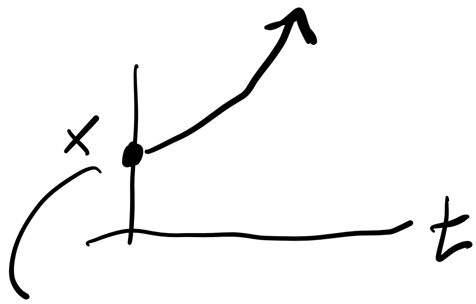
PARAMETRIC NOISE / HETEROGENEITY

EX $$\frac{dy}{dt} = Ay$$

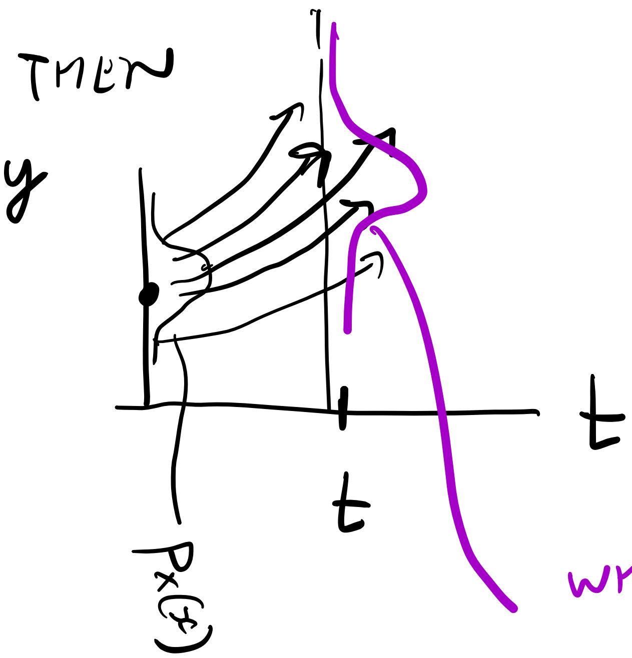


$$y(0) = X \rightsquigarrow y(t) = Xe^{+At}$$

SUPPOSE $X \sim p_X(x)$

X IS RANDOM!

THEN



WHAT IS

$y(t) \sim p_y(y; t)$

SUPPOSE $Y = g(X)$,

SOME FUNCTION

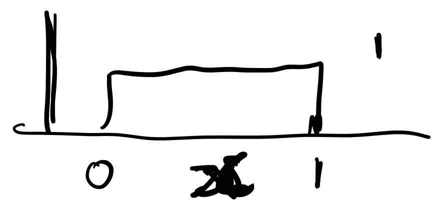
RANDOM VARIABLE

$X \sim p_X(x)$

WHAT IS $p_Y(y)$?

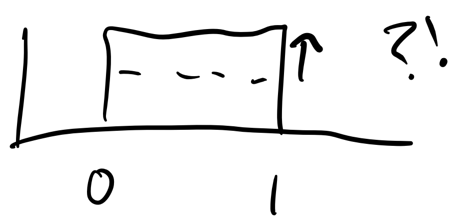
SUPPOSE $Y = cX$

$X \sim \text{UNIF}(0,1)$



IT IS **NOT** TRUE THAT

$p_Y(y) = c\, p_X(x)$!



