Fourier 级数　　$f(x) \rightarrow$ 向量空间（性质：线性、范数、内积、度量、基底分解）

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{n} (a_n \cdot \cos nx + b_n \cdot \sin nx)$$　　$\cos n\omega t$　　$\sin n\omega t$.　　2无为周期

基底：$(1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots)$　　坐标：$(a_0, a_1, b_1, a_2, b_2, \ldots)$

基底正交证明：　区间只需保证在一个周期内就行　　$\cos x$、$\sin x$ 的周期 最大的周期，不能是 $\cos 2x$ 的.

① $\int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot \cos nx \, dx = 0$　　（$\cos nx$ 在一个周期内积分为0）　n个周期同理

② $\int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot \sin nx \, dx = 0$　　（$\sin nx$ 是奇函数，对称区间积分为0，而且周期内为0）

③ $\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos nx \cdot \cos mx = ?$　　$n \neq m$

$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$.

$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$\therefore \cos A \cdot \cos B = [\cos(A+B) + \cos(A-B)] / 2$　　偶函数 $\cos(-kx) = \cos kx$，若 $n-m$ 为负.

$\therefore \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} \cdot \left(\cos(n+m)x + \cos(n-m)x \right) dx = 0$　　（$\cos kx$ 积分同上）

④ $\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin nx \cdot \sin mx = ?$　　$n \neq m$

由上可知：$\sin A \cdot \sin B = [\cos(A-B) - \cos(A+B)] / 2$

$\therefore \sin nx \cdot \sin mx = [\cos(n-m)x - \cos(n+m)x] / 2$

$\therefore \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin nx \cdot \sin mx \, dx = 0$　　（$\cos kx$ 积分同上）

⑤ $\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin nx \cdot \cos mx = ?$　　$n = m$ 也成立.

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$\therefore \sin A \cdot \cos B = [\sin(A+B) + \sin(A-B)] / 2$

$\therefore \sin nx \cdot \cos mx = [\sin(n+m)x + \sin(n-m)x] / 2$　　奇函数. $\sin(-kx) = -\sin kx$

$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin nx \cdot \cos mx = 0$　　$\sin kx$ 奇函数，当 $n-m=-k$ 为负，积分为0.

模长（范数）　　内积 $\xrightarrow{\text{诱导}}$ 范数　　$\int f(x) \cdot f(x) dx = ||f(x)||^2$

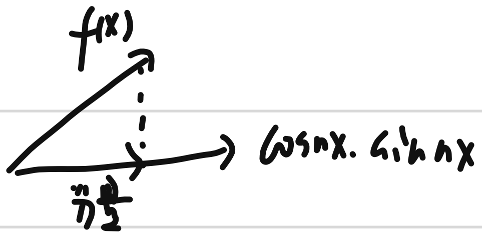
① $\int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot 1 \, dx = T$　　　$\cos kx$ 积分为0.　　1

② $\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos nx \cdot \cos nx \, dx = \frac{1}{2} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} (\cos 2nx + \cos(0x)) dx$

$$= \frac{1}{2}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} 1\, dx = \frac{T}{2}$$

③ $$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin nx\cdot\sin nx\, dx = \frac{1}{2}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} (\cos(0x) - \cos(2nx))\, dx$$
$$= \frac{T}{2}$$

系数 (a_n, b_n)



分量
$$a_n\cdot\cos nx = \frac{\langle f(x).\cos nx\rangle}{\langle\cos nx\cdot\cos nx\rangle}\cdot\cos nx$$

$$\therefore a_n = \frac{\langle f(x).\cos nx\rangle}{\langle\cos nx.\cos nx\rangle} = \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos nx\, dx}{\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos nx\cdot\cos nx\, dx}$$
$$= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\underset{\cos n\omega t}{\cos nx}\,.\,dx$$

$$b_n\cdot\sin nx = \frac{\langle f(x)\cdot\sin nx\rangle}{\langle\sin nx\cdot\sin nx\rangle}\cdot\sin nx$$

$\sin n\omega t.$

$$b_n = \frac{\langle f(x)\cdot\sin nx\rangle}{\langle\sin nx\cdot\sin nx\rangle} = \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin nx\, dx}{\int_{-\frac{T}{2}}^{\frac{T}{2}}\sin nx\cdot\sin nx\, dx} = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin nx\cdot dx$$

$$\frac{a_0}{2}\cdot 1 = \frac{\langle f(x)\cdot 1\rangle}{\langle 1\cdot 1\rangle}\cdot 1 = \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\, dx$$

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\, dx$$ → $\frac{a_0}{2}$ 统一前面系数都为 $\frac{2}{T}$

复数形式.

