Extension of Pascal's Rule and Higher Order Total Differential Equations using the Pascal's Triangle

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Framework

The Pascal's Rule is a combinatoric rule about binomial coefficients which states that $\forall n, r \in \mathbb{N}$:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \tag{1}$$

$$\text{where } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This rule served as the basic for the development of the Pascal triangle which has diverse applications in solving complex problems in mathematics. However, further exploration of the pascal triangle shows that:

$$\binom{n}{r} = \sum_{l=0}^{k} \binom{k}{l} \binom{n-k}{r-k+l} \tag{2}$$

$$\text{where } n-k \geq r,\ r \geq k,\ \text{n, r, k, l} \in \mathbb{N}$$

Proof.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\Rightarrow \binom{n}{r} = \binom{n-2}{r-2} + \binom{n-2}{r-1} + \binom{n-2}{r-1} + \binom{n-2}{r}$$

$$\therefore \binom{n}{r} = \binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r}$$

Further extrapolation of eqn (3) and subsequent equations shows that:

$$\binom{n}{r} = \binom{n-3}{r-3} + 3\binom{n-3}{r-2} + 3\binom{n-3}{r-1} + \binom{n-3}{r}$$

$$\binom{n}{r} = \binom{n-4}{r-4} + 4\binom{n-4}{r-3} + 6\binom{n-4}{r-2} + 4\binom{n-4}{r-1} + \binom{n-4}{r}$$

$$\binom{n}{r} = \binom{n-5}{r-5} + 5\binom{n-5}{r-4} + 10\binom{n-5}{r-3} + 10\binom{n-5}{r-2} + 5\binom{n-5}{r-1} + \binom{n-5}{r}$$

$$\binom{n}{r} = \binom{n-6}{r-6} + 6\binom{n-6}{r-5} + 15\binom{n-6}{r-4} + 20\binom{n-6}{r-3} + 15\binom{n-6}{r-2} + 6\binom{n-6}{r-1} + \binom{n-6}{r}$$

$$\vdots$$

$$\binom{n}{r} = a_0\binom{n-k}{r-k} + a_1\binom{n-k}{r-k+1} + a_2\binom{n-k}{r-k+2} + \cdots + a_2\binom{n-k}{r-2} + a_1\binom{n-k}{r-1} + a_0\binom{n-k}{r}$$

$$\text{where } a_l \, (0 \leq l \leq k) = \binom{k}{l}$$

$$\Rightarrow \binom{n}{r} = \sum_{l=0}^{k} \binom{k}{l}\binom{n-k}{r-k+l}$$

$$\text{where } n-k \geq r \text{ and } r \geq k$$

□

This also imply that:

$$\binom{k}{l} = \sum_{n=q}^{p} \binom{p}{q}\binom{k-p}{l-p+q}$$

$$\text{where } k-p \geq l \text{ and } l \geq p$$

$$\therefore \binom{n}{r} = \sum_{l=0}^{k}\left(\sum_{n=q}^{p}\binom{p}{q}\binom{k-p}{l-p+q}\right)\binom{n-k}{r-k+l}$$

$$\binom{n}{r} = \sum_{l=0}^{k}\sum_{n=q}^{p}\binom{p}{q}\binom{k-p}{l-p+q}\binom{n-k}{r-k+l} \tag{3}$$

$$\text{where } n-k \geq r,\ r \geq k,\ k-p \geq l, \text{ and } l \geq p$$

$$\text{If } \binom{p}{q} = \sum_{t=0}^{s}\binom{s}{t}\binom{p-s}{q-s+t}$$

$$\text{where } p-s \geq q \text{ and } q \geq s$$

$$\therefore \binom{n}{r} = \sum_{l=0}^{k}\sum_{n=q}^{p}\sum_{t=0}^{s}\binom{s}{t}\binom{p-s}{q-s+t}\binom{k-p}{l-p+q}\binom{n-k}{r-k+l} \tag{4}$$

This can also be further extended to obtain other equations of $\binom{n}{r}$. The Pascal triangle can also help us obtain the right formulae for higher order Total Differential equation. According to the Total Differencial formular, if

$$\omega = f(x, y)$$

$$\text{Then } d\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

$$\begin{aligned}
d^2\omega = d(d\omega) &= \frac{\partial(d\omega)}{\partial x}dx + \frac{\partial(d\omega)}{\partial y}dy \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dy \\
&\frac{\partial(dx)}{\partial x} = 1,\ \frac{\partial(dy)}{\partial x} = 0,\ \frac{\partial(dx)}{\partial y} = 0,\ \frac{\partial(dy)}{\partial x} = 1 \\
&= \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx + \frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial y}dy \\
d^2\omega &= \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy
\end{aligned}$$

$$\text{Let } \delta^n f = \sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n f}{\partial x^k\partial y^{n-k}}dx^k dy^{n-k} \tag{5}$$

