CH 7.1

Definition [Continuous Function]: Let $\mathcal{X}, Y$ be a topological space, a function $f: \mathcal{X} \to Y$ is a continuous function or continuous map iff $\forall$ open $U \in Y$, $f^{-1}(U)$ is open in $\mathcal{X}$.

Comment: Analogous of continuity in calculus $\to$ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon$, then f is continuous at x_0. The δ & ε neighborhood is indeed the open sets in Tstd.

Theorem 7.1 Let $\mathcal{X}$ and Y be topogical space, let $f: \mathcal{X} \to Y$ be a function, then the following are equivalent.

(1) Function f is continuous.

(2) For every closed set K in Y, the inverse image $f^{-1}(K)$ is closed in $\mathcal{X}$

(3) For every limit point p of a set A in $\mathcal{X}$, $f(p) \in \overline{f(A)}$

(4) For every $x \in \mathcal{X}$ and open set V containing $f(x)$, $\exists U \in T$ s.t. $x \in U$ s.t. $f(U) \subset V$.

Proof:

(1) $\to$ (2):

- If $\forall U$ that's closed in T_Y, take the complement of U in Y or $Y-$, $Y - U$ is an open set.
- Then by definition of continuous function, $f^{-1}(Y-U)$ is also open in $\mathcal{X}$.
- Thus $\mathcal{X} - f^{-1}(Y-U)$ is closed, next we prove why $\mathcal{X} - f^{-1}(Y-U) = f^{-1}(U)$

$\subseteq$: $[x \in f^{-1}(Y-U)] \leftrightarrow [f(x) \in Y - U]$ Hence $x \notin f^{-1}(Y-U) \leftrightarrow f(x) \notin Y - U$. Lemma

Additionally $f(x) \notin Y - U \leftrightarrow f(x) \in U$. Thus we know that $x \in f^{-1}(U)$.

$\supseteq$: If $x \in f^{-1}(U)$, means $f(x) \in U$. $\leftrightarrow$ $f(x) \notin Y - U$. $\to$ $x \in f^{-1}(Y-U)$

- Therefore $f^{-1}(U) = \mathcal{X} - f^{-1}(Y-U)$ is closed.

(2) $\to$ (3):

- If p is a limit point of A. Then $\forall V \in T$ s.t. $p \in U$, we know that $(U \cap A \setminus \{p\} \neq \emptyset)$
- Let $K = Y \setminus \overline{f(A)}$. If $f(p) \notin \overline{f(A)}$, then $f(p) \in K$.

Now by (2), since $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed

$\to$ As $p \in f^{-1}(Y - \overline{f(A)})$ obviously $p \in \mathcal{X} - f^{-1}(\overline{f(A)})$ (using the Lemma proved above).

- Because $f^{-1}(\overline{f(A)})$ is closed $\mathcal{X} - f^{-1}(\overline{f(A)})$ is open, p is an interior point of $\mathcal{X} - f^{-1}(\overline{f(A)})$ in other word, $\exists V \in T$ s.t. $p \in V$, $V \subset \mathcal{X} - f^{-1}(\overline{f(A)})$ or $V \cap f^{-1}(\overline{f(A)}) = \emptyset$

- Therefore $p \in V$ isn't contained in $f^{-1}(\overline{f(A)})$, $f(p) \notin \overline{f(A)}$. Which is exactly the contrapositive argument of (2) → (3).

(3) → (4):

- Let $x \in T_X$ and $V \in T_Y$ s.t. $f(x) \in V$. WTS. $\exists\, U \in T_X$ s.t. $f(U) \subset V$.
- If not such U exist, $\forall\, U \in T_X$ s.t. $x \in U$, $\exists\, y \in U$ s.t. $f(y) \notin V$ (taking the negation of the last argument).

→ Thus is an limit point of $X - f^{-1}(U)$, now by (3) we know that $f(x) \in \overline{f(X - f^{-1}(U))} = \overline{Y - U}$ (using lemma proved in the first part).

- Hence $f(x) \notin Y - U \subset Y - V$.

(4) → (1): It's obvious. ☒.

Theorem 7.2 Let X, Y be topological spaces, $y_0 \in Y$. The constant map $f: X \to Y$ defined by $f(x) = y_0$ is continuous.

Proof:

- For a constant behavior $f(x) = y_0$, the image of $f(x)$ is always $\{y_0\} \subset Y$.
- For any open set $V \subseteq Y$

$$\begin{cases} y_0 \in V, & \text{then } f^{-1}(V) = X \quad (\text{since all } x \in X \text{ map to } y_0 \text{ and } y_0 \in V). \\ y_0 \notin V, & \text{then } f^{-1}(V) = \phi \end{cases}$$

→ In both cases, both preimage set is open in the topology. Thus the map is continuous. ☒.

Theorem 7.3 Let $X \subset Y$ be topological spaces, the inclusion map $i: X \to Y$ defined by $i(x) = x$ is continuous.

Proof:

- $i^{-1}(V) = \{x \in X \mid x \in V\}$

If $V \subseteq Y$ is open, then $i^{-1}(V) = V \cap X$. The intersection of an open set in Y with the subspace topology in X. $i^{-1}(V)$ is therefore open in X. Hence continuous. ☒.

Theorem 7.4: Let $f: X \to Y$ be a continuous map between topological spaces and let A be a subset of X. The the restriction map $f|_A: A \to Y$ defined by $f|_A(a) = f(a)$ is continuous.

Proof:

Expanding the word continuous: if for every open set $V \subseteq Y$, the preimage $(f|_A)^{-1}(V)$ is open in A (with subspace topology).

The Preimage of an open set $V \subseteq Y$ under $f|_A$ is $(f|_A)^{-1}(V) = \{a \in A \mid f(a) \in V\}$

Obviously $(f|_A)^{-1}(V) = A \cap f^{-1}(V)$, by subspace topology, is indeed open. ☒.

Definition: Let $f: X \to Y$ be a function between topology space between X and Y and let $x \in X$. f is continuous at the point x iff every open set V containing $f(x)$, there is an open set U in X s.t. $f(U) \subset V$.

- A function $f: X \to Y$ is continuous iff it's continuous at each point.

Theorem 7.5: A function $f: \mathbb{R}_{std} \to \mathbb{R}_{std}$ is continuous if and only if for every point $x \in \mathbb{R}$ and $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall y \in \mathbb{R}$ s.t. $d(x,y) < \delta$, $d(f(x), f(y)) < \varepsilon$.

Proof: (Take the definition of open sets and turn it into the concept of ε-neighborhood, then we completed the proof).

Theorem 7.6: Let X be a 1^{st} countable space, and let Y be a topological space, then a function $f: X \to Y$ is continuous iff for each convergent sequence $x_n \to x$ in X. $f(x_n)$ converge to $f(x)$ in Y.

- It again shows that 1^{st} countable space is modelling metric space.

Proof:

- X is countable, means $x \in X$ has a countable neighborhood basis. Let the neighborhood basis be denoted $(B_1, B_2, \dots, B_n, \dots)$.
- If f is continuous, then f is continuous at any $x \in X$ means for every open set V containing $f(x)$, $\exists U \in X$ s.t. $f(U) \subset V$.

→ For a convergence sequence in X, meaning for any U contain x, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $x_n \in U$

$\Rightarrow$:

- For any sequence $x_n \to x \in X$.
- Let $V \subset Y$ be a neighborhood of $f(x)$. f is continuous, $f^{-1}(V)$ is open in X and $f^{-1}(V)$ contains x.

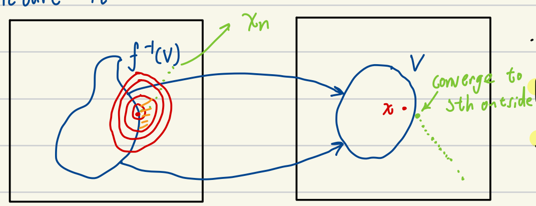
- Since $x_n \to x$, $\exists N \in \mathbb{N}$ s.t. $x_n \in f^{-1}(V)$ $\forall n \geq N$.
  Thus $f(x_n) \in V$ for every $n \geq N$. Thus $f(x_n) \to f(x)$. Then $f(x_n) \to f(x)$ for every $n \geq N$. Proving $f(x_n) \to f(x)$.

