一元傅里叶变换的应用

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1　方程的频率方法（常微分方程）

$$\mathcal{F}D^\alpha = (i\omega)^\alpha \mathcal{F}, \tag{1.1}$$

一维时，

$$(f')^\wedge(\omega) = (i\omega)\hat{f}(\omega) \tag{1.2}$$

其中，$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$，$D_j = \frac{\partial}{\partial x_j}$，$X = (x_1, \ldots, x_n)$，$X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

例题 1.1.

$$\int_0^{+\infty} g(w)\sin wx dw = f(x) \tag{1.3}$$

求积分方程(1.3)的解 $g(w)$

其中

$$f(x) = \begin{cases} \frac{\pi}{2}\frac{e^{-x}}{x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{1.4}$$

解 1.1. 方法一 ： *用正弦积分变换（略），见教材 P99 例 1.6*

方法二 假设 g 为奇函数，则：

$$\begin{aligned} g(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{g}(w)e^{-iwx}dw = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{g}(w)(-i)\sin wx dw \\ &= \frac{-i}{\sqrt{2\pi}}\int_0^{\infty} g(w)\sin wx dw, \end{aligned} \tag{1.5}$$

由上可得

$$\frac{\sqrt{2\pi}}{-2i}\hat{g}(x) = f(x) \tag{1.6}$$

又 g 为奇函数，有 $\hat{g}(-w) = -\hat{g}(w)$，故方程变为：

$$\frac{\sqrt{2\pi}}{-2i}\hat{g}(t) = f(t) \tag{1.7}$$

最后解得 $g(w)$ 为奇函数，仍记为 f.

$$
\begin{aligned}
f(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(w)e^{-iwt}dw = \frac{-2i}{\sqrt{2\pi}}\int_{0}^{+\infty} f(w)\sin wtdw \\
&= \frac{-2i}{\sqrt{2\pi}}\int_{0}^{\pi} \frac{\pi}{2}\sin wt\sin wtdw \\
&= \frac{-2i}{\sqrt{2\pi}}\cdot\frac{\pi}{4}\int_{0}^{\pi}[\cos(1-t)w-\cos(1+t)w]dw \\
&= \frac{-2i}{\sqrt{2\pi}}\cdot\frac{\pi}{4}\left(\frac{\sin(1-t)\pi}{1-t}+\frac{\sin(1+t)\pi}{1+t}\right) \\
&= \frac{-2i}{\sqrt{2\pi}}\cdot\frac{\pi}{4}\cdot 2\cdot\frac{\sin\pi t}{1-t^2} = \frac{-2i}{\sqrt{2\pi}}\cdot\frac{\pi}{2}\cdot\frac{\sin\pi t}{1-t^2}
\end{aligned} \tag{1.8}
$$

对(1.7)两边做傅里叶变换

$$\frac{\sqrt{2\pi}}{-2i}\hat{g}(t) = f(t) \tag{1.9}$$

利用做两次傅里叶变换是镜像对称变换的性质

$$\mathcal{F}^2\{f(t)\} = \mathcal{F}\{\mathcal{F}\{f(t)\}\} = f(-t) \tag{1.10}$$

$$\frac{\sqrt{2\pi}}{-2i}\hat{g}(-t) = \hat{f}(t) \tag{1.11}$$

而且 g 为奇函数可知

$$g(-t) = \frac{-2i}{\sqrt{2\pi}}\cdot\frac{\pi}{2}\cdot\frac{\sin\pi t}{1-t^2} \tag{1.12}$$

$$g(t) = \frac{\sin\pi t}{1-t^2} \tag{1.13}$$

例题 1.2. 积分方程解

$$g(t) = h(t) + \int_{-\infty}^{\infty} f(t)g(t-x)dx \tag{1.14}$$

h、f 已知，且 g、h、f 的傅里叶变换存在。

解 1.2. 由傅里叶变换的卷积公式：

$$\mathcal{F}\{f*g\} = \sqrt{2\pi}\cdot\hat{f}\cdot\hat{g} \tag{1.15}$$

原方程可化为：

$$\hat{g}(w) = \hat{h}(w) + \hat{f}(w)\cdot\hat{g}(w)\cdot\sqrt{2\pi} \tag{1.16}$$

