

# 3X03

# Assignment 2

Johnson Ji

400499564

P1:

Finished in Julia file.

P2:

$$P_2: \|v\|_2 = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = 1$$

$V \triangleq vv^T = \|v\|_2^2$  and  $V$  is symmetric matrix

Note:  $(vv^T)^T = vv^T \therefore A = Q \Lambda Q^{-1}$ ,  $Q$  is orthogonal matrix

$$Vv = \lambda v \Leftrightarrow (Vv^T)v = \lambda v$$

$$\text{LHS} = v(v^Tv) = v$$

$\therefore \lambda = 1$  is an eigenvalue of  $V$ ,  $v$  is an eigenvector

Note, since for all column vectors of  $Q$ , are the eigenvectors of the matrix  $A$  and they are orthogonal.

since  $v$  is one of the eigenvectors, so other eigenvectors  $u$  should  $\perp$  with  $v$

$$\therefore (Vv^T)u = \lambda u$$

$$v(v^Tu) = \lambda u, v^Tu = 0.$$

$$\therefore \text{for any } \vec{u} \neq \vec{0}, \lambda = 0.$$

$\therefore$  for all eigenvectors of  $A$  and their eigenvalues:

$$\begin{cases} E.v = v, E.\text{value} = 1 \\ E.v = u \text{ that } uv^T = 0, E.\text{value} = 0 \end{cases}$$

P3:

Finished in Julia.

P4:

P4:

$$F(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \text{ while } f_i(x) = \|\mathbf{x} - \mathbf{p}_i\| - d_i$$

Note:  $J(x) = \begin{bmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_n(x)^T \end{bmatrix}$  while  $f_i(x) = \left( \sum_{k=1}^n (x_k - p_{ik})^2 \right)^{\frac{1}{2}} - d_i$

$$\begin{aligned} \nabla f_i(x) &= \begin{bmatrix} \frac{\partial f_i}{\partial x_1} \\ \vdots \\ \frac{\partial f_i}{\partial x_n} \end{bmatrix}, \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \left( \sum_{k=1}^n (x_k - p_{ik})^2 \right)^{\frac{1}{2}} - d_i \right] \\ &= \frac{1}{2} \left( \sum_{k=1}^n (x_k - p_{ik})^2 \right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x_j} \left( \sum_{k=1}^n (x_k - p_{ik})^2 \right) \\ &= \frac{1}{2} \left( \sum_{k=1}^n (x_k - p_{ik})^2 \right)^{-\frac{1}{2}} \cdot 2(x_j - p_{ij}) \\ &= \frac{1}{2} \frac{2}{\|\mathbf{x} - \mathbf{p}_i\|} \cdot 2(x_j - p_{ij}) \\ &= \frac{x_j - p_{ij}}{\|\mathbf{x} - \mathbf{p}_i\|} \rightarrow 2 \text{ norm} \end{aligned}$$

$$\therefore \nabla f_i(x) = \frac{\mathbf{x} - \mathbf{p}_i}{\|\mathbf{x} - \mathbf{p}_i\|_2}$$

$$\therefore J(x) = \begin{bmatrix} \frac{(\mathbf{x} - \mathbf{p}_1)^T}{\|\mathbf{x} - \mathbf{p}_1\|_2} \\ \vdots \\ \frac{(\mathbf{x} - \mathbf{p}_n)^T}{\|\mathbf{x} - \mathbf{p}_n\|_2} \end{bmatrix} \text{ until } n,$$

P5:

Finished in Julia.

P6:

P6:

$$f(x) \triangleq \sum_{i=1}^m |f_i(x)|^2 = \sum_{i=1}^m (f_i(x))^2, \quad f_i(x) = \|x - p_i\|_2 - d_i$$

$$\begin{aligned} \nabla f(x) &= \nabla \left( \sum_i^m f_i(x)^2 \right) \\ &= 2 \sum_i^m \nabla f_i(x)^2 = 2 \sum_i^m \nabla f_i(x) f_i(x) \\ &= 2 \sum_i^m \left( (\|x - p_i\|_2 - d_i) \cdot \frac{x - p_i}{\|x - p_i\|_2} \right) \end{aligned}$$