$\Leftarrow$:

→ We WTS $\forall V \in Y$, $f^{-1}(V)$ is open in X.

- If $f^{-1}(V)$ is not open, then there is a limit point of $X - f^{-1}(V)$ is situated in $f^{-1}(V)$. Call the point $x \in X$.
- Because it's a limit point, and it's in X, a first countable space.
  1) Every neighborhood of X intersect $X - f^{-1}(V)$
  2) Exist a countable nested local basis of x, call it $\{U_i\}_{i \in \mathbb{Z}^+}$ $U_i \supset U_{i+1}$

To Picture it:



- A little logic summary:
- We are proving using contrapositive
  WTS → if it's not continuous, then $\exists$ a sequence that's $f(x_n)$ but do not converge to $f(x)$

- Here's how I construct the sequence, the points $x_n \in U_n \setminus f^{-1}(V)$. This's x_n exists because x is a limit point of $X - f^{-1}(V)$
- It's clearly a convergent sequence because by definition of local basis, every open set containing x will have some local basis element contained in it. Hence infinitely many points in it.
- Now because $f(x_n) \to f(x)$. As $f(x) \in V$ and V is open, $\exists N$ s.t. $f(x_n) \in V$ for every $n \geq N$. But clearly since $x_n \notin f^{-1}(V)$, then $f(x_n) \notin V$ $\forall n$.
- Therefore $f^{-1}(V)$ is open, therefore f is continuous. ☒.

(It's fundamentally because sequence is a countable & pointwise "existence").

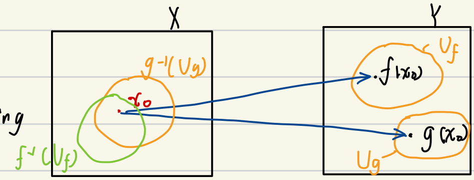
Theorem 7.7: Let X be a space with a dense set D and let Y be Hausdorff. Let $f: X \to Y$ and $g: X \to Y$ be continuous functions such that $\forall d \in D$, $f(d) = g(d)$, then for any $x \in X$ $f(x) = g(x)$

Proof:

- If there exists some x_0 such that $f(x_0) \neq g(x_0)$, where $x_0 \in D$.

Logic: WTS f is discontinuous, or there exists an open set $U \in Y$ s.t. $f^{-1}(U)$ is not open in X.

(We can prove using contrapositive).



- Our strategy is to separate the 2 sets using Hausdorff property.
- Since $f(x_0) \neq g(x_0)$ and Y is Hausdorff, $\exists\, U_f$ and U_g s.t. $U_f \cap U_g = \phi$ and $f(x_0) \in U_f$ and $g(x_0) \in U_g$. Since we know g is continuous, we know that $g^{-1}(U_g)$ is open.
- What about $f^{-1}(U_f)$? It also contain x_0. Thus the set $f^{-1}(U_f) \cap g^{-1}(U_g)$ are is a set that contains x_0.

→ Suppose it has some point $d \in D$ s.t. $d \in f^{-1}(U_f) \cap g^{-1}(U_g)$ but then $f(d) = g(d)$ are both in U_f and U_g, which are 2 disjoint sets. Contradiction.

→ Thus $f^{-1}(U_f) \cap g^{-1}(U_g)$ is not open. Since we know $g^{-1}(U_g)$ is open, $f^{-1}(U_f)$ has to be not open. ⊠.

Theorem 7.8: The cardinality of the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is the same as cardinality of $\mathbb{R}$.

- By the last theorem, we know the value of function in a dense set can certify function value of the entire function. Continuous function $\mathbb{R}$ can be certified by $f(\mathbb{Q})$.
- Hence it's an uncountable set, each element consist of countable element
- Just like how element of $\mathbb{R}$ is an uncountable set of countable sequence.

Theorem 7.9: If $f: X \to Y$ and $g: Y \to Z$ are both continuous, then the function $g \circ f: X \to Z$ is also continuous.

Proof:

- If g is continuous, then for some $U \subset Z$, $f^{-1}(U) \subset Y$ is open.
- Then $g^{-1}(f^{-1}(U)) = (g \circ f)^{-1}(U) \subset X$ is also open as g is also continuous and hence $(g \circ f)$ is continuous. ⊠

Theorem 7.10 (Pasting Lemma): Let $X = A \cup B$, where A, B are closed in X.

Let $f: A \to Y$ and $g: B \to Y$ be continuous function that agrees on $A \cap B$. Then the function $h: A \cup B \to Y$ defined by $h = f$ on A and $h = g$ on B is continuous.

Comment: This theorem is saying that f and g combines to give a continuous function $h: X \to Y$.

Proof:

- To show why h is continuous, we verify the preimage of every closed set in Y is going to be closed in X.
- Suppose C is closed in Y, the preimage of $h^{-1}(C)$ is

$h^{-1}(C) = \{x \in A \mid f(x) \in C\} \cup \{x \in B \mid g(x) \in C\} = f^{-1}(C) \cup g^{-1}(C)$.

- Because f is continuous on A, $f^{-1}(C)$ is closed in A (or in a better word, A is closed under subspace topology that X inherit on A)

→ Since A is also closed in X, we know that $f^{-1}(C)$ is closed in X. (Using theorem 3.28, which states that if there (C is closed in Y, Y is a subspace of X) $\Leftrightarrow \exists D \subset X$ closed in X s.t. $C = D \cap Y$). Here $C \subset A$ is D.

→ For the same reason, $g^{-1}(C)$ is also closed in X. And hence $g^{-1}(C) \cup f^{-1}(C)$ is also closed in Y. ☒.

Comment: Pasting lemma fails if A & B are not closed, because if A or B is not closed, the $f^{-1}(U) \cup g^{-1}(U)$ is not necessarily in the first place. An counterexample:

- Consider $X = \mathbb{R}$, $A = \mathbb{Q}$ (rational numbers) and $B = \mathbb{R} \setminus \mathbb{Q}$ (irrational numbers).

Both A and B are not closed in $\mathbb{R}$. But $\mathbb{R} = A \cup B$.

$f: A \to \mathbb{R}$ by $f(x) = 0$. Since $A \cap B = \phi$ f and g agree on intersection

$f: B \to \mathbb{R}$ by $f(x) = 1$ $h(x) = 0$ if $x \in \mathbb{Q}$ and 1 if $x \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 7.11 (Pasting Lemma): Let $X = A \cup B$, where A, B are open in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous functions which agrees on $A \cap B$. Then the function $h: A \cup B \to Y$ defined by $h = f$ on A and $h = g$ on B is continuous

Proof:

- Similar to last proof, we show the preimage of every open set is open.
- Let $U \subset Y$ be open

The preimage of $h^{-1}(U)$ is : (Same as last theorem)
$\rightarrow$ $h^{-1}(U) = \{x \in A \mid f(x) \in U\} \cup \{x \in B \mid g(x) \in U\} = f^{-1}(U) \cup g^{-1}(U)$.
- Because f is continuous on A, $f^{-1}(U)$ is open in A, since A is open in X, $f^{-1}(U)$ is also open in A. (By definition of subspace).

$\rightarrow$ Similarly $g^{-1}(U)$ is open in B and hence open in X.
The union of $f^{-1}(U) \cup g^{-1}(U)$ is open in X. Hence $h^{-1}(U)$ is open in X $\forall U \subset Y$, then h is continuous. $\boxtimes$.

Theorem 7.13/14: Let $f: X \to Y$ be a function and let B be a basis in Y. Then f is continuous iff every open set U_B in B. $f^{-1}(U_B)$ is open in X. (Same thing every subbasis open set.
Proof: It's obvious.