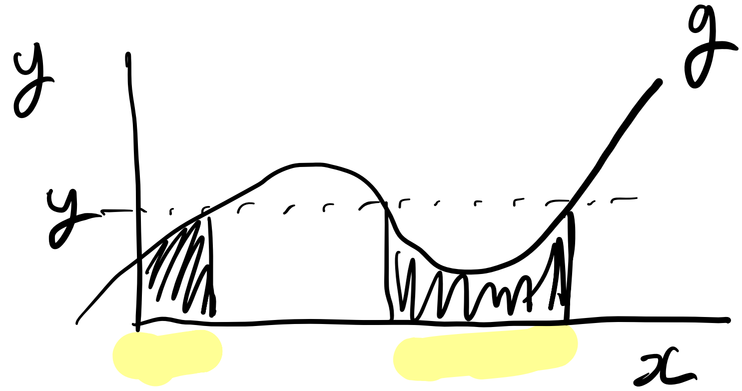
FOR A GENERAL FUNCTION g,

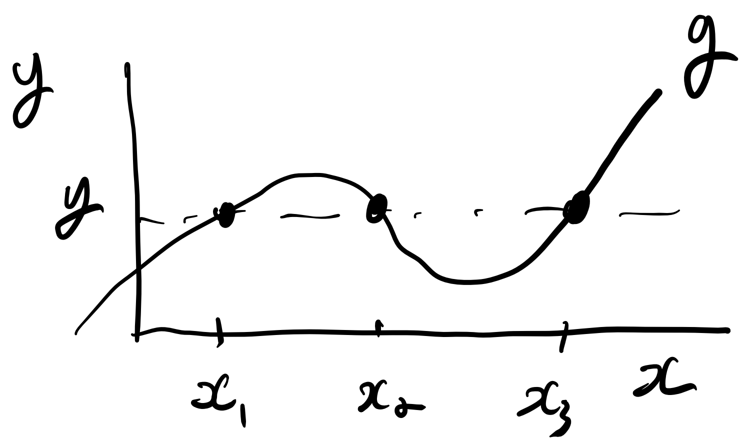
DEFINE $I_y = \{x : g(x) \le y\}$

INDEX SET OF y

$$\{x_k\} = \{x : g(x) = y\}$$

PRE-IMAGE OF y





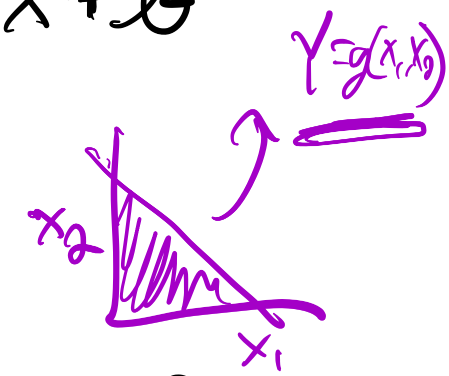
$$\{x_k\} = \{x_1, x_2, x_3\}$$

CUMULATE $F_Y(y) = \mathbb{P}(Y \le y)$

$$= \mathbb{P}(x \in I_y)$$

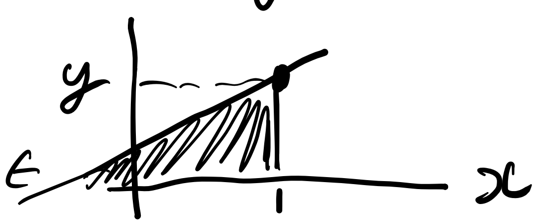
$$= \int_{I_y} p_x(x)\,dx$$

EX $y = ax + b$ $\qquad Y = aX + b$



$a > 0$

$g(x) = ax + b$



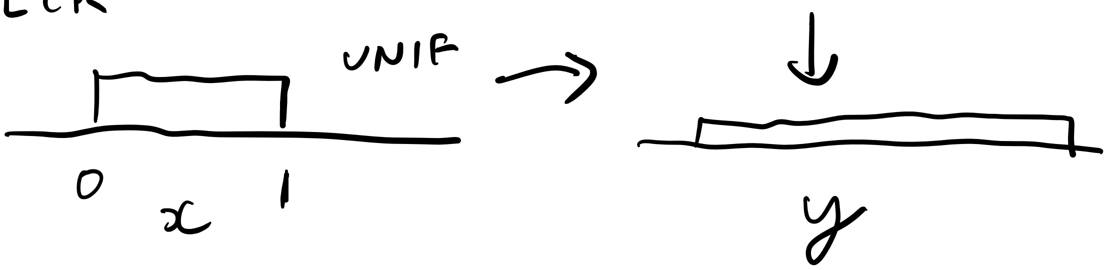
$$I_y = \left\{x \le \frac{y-b}{a}\right\}$$

GIVEN $F_X(x)$, THEN

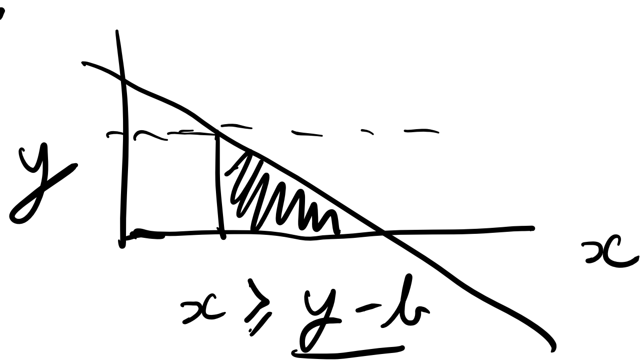
$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$$p_Y(y) = \frac{1}{a} p_X\left(\frac{y-b}{a}\right)$$

CHECK



EX



$x \geq \frac{y-b}{a}$

$y = ax + b$

$a < 0$

$$F_Y(y) = \mathbb{P}\left(x \geq \frac{y-b}{a}\right)$$

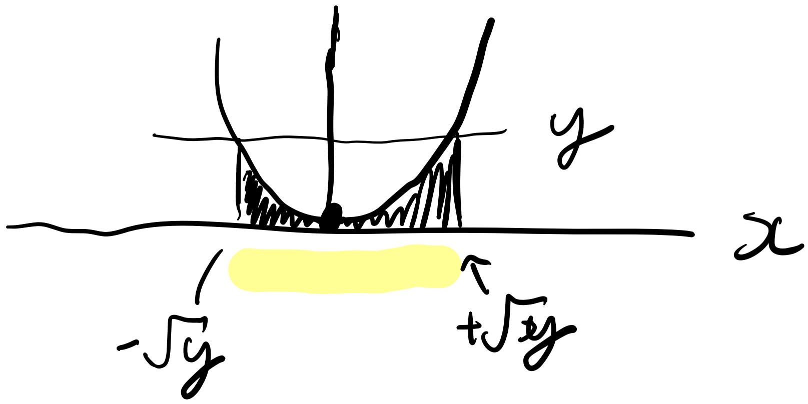
$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

$$p_Y(y) = -\frac{1}{a} p_X\left(\frac{y-b}{a}\right)$$

RECALL $a < 0$

$$= \left|\frac{1}{a}\right| p_X\left(\frac{y-b}{a}\right)$$

EX $\underline{Y = X^2} \rightsquigarrow g(x) = x^2$



$$I_y = \begin{cases} [-\sqrt{y}, \sqrt{y}] & \text{IF } y \geq 0 \\ \text{EMPTY} & y < 0 \end{cases}$$

