

Francis X. Giraldo

An Introduction to Element-
based Galerkin Methods on
Tensor-Product Bases: Analysis,
Algorithms, and Applications

– Monograph –

May 30, 2020

Springer Nature

Preface

The focus of this book is on applying element-based Galerkin (EBG) methods to solve hyperbolic and elliptic equations with an emphasis on the resulting matrix-vector problems. Part I introduces the main topics of the book, followed by Part II which discusses one-dimensional problems. Part III treats multi-dimensional problems. In Part III, the ideas are primarily discussed for two dimensions but some concepts, such as the construction of the tensor-product basis functions, numerical integration, and metric terms, are extended to three dimensions. Part IV discusses advanced topics such as stabilization methods, adaptive mesh refinement, time-integration, and the hybridized discontinuous Galerkin (HDG) method. The contents of each part are described in more detail below and at the very end of this *preface* we include a discussion on how to use this book for teaching a course on this topic.

Because the basic building-blocks of EBG methods rely on interpolation and integration, Chs. 3 and 4 (Part II) cover these two topics in one dimension and Chs. 10 and 11 in multiple dimensions (Part III). These chapters rely on the theory of Jacobi polynomials which is covered in Appendix B.

The EBG methods discussed in this book include the continuous (CG) and discontinuous Galerkin (DG) methods. These methods are introduced for one-dimensional hyperbolic equations in Chs. 5 and 6 for explicit time-integration (Part II). We need to wait until Ch. 21 (Part IV) to discuss the hybridized discontinuous Galerkin (HDG) method because HDG only makes sense in the context of implicit time-integration, which is not introduced until Ch. 20. Chapter 7 is the heart of the manuscript where the idea of unified CG/DG methods is introduced; this chapter also presents the dissipation-dispersion analysis of both methods, and applications of unified CG/DG methods for systems of nonlinear partial differential equations including the shallow water and Euler equations. The application of CG and DG methods for one-dimensional problems is completed in Chs. 8 and 9 with a discussion on the application of these methods for elliptic equations.

Chapters 12, 13, and 14 (Part III) introduce CG, DG, and unified CG/DG methods in two dimensions for elliptic equations. Elliptic equations are handled first in order to focus on spatial discretization which, in multiple dimensions, requires a detailed discussion on the construction of metric terms (Ch. 12). After the basics of EBG

methods in multiple dimensions are covered, we then move to a discussion of these methods for hyperbolic equations in Chs. 15, 16, and 17 for CG, DG, and unified CG/DG. These chapters discuss the efficient construction of CG and DG methods through complexity analysis and show how tensor-product bases allow such efficiencies. Furthermore, Ch. 17 extends further the heart of the book whereby the unified CG/DG approach presented in Ch. 7 for one-dimensional hyperbolic equations is extended to two (and multiple) dimensions.

Part IV covers advanced topics that, while important for the construction of industrial-type codes, is not strictly necessary for understanding the basics of EBG methods. Chapter 18 describes stabilization methods including filters, artificial dissipation (i.e., hyper-diffusion), Riemann solvers, limiters, and entropy-stable methods. Chapter 19 describes the three types of mesh refinement which are h -, p -, and r -refinement. Chapter 20 discusses explicit, implicit, implicit-explicit, semi-Lagrangian, and multirate time-integration methods. Part IV ends with a discussion of the hybridized discontinuous Galerkin method in one dimension in Ch. 21.