CH 7.2

Comment: This chapter discusses what properties are preserved under continuity.
Theorem 7.15: If X is compact, and $f: X \to Y$ is continuous, then Y is compact.
Proof:
- Suppose Y has a open cover $\{U_\alpha\}_{\alpha \in \lambda}$. Then for any $\alpha \in \lambda$ $f^{-1}(U_\alpha)$ is open in X. The Union of $f^{-1}(U_\alpha)$, $\bigcup_{\alpha \in \lambda} f^{-1}(U_\alpha)$ forms an open cover of X.
- Then because X is compact, we may find a finite subcover, $f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})$ covers X. We now show $U_{\alpha_1}, \dots, U_{\alpha_n}$ covers Y.

For any y, $f^{-1}(y) \in X$ and $\exists i \in \{1, \dots, n\}$ s.t. $f^{-1}(y) \in f^{-1}(U_{\alpha_i})$ and hence $y \in U_{\alpha_i}$. Therefore U_{α_i} is a subcover of Y. Thus Y is compact. $\boxtimes$.
Comment: Similar properties that are preserved under continuity are:
Theorem 7.16 Lindelöf space, Theorem 7.17: countable compactness but we aren't proving them over here.

Theorem 7.18: Let D be a dense set of a topological space on X. Let $f: X \to Y$ be continuous and surjective. Then $f(D)$ is dense in Y.
Proof:

- For every open set $U \in Y$, $f^{-1}(U)$ is open in X because f is continuous. Since D is dense in X, $\exists x_0 \in D$ s.t. $x_0 \in f^{-1}(U)$.
- Thus $f(x_0) \in U \rightarrow f(D)$ is dense in Y. ∎

Corollary 7.19: Let X be a separable space, let $f: X \to Y$ be continuous and surjective then Y is separable.

Definition [Closed and Open function]:

1) A function $f: X \to Y$ is closed iff for every closed $A \in X$, $f(A)$ is closed in Y.
2) A function $f: X \to Y$ is open iff for every open $U \in X$, $f(U)$ is open in Y.

- To get a sense how an open & closed function are not related to be being continuous or not, we consider some examples:

① Open function that's not continuous:

- The identity function $\mathbb{R}_{std} \longrightarrow \mathbb{R}_{discrete}$. It's open since every set in $\mathbb{R}_{discrete}$ is open. But the preimage of open set (say a singleton) is not necessarily open.

② Closed function that's not continuous:

- The identity function $\mathbb{R}_{cofinite} \to \mathbb{R}_{std}$. It's closed since every closed set (which are finite sets in cofinite topology) is closed (since in $\mathbb{R}_{std}$, which is T_4 hence T_1, singleton are closed, hence finite union of singleton are closed.)
- But the preimage of a closed set, say $[0,1]$, which is closed in $\mathbb{R}_{std}$, is not having a closed preimage in $\mathbb{R}_{cofinite}$.

③ Continuous function that's neither open nor closed

- The function $f(x) = \frac{1}{1+x^2}$ is clearly continuous on $\mathbb{R}_{std}$
- Preimage of $(-1,1)$ is $(\frac{1}{2}, 1]$ $\rightarrow$ not open
- Preimage of $\mathbb{R}$ is $(0,1]$ $\rightarrow$ not closed.

④ Continuous function that's open but not closed:

- The projection function $\pi(x,y) \to x$ that maps $\mathbb{R}^2_{std} \to \mathbb{R}$ is obvious continuous.
- The open sets in product topology is of the form (open set) × (open set). Hence the map is open.
- The closed set $\{(x, \frac{1}{x}) \mid x \neq 0\} \subset \mathbb{R}^2$ maps to $(-\infty, 0) \cup (0, \infty)$ is not closed

⑤ Continuous function that is closed but not open:

- The function $f(x) = |x|$, is clearly continuous
- It's closed because any closed set maps to $[0, \infty)$
- It's not open since the image of $(-1, 1)$ is $[0, 1)$ is not open.

Theorem 7.21: If X is normal and $f: X \to Y$ is continuous, surjective and closed then Y is normal.

Proof:

- Let A and B be two disjoint sets. Because f is continuous, their preimages $f^{-1}(A)$ and $f^{-1}(B)$ are closed in X.
- Since X is normal, we may find 2 disjoint open sets $U, V \subseteq X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$ with $U \cap V = \phi$. The picture looks like below.

- Now consider the sets

$W = Y \setminus f(\underbrace{X \setminus \underset{\text{open}}{U}}_{\text{closed}})$ $\quad Z = Y \setminus f(X \setminus V)$

As f is a closed map $\to$ $Y \setminus f(X \setminus U)$ is open

For similar reason as above $Y \setminus f(X \setminus V)$ is open.

(Blue part on left is $X \setminus V$, blue part on the right is $Y \setminus f(X \setminus U)$

- The orange part on the right is $Y \setminus f(X \setminus V)$.

- $\forall a \in A$, $f^{-1}(a) \subseteq f^{-1}(A) \subset U \to f^{-1}(a) \notin X \setminus U$ hence $a \notin f(X \setminus U)$. $\to a \in W$.
  $\forall b \in A$, $f^{-1}(B) \subseteq f^{-1}(B) \subset V \to f^{-1}(b) \notin X \setminus V$ hence $b \notin f(X \setminus U) \to b \in Z$
- Thus we prove $A \subseteq W$ and $B \subseteq Z$

Why $W \cap Z = \emptyset$:

- Suppose $y \in W \cap Z$. Then $y \notin f(X \setminus U)$ and $f(X \setminus V)$.
  - Because f is surjective, $y = f(x)$ for some $x \in X$. Then by preimage of the two closed set above, $x \notin X \setminus V$ and $x \notin X \setminus V$.
  - But $U \cap V = \phi$, which is a contradiction. Thus W & Z are disjoint.
- Hence we found 2 disjoint open set in Y separating A & B $\to$ Y is normal.

Theorem 7.22: If $\{B_\alpha\}_{\alpha \in \lambda}$ is a basis for X and $f: X \to Y$ is continuous, surjective and open, then $\{f(B_\alpha)\}_{\alpha\in\lambda}$ is a basis for Y.

Proof:

- If $\{B_\alpha\}_{\alpha\in\lambda}$ is a basis for $f: X \to Y$ is continuous, surjective and open, then $\{f(B_\alpha)\}_{\alpha\in\lambda}$ is a basis for Y.
- We use theorem 3.1, to show it is a basis we show it covers Y and we show it has the intersection property.

- Since f is surjective, every $y \in Y$ has $y = f(x)$ for some $x \in X$.

→ Because $\{B_\alpha\}$ is a basis for X, there exists B_α containing x. Thus, $y \in f(B_\alpha)$

Hence $\{f(B_\alpha)\}$ covers Y.

- Let $y \in f(B_\alpha) \cap f(B_\beta)$. Then $y = f(x_1) = f(x_2)$ for $x_1 \in B_\alpha$ and $x_2 \in B_\beta$
- The set $U = f(B_\alpha) \cap f(B_\beta)$ is open in Y

  Since f is continuous $f^{-1}(U)$ is open in X and contains x_1 and x_2.

  By the basis property of $\{B_\alpha\}$. There exists $B_\gamma \subset f^{-1}(U)$ containing x_1.
- Thus $f(B_\gamma) \subset V$ and $y \in f(B_\gamma)$.

- And therefore $\{f(B_\alpha)\}$ satisfy the basis condition. $\boxtimes$.

Corollary 7.23: If X is 2^{nd} Countable and $f: X \to Y$ is continuous, surjective and open then Y is 2^{nd} countable.

Proof: The theorem directly follows from 7.22, changing the basis to be countable.

Theorem 7.24: Let X be compact, and let Y be Hausdorff. Then any continuous function $f: X \to Y$ is closed.