$$\text{then } \delta^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 \text{ and } \delta f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\therefore d^2\omega = \delta^2 f + \delta f$$

$$\text{Also, } d^3\omega = d(d^2\omega) = \frac{\partial(d^2\omega)}{\partial x}dx + \frac{\partial(d^2\omega)}{\partial y}dy$$

$$\begin{aligned}
\frac{\partial(d^2\omega)}{\partial x}dx =& \frac{\partial^3 f}{\partial x^3}dx^3 + 2\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^3 f}{\partial x^2\partial y}dx^2dy + 2\frac{\partial^2 f}{\partial x\partial y}dxdy \\
&+ \frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx + \frac{\partial^2 f}{\partial x\partial y}dxdy
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(d^2\omega)}{\partial y}dy =& \frac{\partial^3 f}{\partial x^2\partial y}dx^2dy + 2\frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^3 f}{\partial y^3}dy^3 \\
&+ 2\frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial y}dy
\end{aligned}$$

$$\begin{aligned}
\therefore d^3\omega =& \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 \\
&+ 3\frac{\partial^2 f}{\partial x^2}dx^2 + 6\frac{\partial^2 f}{\partial x\partial y}dxdy + 3\frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy
\end{aligned}$$

$$
\begin{aligned}
d^3\omega =& \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dx dy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 \\
&+ 3\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx dy + \frac{\partial^2 f}{\partial y^2}dy^2\right) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy
\end{aligned}
$$

$$
\text{From (5), } \delta^3 f = \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dx dy^2 + \frac{\partial^3 f}{\partial y^3}dy^3
$$

$$
\therefore d^3\omega = \delta^3 f + 3\ \delta^2 f + \delta f
$$

The coefficients of $\delta^n f$ follows the pascal triangle and one can find the coefficient for $d^n\omega$ in the equation by finding the coefficient of $\frac{\partial^n f}{\partial x^n}dx^n$ and then extending it to other coefficients using the Pascal's triangle.

$$
\begin{gathered}
\text{For } d^4\omega \\
\frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial x^3}dx^3\right)dx = \frac{\partial^4 f}{\partial x^4}dx^4 + 3\frac{\partial^3 f}{\partial x^3}dx^3 \\
3\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}dx^2\right)dx = 3\frac{\partial^3 f}{\partial x^3}dx^3 + 6\frac{\partial^2 f}{\partial x^2}dx^2 \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx\right)dx = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx \\
\text{sum } = \frac{\partial^4 f}{\partial x^4}dx^4,\ 6\frac{\partial^3 f}{\partial x^3}dx^3,\ 7\frac{\partial^2 f}{\partial x^2}dx^2,\ \frac{\partial f}{\partial x}dx
\end{gathered}
$$

$$
\therefore d^4\omega = \delta^4 f + 6\ \delta^3 f + 7\ \delta^2 f + \delta f
$$

$$
\begin{gathered}
\text{For } d^5\omega, \\
\frac{\partial}{\partial x}\left(\frac{\partial^4 f}{\partial x^4}dx^4\right)dx = \frac{\partial^5 f}{\partial x^5}dx^5 + 4\frac{\partial^4 f}{\partial x^4}dx^4 \\
6\frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial x^3}dx^3\right)dx = 6\frac{\partial^4 f}{\partial x^4}dx^4 + 18\frac{\partial^3 f}{\partial x^3}dx^3 \\
7\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}dx^2\right)dx = 7\frac{\partial^3 f}{\partial x^3}dx^3 + 14\frac{\partial^2 f}{\partial x^2}dx^2 \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx\right)dx = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx \\
sum = \frac{\partial^5 f}{\partial x^5}dx^5, 10\ \frac{\partial^4 f}{\partial x^4}dx^4,\ 25\frac{\partial^3 f}{\partial x^3}dx^3,\ 15\frac{\partial^2 f}{\partial x^2}dx^2,\ \frac{\partial f}{\partial x}dx \\
\therefore d^5\omega = \delta^5 f + 10\ \delta^4 f + 25\ \delta^3 f + 15\ \delta^2 f + \delta f
\end{gathered}
$$

Following the trend in the expressions or the method of differentiation used above, one would realised that:

$$d^6\omega = \delta^6 f + 15\ \delta^5 f + 65\ \delta^4 f + 90\ \delta^3 f + 31\ \delta^2 f +\ \delta f$$

$$d^7\omega = \delta^7 f + 21\ \delta^6 f + 140\ \delta^5 f + 350\ \delta^4 f + 301\ \delta^3 f + 63\ \delta^2 f +\ \delta f$$