Let us now briefly discuss how one might use this book to teach a course on element-based Galerkin methods. Each of the chapters can be treated as lectures that build upon each other and so can be used to construct lectures that can be delivered sequentially. The author's book website contains a sample syllabus for a 10-week quarter-based course that can be easily extended to a 14-week semester-based course (more projects can be assigned and the advanced topics can be discussed in more detail); sample project assignments are also available on the book website. For a 10-week quarter-based course, I recommend assigning 4 projects. Project 1 treats interpolation and integration in one-dimension (Chs. 3 and 4). This then allows the students to tackle Project 2 which deals with solving a one-dimensional scalar hyperbolic equation (Chs. 5 and 6). I recommend having the students first write two different codes that treat CG and DG separately. Then they can write a unified code following the concepts presented in Ch. 7. Project 3 can then focus on solving a two-dimensional elliptic equation with CG and DG as presented in Chs. 12, 13, and 14. Project 4 would then consist of building a unified CG/DG code for solving a two-dimensional hyperbolic equation as presented in Chs. 15, 16, and 17. Assigning such a project is critical for the student to learn how to write efficient code with EBG methods since many of the optimization strategies described in the book can only be fully exploited for time-dependent problems in multi-dimensions (such as sum factorization and constructing the right-hand side vector without storing full matrices).

For a 14-week semester-based course, I recommend adding two more projects. The new Project 3 would consist of solving systems of equations as discussed in Ch. 7; this project is challenging and extremely helpful in preparing the student to tackle more interesting research problems. The new Project 4 consists of solving one-dimensional elliptic equations as described in Chs. 8 and 9; this project should be relatively simple for the student. Project 5 consists of solving two-dimensional elliptic equations and Project 6 focuses on solving two-dimensional hyperbolic equations. For ambitious students, I recommend a project on solving systems of two-dimensional equations

(extension of Ch. 7); this project can be combined with evolving the equations forward in time using implicit methods as presented in Ch. 20.

Contents

Part I Introduction

1	Motivation and Background	3
1.1	Introduction	3
1.2	Continuous Governing Equations	5
1.3	Analytic Tools for Deriving Solutions to the Continuous Problem	7
1.4	Basic Mathematical Concepts	7
1.4.1	Taylor Series	7
1.4.2	Product Rule of Differentiation	8
1.4.3	Fundamental Theorem of Calculus	8
1.4.4	Integration by Parts	9
1.4.5	Green's First Identity	9
1.5	Computational Tools for Obtaining Solutions to the Discrete Problem	10
1.5.1	From Continuous to Discrete Formulations	10
1.5.2	Validity of the Numerical Solution and the Lax Equivalence Theorem	11
1.5.3	Consistency of a Numerical Method	11
1.5.4	Stability of a Numerical Method: Von Neumann Stability Analysis	16
1.5.5	Convergence	20
1.6	Efficiency of Galerkin Methods on Distributed-Memory Computers	21
2	Overview of Galerkin Methods	27
2.1	Introduction	27
2.2	Differential Form: Finite Differences	27
2.3	Integral Form: Galerkin Methods	28
2.3.1	Continuous Galerkin Method	30
2.3.2	Discontinuous Galerkin Method	31
2.4	A Formal Introduction to Galerkin Methods	35
2.4.1	Problem Statement	35
2.4.2	Function Spaces	36

2.4.3	Sturm-Liouville Operator	38
2.4.4	Basis Functions	39
2.5	Global Galerkin Methods	40
2.6	Element-based Galerkin Methods	41

Part II One-Dimensional Problems

3	Interpolation in One Dimension	47
3.1	Introduction	47
3.2	Modal Interpolation	47
3.2.1	Monomial Expansion	48
3.2.2	Sturm-Liouville Operator	48
3.2.3	Fourier Functions	48
3.2.4	Jacobi Polynomials	49
3.2.5	Legendre Polynomials	50
3.2.6	Chebyshev Polynomials	51
3.2.7	More on Legendre Polynomials	51
3.2.8	Vandermonde Matrix	52
3.3	Nodal Interpolation	54
3.3.1	Lebesgue Function and Lebesgue Constant	55
3.3.2	Lagrange Polynomials	55
3.3.3	Quality of Lagrange Polynomial Interpolation	58
3.4	Example of Interpolation Quality	62
3.4.1	Equi-Spaced Points	64
3.4.2	Chebyshev Points	65
3.4.3	Legendre Points	65
3.4.4	Lobatto Points	66
3.5	Summary of Interpolation Points	67
4	Numerical Integration in One Dimension	69
4.1	Introduction	69
4.2	Divided Differences	69
4.3	Numerical Integration	72
4.3.1	Introduction	72
4.3.2	Quadrature Roots	72
4.3.3	Quadrature Weights	73
4.4	Example of Integration Quality	75
4.5	Summary of Integration Points	77
5	1D Continuous Galerkin Method for Hyperbolic Equations	79
5.1	Introduction	79
5.2	Continuous Galerkin Representation of the 1D Wave Equation	80
5.3	Basis Functions and the Reference Element	81
5.4	Mass Matrix	82
5.4.1	General Form of the Mass Matrix	84
5.5	Differentiation Matrix	84