$$\nabla^2 f(x) = J(\nabla f)$$

$$\begin{aligned} &= 2 \sum_i^m J(f_i(x) \nabla f_i(x)) \\ &= 2 \sum_i^m \left[ \nabla f_i(x) (\nabla f_i(x))^T + f_i(x) J(\nabla f_i(x)) \right] \\ &= 2 \sum_i^m \left[ \nabla f_i(x) (\nabla f_i(x))^T + f_i(x) + H_i(f) \right] \end{aligned}$$

$$\begin{aligned} H_i(f) &= J(\nabla f_i) = J\left(\frac{x - p_i}{\|x - p_i\|_2}\right) \\ &= \frac{I \cdot \|x - p_i\|_2 - (x - p_i) \left(\frac{x - p_i}{\|x - p_i\|_2}\right)^T}{\|x - p_i\|_2^2} \end{aligned}$$

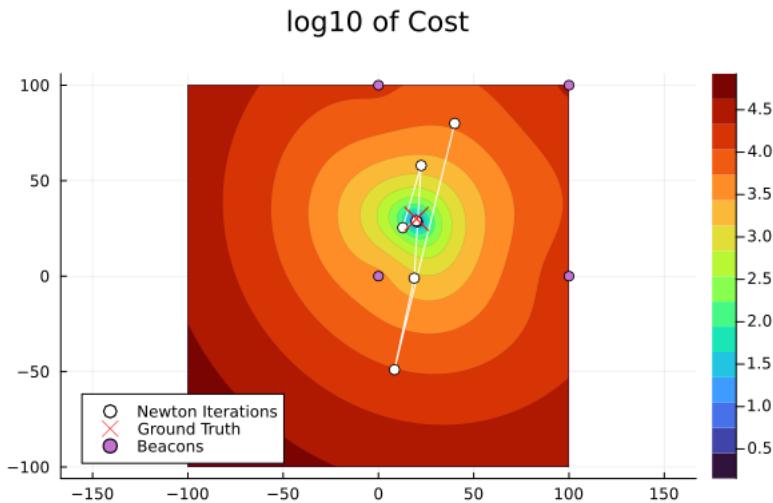
$$\therefore \nabla^2 f(x) = 2 \sum_i^m \left[ \frac{(x - p_i)(x - p_i)^T}{\|x - p_i\|_2^2} + (\|x - p_i\|_2 - d_i) \cdot \left( \frac{I}{\|x - p_i\|_2} - \frac{(x - p_i)(x - p_i)^T}{\|x - p_i\|_2^3} \right) \right]$$

P7:

The function `newton_optimizer()` is finished in Julia.

Note: the definition of the local min should satisfy: gradient is 0 and Hessian matrix is PD,

(a)



In order to give the numerical evidence, I add these code lines under the test file:

```
# ----- P7 test -----

x_hat = x_trace[end]

n = length(x_hat)

gradient_f(xc) = begin
    g = zeros(n)
    m = size(P, 2)
    @inbounds for i in 1:m
        dx = xc .- P[:, i]
        nd = norm(dx)
        if nd > 1e-12
            g .+= 2 * (nd - d[i]) * (dx / nd)
        end
    end
end
```

```

g

end

hessian_f(xc) = begin

    H = zeros(n, n)

    m = size(P, 2)

    I_n = Matrix{Float64}(I, n, n)

    @inbounds for i in 1:m

        dx = xc .- P[:, i]

        nd = norm(dx)

        if nd > 1e-12

            term1 = (dx * dx') / nd^2

            term2 = (nd - d[i]) * ((I_n / nd) - (dx *
            dx') / nd^3)

            H .+= 2 * (term1 + term2)

        end

    end

    H

end

g_final = gradient_f(x_hat)

```

```

println("\n--- Problem 7 Test (same data as plot) ---")
--")

println("Estimated x^      = ", x_hat)

println("Ground truth x_gt = ", x_gt)

println("Final gradient norm ||∇f(x^)|| = ",
norm(g_final))

println("Final gradient = ", g_final)

H_final = hessian_f(x_hat)

eigvals_H = eigvals(H_final)

println("Eigenvalues of Hessian at x^: ",
eigvals_H)

println("λ_min = ", minimum(eigvals_H), ", λ_max = ",
maximum(eigvals_H))

println("-----")
-----")