And so on

This imply that if

$$d^n\omega = a_n\ \delta^n f + a_{n-1}\ \delta^{n-1} f + a_{n-2}\ \delta^{n-2} f + a_{n-3}\ \delta^{n-3} f + \cdots + a_2\ \delta^2 f + a_1\ \delta f$$

where $n \in \mathbb{N}$, and $a_n = a_1 = 1$, then

$$\begin{aligned} d^{n+1}\omega =& \delta^{n+1} f + (na_n + a_{n-1})\ \delta^n f + ((n-1)a_{n-1} + a_{n-2})\ \delta^{n-1} f + \\ & ((n-2)a_{n-2} + a_{n-3})\ \delta^{n-2} f + ((n-3)a_{n-3} + a_{n-4})\ \delta^{n-3} f + \cdots \\ & + ((3)a_3 + a_2)\ \delta^3 f + ((2)a_2 + a_1)\ \delta^2 f +\ \delta f \end{aligned}$$

People often confuse $d^n\omega$ with $d\omega^n$ and think one can simply replace $d^n\omega$ with $d\omega^n$. However, these two notations are never the same. $d\omega^n$ is equal to $(d\omega)^n$ which means taking the nth power of $d\omega$. $d^n\omega$ on the other hand means taking the total differential of function(ω) $n - times$ or taking the nth power of the operator d. Therefore, the right formula for $d\omega^n$ should actually be:

$$d\omega^n = (d\omega)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{(\partial f)^n}{(\partial x)^k (\partial y)^{n-k}} (dx)^k (dy)^{n-k}$$

$$\Rightarrow d\omega^n = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial f^n}{\partial x^k \partial y^{n-k}} dx^k dy^{n-k} \tag{6}$$

Proof.

If $\omega = f(x,y)$

$$d\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\begin{aligned} d\omega^2 = (d\omega)^2 &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)^2 \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \\ &= \frac{\partial f}{\partial x} dx \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) + \frac{\partial f}{\partial y} dy \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \\ &= \frac{(\partial f)^2}{(\partial x)^2} (dx)^2 + \frac{(\partial f)^2}{(\partial x)(\partial y)} (dx)(dy) + \frac{(\partial f)^2}{(\partial x)(\partial y)} (dx)(dy) + \frac{(\partial f)^2}{(\partial y)^2} (dy)^2 \end{aligned}$$

$$d\omega^2 = \frac{\partial f^2}{\partial x^2} dx^2 + 2\frac{\partial f^2}{\partial x \partial y} dx dy + \frac{\partial f^2}{\partial y^2} dy^2$$

$$d\omega^3 = (d\omega)^3 = d\omega(d\omega)^2$$
$$= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)\left(\frac{\partial f^2}{\partial x^2}dx^2 + 2\frac{\partial f^2}{\partial x \partial y}dxdy + \frac{\partial f^2}{\partial y^2}dy^2\right)$$
$$d\omega^3 = \frac{\partial f^3}{\partial x^3}dx^3 + 3\frac{\partial f^3}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial f^2}{\partial x \partial y^2}dxdy^2 + \frac{\partial f^3}{\partial y^3}dy^3$$

$$d\omega^4 = (d\omega)^4 = d\omega(d\omega)^3$$
$$= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)\left(\frac{\partial f^3}{\partial x^3}dx^3 + 3\frac{\partial f^3}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial f^3}{\partial x \partial y^2}dxdy^2 + \frac{\partial f^3}{\partial y^3}dy^3\right)$$
$$d\omega^4 = \frac{\partial f^4}{\partial x^4}dx^4 + 4\frac{\partial f^4}{\partial x^3 \partial y}dx^3dy + 6\frac{\partial f^4}{\partial x^2 \partial y^2}dx^2dy^2 + 4\frac{\partial f^4}{\partial x \partial y^3}dxdy^3 + \frac{\partial f^4}{\partial y^4}dy^4$$
$$\vdots$$
$$\Rightarrow d\omega^n = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial f^n}{\partial x^k \partial y^{n-k}} dx^k dy^{n-k}$$

□

Regardless, for any finite number of independent variables, $\delta^n f$ is equal to expansion of $(d\omega)^n$ where ∂f^n has been replaced by $d^n f$. Also $d^n\omega = \delta^n f$ if:

$$\frac{\partial(dx)}{\partial x} = 0,\ \frac{\partial(dy)}{\partial x} = 0, \cdots\ \frac{\partial(dx)}{\partial y} = 0,\ \frac{\partial(dy)}{\partial y} = 0, \cdots$$

Proof.