5.5.1	General Form of the Differentiation Matrix	87
5.6	Resulting Element Equations	88
5.7	Element Contribution to Gridpoint I	88
5.7.1	Left Element	88
5.7.2	Right Element	89
5.7.3	Total Contribution	90
5.7.4	High-Order Approximation	91
5.8	Direct Stiffness Summation	93
5.9	Analysis of the Matrix Properties of the Spatial Operators	96
5.9.1	Sparsity Pattern of the Mass Matrix	96
5.9.2	Sparsity Pattern of the Differentiation Matrix	97
5.9.3	Eigenvalue Analysis of the Spatial Operator	98
5.10	Example of 1D Wave Equation Problem	99
5.10.1	Initial Condition	100
5.10.2	Boundary Condition	100
5.10.3	Error Norm	100
5.10.4	Time-Integrator	101
5.10.5	Construction of the CG Solution	101
5.10.6	Solution Accuracy for a Smooth Problem	102
5.10.7	Solution Accuracy for a Non-Smooth Problem	105
6	1D Discontinuous Galerkin Methods for Hyperbolic Equations	107
6.1	Discontinuous Galerkin Representation of the 1D Wave Equation	107
6.2	Mass Matrix	109
6.3	Differentiation Matrix	109
6.3.1	General Form of the Weak Differentiation Matrix	110
6.4	Flux Matrix	110
6.4.1	General Form of the Flux Matrix	111
6.5	Resulting Element Equations	111
6.5.1	Left Element	112
6.5.2	Center Element	113
6.5.3	Right Element	113
6.5.4	Total Contribution	114
6.5.5	centered Numerical Flux	117
6.5.6	Rusanov Numerical Flux	119
6.6	High-Order Approximation	123
6.7	Analysis of the Matrix Properties of the Spatial Operators	125
6.7.1	Sparsity Pattern of the Mass Matrix	126
6.7.2	Sparsity Pattern of the Differentiation Matrix	127
6.7.3	Eigenvalue Analysis of Spatial Operator	127
6.8	Conservation Property of DG	130
6.8.1	Exact Integration	130
6.8.2	Inexact Integration	131
6.8.3	Conservation Property of CG	132
6.9	Example of 1D Wave Equation Problem	132