$$e^{ikx} = \cos kx + i\cdot\sin kx$$

$$e^{-ikx} = \cos kx - i\cdot\sin kx$$

$$\therefore \cos kx = \frac{1}{2}\cdot(e^{ikx} + e^{-ikx})$$

$$\therefore \sin kx = \frac{1}{2i}\cdot(e^{ikx} - e^{-ikx})$$

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{n}(a_n\cdot\cos nx + b_n\cdot\sin nx)$$
$$= \frac{a_0}{2} + \sum_{i=1}^{n}\left[\left(\frac{1}{2}a_n\cdot(e^{inx} + e^{-inx}\right) + \frac{1}{2i}\cdot(b_n\cdot(e^{inx} - e^{-inx})\right]$$
$$= \frac{a_0}{2} + \sum_{i=1}^{n}\cdot\left[e^{inx}\cdot\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) + e^{i(-nx)}\cdot\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)\right]$$

$\therefore C_n = \frac{a_n}{2} + \frac{b_n}{2i} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos nx\,dx - \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin nx\,dx\cdot i = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-inx}\cdot dx$

$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{inx}\,dx$

$C_{-n} = \frac{a_n}{2} - \frac{b_n}{2i} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos nx\,dx + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin nx\,dx\cdot i = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos nx + f(x)\cdot\sin nx\cdot i\,dx$

$C_n = \overline{C_{-n}}$ $\qquad = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos(-nx)\,dx - \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin(-nx)\cdot dx\cdot i$

$e^0 = 1 \qquad C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-inx}\cdot dx$

$f(x) = \sum_{-\infty}^{+\infty} C_n\cdot e^{inx} \qquad C_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx$

① $T = 2\pi$. $\quad e^{inx} = \cos nx + \sin nx\cdot i$

$e^{in(x+2\pi)} = e^{inx}\cdot e^{2\pi n\cdot i} = e^{inx}$

$e^{2\pi n\cdot i} = \cos 2\pi n + i\cdot\sin 2\pi n = 1$

$\langle e^{inx}\cdot e^{imx}\rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(n-m)x}\,dx = 0 \quad (n \neq m)$

$\qquad\qquad = T \quad (n = m)$

② $\sum e^{inx}$，求 C_n (从法出上看)

法一

$\therefore C_n = \frac{\langle f(x), e^{inx}\rangle}{\langle e^{inx}\cdot e^{inx}\rangle} = \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{inx}\,dx}{\int_{-\frac{T}{2}}^{\frac{T}{2}} 1\,dx} = \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{inx}\,dx$

法二：(直接法)

$f(x) = \sum_{-\infty}^{+\infty} C_n\cdot e^{inx} \qquad \therefore C_n = e^{-inx}\cdot f(x) - \sum_{k\neq n} C_k\cdot e^{i(k-n)x}$

$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} C_n\,dx = \frac{1}{T}\cdot\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-inx}\cdot f(x)\,dx - \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k\neq n} C_k\cdot e^{i(k-n)x}\,dx\right]$

$= \frac{1}{T}\cdot\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-inx}\,dx - \sum_{k\neq n} C_k\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(k-n)x}\,dx\right]$

积分为0.

$= \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-inx}\,dx$

积分求和互换 (Lbg 控制收敛.

① $\sum\int < \infty$

② $|C_k\cdot e^{i(k-n)x}| = |C_k| \qquad g(t) = \sum|C_k| < \infty.$

$\therefore$ 前提是 $\sum|C_n| < \infty$.

$\therefore C_n$ 必有衰减性.

法三：间接法

$$f(x)=\sum_{-\infty}^{\infty} C_n\cdot e^{inx}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-imx}dx=\int_{-\frac{T}{2}}^{\frac{T}{2}}\sum_{-\infty}^{\infty} C_n\cdot e^{i(n-m)x}dx=\sum_{-\infty}^{\infty} C_n\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(n-m)x}dx$$

当 $n\neq m$ 时 $\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(n-m)x}dx=0$　　当 $n=m$ 时

$\therefore \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-imx}dx=C_n\cdot T$　　$\therefore C_n=\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot e^{-imx}dx$

综：即如何 $\sum|C_n|$ 收敛．$f(t)$ 要如何才能满足．

① 高阶光滑性（如 C^k．C^∞）

② 解析性．（指数衰减）

③ Dini 条件．

④ S 空间

周期函数已解决．

非周期函数．

① 定义域有限，延拓定义域．

② 定义域无限 $(-\infty,+\infty)$　令 $T\to\infty$ 无限．

$$\lim_{T\to\infty} f_T(t)=f(t)$$

$|z| = (x^2+y^2)^{\frac{1}{2}}$

$|z|^2 = z\cdot\bar{z}$

$|z_n - z_m| = [(x_n - x_m)^2 + (y_n - y_m)^2]^{\frac{1}{2}}$ 范数. 有范数的所有性质

$\lim_{n\to\infty} z_n = z$ z唯一

1. 收敛 $\Leftrightarrow$ Cauchy列.

$\Rightarrow$ $z_n \to z$ $n \to +\infty$ 实部收敛. 虚部收敛.

$\therefore |z_n - z_m| \le |z_n - z| + |z - z_m| < \varepsilon$

$\Leftarrow$ 设 $\{z_n\}$ 为Cauchy列.