$$\text{For } \omega = f(x, y)$$
$$d^2\omega = d(d\omega) = \frac{\partial(d\omega)}{\partial x}dx + \frac{\partial(d\omega)}{\partial y}dy$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dy$$
$$\text{If } \frac{\partial(dx)}{\partial x} = 0,\ \frac{\partial(dy)}{\partial x} = 0,\ \frac{\partial(dx)}{\partial y} = 0,\ \frac{\partial(dy)}{\partial x} = 0 \text{ then:}$$

$$\begin{aligned} d^2\omega &= \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 \\ &= \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 \\ &= \delta^2 f \end{aligned}$$

$$\text{Also, } d^3\omega = d(d^2\omega) = \frac{\partial(d^2\omega)}{\partial x}dx + \frac{\partial(d^2\omega)}{\partial y}dy$$

$$\begin{aligned}\therefore d^3\omega =& \frac{\partial^3 f}{\partial x^3}dx^3 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + \frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial x^2\partial y}dx^2dy \\ &+ 2\frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 \\ =& \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 \\ =& \ \delta^3 f\end{aligned}$$

$$\begin{aligned}\text{Similarly, } d^4\omega =& \frac{\partial^4 f}{\partial x^4}dx^4 + 3\frac{\partial^4 f}{\partial x^3\partial y}dx^3dy + 3\frac{\partial^4 f}{\partial x^2\partial y^2}dx^2dy^2 + \frac{\partial^4 f}{\partial x\partial y^3}dxdy^3 \\ &+ \frac{\partial^4 f}{\partial x^3\partial y}dx^3dy + 3\frac{\partial^4 f}{\partial x^2\partial y^2}dx^2dy^2 + 3\frac{\partial^4 f}{\partial x\partial y^3}dxdy^3 + \frac{\partial^4 f}{\partial y^4}dy^4 \\ =& \frac{\partial^4 f}{\partial x^4}dx^4 + 4\frac{\partial^4 f}{\partial x^3\partial y}dx^3dy + 6\frac{\partial^4 f}{\partial x^2\partial y^2}dx^2dy^2 + 4\frac{\partial^4 f}{\partial x\partial y^3}dxdy^3 + \frac{\partial^4 f}{\partial y^4}dy^4 \\ =& \ \delta^4 f\end{aligned}$$

$$\begin{aligned}d^5\omega =& \frac{\partial^5 f}{\partial x^5}dx^5 + 4\frac{\partial^5 f}{\partial x^4\partial y}dx^4dy + 6\frac{\partial^5 f}{\partial x^3\partial y^2}dx^3dy^2 + 4\frac{\partial^5 f}{\partial x^2\partial y^3}dx^2dy^3 + \frac{\partial^5 f}{\partial x\partial y^4}dxdy^4 \\ &+ \frac{\partial^5 f}{\partial x^4\partial y}dx^4dy + 4\frac{\partial^5 f}{\partial x^3\partial y^2}dx^3dy^2 + 6\frac{\partial^5 f}{\partial x^2\partial y^3}dx^2dy^3 + 4\frac{\partial^5 f}{\partial x\partial y^4}dxdy^4 + \frac{\partial^5 f}{\partial y^5}dy^5 \\ =& \frac{\partial^5 f}{\partial x^5}dx^5 + 5\frac{\partial^5 f}{\partial x^4\partial y}dx^4dy + 10\frac{\partial^5 f}{\partial x^3\partial y^2}dx^3dy^2 + 10\frac{\partial^5 f}{\partial x^2\partial y^3}dx^2dy^3 + 5\frac{\partial^5 f}{\partial x\partial y^4}dxdy^4 \\ &+ \frac{\partial^5 f}{\partial y^5}dy^5 \\ =& \ \delta^5 f\end{aligned}$$

$$\vdots$$

$$\text{For } \omega = f(x,y,z)$$

$$d\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \text{ and}$$

$$\text{If } \frac{\partial(dx)}{\partial x} = \frac{\partial(dy)}{\partial y} = \frac{\partial(dz)}{\partial z} = \frac{\partial(dy)}{\partial x} = \frac{\partial(dz)}{\partial x}$$

$$= \frac{\partial(dx)}{\partial y} = \frac{\partial(dz)}{\partial y} = \frac{\partial(dx)}{\partial z} = \frac{\partial(dy)}{\partial z} = 0,$$

then

$$\begin{aligned}d^2\omega =& \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial x\partial z}dxdz + \frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial^2 f}{\partial y\partial z}dydz \\ &+ \frac{\partial^2 f}{\partial x\partial z}dxdz + \frac{\partial^2 f}{\partial y\partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2 \\ =& \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + 2\frac{\partial^2 f}{\partial x\partial z}dxdz + 2\frac{\partial^2 f}{\partial y\partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2 \\ =& \delta^2 f\end{aligned}$$

$$
\begin{aligned}
d^3\omega =& \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 + 3\frac{\partial^3 f}{\partial x^2 \partial z}dx^2dz \\
&+ 6\frac{\partial^3 f}{\partial x \partial y \partial z}dxdydz + 3\frac{\partial^3 f}{\partial y^2 \partial z}dy^2dz + 3\frac{\partial^3 f}{\partial x \partial z^2}dxdz^2 + 3\frac{\partial^3 f}{\partial y \partial z^2}dydz^2 + \frac{\partial^3 f}{\partial z^3}dz^3 \\
=& \delta^3 f
\end{aligned}
$$