6.9.1	Initial Condition	133
6.9.2	Boundary Condition	133
6.9.3	Error Norm	133
6.9.4	Time-Integrator	133
6.9.5	Construction of the DG Solution	133
6.9.6	Solution Accuracy for a Smooth Problem	134
6.9.7	Solution Accuracy for a Non-Smooth Problem	139
7	1D Unified Continuous and Discontinuous Galerkin Methods for Systems of Hyperbolic Equations	141
7.1	Introduction	141
7.2	CG and DG Storage of Data	141
7.2.1	From DG to CG Storage	142
7.2.2	From CG to DG Storage	143
7.3	1D Wave Equation	143
7.3.1	Communicator	145
7.3.2	Construction of the Unified CG/DG Solution	146
7.3.3	Face Data Structure	151
7.4	1D Shallow Water Equations	152
7.4.1	Example of Linearized 1D Shallow Water Equations	154
7.4.2	Analytic Solution and Initial Condition	154
7.4.3	Boundary Condition	155
7.4.4	Error Norm	155
7.4.5	Time-Integrator	156
7.4.6	CG Solution Accuracy	156
7.4.7	DG Solution Accuracy	158
7.5	1D Euler Equations	160
7.6	Dissipation and Dispersion Analysis	162
7.6.1	Continuous Galerkin Method	163
7.6.2	Discontinuous Galerkin Method	165
7.7	Dispersion and High-Frequency Waves	168
7.7.1	Multi-scale Test Problem	168
7.7.2	CG and DG Solutions	169
8	1D Continuous Galerkin Methods for Elliptic Equations	173
8.1	Introduction	173
8.2	Elliptic Equations	173
8.3	Finite Difference Method	174
8.4	Continuous Galerkin Method	175
8.5	First Derivatives in their Strong and Weak Forms	177
8.5.1	Strong Form	177
8.5.2	Weak Form	180
8.6	Second Derivatives in their Weak Form	183
8.6.1	Laplacian Matrix	183
8.6.2	Resulting Element Equations	185

8.6.3	Element Contribution to Gridpoint I	185
8.6.4	1D Elliptic Equation	186
8.7	Analysis of the Matrix Properties of the Spatial Operators	191
8.7.1	Sparsity Pattern of the Mass Matrix	192
8.7.2	Sparsity Pattern of the Laplacian Matrix	192
8.7.3	Eigenvalue Analysis of the Laplacian Operator	193
8.8	Example of 1D Poisson Equation Problem	194
8.8.1	Error Norm	194
8.8.2	Solution Accuracy	194
9	1D Discontinuous Galerkin Methods for Elliptic Equations	197
9.1	Introduction	197
9.2	Elliptic Equations	197
9.3	Discontinuous Galerkin Method	198
9.4	First Derivatives in Weak Form	200
9.4.1	Resulting Element Equations	201
9.4.2	Element Derivative at the Gridpoint I	202
9.5	Second Derivatives	204
9.5.1	First Derivative: Auxiliary Variable	205
9.5.2	Resulting Element Equations	206
9.5.3	Element Derivative at the Gridpoint I	206
9.5.4	Possible Choices for the Numerical Flux	207
9.5.5	Resulting Auxiliary Variable at the Gridpoint I	208
9.5.6	Second Derivative	209
9.5.7	Resulting Element Equations	210
9.5.8	Element Second Derivative at the Gridpoint I	210
9.5.9	Resulting Second Derivative at the Gridpoint I	211
9.5.10	1D Elliptic Equation	212
9.6	Analysis of the Matrix Properties of the Spatial Operators	216
9.6.1	Sparsity Pattern of the Mass Matrix	216
9.6.2	Sparsity Pattern of the Laplacian Matrix	216
9.6.3	Eigenvalue Analysis of the Laplacian Operator	217
9.7	Example of 1D Poisson Equation Problem	218
9.7.1	Error Norm	218
9.7.2	Solution Accuracy	218

Part III Multi-Dimensional Problems

10	Interpolation in Multiple Dimensions	223
10.1	Introduction	223
10.2	Interpolation on the Quadrilateral	224
10.2.1	Modal Interpolation	224
10.2.2	Nodal Interpolation	224
10.2.3	Popularity of Quadrilateral Elements	225
10.2.4	Example of Quadrilateral Basis Functions	225