```

```

--- Problem 7 Test (same data as plot) ---
Estimated x^      = [20.009970633687892, 31.78856799824912]
Ground truth x_gt = [20.0, 30.0]
Final gradient norm ||∇f(x^)|| = 0.0005846087561055961
Final gradient = [0.0004339448970226978, 0.0003917387191270383]
Eigenvalues of Hessian at x^: [3.155490585403337, 4.719091383691101]
λ_min = 3.155490585403337, λ_max = 4.719091383691101
-----
```

For a, I print the norm of the gradient to prove the point is a critical point, notice

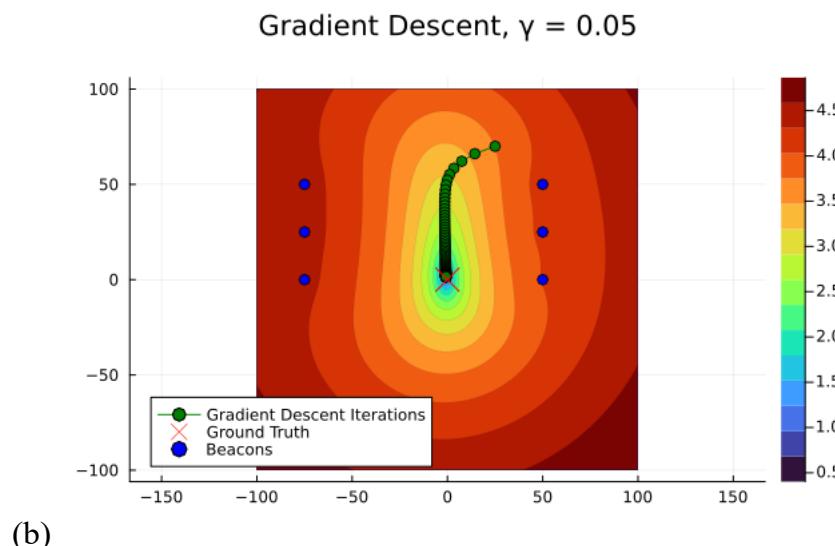
that the result is 0.00058, which is very close to 0. So this is a local min.

**(b)**

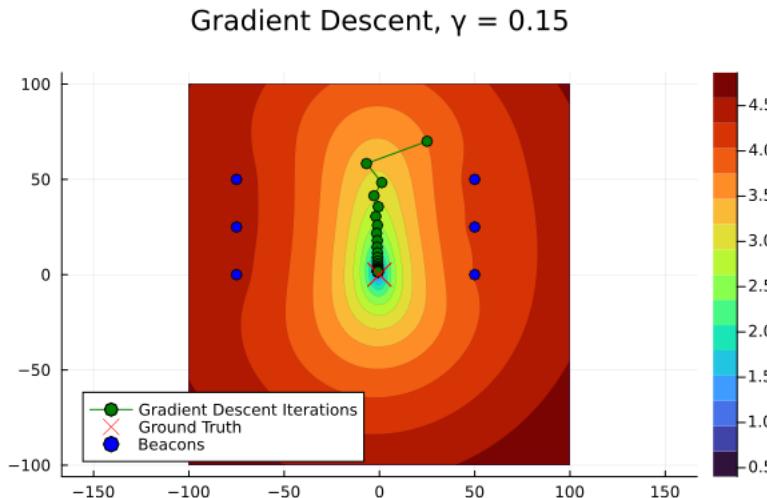
The second derivative for a multivariable function is Hessian matrix, so we measure all of the eigenvalue of the Hessian matrix, we find all of the eigenvalues are positive so which means the local behavior near this critical point is concave up so this point is local min (since H is positive definite).