EX
$Y = X^2 + a$
$g(x) = x^2 + a$

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} p_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} p_X(-\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

GENERAL FORMULA

LET $X \sim p_X(x)$ AND $Y = g(X)$

THEN

$$p_Y(y) = \sum_k p_X(x_k(y)) \cdot \left| \frac{dx_k}{dg} \right|$$

WHERE $\{x_k\}$ IS THE PRE-IMAGE

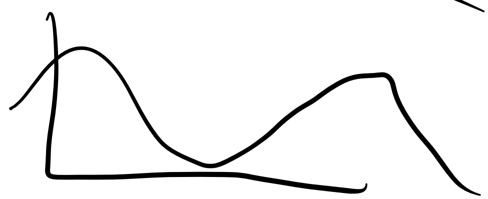
IF $\frac{dg}{dx} \neq 0$, THEN

$$p_Y(y) = \sum_k p_X(g_k^{-1}(y)) \left(\left| \frac{dg}{dx} \right| \right)^{-1}$$

IF g IS MONOTONIC

OR

NOT

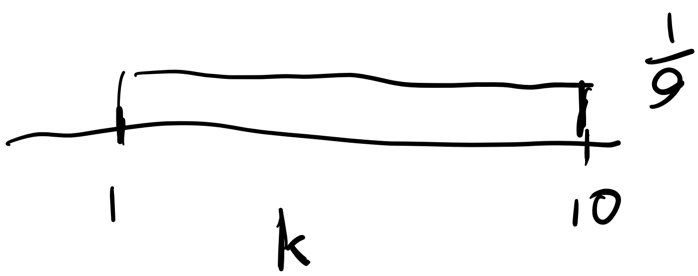


THEN

$$p_Y(y) = p_X(g^{-1}(y)) \left| \left(\frac{dg}{dx}\right)^{-1} \right|$$

EX $K \sim \text{UNIF}(1, 10)$

K - RATE CONSTANT $\left(\frac{1}{\text{TIME}}\right)$



$$p_K(k) = \begin{cases} \frac{1}{9} & 1 \le k \le 10 \\ 0 & \text{ELSE} \end{cases}$$

$T = \frac{1}{K}$ MEAN TIME (TIME)

$g(k) = \frac{1}{k}$ $\frac{dg}{dk} = -\frac{1}{k^2}$

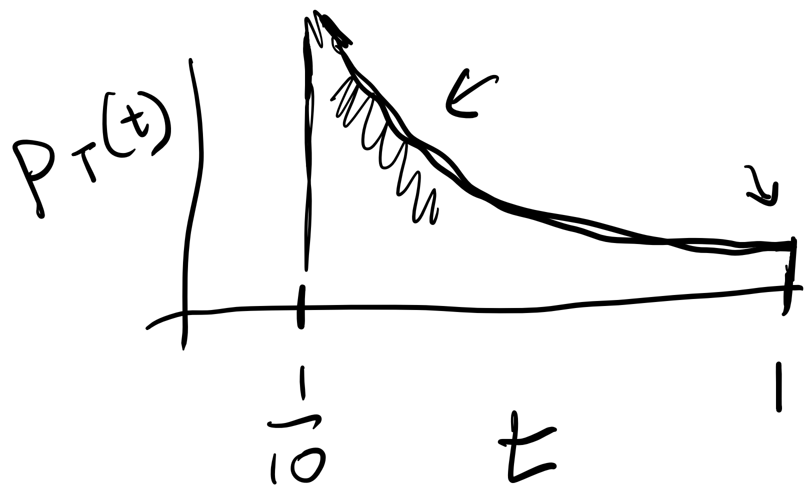
$g^{-1}(t) = \frac{1}{t}$

$$p_T(t) = p_K(g^{-1}(t)) \left(\left| \frac{dg}{dk} \right| \right)^{-1}$$

$$= \begin{cases} \frac{1}{9} \left(\frac{1}{k^2} \right)^{-1} & \frac{1}{10} < t < 1 \\ 0 & \text{ELSE} \end{cases}$$

$t = \frac{1}{k}$

$$= \begin{cases} \frac{1}{9} \frac{1}{t^2} & \frac{1}{10} < t < 1 \\ 0 & \text{ELSE} \end{cases}$$



P_T(t)
1/10
t
1