10.3	Interpolation on the Hexahedron	228
10.3.1	Modal Interpolation	229
10.3.2	Nodal Interpolation	230
10.3.3	Indexing of the Basis Functions	231
11	Numerical Integration in Multiple Dimensions	233
11.1	Introduction	233
11.2	Numerical Integration on the Quadrilateral	234
11.3	Numerical Integration on the Hexahedron	235
11.4	Types of Integrals Required	236
12	2D Continuous Galerkin Methods for Elliptic Equations	237
12.1	Introduction	237
12.2	Problem Statement for the Elliptic Equation	238
12.3	Integral Form	238
12.4	Basis Functions and the Reference Element	239
12.5	Basis Functions in Two Dimensions	239
12.6	Basis Functions in Three Dimensions	240
12.7	Metric Terms of the Mapping	241
12.7.1	Metric Terms in Two Dimensions	241
12.7.2	Metric Terms in Three Dimensions	245
12.7.3	Normal Vectors	249
12.7.4	Algorithm for the Metric Terms	250
12.8	Element Equations on a Single Element	255
12.8.1	Integration by Parts for the Diffusion Operator	255
12.8.2	Matrix-Vector Problem Resulting from Exact Integration ..	255
12.8.3	Matrix-Vector Problem Resulting from Inexact Integration ..	256
12.8.4	Algorithms for the Element Matrices	257
12.9	Global Matrix-Vector Problem	260
12.9.1	Direct Stiffness Summation	260
12.9.2	Boundary Condition	263
12.10	Second Order Operators without Integration by Parts	264
12.11	Example of 2D CG for Linear Elements	265
12.11.1	2D Basis Functions on Quadrilaterals	265
12.11.2	Metric Terms	266
12.11.3	Derivatives in Physical Space	267
12.11.4	Laplacian Matrix	268
12.11.5	Mass Matrix	269
12.11.6	Matrix Equations on the Reference Element	270
12.11.7	Difference Equation for the Laplacian Operator	270
12.12	2D Elliptic Equation	274
12.12.1	Algorithm for the 2D Elliptic Equation	276
12.13	Analysis of the Matrix Properties of the Spatial Operators	277
12.13.1	Sparsity Pattern of the Mass Matrix	277
12.13.2	Sparsity Pattern of the Laplacian Matrix	277

12.13.3 Eigenvalue Analysis of Spatial Operator	278
12.14 Example of 2D Poisson Equation	279
12.14.1 Error Norm	279
12.14.2 Solution Accuracy	280
12.15 Computational Cost of High-Order	281
12.15.1 Solution Accuracy	281
13 2D Discontinuous Galerkin Methods for Elliptic Equations	285
13.1 Introduction	285
13.2 2D Elliptic Equation	285
13.3 Weak Integral Form	286
13.4 Basis Functions and the Reference Element	287
13.5 Element Equations on a Single Element	287
13.5.1 First Step: Evaluating the Auxiliary Variable	288
13.5.2 Second Step: Evaluate the Poisson Problem	289
13.6 Solution Strategy	290
13.6.1 Approach I	290
13.6.2 Approach II	292
13.7 Algorithm for LDG	293
13.7.1 Element Differentiation Matrix	293
13.7.2 Global Flux Matrix	295
13.8 Analysis of the Matrix Properties of the Spatial Operators	295
13.8.1 Sparsity Pattern of the Mass Matrix	295
13.8.2 Sparsity Pattern of the Laplacian Matrix	297
13.8.3 Eigenvalue Analysis of Spatial Operator	297
13.9 Example of 2D Poisson Equation	298
13.9.1 Solution Accuracy	298
14 2D Unified Continuous and Discontinuous Galerkin Methods for Elliptic Equations	301
14.1 Introduction	301
14.2 Elliptic Equation	301
14.3 Flux Formulation	302
14.4 Primal Formulation	303
14.5 Symmetric Interior Penalty Galerkin (SIPG) Method	304
14.6 Algorithm for the SIPG Method	305
14.7 Analysis of the Matrix Properties of the Spatial Operators	306
14.7.1 Sparsity Pattern of the Laplacian Matrix	307
14.7.2 Eigenvalue Analysis of Spatial Operator	308
14.8 Example of 2D Poisson Equation	308
14.8.1 Solution Accuracy	308