Proof:

- For a continuous function $f: X \to Y$. Suppose there is a closed set in X, call it C, then we want to show that $f(C)$ is closed.
- Since C is closed in X, a compact space, by theorem 6.8. C is also compact.

  Now by theorem 7.15 the set $f(C)$ is also compact (because f is continuous).
- Thus we know that $f(C)$ is a compact subspace of Hausdorff space (Y), thus we know that $f(C)$ is closed. Hence f is closed, by theorem 6.9. $\boxtimes$.

(Note that in 2^{nd} step we also used theorem 7.4, the restriction map of a continuous map is also continuous, hence $f|_C$ is continuous).

Theorem 7.25: Let X be compact and 2^{nd} countable and let Y be Hausdorff. If $f: X \to Y$ is continuous and surjective, Y is 2^{nd} countable.

Proof:

- Since X is second countable, there exists a countable basis $\{B_i\}_{i \in \mathbb{N}}$
- We claim that $\{f(B_i)\}_{i \in \mathbb{N}}$ is the countable basis for Y. It directly follows from theorem 7.22. ☒.

CH 7.3

Definition: A function $f: X \to Y$ is a homeomorphism iff f is continuous bijection and the inverse map $f^{-1}: Y \to X$ is also continuous.

Definition: Two topological space X and Y are homeomorphic or topologically equivalent iff there exists a homeomorphism $f: X \to Y$.

Comment: A homeomorphism f provides both a bijection between sets and a bijection between topologies. As open sets are preserved by f & f^{-1}, then all the topological properties (Hausdoroff, regular, normal, compact, dense, basis) are all preserved by a homeomorphism.

- Indeed a very important property of homeomorphism is that:

Theorem 7.26: Being homeomorphic is an equivalence relation on topological spaces.

Proof:

1) Reflexivity: The identity map $id_X: X \to X$ defined by $id_X(x) = x$, is a continuous bijection its inverse is still id_X which is continuous.

2) Symmetry: Let $f: X \to Y$ be a homeomorphism. $f^{-1}: Y \to X$ is also a homeomorphism. Because f is a continuous bijection with f^{-1} is continuous.)

3) Transtivity: Let $f: X \to Y$ and $g: Y \to Z$ be homeomorphisms, the composite function $g \circ f: X \to Z$ is:

① Bijection (as both f & g are bijective)

② Continuous by theorem 7.9.

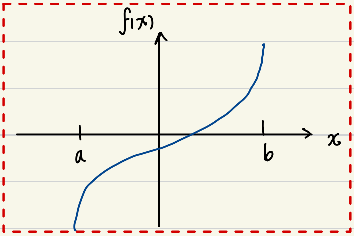
③ It's inverse is continuous as composition of continuous functions are continuous.

- As needed. ☒.

Example : Let a, b be points in $\mathbb{R}^1$ with $a < b$. Show that (a, b) with the subspace topology from $\mathbb{R}^1_{std}$ is homeomorphic to $\mathbb{R}^1_{std}$.

E.g. $f(x) = \tan\left(\frac{\pi}{a-b}\left(x - \frac{a+b}{2}\right)\right)$

- We used translation & stretch to make the domain (a, b) and the function looks like



- By property of tangent it's obviously a continuous bijection and we know that arctan is also a continuous function.
- Thus we find a homeomorphism between (a, b) and $\mathbb{R}$. Hence they have the same cardinality.

Theorem 7.28 If $f: X \to Y$ is continuous, the following are equivalent.

(a) f is a homeomorphism.

(b) f is a closed bijection.

(c) f is an open bijection.

Proof :

(a) $\to$ (b) :

Because f is a continuous bijection.

- If f has a continuous inverse, then any closed set $C \subset X$, $(f^{-1})^{-1}(C) = C$ is closed. Thus $f^{-1}(C)$ is closed iff C is closed $\to$ it's a closed map.
- It's a bijection, since f is a homeomorphism.

(a) $\to$ (c) : Similar Reason

(b) $\to$ (a) : Let $C \subset X$ be closed. Since f is closed, $f(C)$ is closed in Y. Then the preimage of $f(C)$ under f^{-1} is C. Hence f^{-1} is continuous. As f is bijection by hypothesis, hence f is a homeomorphism. □.

(c) $\to$ (a) : Similar Reason

Theorem 7.29: Suppose $f: X \to Y$ is continuous bijection where X is compact and Y is Hausdorff. Then f is a homeomorphism.

Proof:

- Because X is compact and Y is Hausdorff, by theorem 7.24, the map f

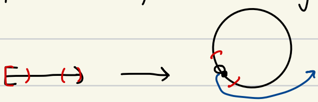
is closed hence f is a closed bijection.

- Additionally because f is continuous, by theorem 7.28 (b) $\to$ (a), f is a homeomorphism. $\square$.

Example: Illustrate why theorem 7.29's condition being compact and Hausdorff is necessary. (We are essentially illustrating why a continuous bijection doesn't generally have a continuous bijection).

- When X isn't compact:

We map the clopen line segment in $\mathbb{R}^2$ to a circle in $\mathbb{R}^2$



it's where the two endpoints intersect.

$\to$ It's obviously a continuous bijection. But the inverse is not necessarily continuous. In particular, the red part is an open under subspace topology on the circle but not an open set on the line.

- When Y is not Hausdorff:

For example the identity map from $\mathbb{R}$ discrete to $\mathbb{R}$ indiscrete is a continuous bijection because any set in discrete topology is an open set but the inverse is never open. $\square$.

Definition [Embedding]: A function $f: X \to Y$ is an embedding iff $f: X \to f(X)$ is a homeomorphism from X to $f(X)$. $f(X)$ has the subspace topology from Y.

- And with that information in mind, we have a corollary directly from the previous theorem.

Corollary 7.31: Let X be a compact space and let Y be Hausdorff. If $f: X \to Y$ is a continuous injective map, then f is an embedding.

Proof: Directly follow from 7.29.

CH 7.4

Definition [Projection Map]: The projection maps $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are defined as $\pi_X(x,y) = x$, $\pi_Y(x,y) = y$.

Comment: Product spaces have natural projection function and the continuity of those projection mapping, which characterizes topology of the product space.

Theorem 7.32: Let X and Y be topological spaces, the projection maps π_X, π_Y on $X \times Y$ are continuous, surjective and open.

Proof:

- WLOG, we do the proof for π_X

1. Continuous ($\{\pi_X^{-1}(A) \mid A \in X \times Y\}$ is open under product topology): It's obvious because $\pi_X^{-1}(A)$ is exactly the subbasic element under definition of product topology.
2. Surjective ($\forall x \in Y, \exists x' \in X \times Y$ s.t. $f(x') = x$): It's also obvious because if $x \in Y$, then for any $x_0 \in X$, (x, x_0) is obviously an open set in X.
3. Open (A recap of definition: A function $f: X \to Y$ is open iff $\forall U \in X$, $f(U)$ is open in Y)
   Here it means $\forall U \in X \times Y$ $\pi^{-1}(U)$ is open in Y.
   Under finite product, the box topology is same as product topology.

- $f(U)$ is open because by definition of box topology, $U = U_X \times U_Y$ where $U_X \in T_X$ and $U_Y \in T_Y$. Hence $\pi_X(U) = U_X \in T_X$ as needed. $\square$.

Theorem 7.33: Let $X \times Y$ be topological spaces, the product topology is the coarsest topology that makes π_X, π_Y on $X \times Y$ continuous.

Proof:

- We show that the T is a topology on $X \times Y$ that make π_X and π_Y continuous, then T must contain product topology.