$|z_m - z_n| < \varepsilon$

$\therefore (a_m - a_n)^2 + (b_m - b_n)^2 < \varepsilon^2$

$\therefore |a_m - a_n| < \varepsilon$ $|b_m - b_n| < \varepsilon$

$\{a_n\}, \{b_n\}$ 为 R^1 Cauchy列.

$\therefore$ 皆收敛.

$\therefore a_n \to a$ $b_n \to b$

$\therefore |a_n - a| < \varepsilon \cdot \frac{\sqrt{2}}{2}$ $|b_n - b| < \varepsilon \cdot \frac{\sqrt{2}}{2}$ $\therefore |z - z_n| < \varepsilon$

$\therefore z_n \to z$.

2. $\sum_{n=1}^{\infty} z_n$ 收敛条件:

2.1 $|z_n| \le a_n$ a_n 为正实数列. $\sum a_n$ 收敛.

$S_N = \sum_{n=1}^{N} z_n$ $m < N$ 有限

$\therefore |S_N - S_M| = \left|\sum_{n=M+1}^{N} z_n\right| \le \sum_{n=M+1}^{N} |z_n| \le \left|\sum_{n=M+1}^{N} a_n\right| < \varepsilon$

$\therefore \{S_N\}$ 为Cauchy列 复平面完备.

$\therefore \lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} z_n = S$ 收敛.

复指数的定义.

1. $e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!}$

比值判别法.

$\frac{|z|^{n+1}}{(n+1)!} \cdot \frac{n!}{|z|^n} = \frac{|z|}{n+1} \xrightarrow{n\to\infty} 0$

$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = 0 < 1$

$\therefore e^{z}$收敛.

在闭圆盘内

$|z| \le M \Rightarrow \left|\frac{z^n}{n!}\right| \le \frac{M^n}{n!}$

回 $\frac{z^n}{n!}$ 一致收敛 $\exists\, n, m > N$

M-判别法. 正项: $\sum M_n$ 收敛 $+ |z_n| \le M_n \Rightarrow z_n$ 一致收敛.

2. 定义指数乘法. 柯西乘积. $\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$

$(a_1 + a_2 + \cdots)(b_1 + \cdots)$ 对角线相加.

$a_1 b_1 \quad a_1 b_2$

$a_2 b_1$

$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

$e^{z_1} \cdot e^{z_2} = \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right)$

$C_n = \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} z_1^k \cdot z_2^{n-k} = \frac{1}{n!} \cdot \sum_{k=0}^{n} C_n^k z_1^k z_2^{n-k}$

$= \frac{1}{n!} \cdot (z_1 + z_2)^n$ 收敛.

3. $e^{iy} = \cos y + \sin y \cdot i$

$e^{iy} = \sum_{n=0}^{\infty} \left(\frac{(iy)^n}{n!}\right) = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \cdots\cdots$

绝对收敛可改其顺序.

$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} \cdots\cdots\right) + i \cdot \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right)$

$= \cos y + i \cdot \sin y$

$\therefore$ 4. $e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y)$

$|e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x| \cdot |e^{iy}| = |e^x| \cdot |\cos y + i \cdot \sin y| = |e^x| (\cos^2 y + \sin^2 y) = e^x$

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$z_1 = (a_1 + b_1 i) \qquad z_2 = (a_2 + b_2 i)$

$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_2 b_1 + a_1 b_2) \cdot i$

$$|z_1 \cdot z_2| = \left[(a_1a_2 - b_1b_2)^2 + (a_2b_1 + a_1b_2)^2\right]^{\frac{1}{2}} = \left[a_1^2a_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + b_1^2b_2^2\right]^{\frac{1}{2}}$$

$$|z_1| \cdot |z_2| = (a_1^2 + b_1^2)^{\frac{1}{2}} \cdot (a_2^2 + b_2^2)^{\frac{1}{2}} = \left[(a_1^2 + b_1^2) \cdot (a_2^2 + b_2^2)\right]^{\frac{1}{2}}$$

5. $e^{2k\pi i} = \cos 2k\pi + \sin 2k\pi \cdot i = 1$

$$|e^{2k\pi \cdot i}| = |e^0| = 1$$

6. $z = r \cdot e^{i\theta}$ $\quad \theta = \tan \frac{y}{x}$ $\quad r = |z|$

$|z| = r = (x^2 + y^2)^{\frac{1}{2}}$ $\quad \therefore$ 令 $x = r \cdot \cos\theta$ $\quad y = r \cdot \sin\theta$.

$$i \cdot r \cdot e^{i\theta} = r \cdot e^{i(\theta + \frac{\pi}{2})} = r\left[e^{i\theta} \cdot e^{\frac{\pi}{2} i}\right] = i \cdot r \cdot e^{i\theta}$$

$$e^{\frac{\pi}{2} i} = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} \cdot i = i$$

柯西乘积

$$\left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} c_n$$

$(a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots)$

$a_1b_1 \quad a_1b_2 \quad a_1b_3$ $\qquad \therefore \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_k b_{n-k}\right)$

$a_2b_1 \quad a_2b_2$

a_3b_1

$$\therefore c_n = \sum_{k=1}^{n} a_k b_{n-k}.$$

收敛性：

若 $\sum_{n=1}^{\infty} a_n = A$ $\quad \sum_{n=1}^{\infty} b_n = B$.

$$\therefore \sum_{n=1}^{\infty} c_n = A \cdot B$$

若 $\sum_{n=1}^{\infty} c_n$ 收敛，$\sum_{n=1}^{\infty} a_n$ 与 $\sum_{n=1}^{\infty} b_n$ 不一定收敛.