15	2D Continuous Galerkin Methods for Hyperbolic Equations	311
15.1	Introduction	311
15.2	2D Advection-Diffusion Equation	311
15.3	Integral Form	312
15.4	Element Equations on the Reference Element	312
15.4.1	Integration by Parts for the Diffusion Operator	313
15.4.2	Matrix-Vector Problem Resulting from Exact Integration	314
15.4.3	Matrix-Vector Problem Resulting from Inexact Integration	315
15.5	Global Matrix-Vector Problem	316
15.6	Example of 2D CG for Linear Elements	318
15.6.1	2D Basis Functions	318
15.6.2	Metric Terms	319
15.6.3	Derivatives in Physical Space	319
15.6.4	Mass and Laplacian Matrices	320
15.6.5	Advection Matrix	320
15.6.6	Matrix Equations on the Reference Element	322
15.7	Algorithms for the CG Global Matrix-Vector Problem	323
15.7.1	Non-Tensor-Product Approach	323
15.7.2	Tensor-Product Approach	326
15.8	Example of 2D Hyperbolic Equation Problem	329
15.8.1	Solution Accuracy	330
15.8.2	Computational Cost of High-Order	330
16	2D Discontinuous Galerkin Methods for Hyperbolic Equations	333
16.1	Introduction	333
16.2	2D Advection-Diffusion Equation	333
16.3	Integral Form	334
16.4	Basis Functions and the Reference Element	335
16.5	Element Equations on a Single Element: Weak Form	336
16.5.1	Local Discontinuous Galerkin Method for the Diffusion Operator	336
16.5.2	Matrix-Vector Problem Resulting from Exact Integration	337
16.5.3	Matrix-Vector Problem Resulting from Inexact Integration	338
16.6	Element Equations on a Single Element: Strong Form	339
16.6.1	Matrix-Vector Problem Resulting from Exact Integration	340
16.6.2	Matrix-Vector Problem Resulting from Inexact Integration	341
16.7	Example of 2D DG for Linear Elements	342
16.7.1	2D Basis Functions	342
16.7.2	Metric Terms	342
16.7.3	Derivatives in Physical Space	343
16.7.4	Mass Matrix	343
16.7.5	Differentiation Matrix	343
16.7.6	Flux Matrix	344
16.7.7	Numerical Flux Function	347
16.8	Algorithms for the DG Matrix-Vector Problem	349

16.8.1 Non-Tensor-Product Approach	349
16.8.2 Tensor-Product Approach	356
16.9 Example of 2D Hyperbolic Equation Problem	360
16.9.1 Error Norm	360
16.9.2 Lobatto Points	360
16.9.3 Legendre Points	362
17 2D Continuous/Discontinuous Galerkin Methods for Hyperbolic Equations	363
17.1 Introduction	363
17.2 2D Advection-Diffusion Equation	363
17.3 Weak Integral Form	364
17.4 2D Basis Functions and the Reference Element	365
17.5 Element Equations on a Single Element: Weak Form	365
17.5.1 Matrix-Vector Problem Resulting from Exact Integration ...	365
17.5.2 Matrix-Vector Problem Resulting from Inexact Integration ..	367
17.6 Element Equations on a Single Element: Strong Form	368
17.6.1 Matrix-Vector Problem Resulting from Exact Integration ...	368
17.6.2 Matrix-Vector Problem Resulting from Inexact Integration ..	369
17.7 Algorithms for a Unified CG/DG Matrix-Vector Problem	370
17.7.1 First Order Operator	370
17.7.2 Second Order Operator	372
17.7.3 Construction of the Global Matrix Problem	376
17.8 Example of 2D Hyperbolic Equation Problem	376
 Part IV Advanced Topics	
18 Stabilization of High-Order Methods	383
18.1 Introduction	383
18.2 Aliasing Error	384
18.3 Spectral Filters	388
18.4 Riemann Solvers and Upwinding	392
18.5 Limiters	394
18.5.1 Stabilization Limiters	394
18.5.2 Positivity-Preserving Limiters	399
18.6 Local Adaptive Viscosity Methods	403
18.6.1 Diffusion Operators	403
18.6.2 Streamline Upwind Petrov-Galerkin	407
18.6.3 Variational Multi-Scale Method	410
18.6.4 Dynamic Sub-Grid Scales	412
18.7 Provably Stable Methods	414
18.7.1 Classical DG Solution of the 1D Burgers Equation	415
18.7.2 Energy Stable Solution of the 1D Burgers Equation	416
18.7.3 Entropy Stable Methods	418