$$
\begin{aligned}
d^4\omega =& \frac{\partial^4 f}{\partial x^4}dx^4 + 4\frac{\partial^4 f}{\partial x^3 \partial y}dx^3dy + 6\frac{\partial^4 f}{\partial x^2 \partial y^2}dx^2dy^2 + 4\frac{\partial^4 f}{\partial x \partial y^3}dxdy^3 + \frac{\partial^4 f}{\partial y^4}dy^4 + 4\frac{\partial^4 f}{\partial x^3 \partial z}dx^3dz \\
&+ 12\frac{\partial^4 f}{\partial x^2 \partial y \partial z}dx^2dydz + 12\frac{\partial^4 f}{\partial x \partial y^2 \partial z}dxdy^2dz + 4\frac{\partial^4 f}{\partial y^3 \partial z}dy^3dz + 6\frac{\partial^4 f}{\partial x^2 \partial z^2}dx^2dz^2 \\
&+ 12\frac{\partial^4 f}{\partial x \partial y \partial z^2}dxdydz^2 + 6\frac{\partial^4 f}{\partial y^2 \partial z^2}dy^2dz^2 + 4\frac{\partial^4 f}{\partial x \partial z^3}dxdz^3 + 4\frac{\partial^4 f}{\partial y \partial z^3}dydz^3 + \frac{\partial^4 f}{\partial z^4}dz^4 \\
=& \delta^4 f
\end{aligned}
$$

$$\vdots$$

And so on.

$\square$

If $\omega = f(x, y, z)$,

$$d\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \text{ and}$$

$$\frac{\partial (dx)}{\partial x} = \frac{\partial (dy)}{\partial y} = \frac{\partial (dz)}{\partial z} = 1, \; \frac{\partial (dy)}{\partial x} = \frac{\partial (dz)}{\partial x} =$$

$$\frac{\partial (dx)}{\partial y} = \frac{\partial (dz)}{\partial y} = \frac{\partial (dx)}{\partial z} = \frac{\partial (dy)}{\partial z} = 0,$$

then

$$d^2\omega = d(d\omega) = \frac{\partial (d\omega)}{\partial x}dx + \frac{\partial (d\omega)}{\partial y}dy + \frac{\partial (d\omega)}{\partial z}dz$$

$$
\begin{aligned}
\frac{\partial (d\omega)}{\partial x}dx =& \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)dx \\
=& \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx + \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial x \partial z}dxdz \\
\frac{\partial (d\omega)}{\partial y}dy =& \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)dy \\
=& \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial y}dy + \frac{\partial^2 f}{\partial y \partial z}dydz
\end{aligned}
$$

$$\begin{aligned}\frac{\partial(d\omega)}{\partial z}dz =& \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)dz\\ =& \frac{\partial^2 f}{\partial x \partial z}dxdz + \frac{\partial^2 f}{\partial y \partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2 + \frac{\partial f}{\partial z}dz\end{aligned}$$

$$\begin{aligned}\therefore d^2\omega =& \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + 2\frac{\partial^2 f}{\partial x \partial z}dxdz + 2\frac{\partial^2 f}{\partial y \partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2\\ &+ \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\\ \delta^2 f =& \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + 2\frac{\partial^2 f}{\partial x \partial z}dxdz + 2\frac{\partial^2 f}{\partial y \partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2\\ \text{and } \delta f =& \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\end{aligned}$$

$$\therefore d^2\omega = \delta^2 f + \delta f$$

$$d^3\omega = d(d^2\omega) = \frac{\partial(d^2\omega)}{\partial x}dx + \frac{\partial(d^2\omega)}{\partial y}dy + \frac{\partial(d^2\omega)}{\partial z}dz$$

$$\begin{aligned}\frac{\partial(d^2\omega)}{\partial x}dx =& \frac{\partial^3 f}{\partial x^3}dx^3 + 2\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^3 f}{\partial x \partial y^2}dxdy^2\\ &+ 2\frac{\partial^3 f}{\partial x^2 \partial z}dx^2dz + 2\frac{\partial^2 f}{\partial x \partial z}dxdz + 2\frac{\partial^3 f}{\partial x \partial y \partial z}dxdydz + \frac{\partial^3 f}{\partial x \partial z^2}dxdz^2 + \frac{\partial^2 f}{\partial x^2}dx^2\\ &+ \frac{\partial f}{\partial x}dx + \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial x \partial z}dxdz\\ =& \frac{\partial^3 f}{\partial x^3}dx^3 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + \frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + 2\frac{\partial^3 f}{\partial x^2 \partial z}dx^2dz + 2\frac{\partial^3 f}{\partial x \partial y \partial z}dxdydz\\ &+ \frac{\partial^3 f}{\partial x \partial z^2}dxdz^2 + 3\frac{\partial^2 f}{\partial x^2}dx^2 + 3\frac{\partial^2 f}{\partial x \partial y}dxdy + 3\frac{\partial^2 f}{\partial x \partial z}dxdz + \frac{\partial f}{\partial x}dx\end{aligned}$$