$\to$ $\forall U \subseteq X$ $\quad \pi_X^{-1}(U) = U \times Y$ must be open in $X \times Y$

$\quad \forall V \subseteq Y$ $\quad \pi_Y^{-1}(V) = X \times V$ must be open in $X \times Y$

- Basis of product topology ensures $\pi_X^{-1}(U) \cap \pi_Y^{-1}(U)$ is open (a finite intersection).
- Conversely, if there is a topology on $X \times Y$ that is coarser than product topology. $\exists U \notin \{$The coarser topology$\}$ s.t. $U \in T_{X \times Y}$. $\leftarrow$ U can be written as $U_X \times U_Y$ where $U_X \in X$, $U_Y \in Y$ in product topology.

- Then the subbasis $\{\pi_X^{-1}(U_X) = U_X \times Y,\ \pi_Y^{-1}(U_Y) = X \times U_Y\}$ has at least one not open under product topology, or else $\pi_X^{-1}(U_X) \cap \pi_Y^{-1}(U_Y)$ is open, which contradicts our assumption.

$\rightarrow$ Thus either π_X is discontinuous or π_Y is discontinuous. $\boxtimes$

Example: Where $\pi_X : X \times Y \to X$ is not necessarily a closed map.

- Define the projection map $\pi_X : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\pi_X(x, y) = x$
- Consider the closed set $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\}$.
  It's closed since it's the zero-set of $f(x, y) = xy - 1$, a continuous mapping.
- The image of C under π_X is $\{x \in \mathbb{R} \mid \exists\, y \in \mathbb{R},\ xy = 1\}$

$\rightarrow$ It simplifies to $\pi_X(C) = \mathbb{R} \setminus \{0\}$, which is clearly not a closed set since $\{0\}$ is not an open set

---

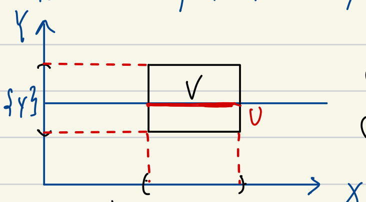
Theorem 7.35 Let X and Y be topological space. $\forall\, y \in Y$, the subspace $X \times \{y\}$ of $X \times Y$ is homeomorphic to X.

(Recap: A function $f : X \to Y$ is a homeomorphism iff f is a continuous bijection and $f^{-1} : Y \to X$ is also continuous).

Proof:

- The subspace topology, by definition, is given by $\{U \mid U = V \cap X \times \{y\},\ V \in X \times Y\}$.

(As an example, we may view it as the following diagram



- Therefore, the question want us to show

① $\exists\, f : X \times \{y\} \to X$ s.t. f is a continuous bijection and

② $f^{-1} : X \to X \times \{y\}$ is continuous.

- We claim the function $f(x, y) = x$

① : The function is a bijection, obviously.

The function is continuous: $f^{-1}(U) = U \times \{y\}$.

$\rightarrow$ Suppose U is open in X. $U \times \{y\} = \bigcup \{U \times P : P \in T_Y\} \cap X \times \{y\}$

Since $P \in T_Y$, $U \in T_X$, $U \times P$ is open in $X \times Y$.

- If $\{y\}$ exist in the union, then the equation true.
- If $\{y\}$ doesn't exist in the union, meaning $\{y\}$ itself is an isolated point,

then $\{y\}$ is an open set itself. Therefore we may conclude either way
→ the preimage of U is open.

② : Suppose there is an open set $U \in X \times \{y\}$, by definition of subspace topology,
$\{ U \mid U = V \cap X \times \{y\}, V \in X \times Y \}$

Since $U \in X \times \{y\}$ → $U = U_x \times \{y\}$

(Note since f is a bijection, $(f^{-1})^{-1} = f$)

$(f^{-1})^{-1}(U_x \times \{y\}) = U_x$. By box topology on $X \times \{y\}$, U_x is an open set in X. ☐

Theorem 7.36 : Let X, Y, Z be topological spaces. A function $g: Z \to X \times Y$ is continuous iff $\pi_X \circ g$ and $\pi_Y \circ g$ is continuous.

Comment: An example is that the parametric function $f: \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, a parametric curve $(x^2, \cos x)$, both $x^2 = \pi_X(x^2, \cos x)$ $\cos x = \pi_Y(x^2, \cos x)$ has to be continuous for the curve to be continuous.

Proof:

$\Rightarrow$: $g: Z \to X \times Y$ is continuous $\longrightarrow$ both $(\pi_X \circ g$ is continuous$) \wedge (\pi_Y \circ g)$ is contin.

- Since $\pi_X \circ g$ is continuous, Let A be an open set in X
$(\pi_X \circ g)^{-1}(A) = (g^{-1} \circ \pi_X^{-1})(A) = g^{-1}(\pi_X^{-1}(A))$
→ We know the set $\pi_X^{-1}(A) = A \times Y$. which is a product of 2 open sets. In the definition of product (finite box) topology, is an open set in $X \times Y$.
- Since $g: Z \to X \times Y$ is continuous, meaning if $\exists U \in T_{X \times Y}$, $g^{-1}(U)$ is open.
Thus $g^{-1}(\pi_X^{-1}(A)) = g^{-1}(A \times Y)$ is open. Thus $\pi_X \circ g$ is continuous.

→ We repeat the steps and the same argument works.

$\Leftarrow$: Both $\pi_X \circ g$ & $\pi_Y \circ g$ is continuous $\longrightarrow$ $g: Z \to X \times Y$ is continuous.

- For some open sets in $X \times Y$.
- By definition of the product topology, $X \times Y = \bigcup$ (basic elements) where basic elements are given by $U \times V$, $U \in T_X$ and $V \in T_Y$.

→ U can also be written as $\pi_X(U \times Y)$
V ~ $\pi_Y(V \times X)$ → $\boxed{\begin{array}{l} U \times Y = \pi_X^{-1}(U) \\ V \times X = \pi_Y^{-1}(V) \end{array}}$

- Therefore $U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$

$$g^{-1}(U \times V) = g^{-1}(\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)) = \underbrace{(\pi_X \circ g)^{-1}(U) \cap (\pi_Y \circ g)^{-1}(V)}_{\text{both mapping are continuous thus both sets are open.}}$$

→ Now we know that the intersection is open. Proved g^{-1} is continuous. ☒

Remark: The map $f: X \times Y \to Z$, map out of a product space is continuous if map in both directions are continuous.

· Proof:

If $f_X: X \to Z$ and $f_Y: Y \to Z$ are continuous, meaning that for every open $U \in Z$, $f_X^{-1}(U)$ is open in X and $f^{-1}{}_Y(U)$ is open in Y.

· Hence by box topology. The set $f_X^{-1}(U) \times f^{-1}{}_Y(U)$ is an open set in the product topology.

Theorem 7.38 Let $\prod_{\alpha\in\lambda} X_\alpha$ be the product topology of topological space $\{X_\alpha\}_{\alpha\in\lambda}$. the projection map $\pi_\beta: \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ is a continuous, surjective and open map.

Proof: The proof directly follows from definition of product topology.

1) It's continuous because for every open set of X_β, the inverse projection in product topology is just a subbasic open set, which is of course open.

2) It's of course surjective, since it's a projection

3) For every open set under product topology, it's union of basic open set. Each coordinate of a basis is X_α or an open set. The projection onto β is at least a union of open set, which is open.

Theorem 7.39: The product topology is the coarest topology on $\prod_{\alpha\in\lambda} X_\alpha$ that makes every projection map continuous.

Proof:

· Suppose there is a topology that's coarser, then there is an open set that is originally open in product topology, but is not open in the new topology.

· In product topology, the open set can be written as union of basic open set, then at least one of them is not open under the new topology.

· The basic open set can be written as product of X_α at every α except for open set $\neq X_\alpha$ at finitely many coordinate. For the basic open set to be not open. One of the finite $U_{\alpha_0} \neq X_{\alpha_0}$ is not open. But under topology of X_α,

U_α is open.

- Hence the inverse projection $\pi_{\alpha_0}^{-1}(U_{\alpha_0}) = \prod_{\substack{\alpha\in\lambda \\ \alpha\neq\alpha_0}} X_\alpha \times U_{\alpha_0}$ is not open. Thus the projection is not continuous. ☒.