19	Adaptive Mesh Refinement	421
19.1	Introduction	421
19.2	Conforming vs non-conforming mesh	422
19.3	H-Refinement Method	423
19.3.1	Data Structures	424
19.3.2	H-Refinement Algorithm	425
19.3.3	Gather and Scatter Matrices	427
19.3.4	H-Refinement Example	430
19.4	P-Refinement Method	431
19.4.1	P-Refinement for Modal DG	432
19.4.2	P-Refinement for Nodal CG and DG	433
19.4.3	Additional Matrices for Nodal P-Refinement	433
19.5	R-Refinement Method	435
19.6	H-Refinement Example in 2D	436
19.6.1	Quadtree Data Structure	437
19.6.2	Space filling curve and data storage	437
19.6.3	2:1 balance	438
19.6.4	Handling of non-conforming faces for DG	439
19.6.5	Handling of non-conforming faces for CG	442
19.6.6	2D Test Problem	447
20	Time Integration	449
20.1	Introduction	449
20.2	Explicit Methods	450
20.2.1	Single-step Multi-stage Methods	450
20.2.2	Multi-step Methods	452
20.3	Fully-Implicit Methods	452
20.3.1	Single-step Multi-stage Methods	454
20.3.2	Multi-step Methods	455
20.4	Implicit-Explicit (IMEX) Methods	457
20.4.1	Single-step Multi-stage Methods	457
20.4.2	Multi-step Methods	458
20.4.3	Stability of IMEX Methods	459
20.5	Semi-Lagrangian Methods	460
20.5.1	Non-Conservative Semi-Lagrangian Method	460
20.5.2	Conservative Semi-Lagrangian Method	464
20.5.3	Semi-Lagrangian Example	466
20.5.4	Semi-Lagrangian Method for Balance Laws	467
20.6	Multirate Methods	468
20.6.1	Multirate Infinitesimal Step	468
20.6.2	Convergence Rate	470
20.6.3	Speedup	471

21	1D Hybridizable Discontinuous Galerkin Method	475
21.1	Introduction	475
21.2	Implicit Time-Integration for CG/DG	476
21.2.1	Continuous and Discontinuous Galerkin Discretizations	477
21.2.2	Backward Euler	480
21.2.3	Runge-Kutta Method	481
21.3	Implicit Time-Integration for HDG	482
21.3.1	Spatial Discretization with HDG	482
21.3.2	HDG Solution Procedure	488
21.4	HDG Example	494
21.5	Complexity Analysis of HDG	495
21.5.1	HDG versus CGc	496
21.5.2	HDG versus DG	497
21.5.3	Degrees of Freedom as a Function of N and N_e	497
21.5.4	Operation Count as a Function of N and N_e	498
A	Classification of Partial Differential Equations	501
A.1	Scalar Second Order Equation	501
B	Jacobi Polynomials	505
B.1	Chebyshev Polynomials	505
B.2	Legendre Polynomials	506
B.3	Lobatto Polynomials	507
C	Boundary Conditions	509
C.1	Transmissive Boundary Condition	510
C.2	Inflow/Outflow Boundary Condition	510
C.3	Impermeable Wall Boundary Condition	510
C.4	Gradient Boundary Condition	511
C.5	Summary	511
D	Contravariant Form	513
D.1	Summary	516
	References	519
	Index	541