$$\begin{aligned}\frac{\partial(d^2\omega)}{\partial y}dy =& \frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 2\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^3 f}{\partial y^3}dy^3 + 2\frac{\partial^2 f}{\partial y^2}dy^2\\ &+ 2\frac{\partial^3 f}{\partial x \partial y \partial z}dxdydz + 2\frac{\partial^3 f}{\partial y^2 \partial z}dy^2dz + 2\frac{\partial^2 f}{\partial y \partial z}dydz + \frac{\partial^3 f}{\partial y \partial z^2}dydz^2 + \frac{\partial^2 f}{\partial x \partial y}dxdy\\ &+ \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial y}dy + \frac{\partial^2 f}{\partial y \partial z}dydz\\ =& \frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 2\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 + 2\frac{\partial^3 f}{\partial x \partial y \partial z}dxdydz + 2\frac{\partial^3 f}{\partial y^2 \partial z}dy^2dz\\ &+ \frac{\partial^3 f}{\partial y \partial z^2}dydz^2 + 3\frac{\partial^2 f}{\partial x \partial y}dxdy + 3\frac{\partial^2 f}{\partial y^2}dy^2 + 3\frac{\partial^2 f}{\partial y \partial z}dydz + \frac{\partial f}{\partial y}dy\end{aligned}$$

$$\begin{aligned}
\frac{\partial(d^2\omega)}{\partial z}dz =& \frac{\partial^3 f}{\partial x^2\partial z}dx^2dz + 2\frac{\partial^3 f}{\partial x\partial y\partial z}dxdydz + \frac{\partial^3 f}{\partial y^2\partial z}dy^2dz + 2\frac{\partial^3 f}{\partial x\partial z^2}dxdz^2 + 2\frac{\partial^2 f}{\partial x\partial z}dxdz \\
&+ 2\frac{\partial^3 f}{\partial y\partial z^2}dydz^2 + 2\frac{\partial^2 f}{\partial y\partial z}dydz + \frac{\partial^3 f}{\partial z^3}dz^3 + 2\frac{\partial^2 f}{\partial z^2}dz^2 + \frac{\partial^2 f}{\partial x\partial z}dxdz + \frac{\partial^2 f}{\partial y\partial z}dydz \\
&+ \frac{\partial^2 f}{\partial z^2}dz^2 + \frac{\partial f}{\partial z}dz \\
=& \frac{\partial^3 f}{\partial x^2\partial z}dx^2dz + 2\frac{\partial^3 f}{\partial x\partial y\partial z}dxdydz + \frac{\partial^3 f}{\partial y^2\partial z}dy^2dz + 2\frac{\partial^3 f}{\partial x\partial z^2}dxdz^2 + 2\frac{\partial^3 f}{\partial y\partial z^2}dydz^2 \\
&+ \frac{\partial^3 f}{\partial z^3}dz^3 + 3\frac{\partial^2 f}{\partial x\partial z}dxdz + 3\frac{\partial^2 f}{\partial y\partial z}dydz + 3\frac{\partial^2 f}{\partial z^2}dz^2 + \frac{\partial f}{\partial z}dz
\end{aligned}$$

$$\begin{aligned}
\therefore d^3\omega =& \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 + 3\frac{\partial^3 f}{\partial x^2\partial z}dx^2dz + 6\frac{\partial^3 f}{\partial x\partial y\partial z}dxdydz \\
&+ 3\frac{\partial^3 f}{\partial y^2\partial z}dy^2dz + 3\frac{\partial^3 f}{\partial x\partial z^2}dxdz^2 + 3\frac{\partial^3 f}{\partial y\partial z^2}dydz^2 + \frac{\partial^3 f}{\partial z^3}dz^3 + 3\frac{\partial^2 f}{\partial x^2}dx^2 + 6\frac{\partial^2 f}{\partial x\partial y}dxdy \\
&+ 3\frac{\partial^2 f}{\partial y^2}dy^2 + 6\frac{\partial^2 f}{\partial x\partial z}dxdz + 6\frac{\partial^2 f}{\partial y\partial z}dydz + 3\frac{\partial^2 f}{\partial z^2}dz^2 + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz
\end{aligned}$$

$$\begin{aligned}
=& \Bigg(\frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 + 3\frac{\partial^3 f}{\partial x^2\partial z}dx^2dz \\
&+ 6\frac{\partial^3 f}{\partial x\partial y\partial z}dxdydz + 3\frac{\partial^3 f}{\partial y^2\partial z}dy^2dz + 3\frac{\partial^3 f}{\partial x\partial z^2}dxdz^2 + 3\frac{\partial^3 f}{\partial y\partial z^2}dydz^2 + \frac{\partial^3 f}{\partial z^3}dz^3\Bigg) \\
&+ 3\Bigg(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + 2\frac{\partial^2 f}{\partial x\partial z}dxdz + 2\frac{\partial^2 f}{\partial y\partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2\Bigg) \\
&+ \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz
\end{aligned}$$