Theorem 7.40: Let $\prod_{\alpha\in\lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha\in\lambda}$, and let Z be a topological space. A function $g: Z \to \prod_{\alpha\in\lambda} X_\alpha$ is continuous iff $\pi_\beta \circ g$ is continuous $\forall \beta \in \lambda$.

Proof:

$\Rightarrow$: $(g: Z \to \prod_{\alpha\in\lambda} X_\alpha$ is continuous$) \to (\pi_\beta \circ g$ is continuous $\forall \beta\in\lambda)$.

- For every $\pi_\beta \circ g$ takes an element from Z to X_β. Suppose there is γ s.t. $\pi_\gamma \circ g$ is not a continuous mapping.
- Then by definition, there exists an open set $U_\gamma \subset X_\gamma$ s.t. $g^{-1} \circ \pi_\gamma^{-1}(U_\gamma)$ is not open.
- Then consider the set $U_\gamma \times \prod_{\substack{\alpha\in\lambda \\ \alpha\neq\gamma}} X_\alpha$, a basic open set under product topology $g^{-1}(U_\gamma \times \prod_{\substack{\alpha\in\lambda \\ \alpha\neq\gamma}} X_\alpha)$ is not open. (Note it's because $\pi_\gamma^{-1}(U_\gamma) = U_\gamma \times \prod_{\substack{\alpha\in\lambda \\ \alpha\neq\gamma}} X_\alpha)$.

$\Leftarrow$: $(\pi_\beta \circ g$ is continuous $\forall \beta\in\lambda) \to (g: Z \to \prod_{\alpha\in\lambda} X_\alpha$ is continuous$)$.

- For every basic open set B in $\prod_{\alpha\in\lambda} X_\alpha$, under product topology is in the form $\prod_{\alpha\in\lambda} U_\alpha$ as $U_\alpha \subset X_\alpha$ and $U_\alpha = X_\alpha$ for all but finitely many coordinates.
- We know that for any open set at the β coordinate, call it U_β $g^{-1} \circ \pi_\beta^{-1}(U_\beta)$ is open. (Because $\pi_\beta \circ g$ is continuous).

$\to$ Hence $g^{-1}(B) = g^{-1}(\bigcap_{i\in\xi} S_i) = g^{-1}(\bigcap_{i\in\xi} \pi_i^{-1}(U_i)) = \bigcap_{i\in\xi} \underbrace{(g^{-1} \circ \pi_i^{-1}(U_i))}_{\text{an open set.}}$

Where ξ is the coordinates where $U_\alpha \neq X_\alpha$

S_i is the subbasic open set where $U_i \neq X_i$ at ith coordinate

- Hence $g^{-1}(B)$ is open since it's an intersection of open sets. ☒.

Example: Another distinguishment between box and product topology.

- Let $\mathbb{R}^\omega$ be a countable product of $\mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}^\omega$ by $f(x) = (x, x, \dots)$
- Then $f(x)$ is continuous if $\mathbb{R}^\omega$ is given product topology, but not when $\mathbb{R}^\omega$ is given box topology.

Proof:

- By definition, for some open set in any product space $U = U_1 \times U_2 \times U_3 \times \cdots$ $f^{-1}(U)$ is the set that $\bigcap U_i$, as f is the "identity map".

1. Why is it continuous under product topology.

- Because the preimage of an open set U can be written as

$$f^{-1}(U) = f^{-1}\left(\bigcup_{\beta \in \lambda} B_\beta\right) \quad B_\beta \text{ being some basic open set.}$$

$$= \bigcup_{\beta\in\lambda}\left(f^{-1}(B_\beta)\right) = \bigcup_{\beta\in\lambda}\left(f^{-1}\left(\underline{\bigcap_{i\in\zeta_\beta}}(S_i)\right)\right)$$

It's open, as it's finite intersection of open set.

$$= \bigcup_{\beta\in\lambda}\bigcap_{i\in\zeta_\beta}\left(f^{-1}(S_i)\right) = \bigcup_{\beta\in\lambda}\bigcap_{i\in\zeta_\beta} S_i \quad \text{is open.}$$

2. Why is it not continuous under box topology.

- The proof for product topology doesn't work because it's now infinite intersection.
- As an counterexample: the set $\prod_{n\in\mathbb{Z}^+}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open under box topology, but its preimage under f is $\{0\}$, which is not open. (In $\mathbb{R}_{std}$).

Definition [Cantor Set]: Consider the subset of $\mathbb{R}$. $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ (the middle third removed). $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

And each C_{n+1} removes the middle $\frac{1}{3}$ for each interval of the last C_n.

$C = \bigcap_{i=1}^{\infty} C_i$ is called the Cantor Set & Standard Cantor Set.

Theorem 7.42: The Cantor Set is homeomorphic to $\prod_{n\in\mathbb{N}}\{0,1\}$ where $\{0,1\}$ has discrete topology.

Comment: The topology $2^{\mathbb{N}}$ has the Cardinality of $\mathbb{R}$ and all continuous function of $\mathbb{R}$ (theorem 7.8).

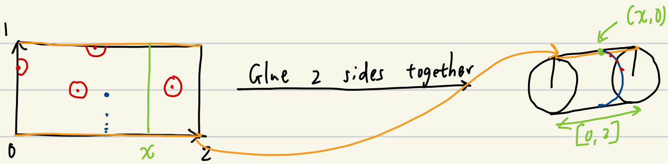
Proof: The logic is letting each $\{0,1\}$ gets assigned to the first & second $\frac{1}{3}$ of the partition.

CH7.5

- Example: An illustration of Quotient

→ Suppose we have a sheet of paper with its subspace topology given by its embedding as a subset of a plane. Call it $\mathcal{X}$.

- In Particular, points in the interior has its normal neighborhood and point on the boundary has half sphere as the enclosing open set.

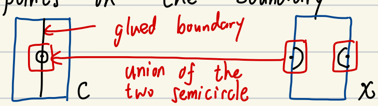


$\forall\, x \in [0,2]$, identify $(x,0)$, $(x,1)$ they are the same point.

- Now consider the sequence of points in X, $\{(\frac{1}{2}, \frac{1}{n})\}$, it is now converge to $(\frac{1}{2}, 1)$ since $(\frac{1}{2}, 1)$ is now also $(\frac{1}{2}, 0)$. Now:

1. We can describe the new space C by X transformed using a gluing function.
   $g: X \to C$, $g(x,y) = (x, \sin 2\pi y, \cos 2\pi y)$.
   - Obviously $(x,0)$ & $(x,1)$.
2. Gluing function is usually continuous. The topology that's natural to define U to be open in C iff inverse image $g^{-1}(U)$ is open in X.
   - For example, points on the boundary:

Looking down:



Hence any open set's preimage under gluing function is open.

Definition [Identification Map & Identifcation space]: Let X be a topological space and let X into disjoint subsets whose union is X. Let $f: X \to X^*$ be the surjective map that carries each $x \in X \to x^* \in X^*$.

- Define topology on X^* to be the subset U in X^* is open iff $f^{-1}(U)$ is open in X.

1) The map is called an identification map.
2) X^* is an identification space.

Definition [Equivalence Relation]: Any equivalence relation $\sim$ on X yields a partition of X into equivalence classes. We denote the identification space $X/\sim$.

Example of Identification Map:

- 1^{st} Example: The cylinder Example.

The cylinder C doesn't need to be embedded in $\mathbb{R}^3$. Instead we may define it as an identification space $X = [0,2] \times [0,1]$ with the partition whose sets are singleton or pairs.

$C^* = \{\{(x,y)\} : x \in [0,2],\ y \in (0,1)\} \cup \{\{(x,0), (x,1)\} : x \in [0,2]\}.$

- The identification map $f: X \to C^*$ can be written as

$$f(x,y) = \begin{cases} (x,y) & \text{if } y \neq \{0,1\} \\ \{\underline{(x,0)}, \underline{(x,1)}\} & \text{if } y = 0 \text{ or } y = 1. \end{cases}$$

One point maps to 2 points suggest now they are the same point.