解得：

$$\hat{g}(w) = \frac{\hat{h}(w)}{1-\sqrt{2\pi}\cdot\hat{f}(w)} \tag{1.17}$$

因此：

$$g(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{h}(w)}{1-\sqrt{2\pi}\cdot\hat{f}(w)}e^{iwt}dw \tag{1.18}$$

例题 1.3. 常微分非齐次线性积分方程：

$$y''-y=-f \tag{1.19}$$

其中 f 为已知函数

解 1.3. 对两边取傅里叶变换：

$$(iw)^2\hat{y}(w)-\hat{y}(w)=-\hat{f}(w) \tag{1.20}$$

解得：

$$\hat{y}(w)=\frac{\hat{f}(w)}{1+w^2} \tag{1.21}$$

因此：

$$y(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{f}(w)}{1+w^2}e^{iwx}dw \tag{1.22}$$

将 $\frac{\hat{f}(w)}{1+w^2}$ 视为 $\hat{f}(w)$ 与 $\frac{1}{1+w^2}$ 的乘积。

由卷积定理：

$$(f*h)(w)=\sqrt{2\pi}\hat{f}(w)\cdot\hat{h}(w),\quad 其中\ \hat{h}(w)=\frac{1}{1+w^2}. \tag{1.23}$$

因此：

$$\hat{y}(w)=\sqrt{2\pi}\hat{f}(w)\cdot\hat{g}(w),\quad 其中\ \hat{g}(w)=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1+w^2}. \tag{1.24}$$

所以：

$$y(t)=(f*g)(t),\quad 其中\ g(w)=\frac{1}{2}e^{-|t|}. \tag{1.25}$$

因此：

$$y(t)=\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-|t-x|}dx. \tag{1.26}$$

例题 1.4. 求解积分方程：

$$ax'(t)+bx(t)+c\int_0^t x(t)dt=h(t), \tag{1.27}$$

其中 $a,b,c\in\mathbb{R}$, h 已知

解 1.4. 关键是通过傅里叶变换求解。设 $G'=g$，且 G 有傅里叶变换。

对两边取傅里叶变换：

$$(iw)\hat{G}(w)=\hat{g}(w)\Rightarrow\hat{G}(w)=\frac{\hat{g}(w)}{iw}. \tag{1.28}$$

利用此公式，原方程可变换为：

$$a(iw)\hat{x}(w)+b\hat{x}(w)+c\frac{\hat{x}(w)}{iw}=h(w) \tag{1.29}$$

解得：

$$\hat{x}(w)=\frac{h(w)}{iaw+b+\frac{c}{iw}}. \tag{1.30}$$

2 PDE 的傅里叶变换

2.1 一维波动方程初值问题

例题 2.1. 求解一维波动方程初值问题：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}, \quad x\in\mathbb{R}, t>0 & (2.1a) \\ u|_{t=0}=\varphi_0(x) & (2.1b) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\varphi_1(x) & (2.1c) \end{cases}$$

解 2.1. 对二元函数 $u(x,t)$ 的 x 变量作傅里叶变换，记之为 $V(w,t)$。

则：

$$V(w,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-iwx}dx \tag{2.2}$$

于是：

$$\frac{\partial V}{\partial t}(w,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}(x,t)e^{-iwx}dx=\mathcal{F}_1\left(\frac{\partial u}{\partial t}(x,t)\right) \tag{2.3}$$