$$\therefore d^3\omega = \delta^3 f + 3\ \delta^2 f + \delta f$$

Expanding the equation this way is quit tedious and tasking. However, we can just find the coefficients of $\frac{\partial^n f}{\partial x^n}dx^n$, just like we did earlier for the two independent variable equation, from the coefficients of $d^{(n-1)}\omega$ then extend the full expression using the pascals triangle or multinomial expansion.

Since $d^2\omega = \delta^2 f + \delta f$,

coefficients of $\dfrac{\partial^2 f}{\partial x^2}dx^2 = \dfrac{\partial f}{\partial x}dx = 1$ for $d^2\omega$. Therefore for $d^3\omega$:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}dx^2\right)dx = \frac{\partial^3 f}{\partial x^3}dx^3 + 2\frac{\partial^2 f}{\partial x^2}dx^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx\right)dx = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx$$

By finding the sum, coefficient of

$$\frac{\partial^3 f}{\partial x^3}dx^3 = 1, \frac{\partial^2 f}{\partial x^2}dx^2 = 3, \frac{\partial f}{\partial x}dx = 1$$

$$\therefore d^3\omega = \delta^3 f + 3\ \delta^2 f + \delta f$$

Also, for $d^4\omega$

coefficients of $\frac{\partial^3 f}{\partial x^3}dx^3 = 1, \frac{\partial^2 f}{\partial x^2}dx^2 = 3, \frac{\partial f}{\partial x}dx = 1$ for $d^3\omega$. Therefore for $d^4\omega$:

$$\frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial x^3}dx^3\right)dx = \frac{\partial^4 f}{\partial x^4}dx^4 + 3\frac{\partial^3 f}{\partial x^3}dx^3$$

$$3\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}dx^2\right)dx = 3\frac{\partial^3 f}{\partial x^3}dx^3 + 6\frac{\partial^2 f}{\partial x^2}dx^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx\right)dx = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx$$

By finding the sum, coefficient of

$$\frac{\partial^4 f}{\partial x^4}dx^4 = 1,\ \frac{\partial^3 f}{\partial x^3}dx^3 = 6,\ \frac{\partial^2 f}{\partial x^2}dx^2 = 7,\ \frac{\partial f}{\partial x}dx = 1$$

$$\therefore d^4\omega = \delta^4 f + 6\ \delta^3 f + 7\ \delta^2 f +\ \delta f$$

For $d^5\omega$,

coefficients of $\frac{\partial^4 f}{\partial x^4}dx^4 = 1, \frac{\partial^3 f}{\partial x^3}dx^3 = 6, \frac{\partial^2 f}{\partial x^2}dx^2 = 7, \frac{\partial f}{\partial x}dx = 1$ for $d^4\omega$. Therefore for $d^5\omega$:

$$\frac{\partial}{\partial x}\left(\frac{\partial^4 f}{\partial x^4}dx^4\right)dx = \frac{\partial^5 f}{\partial x^5}dx^5 + 4\frac{\partial^4 f}{\partial x^4}dx^4$$

$$6\frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial x^3}dx^3\right)dx = 6\frac{\partial^4 f}{\partial x^4}dx^4 + 18\frac{\partial^3 f}{\partial x^3}dx^3$$

$$7\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}dx^2\right)dx = 7\frac{\partial^3 f}{\partial x^3}dx^3 + 14\frac{\partial^2 f}{\partial x^2}dx^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx\right)dx = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x}dx$$

By finding the sum, coefficient of

$$\frac{\partial^5 f}{\partial x^5}dx^5 = 1,\ \frac{\partial^4 f}{\partial x^4}dx^4 = 10,\ \frac{\partial^3 f}{\partial x^3}dx^3 = 25,\ \frac{\partial^2 f}{\partial x^2}dx^2 = 15,\ \frac{\partial f}{\partial x}dx = 1$$

$$\therefore d^5\omega = \delta^5 f + 10\ \delta^4 f + 25\ \delta^3 f + 15\ \delta^2 f +\ \delta f$$

And so on.

The coefficients obtained are the same coefficients that was obtained for $\omega = f(x, y)$ and will be the same coefficient for any finite number of dependent variables.

$$\therefore \text{If } \omega = f(x_1, x_2, x_3, ..., x_m) \text{ and}$$

$$d\omega = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 + \ldots + \frac{\partial f}{\partial x_m} dx_m$$

$$\text{then } d^n\omega == a_n\ \delta^n f + a_{n-1}\ \delta^{n-1} f + a_{n-2}\ \delta^{n-2} f + a_{n-3}\ \delta^{n-3} f + \cdots + a_2\ \delta^2 f + a_1\ \delta f$$

where $a_n \cdots a_1$ are the coefficients obtained from $d^{n-1}\omega$ and $a_n = a_1 = 1$.