① $a_n \to a$ ，则 $\frac{1}{n}\sum_{n=1}^{\infty} a_n = a$

对于 $\forall \varepsilon > 0$，$\exists N$，当 $n > N$ 时 $|a_n - a| < \varepsilon$

$$\left|\frac{1}{n} \cdot \sum_{n=1}^{n} a_n - a\right| = \left|\frac{1}{n} \cdot \sum_{n=1}^{N} a_n + \frac{1}{n} \sum_{n=N+1} a_n - a\right|$$

$\leq \frac{1}{n}\cdot\left(|a_1-a|+|a_2-a|+\cdots+|a_{N+1}-a|+\cdots\cdots\right)$

$< \frac{1}{n}\cdot(|a_1-a|+\cdots|a_N-a|)+\frac{1}{n}\cdot(n-N)\cdot\varepsilon.$

$\lim_{n\to\infty}\left|\frac{1}{n}\cdot\sum a_n-a\right|=0.$

即 $\frac{a_1+\cdots+a_n}{n}\xrightarrow{n\to\infty}a$

② $a_n\to a$ $\quad b_n\to b.$

$\therefore \lim_{n\to\infty}\frac{c_n}{n}=ab$

第二章：收敛性.

$\hat{f}(n)=\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cdot e^{-inx}dx.$ 　　周期为 2π. $[-\pi,\pi]$

周期在 $[a,b]$

设周期为 L.

$\therefore \frac{t}{L}\cdot 2\pi=x$

$\therefore \hat{f}(n)=\frac{1}{L}\cdot\int_a^b f(t)\cdot e^{-2\pi int/L}\cdot dt.$

$\therefore f(t)\sim\sum_{n=-\infty}^{\infty}\hat{f}(n)\cdot e^{2\pi int/L}.$

$f(x)$ 为周期函数. $T=2\pi.$

$\therefore \int_a^b f(x)dx=\int_{a+2\pi}^{b+2\pi}f(x)dx=\int_{a-2\pi}^{b-2\pi}f(x)dx$ 　　平移不变.

且 $\int_{-\pi}^{\pi}f(x+a)dx=\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi+a}^{\pi+a}f(x)dx$

部分和级数 $S_N(f)=\sum_{n=-N}^{N}\hat{f}(n)\cdot e^{2\pi int/L}$

Dirichlet Kernel.

$D_N(x)=\sum_{n=-N}^{N}e^{inx}$ 　$\forall x\in[-\pi,\pi]$ 　　e^-

$D_N(x)=\sum_{n=-N}^{-1}e^{inx}+\sum_{n=0}^{N}\cdot e^{inx}=\sum_{n=1}^{N}\cdot e^{-inx}+\sum_{n=0}^{N}\cdot e^{inx}$

$$= (e^{-ix}) \cdot \frac{1-(e^{-ix})^N}{1-e^{-ix}} + 1 \cdot \frac{1-(e^{ix})^{N+1}}{1-e^{ix}}$$

$$= \frac{(e^{-ix}) \cdot e^{ix} \cdot 1-(e^{-ix})^N}{(1-e^{-ix}) \cdot e^{ix}} + \cdots$$

$$= \frac{1-(e^{-ix})^N}{e^{ix}-1} + \frac{1-(e^{ix})^{N+1}}{1-e^{ix}} = \frac{(e^{-iNx} - e^{i\cdot(N+1)\cdot x})}{1-e^{ix}} = \frac{e^{-i\cdot(N+\frac{1}{2})x} - e^{i\cdot(N+\frac{1}{2})x}}{e^{-\frac{x}{2}\cdot i} - e^{\frac{x}{2}i}}$$

$$= \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

Possion kernel.

绝对收敛 + 一致收敛.

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \cdot e^{in\theta} \qquad \theta \in [-\pi, \pi], \quad 0 \le r < 1$$

$$P_r(\theta) = \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \bar{w}^n \qquad w = r \cdot e^{i\theta} \quad \bar{w} = r \cdot e^{-i\theta} \qquad w^n = r^n \cdot e^{i\theta n} \quad \bar{w}^n = r^n \cdot e^{-i\theta n}$$

$$= \frac{1}{1-w} + \frac{\bar{w}}{1-\bar{w}}$$

$$= \frac{1}{1-r\cdot e^{i\theta}} + \frac{r\cdot e^{-i\theta}}{1-r\cdot e^{-i\theta}} = \frac{1-r\cdot e^{-i\theta} + r\cdot e^{-i\theta} - r^2}{(1-r\cdot e^{i\theta})\cdot(1-r\cdot e^{-i\theta})}$$

$$= \frac{1-r^2}{1+r^2 - r\cdot(e^{i\theta}+e^{-i\theta})} = \frac{1-r^2}{r^2 - 2\cos\theta \cdot r + 1}$$

$r^n \cdot e^{in\theta}$ 公比 $r\cdot e^{i\theta}$ $|r\cdot e^{i\theta}| = |r| < 1$

$\sum_{n=0}^{\infty} a \cdot r^n$ $|r|<1$ 级数收敛.