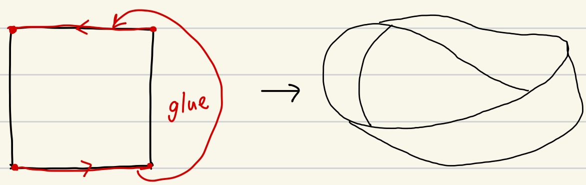
- A basis for topology on C^*:

$\{(x-\varepsilon_x, x+\varepsilon_x) \times (y-\varepsilon_y, y+\varepsilon_y) : x \in [0,2],\ y \in (0,1)\} \cup$

$\{(\varepsilon_x, x+\varepsilon_x) \times (y-\varepsilon_y, y+\varepsilon_y) \cup (1-\varepsilon_x, x+\varepsilon_x) \times (y-\varepsilon_y, y+\varepsilon_y) : x \in [0,2],\ y \in \{0,1\}\}$

- 2^{nd} Example: Möbius band

We construct a Möbius band by gluing one side of a sheet to the other side but in inverted direction.

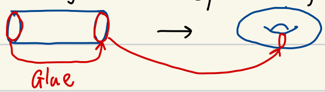


- The identifcation space of $[0,2] \times [0,1]$ can be written as

$C^* = \{\{(x,y)\} : x \in [0,2],\ y \in (0,1)\} \cup \{\{(x,0), (2-x,1)\} : x \in [0,2]\}$

- 3^{rd} Example: A Torus is the surface of a doughnut, construct a torus as:

(1) Identification space of a cylinder C



- Let S^1 be subspace topology of a circle in $\mathbb{R}^2$. Now cylinder can be written as $S^1 \times [0,1]$

- Let S^1 be desribed by polar angle θ.
- The Identification map is $(\theta, 0) \sim (\theta, 1)$. Hence Torus $= C/\sim$.

(2) Identification spare of $X = [0,1] \times [0,1]$

- We may first turn it into first a cylinder, then a torus. Hence we first glue the two side, then the 2 other. The identification map can be written as
- $(0,y) \sim (1,y)\ \forall y \in [0,1]$ $\quad (x,0) \sim (x,1)\ \forall x \in [0,1]$

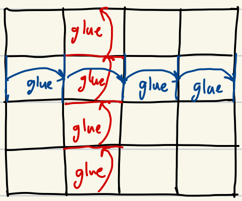
And Torus $= X/\sim$.

(3) Identification space of $\mathbb{R}^2$

· Similarly, we turn $\mathbb{R}^2$ mapped into $X = [0,1] \times [0,1]$, then apply (2).

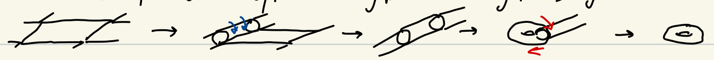
→ We can wrapped $\mathbb{R}^2$ around X. Which is just taking quotient can transfer into identification space $[(x,y)] = \{(x+m, y+n) \mid m, n \in \mathbb{Z}\}$ or

$(x,y) \sim (x+m, y+n)$ → Torus $= \mathbb{R}^2/\sim$



· To visualize, we are wrapping the planing around $[0,1]$ around again and again. Generating an infinite cylinder.

· And wrap the cylinder again and again, generate a torue.



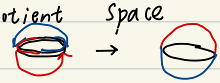
Example 4: We can construct the 3 dimensional ball as an identification space of two discs in $\mathbb{R}^2$.

· Let D_1 & D_2 be 2 copies of closed unit disc

$$\overline{D^2} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$$

· Now Join the ∂D_1 to ∂D_2, define $(x,y) \sim (-x,y)$ $\forall (x,y) \in \partial D_1 \cup \partial D_2$

Form the quotient space $S^2 = D_1 \cup D_2 / \sim$



· Now we generalize how identification space can be constructed from a partition.

Definition [Quotient Topology]: Let $f: X \to Y$ be a surjective map from a topological space X onto a set Y. The quotient topology on Y wrt f is the collection of all sets U s.t. $f^{-1}(U)$ is open in X.

Comment: the map is mapping onto a set, but not a subset of the original space X which is the definition of identification space.

In other word: $\begin{cases} \text{Identification map} \to X^*: \text{set of equivalence class with iden topology.} \\ \text{Quotient topology} \to Y \text{ gets the quotient topology.} \end{cases}$

· How are the 2 definitions connected? (let 1 be identification topology, 2 be quotient)

1 → 2 : Given X & $\sim$ on X. Let $Y = X^*$ = set of equivalence class.

$f: X \to Y$ $f(x) = [x]$ set of point equivalent to x.

Y is the set of $[x]$, each point represents a set of points in X equivalent to x.

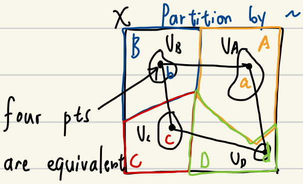
Y is also a subset of X, as each equivalent class can be represent by a point $[x] := x \in X$.

$2 \to 1$: Suppose $f: X \to Y$ is surjective. Define $\sim X$ by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$

Equivalence class of $\sim \quad \longleftrightarrow$ pts on Y.

- Equivalence class of $x = [x] \longmapsto f(x)$, $\quad f^{-1}(\{y\}) \longleftarrow y$.

$\to$ My attempt to visualize



- The identification map $f: X \to A$

$f^{-1}(U_A) = U_A \cup U_B \cup U_C \cup U_D$.

- The quotient map:

$f(a) = f(b) = f(c) = f(d) = Y_a$

- The main difference between the 2 definitions are that points in Y need not be the same representation as point in X.

Definition [Quotient Map & Quotient Space.]: A surjective map $f: X \to Y$ between topological spaces is a Quotient Map and Y is a quotient space iff the topology on Y is the quotient topology with respect to f.

- An alternative definition: it's a quotient map iff $(f^{-1}(U)$ open $\Leftrightarrow U$ open$)$.
- And Y is a quotient space if $\exists$ quotient map $f: X \to Y$

Theorem 7.47 The quotient topology defines a topology.

Comment: The definition of quotient topology $-$ $f^{-1}(U)$ is open set of X is is a construction of a continuous mapping between X & Y.

Proof: (let quotient topology denote T_Q)

1) $\phi \in T_Q$, since $f^{-1}(\phi) = \phi$ is open.

2) $Y \in T_Q$, since $f^{-1}(Y) = X$ since the map is surjective.

3) Suppose $V \in T_Q$, $U \in T_Q \to f^{-1}(V) \in T_X$, $f^{-1}(U) \in T_Q$

$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \in T_X$

4) Same as proof of 3). Change Intersection to Union. $\square$

Theorem 7.48: Let X be a topological space, let Y be a set, and let $f: X \to Y$ be a surjective map. The quotient topological on Y is the finest topology that makes f continuous.

Proof:

- It's obviously true f is continuous under quotient topology definition.
- Now consider a finer topology than quotient topology.
  It has an open set that's not originally open in Y under quotient topology.

→ However since in quotient topology, U is open only if $f^{-1}(U)$ is open.

→ Thus now f^{-1}(The extra open set) is not necessarily open. Hence we know that f is no longer continuous. ☒.

Theorem 7.49: Let X & Y be topological space. A surjective, continuous map $f: X \to Y$ that's an open map is a quotient map.

Proof:

- Y already has a topology, we need to test if it's the quotient topology with respect to f.

→ In other word, we need to show T_Y are the ONLY sets that $f^{-1}(U)$ is open in X.

- Of course, since f is continuous $f^{-1}(U)$ is open.
- For every none open set Y, since f is open $(A \in T_X \to f(A) \in T_Y)$ $(f(A) \notin T_Y \to A \notin T_X)$, the inverse can't be open.
- Additionally since f is surjective, f is a quotient map. ☒.