Or as stated earlier,

$$\text{If } d^n\omega = a_n\ \delta^n f + a_{n-1}\ \delta^{n-1} f + a_{n-2}\ \delta^{n-2} f + a_{n-3}\ \delta^{n-3} f + \cdots + a_2\ \delta^2 f + a_1\ \delta f$$

where $n \in \mathbb{N}$ and $a_n = a_1 = 1$ then

$$\begin{aligned} d^{n+1}\omega =& \delta^{n+1} f + (na_n + a_{n-1})\ \delta^n f + ((n-1)a_{n-1} + a_{n-2})\ \delta^{n-1} f + \\ & ((n-2)a_{n-2} + a_{n-3})\ \delta^{n-2} f + ((n-3)a_{n-3} + a_{n-4})\ \delta^{n-3} f + \cdots \\ & + ((3)a_3 + a_2)\ \delta^3 f + ((2)a_2 + a_1)\ \delta^2 f +\ \delta f \end{aligned} \tag{7}$$

In summary, for

$$\omega = f(x_1, x_2, x_3, ..., x_m):$$

$$d\omega = \delta f$$

$$d^2\omega = \delta^2 f + \delta f$$

$$d^3\omega = \delta^3 f + 3\ \delta^2 f + \delta f$$

$$d^4\omega = \delta^4 f + 6\ \delta^3 f + 7\ \delta^2 f +\ \delta f$$

$$d^5\omega = \delta^5 f + 10\ \delta^4 f + 25\ \delta^3 f + 15\ \delta^2 f + \delta f$$

$$d^6\omega = \delta^6 f + 15\ \delta^5 f + 65\ \delta^4 f + 90\ \delta^3 f + 31\ \delta^2 f +\ \delta f$$

$$d^7\omega = \delta^7 f + 21\ \delta^6 f + 140\ \delta^5 f + 350\ \delta^4 f + 301\ \delta^3 f + 63\ \delta^2 f + \ \delta f$$

$$d^8\omega = \delta^8 f + 28\ \delta^7 f + 266\ \delta^6 f + 1050\ \delta^5 f + 1701\ \delta^4 f + 966\ \delta^3 f + 127\ \delta^2 f + \ \delta f$$

$$d^9\omega = \delta^9 f + 36\ \delta^8 f + 462\ \delta^7 f + 2646\ \delta^6 f + 6951\ \delta^5 f + 7770\ \delta^4 f + 3025\ \delta^3 f + 255\ \delta^2 f + \ \delta f$$

$$\vdots$$

The coefficients form the triangle

$$\begin{array}{c}
1\\
1\ \ 1\\
1\ \ 3\ \ 1\\
1\ \ 6\ \ 7\ \ 1\\
1\ \ 10\ \ 25\ \ 15\ \ 1\\
1\ \ 15\ \ 65\ \ 90\ \ 31\ \ 1\\
1\ \ 21\ \ 140\ \ 350\ \ 301\ \ 63\ \ 1\\
1\ \ 28\ \ 266\ \ 1050\ \ 1701\ \ 966\ \ 127\ \ 1\\
1\ \ 36\ \ 462\ \ 2646\ \ 6951\ \ 7770\ \ 3025\ \ 255\ \ 1\\
1\ \ 45\ \ 750\ \ 5880\ \ 22827\ \ 42525\ \ 34105\ \ 9330\ \ 511\ \ 1\\
\vdots
\end{array}$$

Below is a python code that gives the full nth differential equation for any n.

```python
def D_Equation(n):
    lis1=[1]
    for i in range(n):
        if i<2:
            lis1+=[1]
        else:
            lis2=[1]*(i+1)
            for j in range(i-1):
                lis2[j+1]=(i-j)*lis1[j]+lis1[j+1]
            lis1=lis2
    superscript = str.maketrans("0123456789", "\u2070\u00b9\u00b2\u00b3\u2074\u2075\u2076\u2077\u2078\u2079")
    x=chr(948)
    s=''
    s+=f'd{n}{chr(969)} ='.translate(superscript)
    s+=f' {x}{n}f'.translate(superscript)
    for j in range(1,n-1):
```

```python
        s+=f' + {lis2[j]}'
        s+=f' {x}{n-j}f'.translate(superscript)
    s+=f' + {x}f'
    print(s)
n=int(input('Enter n: '))
D_Equation(n)
```

References

[1] Jon P. Fortney (2018) *A Visual Introduction to Differential Forms and Calculus on Manifolds*, Springer Nature Switzerland AG, https://doi.org/10.1007/978-3-319-96992-3.