$f(\theta) = \lim_{N\to\infty} S_N(f)(\theta)$ 是否一致收敛.

级数收敛 转化成 部分和是否一致收敛.

$\int_0^{2\pi} f(\theta)\cdot e^{-in\theta}\, d\theta.$

f $R \to C^1 \to C^2 \to \cdots C^\infty$ 加光滑性条件.

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n)\cdot(\cos n\theta + i\cdot\sin n\theta)$$

$$= \hat{f}(0) + \sum_{n=1}^{\infty} \hat{f}(n)\cdot(\cos n\theta + i\cdot\sin n\theta) + \sum_{n=1}^{\infty} \hat{f}(-n)\cdot(\cos n\theta - i\cdot\sin n\theta)$$

$$\therefore f(\theta) \sim \hat{f}(0) + \sum_{n=1}^{\infty} [\hat{f}(n)+\hat{f}(-n)]\cdot\cos n\theta + i\cdot[\hat{f}(n)+\hat{f}(-n)]\cdot\sin n\theta$$

思考：若 f 与 g 不同，f 与 g 是否有在相同的 fourier series.

$\hat{f}(n) = 0$ $\qquad \therefore f = 0$

$$\hat{f}(n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\theta)\cdot e^{-in\theta}\, d\theta = 0.$$

$$\therefore \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\theta)\cdot C_n\cdot e^{-in\theta}\, d\theta = 0.$$

$$\therefore \int_{-\frac{T}{2}}^{\frac{T}{2}} f(\theta)\cdot \sum_{k=-n}^{n} C_k\cdot e^{-ik\theta}\cdot d\theta = 0$$

定义内积. $\langle f, g\rangle = \frac{1}{2l}\int_{-l}^{l} f(t)\cdot \overline{g(t)}\, dt$

Parserval's Theorem (帕解量)

$f(x) = \sum_{n=-\infty}^{\infty} C_n\cdot e^{inx}$ $\qquad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\, dx$

$$\Rightarrow \|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi\cdot\sum_{n=-\infty}^{\infty} |C_n|^2$$

$$= 2\pi\cdot\sum_{n=-\infty}^{\infty}\cdot\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}$$

最佳逼近. u_n 为三角多项式. 有限基的线性组合

$S_n(f) = \sum_{k=-n}^{n} \langle f, e_k\rangle\cdot e_n$ $\qquad U_n = \sum_{k=-n}^{n} d_k\cdot e^{ikx}$ $\quad d_k$ 为 $\forall x\in\mathbb{C}$

部分和

$S_n(f) = \underset{u_n\in \Sigma_n}{\arg\min} \|f - u_n\|$ 即 $S_n(f)\to f$ 以三角多项式为基.

$\|f - S_n(f)\|_2^2 = \|f\|_2^2 - \sum_{k=-n}^{n} |C_k|^2.$ 最佳逼近的误差.

$\langle f, f\rangle$ $\qquad \langle U_n, U_n\rangle$

证: $\|f-u_n\|_2^2 = \langle f-u_n, f-u_n\rangle = \|f\|_2^2 - \langle u_n, f\rangle - \langle f, u_n\rangle + \|u_n\|_2^2$

$\langle u_n, f\rangle = \langle \sum_{k=-n}^{n} d_k \cdot e^{ikx}, f\rangle = \sum_{k=-n}^{n} d_k \cdot \underbrace{\langle e^{ikx}, f\rangle}_{\overline{\langle f, e^{ikx}\rangle}} = \sum_{k=-n}^{n} d_k \cdot \bar{c}_k$

$\langle f, u_n\rangle = \overline{\langle u_n, f\rangle} = \sum_{k=-n}^{n} \bar{d}_k \cdot c_k$

$\langle u_n, u_n\rangle = \langle \sum_{k=-n}^{n} d_k \cdot e^{ikx}, \sum_{k} d_k \cdot e^{ikx}\rangle = \sum_{k=-n}^{n} |d_k|^2$

$\langle e^{ikx}, e^{ikx}\rangle = e^0 = 1$

$\therefore \|f-u_n\|_2^2 = \|f\|_2^2 + \sum_{k=-n}^{n} (|d_k|^2 - d_k \bar{c}_k - \bar{d}_k \cdot c_k)$

