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An Introduction to Element-based Galerkin Methods on Tensor-Product Bases: Analysis, Algorithms, and Applications

– Monograph –

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Preface

The focus of this book is on applying element-based Galerkin (EBG) methods to solve hyperbolic and elliptic equations with an emphasis on the resulting matrix-vector problems. Part I introduces the main topics of the book, followed by Part II which discusses one-dimensional problems. Part III treats multi-dimensional problems. In Part III, the ideas are primarily discussed for two dimensions but some concepts, such as the construction of the tensor-product basis functions, numerical integration, and metric terms, are extended to three dimensions. Part IV discusses advanced topics such as stabilization methods, adaptive mesh refinement, time-integration, and the hybridized discontinuous Galerkin (HDG) method. The contents of each part are described in more detail below and at the very end of this *preface* we include a discussion on how to use this book for teaching a course on this topic.

Because the basic building-blocks of EBG methods rely on interpolation and integration, Chs. 3 and 4 (Part II) cover these two topics in one dimension and Chs. 10 and 11 in multiple dimensions (Part III). These chapters rely on the theory of Jacobi polynomials which is covered in Appendix B.

The EBG methods discussed in this book include the continuous (CG) and discontinuous Galerkin (DG) methods. These methods are introduced for one-dimensional hyperbolic equations in Chs. 5 and 6 for explicit time-integration (Part II). We need to wait until Ch. 21 (Part IV) to discuss the hybridized discontinuous Galerkin (HDG) method because HDG only makes sense in the context of implicit time-integration, which is not introduced until Ch. 20. Chapter 7 is the heart of the manuscript where the idea of unified CG/DG methods is introduced; this chapter also presents the dissipation-dispersion analysis of both methods, and applications of unified CG/DG methods for systems of nonlinear partial differential equations including the shallow water and Euler equations. The application of CG and DG methods for one-dimensional problems is completed in Chs. 8 and 9 with a discussion on the application of these methods for elliptic equations.

Chapters 12, 13, and 14 (Part III) introduce CG, DG, and unified CG/DG methods in two dimensions for elliptic equations. Elliptic equations are handled first in order to focus on spatial discretization which, in multiple dimensions, requires a detailed discussion on the construction of metric terms (Ch. 12). After the basics of EBG methods in multiple dimensions are covered, we then move to a discussion of these methods for hyperbolic equations in Chs. 15, 16, and 17 for CG, DG, and unified CG/DG. These chapters discuss the efficient construction of CG and DG methods through complexity analysis and show how tensor-product bases allow such efficiencies. Furthermore, Ch. 17 extends further the heart of the book whereby the unified CG/DG approach presented in Ch. 7 for one-dimensional hyperbolic equations is extended to two (and multiple) dimensions.

Part IV covers advanced topics that, while important for the construction of industrial-type codes, is not strictly necessary for understanding the basics of EBG methods. Chapter 18 describes stabilization methods including filters, artificial dissipation (i.e., hyper-diffusion), Riemann solvers, limiters, and entropy-stable methods. Chapter 19 describes the three types of mesh refinement which are h-, p-, and r-refinement. Chapter 20 discusses explicit, implicit, implicit-explicit, semi-Lagrangian, and multirate time-integration methods. Part IV ends with a discussion of the hybridized discontinuous Galerkin method in one dimension in Ch. 21.

Let us now briefly discuss how one might use this book to teach a course on element-based Galerkin methods. Each of the chapters can be treated as lectures that build upon each other and so can be used to construct lectures that can be delivered sequentially. The author's book website contains a sample syllabus for a 10-week quarter-based course that can be easily extended to a 14-week semester-based course (more projects can be assigned and the advanced topics can be discussed in more detail); sample project assignments are also available on the book website. For a 10-week quarter-based course, I recommend assigning 4 projects. Project 1 treats interpolation and integration in one-dimension (Chs. 3 and 4). This then allows the students to tackle Project 2 which deals with solving a one-dimensional scalar hyperbolic equation (Chs. 5 and 6). I recommend having the students first write two different codes that treat CG and DG separately. Then they can write a unified code following the concepts presented in Ch. 7. Project 3 can then focus on solving a two-dimensional elliptic equation with CG and DG as presented in Chs. 12, 13, and 14. Project 4 would then consist of building a unified CG/DG code for solving a two-dimensional hyperbolic equation as presented in Chs. 15, 16, and 17. Assigning such a project is critical for the student to learn how to write efficient code with EBG methods since many of the optimization strategies described in the book can only be fully exploited for time-dependent problems in multi-dimensions (such as sum factorization and constructing the right-hand side vector without storing full matrices).

For a 14-week semester-based course, I recommend adding two more projects. The new Project 3 would consist of solving systems of equations as discussed in Ch. 7; this project is challenging and extremely helpful in preparing the student to tackle more interesting research problems. The new Project 4 consists of solving one-dimensional elliptic equations as described in Chs. 8 and 9; this project should be relatively simple for the student. Project 5 consists of solving two-dimensional elliptic equations and Project 6 focuses on solving two-dimensional hyperbolic equations. For ambitious students, I recommend a project on solving systems of two-dimensional equations (extension of Ch. 7); this project can be combined with evolving the equations forward in time using implicit methods as presented in Ch. 20.

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