有以下 *6* 个公式

$$\mathcal{F}_1\left(\frac{\partial u}{\partial x}\right)(w,t)=\frac{\partial}{\partial t}V(w,t) \quad \textit{(视 } w \textit{ 为常数)} \tag{2.4}$$

$$\mathcal{F}_1\left(\frac{\partial u}{\partial x}\right)(w,t)=(iw)V(w,t) \tag{2.5}$$

$$\mathcal{F}_1\left(\frac{\partial^2 u}{\partial x^2}\right)(w,t)=\frac{\partial^2}{\partial t^2}V(w,t) \tag{2.6}$$

$$\mathcal{F}_1\left(\frac{\partial^2 u}{\partial x^2}\right)(w,t)=(iw)^2V(w,t) \tag{2.7}$$

$$\mathcal{F}_1(\sin x)(w) = \sqrt{\frac{\pi}{2}} i\left[\delta(w+1) - \delta(w-1)\right] \tag{2.8}$$

$$\mathcal{F}_1(\cos x)(w) = \sqrt{\frac{\pi}{2}}\left[\delta(w+1) + \delta(w-1)\right] \tag{2.9}$$

$$\mathcal{F}_1(\cos x)(w) = \sqrt{\frac{\pi}{2}}\left[\delta(w+1) + \delta(w-1)\right] \tag{2.10}$$

原方程在频率域可转化为常微分方程问题

$$\begin{cases} \dfrac{d^2V}{dt^2} = -w^2V & (2.11a) \\ V|_{t=0} = \sqrt{\dfrac{\pi}{2}}\left[\delta(w+1) + \delta(w-1)\right] & (2.11b) \\ \left.\dfrac{dV}{dt}\right|_{t=0} = \sqrt{\dfrac{\pi}{2}} \cdot i\left[\delta(w+1) - \delta(w-1)\right] & (2.11c) \end{cases}$$

通解如下：

$$V(w,t) = C_1\sin(wt) + C_2\cos(wt) \tag{2.12}$$

注记 2.1. 齐次方程特征方程为 $\lambda^2 + w^2 = 0$，其解为 $\lambda = \pm iw$，对应的解为 $e^{\pm iwt}$，即 $\cos(wt)$ 和 $\sin(wt)$。

由(2.11b)

$$C_2 = \sqrt{\frac{\pi}{2}}\left[\delta(w+1) + \delta(w-1)\right] \tag{2.13}$$

由(2.11c)

$$C_1 = \sqrt{\frac{\pi}{2}} \cdot \frac{i}{w}\left[\delta(w+1) - \delta(w-1)\right] \tag{2.14}$$

最后得

$$\begin{aligned} V(w,t) &= \sqrt{\frac{\pi}{2}} \cdot \frac{i}{w}\left[\delta(w+1) - \delta(w-1)\right]\sin(wt) \\ &\quad + \sqrt{\frac{\pi}{2}}\left[\delta(w+1) + \delta(w-1)\right]\cos(wt) \\ &= \left(\sqrt{\frac{\pi}{2}} \cdot \frac{i}{w}\sin(wt) + \sqrt{\frac{\pi}{2}}\cos(wt)\right)\delta(w+1) \\ &\quad + \left(-\sqrt{\frac{\pi}{2}} \cdot \frac{i}{w}\sin(wt) + \sqrt{\frac{\pi}{2}}\cos(wt)\right)\delta(w-1) \end{aligned} \tag{2.15}$$

做傅里叶逆变换，从频率域变回时间域

$$
\begin{aligned}
u(x,t) &= \mathcal{F}_1^{-1}(V(\cdot,t))(x) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\left[\sqrt{\frac{\pi}{2}}\cdot\frac{i}{w}\sin(wt)+\sqrt{\frac{\pi}{2}}\cos(wt)\right]\delta(w+1)\right. \\
&\quad \left.+\left[-\sqrt{\frac{\pi}{2}}\cdot\frac{i}{w}\sin(wt)+\sqrt{\frac{\pi}{2}}\cos(wt)\right]\delta(w-1)\right\}e^{iwx}dw \\
&= \frac{1}{\sqrt{2\pi}}\left\{\left[\sqrt{\frac{\pi}{2}}\cdot\frac{i}{-1}\sin(-t)+\sqrt{\frac{\pi}{2}}\cos(-t)\right]e^{-ix}\right. \\
&\quad \left.+\left[\sqrt{\frac{\pi}{2}}\cdot\frac{i}{1}\sin(t)+\sqrt{\frac{\pi}{2}}\cos(t)\right]e^{ix}\right\} \\
&= \frac{1}{2}\left\{\left[-i\sin(t)+\cos(t)\right]e^{-ix}+\left[i\sin(t)+\cos(t)\right]e^{ix}\right\} \\
&= \frac{1}{2}\left\{\cos(t)e^{-ix}+\cos(t)e^{ix}+i\sin(t)e^{ix}-i\sin(t)e^{-ix}\right\} \\
&= \frac{1}{2}\left\{\cos(t)\left(e^{ix}+e^{-ix}\right)+i\sin(t)\left(e^{ix}-e^{-ix}\right)\right\} \\
&= \frac{1}{2}\left\{2\cos(t)\cos(x)+2i\sin(t)\sin(x)\right\} \\
&= \cos(t-x)
\end{aligned} \tag{2.16}
$$