Theorem 7.50: Let X and Y be topological space. A surjective, continuous map $f: X \to Y$ that's a closed map is a quotient map.

Proof:

- Since f is continuous, $f^{-1}(U)$ is open.
- For every non-open set Y, V. V^c is not closed. Hence $f^{-1}(V)$ is not closed since $f^{-1}(V^c)$ is not closed.
- Thus $f^{-1}(V)$ is not open. As needed. ☒.

Despite theorem 7.49 & 7.50, Quotient map need not be open or closed map.

- A dummy example:

Define the equivalence relation $\sim$ on $\mathbb{R}$ by letting $x \sim y$ if x and y are both situated in $\underbrace{(0, \infty)}_{A}$ or $\underbrace{(-\infty, 0]}_{B}$. Indeed, they are exactly the equivalence classes.

Name

- The quotient map $f: X \to Y$, $X = \mathbb{R}$ $Y = \{A, B\}$. The topology on Y is $\{\phi, A, Y\}$
  And thus the closed sets are exactly $\{\phi, B, Y\}$.

(Since quotient topology are defined as $U \subset Y$ s.t. $f^{-1}(U) \in T_X$.)
$f^{-1}(B) = B$ - not open $f^{-1}(A) = A$ open $f^{-1}(Y) = Y$ open $f^{-1}(\phi) = \phi$ open).

- It's not an open map : $f((-2, -1)) = B$ not open
  closed map : $f([1, 2]) = A$ not closed

Theorem 7.53 : Let $f: X \to Y$ be a quotient map. Then a map $g: Y \to Z$ is continuous iff $g \circ f$ is continuous.

Proof:

$\Rightarrow$:

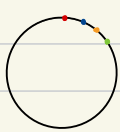
- For every open set $U \subset Z$, WTS $(f^{-1} \circ g^{-1})(U)$ is open in X.
- Since g is continuous, $g^{-1}(U)$ is open in Y. And because f is a quotient map, $g^{-1}(U)$ is only open if $f^{-1}(g^{-1}(U))$ is open.
- Thus we know that $(f^{-1} \circ g^{-1})(U)$ is open.

$\Leftarrow$:

- For every open set $U \subset Z$, WTS $g^{-1}(U)$ is open in Y.
- Because $g \circ f$ is continuous.
  If $g^{-1}(U)$ is not open, then by definition of a quotient map, $f^{-1}(g^{-1}(U))$ isn't open.

$\to$ Hence $f^{-1} \circ g^{-1}$ can't be continuous. $\square$.

Example :

- We identify the $\sim$ relation as $x_1 \sim x_2$ iff $x_1 - x_2 \in \mathbb{Z}$.
  Then : X^* is the identification space of X via $\sim$.
- The equivalence class : 
  The points with same color are consider 1 point.
  Quotient space should look like:  $\leftarrow$ Each color go through 2π rotation with land at same place.
  (Each point is a equivalence class).

- In fact, we may prove that $\mathbb{R}/\sim = S^1$. Since we may define a bijection between S^1 and quotient space. Namely $f(t) = (\cos(2\pi t), \sin(2\pi t))$

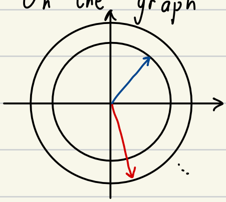
Example: Change $\mathbb{Z}$ in the last example to $\mathbb{Q}$. Then the equivalence classes looks like:

Every color is dense. ($\mathbb{Q}$ is dense).

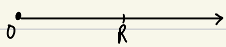
- Then what's the quotient topology?
- By definition, every open set in the quotient has an open preimage under the quotient map. But then, for every open set in the original topology, every equivalent class must have at least one point in it.
- Thus the open set under quotient topology contains all the point (each point represent an equivalence class). Thus the topology is indiscrete. $\boxtimes$.

Example: The identification space of $\mathbb{R}^2$ with $|\vec{x}| \sim |\vec{y}|$ iff $\|\vec{x}\| = \|\vec{y}\|$

- On the graph



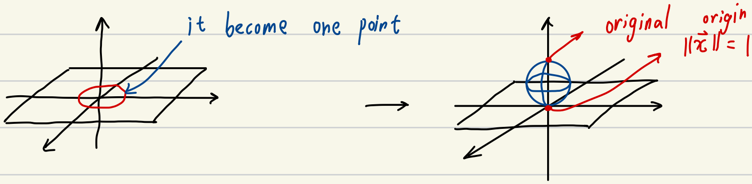
The identification space looks like a one-end bounded



with each point be the equivalence class of the set of class of vector w/ length $\mathbb{R}$.

Example: The identification space of $\mathbb{R}^2$ with $\{|\vec{x}| : \|\vec{x}\| = 1\}$ identified to be a single point.

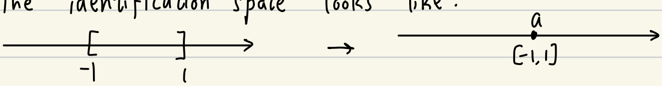
- The identification space indeed looks like:



- We map the $\|\vec{x}\| < 1$ vector onto the sphere
- We map all the point outside the circle onto the entire plane.

Example: The identification space of $\mathbb{R}$ with $[-1, 1]$ defined to be a point.

- The identification space looks like:

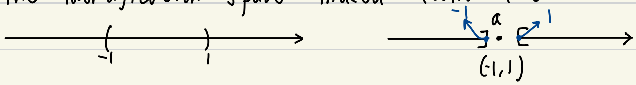


- The identification topology make sense since you can't contain point a with an open set without containing some neighborhood of a.

- Likewise you can't contain $[-1,1]$ in $\mathbb{R}_{std}$ without containing $N_R(0)$, $R>1$. Thus the open set configuration of the sets are the same.

Example : The identification space of $(-1,1)$ defined to be a point.

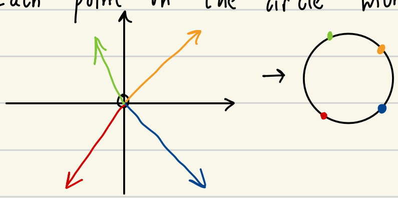
- The identification space indeed looks like:



- The point a is mapped to a central point between -1 & 1.
- We can't containing -1 or 1 without containing a using an open set. It's indeed a non-Hausdorff like topology near the point a.

Example: The identification space of $\mathbb{R}^2 - \{\vec{0}\}$ where $\vec{x} \sim a\vec{x}$ for $a \in \mathbb{R}^+$

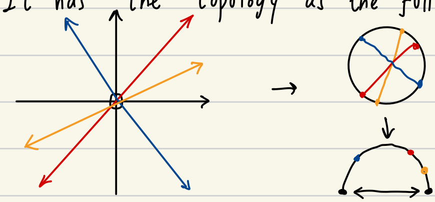
- The identification topology is homeomorphic with S^1, a 2D circle.
- Each point on the circle with coordinate θ is the equivalence class of vector



with the angle θ from positive x-axis.
← Like the diagram on the right.

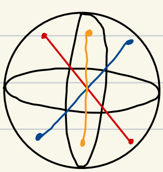
Example: The identification space of $\mathbb{R}^2 - \{\vec{0}\}$ where $\vec{x} - a\vec{x}$ for $\mathbb{R} \setminus \{0\}$
(It's the space of 1 dimension of vector subspace of $\mathbb{R}^2$)

- It has the topology as the following:

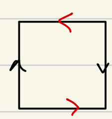


- It's kind of the quotient of S^1 by matching the opposite point
- It's a semi-circle with the 2 end points connected.

- If we extend the example into $\mathbb{R}^3$, the identification topology looks like:



- The sphere in $\mathbb{R}^3$, with the opposite points matching and become one single point.
- It's homemorphic to a "double Mobius band". Where you fold the other sides you fold a Möbius band



- It's called a "projective plane".
