frac{1}{3}(1+1)^2 + \frac{4}{3}(1)^2 + \frac{1}{3}(\emptyset) \equiv \frac{8}{3}$

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{8} \begin{pmatrix} 8/3 & 4/3 \\ 4/3 & 8/3 \end{pmatrix}$$

**Inexact Integration** Let's use $N=Q=1$ which, we know, will not yield exact integration using Lobatto points.

We have: $\xi \in [-1, +1]$ $\omega \in [+1, +1]$

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{2} \sum_{k=0}^{1} \omega_k \psi_i(\xi_k)\psi_j(\xi_k)$$

$$= \frac{\Delta x^{(e)}}{2} \sum_{k=0}^{1} \omega_k \begin{pmatrix} (1-\xi_k)^2 & 1-\xi_k^2 \\ 1-\xi_k^2 & (1+\xi_k)^2 \end{pmatrix}$$

$$= \frac{\Delta x^{(e)}}{8} \begin{pmatrix} (1)(4)+(1)(0) & (1)(0)+(1)(0) \\ (1)(0)+(1)(0) & (1)(0)+(1)(4) \end{pmatrix}$$

$$= \frac{\Delta x^{(e)}}{8} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \frac{\Delta x^{(e)}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boxed{\text{HW}}$ Use $Q = N = 1$ w/ Legendre Points

## Differentiation Matrix

we have:

$$D_{ij}^{(e)} = \int_{-1}^{+1} \psi_i(\xi) \frac{d\psi_j}{d\xi}(\xi)\, d\xi$$

& for $N=1$ we have:

$$\psi_i = \tfrac{1}{2}(1+\xi_i \xi) \qquad \xi_i = -1, +1 \qquad i = 0, 1$$

&

$$\frac{d\psi_i}{d\xi} = \tfrac{1}{2}\xi_i \quad , \text{ which yields:}$$

$$D_{ij}^{(e)} = \int_{-1}^{+1} \begin{pmatrix} \psi_0 \frac{d\psi_0}{d\xi} & \psi_0 \frac{d\psi_1}{d\xi} \\ \psi_1 \frac{d\psi_0}{d\xi} & \psi_1 \frac{d\psi_1}{d\xi} \end{pmatrix} d\xi = \int_{-1}^{+1} \begin{pmatrix} -\frac{1}{4}(1-\xi) & \frac{1}{4}(1-\xi) \\ -\frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) \end{pmatrix}$$

$$D_{ij}^{(e)} = \frac{1}{4}\begin{pmatrix} -\xi+\frac{1}{2}\xi^2 & \xi-\frac{1}{2}\xi^2 \\ -\xi-\frac{1}{2}\xi^2 & \xi+\frac{1}{2}\xi^2 \end{pmatrix}\Bigg|_{-1}^{+1} = \frac{1}{4}\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$$

$$D_{ij}^{(e)} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

$\boxed{\text{Question}}$ Why does the row sum of $D_{ij}^{(e)}$ yield: $\sum_{j=0}^{1} D_{ij}^{(e)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Taylor Series for a deriv gives, e.g.,

$$\frac{f_{i+1}-f_{i-1}}{2\Delta x} = \frac{1}{2\Delta}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_{i-1} \\ f_i \\ f_{i+1} \end{pmatrix}$$

## Differentiation Matrix & Numerical Integration

Note that the polynomial in $D$ is $O(N+N-1) = O(2N-1)$

$\therefore$ we can use Lobatto w/ $Q=N=1$ & get exact integration. Let's try it.

$\omega_k \in [+1,+1]$, $\xi_k = [-1,+1]$

$$D_{ij}^{(e)} = \frac{1}{4}\sum_{k=0}^{1} \omega_k \begin{pmatrix} -(1-\xi_k) & 1-\xi_k \\ -(1+\xi_k) & 1+\xi_k \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (1)(-2)+(1)(0) & (1)(2)+(1)(0) \\ (1)(0)+(1)(-2) & (1)(0)+(1)(2) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \rightarrow \text{Same as in analytic integration}$$

$\boxed{\text{Ex}}$ Try using $Q=2$ w/ $\xi \in [-1, 0, +1]$ & $\omega \in [\frac{1}{3}, \frac{4}{3}, \frac{1}{3}]$