2.2 一维热传导方程

例题 2.2. 求解一维热传导方程初值问题：

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}+f(x,t), x\in\mathbb{R}, t>0 & (2.17a) \\
u|_{t=0} = \varphi(x) & (2.17b)
\end{cases}
$$

解 2.2. 令 $V(w,t)=\mathcal{F}_1(u(\cdot,t))(w)$. 则

$$
\begin{cases}
\mathcal{F}_1\left(\dfrac{\partial u}{\partial t}(\cdot,t)\right) = \dfrac{\partial}{\partial t}(\mathcal{F}_1 u(w,t)) = \dfrac{\partial}{\partial t}V(w,t) = \dfrac{dV}{dt}(w,t) & (2.18a) \\
\mathcal{F}_1\left(\dfrac{\partial^2 u}{\partial x^2}\right)(w) = (iw)^2V(w,t) = -w^2V(w,t) & (2.18b)
\end{cases}
$$

记

$$
\begin{cases}
\hat{f}_1(w,t) = \mathcal{F}_1 f(\cdot,t)(w) & (2.19a) \\
\hat{\varphi}(w) = (\mathcal{F}\varphi)(w) & (2.19b)
\end{cases}
$$

则原方程在频域表示为：

$$
\begin{cases}
\dfrac{dV}{dt} = -a^2 w^2 V + \hat{f}_1(w,t) & (2.20a) \\
V|_{t=0} = \hat{\varphi}(w) & (2.20b)
\end{cases}
$$

方程(2.20a)为一阶非齐次常微分方程。由常数变异法，可得解如下：

$$
V(w,t) = \hat{\varphi}(w) e^{-a^2 w^2 t} + \int_0^t \hat{f}_1(w,\tau) e^{-a^2 w^2 (t-\tau)} d\tau \tag{2.21}
$$

$$
\mathcal{F}^{-1}\left(e^{-a^2 w^2 t}\right)(x) = \sqrt{2\pi} \cdot \frac{1}{2a\sqrt{\pi t}} e^{-\dfrac{x^2}{4a^2 t}} \tag{2.22}
$$

其中，

$$
\mathcal{F}\left(e^{-\beta x^2}\right)(w) = \frac{\sqrt{\pi}}{\sqrt{\beta}} e^{-\dfrac{w^2}{4\beta}} \tag{2.23}
$$

令 $\beta = a^2 t$，则

$$
\mathcal{F}^{-1}\left(e^{-a^2 w^2 t}\right)(x) = \frac{1}{a\sqrt{2\pi t}} e^{-\dfrac{x^2}{4a^2 t}} \tag{2.24}
$$

进一步地，

$$
\mathcal{F}^{-1}\left(e^{-a^2 (w-k)^2 t}\right)(y) = \frac{1}{a\sqrt{2\pi (t-\tau)}} e^{-\dfrac{(y-x)^2}{4a^2 (t-\tau)}} \tag{2.25}
$$

最终得解：

$$
\begin{aligned}
u(x,t) &= \left(\varphi(x) * \frac{1}{a\sqrt{2\pi t}} e^{-\dfrac{x^2}{4a^2 t}}\right)(x) \\
&\quad + \int_0^t \left(f(x,\tau) * \frac{1}{a\sqrt{2\pi (t-\tau)}} e^{-\dfrac{x^2}{4a^2 (t-\tau)}}\right)(x) d\tau
\end{aligned} \tag{2.26}
$$