$\because |c_k - d_k|^2 = |c_k|^2 - c_k \bar{d}_k - \bar{c}_k d_k + |d_k|^2$

$= \|f\|_2^2 + \sum_{k=-n}^{n} |c_k - d_k|^2 - \sum_{k=-n}^{n} |c_k|^2$

当 $d_k = c_k$ 时. 即 $u_n = S_n(f)$

$\|f - S_n(f)\|_2^2 = \|f\|_2^2 - \sum_{k=-n}^{n} |c_k|^2$

推论: $\sum_{k=-n}^{n} |c_k|^2 \le \|f\|_2^2$ $\qquad \sum_{k=1}^{\infty} |(x, e_k)|^2 \le \|x\|^2$

$n\to\infty$ $\sum_{k=-n}^{n} |c_k|^2 \le \|f\|_2^2 < +\infty$ $\qquad$ Bessel 不等式.

证法二 (最佳逼近) $\qquad$ Bessel 不等式. $\qquad$ e_k 为基.

$\forall x \in X$ $\qquad \|f - S_n(f)\|$ $\qquad S_n(f)$ 部分和.

$\forall n$

$\|x - \sum_{k=1}^{n} (x, e_k) e_k\|^2 = (x - \sum_{k=1}^{n} (x, e_k) e_k, x - \sum_{k=1}^{n} (x, e_k) e_k)$

$= (x, x) - (x, \sum_{k=1}^{n} (x, e_k) e_k) - (\sum_{k=1}^{n} (x, e_k) e_k, x) - (\sum_{k=1}^{\infty} (x, e_k) e_k, \sum_{k=1}^{n} (x, e_k) e_k)$

$(x, \sum_{k=1}^{n} (x, e_k) e_k) = \sum_{k=1}^{n} (x, e_k) \cdot (x, e_k) = \sum_{k=1}^{n} |(x, e_k)|^2$

$(\sum_{k=1}^{n} (x, e_k) e_k, x) = \sum_{k=1}^{n} (x, e_k) \cdot (e_k, x) = \sum_{k=1}^{n} (x, e_k) \cdot \overline{(x, e_k)} = \sum_{k=1}^{n} (x, e_k)^2$

$\therefore$ 原式 $= \|x\|^2 - \sum_{k=1}^{n} |(x, e_k)|^2 \geq 0.$ $\qquad$ 当且仅当 e_k 为基.时取等

$\therefore \sum_{k=1}^{n} (x, e_k)^2 \le \|x\|^2$

$n \to +\infty$ 等式仍成立.

∴ 在 L_2 内积空间.

$\|f\| < +\infty$.

即可得出 f 依 L_2 范数收敛

$$\|c_n e_n\| = c_n \|e_n\| = c_n \text{ 或 } c_n \cdot 2\pi$$

X完备 + 范数级数收敛 ⟹ 级数收敛.

$S_n(f)$ x的元素 部分和. 设 $m = n+p$ p为任意自然数 $m, n > N$

$$\|S_m(f) - S_n(f)\| = \|c_{n+1}e_n + c_{n+2}e_n + \cdots + c_m e_n + (-c_{n+1}e_n - \cdots - c_m e_n\|$$

$$\le \|c_{n+1}e_n\| + \|c_{n+2}e_n\| + \cdots < \varepsilon.$$

2P2q.

由柯西收敛准则可知 $\sum \|c_n e_n\|$ 收敛 ⟹ 当 $n > N$ 时. 部分和 $< \varepsilon$.

∴ $S_n(f)$ 为 Cauchy 列.

$\sum \frac{1}{n^2}$ 收敛 $\sum \frac{1}{n}$ 发散

∴ $S_n(f) \overset{L_2}{\Longrightarrow} f$ 一定要(因为 $\sum |a_n|^2$ 收敛 $\not\Rightarrow$ $\sum |a_n|$ 收敛.

$$\|S_m(f) - S_n(f)\|^2 \le \|c_{n+1}e_{n+1}\|^2 + \cdots \quad < \varepsilon.$$

∴ $\|S_m(f) - S_n(f)\| < \varepsilon$.

∴ $S_n(f) \to g$ 依 L_2 范数收敛. $S_n(f) = \sum_{k=1}^{n} (f, e_n) e_n$

证 $g = f$ 内积连续性

$$\langle g, e_k\rangle = \langle \lim_{n\to\infty} S_n, e_k\rangle = \lim_{n\to\infty} \langle S_n, e_k\rangle = \lim_{n\to\infty} \left(\sum_{k=1}^{n} (f, e_n)e_n, e_k\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (f, e_n)\cdot(e_n, e_k)$$

$$= \lim_{n\to\infty} (f, e_k)\cdot(e_k, e_k)$$

$$= (f, e_k)$$

∴ $\langle f, e_k\rangle - (g, e_k) = 0$.

∴ $\langle f-g, e_k\rangle = 0$.

∴ $f = g$

证明思路: Bessel (范数级数收敛) 平方和 + 级数收敛 (完备 + 范数级数收敛) L_2 空间

逐点收敛 $f\in L^1$，第一类间断点.