**P8:**

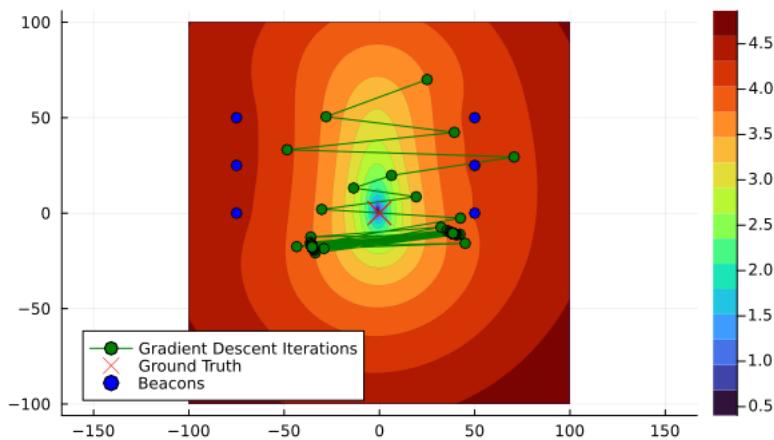
(a) Finished in Julia.



(b)



Gradient Descent,  $\gamma = 0.25$



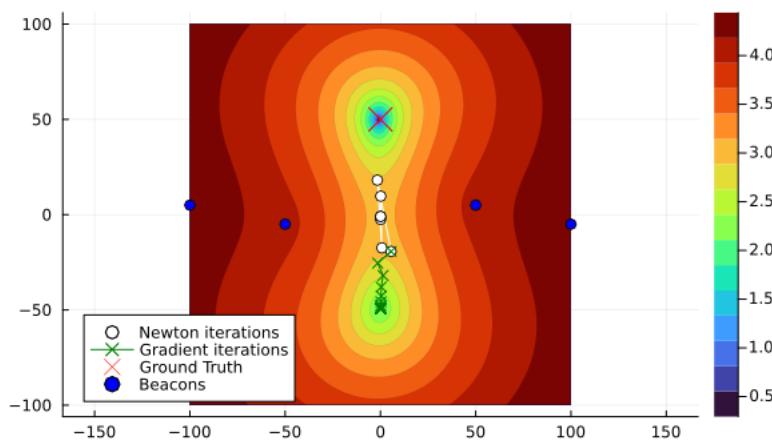
Compare these 3 plots, we find that for the first plot, since the step length is too small so although the gradient method will converge, but it is very slow (note the green points are much more than the second picture).

The second plot is convergent and converged to a good result with less points than first plot.

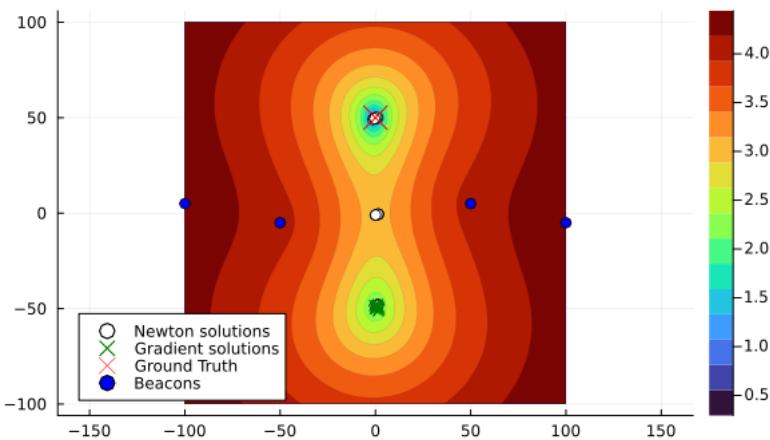
The third plot uses large step length, so it is not convergent to a good result.

**P9:**

Iterations of Newton and Gradient Methods



### Solutions found by Newton and Gradient Methods



(a)

Since the Newton method needs to compute the Hessian matrix and needs to compute its inverse, but when the H is singular (hard to compute the inverse), any small noise will be amplified and that influence the iteration, so the step will be misled. However, the gradient method just simply use the gradient to calculate and guide the next step's direction, which is safe and stable since gradient is always well-defined. So that is the reason why even in the extreme cases, the gradient method converges but not for Newton's method. We can use condition number to examine the status of the Hessian matrix; if the condition number is very high which means the next step might have some problems.

(b)

We should compare the convergence order for each; the order of convergence for Newton method is quadratic and for gradient method is first order, which means  $e_{k+1} \leq C * e_k^2$  for Newton method and  $e_{k+1} \leq C * e_k$  for gradient method, so that is the reason why Newton method converges faster.