2.3 附

例题 2.3. 求解微分方程初值问题：

$$
\begin{cases}
\dfrac{dy}{dx} = Ay + f(x), x \in \mathbb{R} & (2.27a) \\
y(0) = C & (2.27b)
\end{cases}
$$

解 2.3. 齐次方程 $y' = Ay$。解得：

$$\frac{y'}{y} = A \implies (\ln y)' = A \implies \ln y = Ax + C_1 \implies y = e^{C_1}e^{Ax} \tag{2.28}$$

即 $y = Ce^{Ax}$。

令 $y = C(x)e^{Ax}$。代入原方程：

$$y' = C'(x)e^{Ax} + C(x)e^{Ax}A \tag{2.29}$$

代入 $y' = Ay + f(x)$：

$$C'(x)e^{Ax} + C(x)e^{Ax}A = AC(x)e^{Ax} + f(x) \tag{2.30}$$

解得：

$$C'(x) = f(x)e^{-Ax} \tag{2.31}$$

因此：

$$C(x) = \int f(x)e^{-Ax}dx \tag{2.32}$$

特解：

$$C(x) = \int_0^x f(x)e^{-Ax}dx + C \tag{2.33}$$

解为：

$$y = \int_0^x f(x)e^{-Ax}dx \cdot e^{Ax} + Ce^{Ax} \tag{2.34}$$

当 $y(0) = C$ 时，解得 $C = C$。

因此通解：

$$y(x) = Ce^{Ax} + \int_0^x f(x)e^{A(x-x)}dx \tag{2.35}$$

2.4 求解上半平面无源静电场内电势的边值问题

例题 2.4. 求解上半平面无源静电场内电势的边值问题（拉普拉斯方程）：

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, x \in \mathbb{R}, y > 0 & \text{(2.36a)} \\ u|_{y=0} = f(x) & \text{(2.36b)} \end{cases}$$

解 2.4. 记 $V(w, y) = \mathcal{F}_1 u(x, y)(w)$。

将原方程化为：

$$\begin{cases} -w^2V(w,y) + \dfrac{\partial^2}{\partial y^2}V(w,y) = 0 & \text{(2.37a)} \\ V(w,y)|_{y=0} = \hat{f}(w) & \text{(2.37b)} \\ \lim\limits_{y\to+\infty} V(w,y) = 0 & \text{(2.37c)} \end{cases}$$

视 w 为常数，这是一个二阶常微分方程。特征方程为：

$$\lambda^2 - w^2 = 0 \implies \lambda = \pm|w| \tag{2.38}$$

通解为：

$$V(w,y) = C_1 e^{|w|y} + C_2 e^{-|w|y} \tag{2.39}$$

由边界条件：

$$V(w,y)|_{y=0} = \hat{f}(w) \implies C_1 + C_2 = \hat{f}(w) \tag{2.40}$$

以及

$$\lim_{y\to+\infty} V(w,y) = 0 \implies C_1 = 0 \tag{2.41}$$

因此：

$$V(w,y) = \hat{f}(w) e^{-|w|y} \tag{2.42}$$

利用傅里叶逆变换：

$$\mathcal{F}^{-1}\left(e^{-|w|y}\right)(x) = \sqrt{\frac{\pi}{y}} \frac{y}{x^2+y^2} \tag{2.43}$$

上式子可由下式得出

$$\mathcal{F}\left(e^{-\frac{1}{r}}\right) \implies \sqrt{\frac{\pi}{2}} \frac{r}{1+r^2\xi^2} \tag{2.44}$$

最终得解

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2\pi}} \left(f * \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2} \right)(x) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{x^2+y^2} f(x-t)\, dt \end{aligned} \tag{2.45}$$

上半平面 *Poisson* 积分核

$$P_y(x) = \frac{1}{\pi} \cdot \frac{y}{y^2+x^2} \tag{2.46}$$