D核变换. $D_N(t) = \sum_{n=-N}^{N} e^{int}$

$$\int_{-\pi}^{\pi} D_N(t)\,dt = \int_{-\pi}^{\pi}\sum_{n=-N}^{N} e^{int}\,dt = \sum_{n=-N}^{N}\cdot\int_{-\pi}^{\pi} e^{int}\,dt$$
$$= \int_{-\pi}^{\pi} 1\,dt = 2\pi.$$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1$$

$$S_N(t) = \sum_{n=-N}^{N}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{-int}\,dt\cdot e^{inx}$$
$$= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(t)\cdot\sum_{n=-N}^{N}\cdot e^{in(x-t)}\,dt$$
$$= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi}\cdot f(t)\cdot D_N(x-t)\,dt.$$

$S_N(t) =$ $f(t)$ 与 D_N 的卷积. $(f*D_N)(x)$

令 $u = x-t$ $\quad dt = -du$

$t\in(-\pi,\pi)$ $\quad u\in(x-\pi, x+\pi)$

$$S_N(u) = \frac{1}{2\pi}\cdot\int_{x-\pi}^{x+\pi}\cdot f(x-u)\cdot D_N(u)\,du$$

$f(u)$, $D_N(t)$ 都以 2π 为周期.

$$\therefore S_N(u) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(x-u)\cdot D_N(u)\,du$$

$$\therefore S_N(t) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(x-t)\cdot D_N(t)\cdot dt.$$

$$= \frac{1}{2\pi}\cdot\int_{|\delta|<\pi} f(x-t)\cdot D_N(t)\cdot dt + \frac{1}{2\pi}\cdot\int_{-\delta}^{\delta} f(x-t)\cdot D_N(t)\cdot dt.$$

离散部. $\qquad$ 主部分.

$$D_N(t) = \sum_{-N}^{N}\cdot e^{int} = \sum_{n=0}^{N}\cdot e^{int} + \sum_{n=1}^{N}\cdot e^{-int} \qquad t\neq 0.$$
$$= 1\cdot\frac{1-e^{it(n+1)}}{1-e^{it}} + e^{-it}\cdot\frac{1-e^{-int}}{1-e^{-it}}$$
$$= \frac{1-e^{it(n+1)}}{1-e^{it}} + e^{-it}\cdot\frac{e^{it}\cdot(1-e^{-int})}{e^{it}-1} = \frac{1-e^{it(n+1)}}{1-e^{it}} + \frac{e^{-int}-1}{1-e^{it}}$$
$$= \frac{e^{-int}-e^{i(n+1)t}}{1-e^{it}} = \frac{e^{-i(n+\frac{1}{2})t}-e^{i(n+\frac{1}{2})t}}{e^{-\frac{it}{2}}-e^{\frac{it}{2}}} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}$$

$D_N(t) = \sum_{-N}^{N} e^{int} = \sum_{n=1}^{N} e^{int} + \sum_{n=1}^{N} e^{-int} + 1$

$\cos t = \frac{e^{it}+e^{-it}}{2}$ $\quad \therefore 2\cos nt = e^{int} + e^{-int}$

偶函数.

原式 $= 1 + 2\cdot\sum_{n=1}^{N} \cos nt$ $\quad D_N(-t) = 1 + 2\cdot\sum_{n=1}^{N}\cos nt = D_N(t)$

$\therefore$ 尾积分: $\int_{\delta}^{\pi} f(x-t)\cdot \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt = \int_{\delta}^{\pi} \frac{f(x-t)}{\sin\frac{t}{2}}\cdot \sin(n+\frac{1}{2})t\,dt$

$\frac{1}{\sin\frac{t}{2}}$ 在 (δ,π) 有界.

令 $g(t) = f(x-t)\cdot\frac{1}{\sin\frac{t}{2}}$ $\quad f(x-t) \in L^1$ $\quad \therefore g(t) \in L^1$

$\therefore$ 尾积分: $\int_{\delta}^{\pi} g(t)\cdot\sin(n+\frac{1}{2})t\,dt \to 0.$

勒贝格-黎曼引理 $\quad \int_{-\pi}^{\pi} f(t)\cdot \sin nt\,dt \to 0$

$n \to \infty.$ $\quad \int_{-\pi}^{\pi} f(t)\cdot\cos nt\cdot dt \to 0.$

$\int_{-\pi}^{\pi} f(t)\cdot e^{int}\,dt \to 0.$

主积分: $\int_0^{\delta} f(x-t)\cdot D_N(t)\,dt$ $\quad \int_0^{\delta} f(x-t)\cdot D_N(t)\,dt$

当 $\delta \to 0$ 时 $\quad f(x-t) \to f(x^-)$

原式: $f(x^-)\cdot\int_0^{\delta} D_N(t)\,dt = f(x^-)\cdot\pi$

$\int_0^{-\delta} f(x-t)\cdot D_N(t)\cdot dt.$

$\delta < \delta_K'$ $\quad f(x-t) \to f(x^+)$ $\quad$ 有限个第一类间断点，其他都是连续点.

有限个极值点(有界变差)

原式: $f(x^+)\cdot\int_0^{-\delta} D_N(t)\,dt = f(x^+)\cdot\pi.$ $\quad$ Jordan 条件(有界变差必逐点收敛,

$\therefore S_N(t) \to \frac{f(x^-)+f(x^+)}{2}$

间断点. 极值点. (无限)
(可数)

$S_N(t) = \int_{-\pi}^{\pi} f(x-t)\,D_N(t)\cdot dt = \int_{\pi}^{-\pi} f(x+u)\cdot D_N(-u)\cdot du\cdot(-1)$

$u = -t.$ $\quad t \in (-\pi,\pi)$ $\quad u \in (-\pi,\pi)$ $\quad = \int_{-\pi}^{\pi} f(x+u)\cdot D_N(u)\cdot du = \int_{-\pi}^{\pi} f(x+t)\cdot D_N(t)\cdot dt.$

$\therefore S_N(t) = \int_0^{\pi} [f(x+t) + f(x-t)]\cdot D_N(t)\cdot dt$

第二中值定理: $\int_a^b f(t)\cdot g(t)\,dt = f(c)\cdot\int_a^b g(t)\,dt.$ $\quad f \in C[a,b]$ $\quad g$ 单调.

$= f(a^+)\cdot\int_a^c g(x)\,dx + f(b^-)\int_c^b g(x)\,dx$ $\quad$ 或 g 可积, f 有界变差.

$$\int_0^{\delta} [f(x+t)+f(x-t)]\cdot D_n(t)\, dt$$

Dini 条件：$\int_0^{\delta} \frac{|f(x+t)+f(x-t)-2f(x)|}{t}\, dt < +\infty.$

$$\lim_{n\to\infty} \int_0^{\delta} (f(x+0)-f(x+t)) \cdot \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\, dt = 0.$$

$$\varphi(t) = \frac{f(x+0)-f(x+t)}{2\sin\frac{t}{2}} = -\frac{f(x+t)-f(x+0)}{2\sin\frac{t}{2}} = -\frac{f(x+t)-f(x+0)}{t}\cdot\frac{\frac{t}{2}}{\sin\frac{t}{2}}\cdot 2.$$

$\lim_{t\to 0^+} \varphi(t) = \frac{-f'(x+0)}{2}$ 有限个极值点. 保证导数收敛.

导函数绝对可积.

$$\therefore \lim_{n\to\infty} \int_0^{\delta} (f(x+0)-f(x+t))\cdot \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\, dt = \lim_{n\to\infty} \int_0^{\delta} \varphi(t)\cdot \sin(n+\tfrac{1}{2})t\, dt.$$

$$= 0.$$

$$\frac{f(x+0)+f(x-0)}{2} = \lim_{n\to\infty} S_n(x).$$

即：$\lim_{n\to\infty} \left(\frac{f(x+0)+f(x-0)}{2} - S_n(x) \right) = 0.$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos nt + b_n \cdot \sin nt$$

$$\frac{a_0}{2} = f - \sum_{n=1}^{\infty} (a_n \cdot \cos nt + b_n \cdot \sin nt)$$

$$a_0 = 2f - 2\sum_{n=1}^{\infty} (a_n \cdot \cos nt + \Sigma b_n \cdot \sin nt)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} a_0 \, dt = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f \, dt - \sum_{n=1}^{\infty} \frac{1}{\pi} \cdot a_n \cdot \int_{-\pi}^{\pi} \cos nx \, dx - \sum_{n=1}^{\infty} \frac{1}{\pi} \cdot b_n \cdot \int_{-\pi}^{\pi} \cdot \sin nx \cdot dx$$

$$a_0 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f \, dt.$$

$$f \cdot \cos nt = \frac{a_0}{2} \cdot \cos nt + \sum_{k=1}^{\infty} (a_n \cdot \cos kt + b_n \cdot \sin t) \cdot \cos nt$$

$$\therefore \int_{-\pi}^{\pi} f \cdot \cos nt \, dt = \int_{-\pi}^{\pi} \frac{a_0}{2} \cdot \cos nt. + \sum_{k=1}^{\infty} (a_n \cdot \cos kt \cdot \cos nt + b_n \cdot \sin t \cdot \cos t) \, dt.$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2} \cos nt \, dt + a_n \cdot \int_{-\pi}^{\pi} \cdot \cos^2 nt \, dt.$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2} \cdot \cos nt \, dt + a_n \cdot \int_{-\pi}^{\pi} \frac{1}{2} \, dt + \int_{-\pi}^{\pi} \cos 2nt \, dt.$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nt \, dt + \pi \cdot a_n = \pi a_n.$$

$$a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f \cdot \cos nt \, dt.$$

同理可得: $b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f \cdot \sin nt \cdot dt.$

$\sum_{n=1}^{\infty} a_n \cdot \cos nx < \infty$ $\sum_{n=1}^{\infty} b_n \cdot \sin x < \infty.$