CH4.1

Comment : This Chapter discuss the question of whether open sets exist in topology that separate various subsets of the space from another?

Definition [T_1 - space, Hausdorff space (T_2 - space) , regular space, normal space]:

- X is a T_1 - space iff $\forall\, x, y$ s.t. $x \neq y$, $\exists\, U, V \in T_X$ s.t. $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.
- X is a T_2 - space iff $\forall\, x, y$ s.t. $x \neq y$, $\exists\, U, V \in T_X$ s.t. $x \in U$, $y \in V$ and $U \cap V = \phi$.
- X is regular iff $\forall\, x \in X$ and closed $A \subset X$ s.t. $x \notin A$, $\exists\, U, V$ s.t. $x \in U$ and $A \subset V$, $U \cap V = \phi$.
- X is normal iff $\forall$ closed A, B, $\exists\, U, V \in T_X$ where $A \subset U$, $B \subset V$, $U \cap V = \phi$.

Definition [T_3 - Space] : A space is T_3 if the space is both T_1 and regular.

Comment : The most important property of T_1 - space is that points are closed.

Theorem 4.1 : A topological space (X, T) is T_1 iff $\forall\, x \in X$, $\{x\}$ is a closed set.

Proof :

$\Rightarrow$:

- Suppose $y \neq x$, $\exists\, V \in T$ s.t. $y \in V$ and $x \notin V$. Hence $\exists\, X \setminus \{x\}$ contains an open set U' such that $y \in U'$.
- Since it work for any y. We may construct $X \setminus \{x\}$ by a union of open set, which is open. Hence $\{x\}$ is closed.

$\Leftarrow$:

- Let $\{y\}$ and $\{x\}$ be two singleton, then $X \setminus \{x\}$ and $X \setminus \{y\}$ are the U, V we are looking for. $\square$.

E.g.

- Consider $\mathbb{R}^2$ with standard topology. Let $p \in \mathbb{R}^2$ be a point not in closed set A, $\operatorname{Inf}\{d(a,b) \mid a \in A \text{ and } b \in B\} > 0$. (To prove $\mathbb{R}^2_{std}$ is regular).

Proof : Since complement of A is an open set. There always exist an ε - nbhd near p that is disjoint from A.

$\mathbb{R}^2$ has the interesting property that though we can't prove normality in a plausible approach, but it's still normal, such as the following example.

- Two closed disjoint subsets A, B of $\mathbb{R}^2$ s.t. $\inf\{d(a,b) \mid a \in A \text{ and } b \in B\} = 0$.

  $A = \{(x,0) \mid x > 0\}$ · Due to the asymptotic behavior and $\mathbb{R}$ doesn't include ∞ like

  $B = \{(x, \frac{1}{x}) \mid x > 0\}$ $\mathbb{C}$, therefore the limit might be 0.

- Even for this example, we can find open sets that separate them.

  Consider the open sets

  $A' = \{(x,y) \in \mathbb{R}^2 \mid y > \frac{1}{2x}\}$ $B' = \{(x,y) \in \mathbb{R}^2 \mid y < \frac{1}{2x}\}$

  It's obviously $A \in A'$, $B \in B'$ and $A' \cap B' = \phi$.

→ Below is a proof of why $\mathbb{R}^2$ is normal.

Proof:

- For two disjoint closed sets in $\mathbb{R}^2$. Call them A and B.
- Obviously the complement of A is open. Then for every point in B, B is containe in $\mathbb{R}^2 - A$. Therefore for every point $b \in B$, is an interior point of $\mathbb{R}^2 - A$.
- Therefore we know that $\exists \varepsilon > 0$ s.t. $N_\varepsilon(p) \subset \mathbb{R}^2 - A$.

  Then the union of these open balls, it's an open set that contain B in A^c.

→ We can similarly find an open set that contains A in B^c.

- Now we find a way to "shrink" the two union of open sets so that they don't intersect.
- We may find $\forall x \in A$, let $r_x = \inf\{\frac{d(x,y)}{2} : y \in B\} > 0$ let the open

  $\forall y \in B$, $r_y = \inf\{\frac{d(x,y)}{2} : x \in B\} > 0$ neighborhood radius $< r_x$

→ Let $U = \bigcup_x N_{r_x}(x)$, $V = \bigcup_y N_{r_y}(y)$ WTS $U \cap V = \phi$. If it's not, $\exists z \in U \cap V$, for some $x \in A$, $y \in B$. $d(x,y) \leq d(x,z) + d(y,z) < r_x + r_y \leq d(x,y)$ ☒.

Theorem 4.7:

(1) A T_2 space (Hausdorff space) is a T_1 space.

(2) A T_3 space (Regular and T_1) is a T_2 space.

(3) A T_4 space (Normal and T_1) is a T_3 space.

Proof: The proof is pretty obvious, we will skip it over here. (Use theorem 4.1)

Theorem 4.8 A topological space X is regular if and only if for each p in X and open set U containing p, $\exists$ open V s.t. $p \in V$ and $\overline{V} \subset U$.

Theorem 4.9 A topological space X is normal if and only if for each closed A in X and open set U containing A. $\exists V$ s.t. $A \subset V$ and $\overline{V} \subset U$.

Proof: The proof are exactly the same format, so we only prove 4.8 over here.

$\Rightarrow$:

- If X is regular, suppose x and neighborhood of x, U was given. Let $B = X - U$. then B is a closed set.
- By definition of regular set, $\exists V, W \in T$ s.t. $x \in V$ and $B \subset W$ and $V \cap W = \phi$.

$\rightarrow$ If $y \in B$, the set W is a neighborhood of y disjoint from B, thus $\overline{V} \cap B = \phi$. Thus $\overline{V} \subset U$, as needed.

$\Leftarrow$:

- For an $x \in X$ and $x \notin B$, B is closed.
- Take $U = X - B$, $\exists V \in T$ s.t. $x \in V$ s.t. $\overline{V} \subset U$. By hypothesis.

$\rightarrow$ Open sets V and $X - \overline{V}$ are disjoint open sets containing x and B. X is regular. $\square$

Theorem 4.10 A topological space X is normal iff $\forall A, B$ that's closed sets A and B s.t. $A \cap B = \phi$. $A \subset U$ and $B \subset V$, thus $\overline{U} \cap \overline{V} = \phi$.

Proof:

$\Rightarrow$:

- Since X is normal, then $\exists U', V' \in T$ s.t. $A \subset U'$ and $B \subset V'$. $U' \cap V' = \phi$.
- Now by theorem 4.9. $\exists U \in T$ s.t. $A \subset U$ and $\overline{U} \subset U'$

$V \in T$ $B \subset V$ $\overline{V} \subset V'$.

- Since $U' \cap V' = \phi$, for every $p \in U$, U' is a neighborhood of p disjoint from U, then $\overline{U} \cap V = \phi$. Similarly $\overline{V} \cap U = \phi$.

$\rightarrow$ Therefore $\overline{U} \cap \overline{V} = \phi$. As needed.

$\Leftarrow$:

- If $\overline{U} \cap \overline{V} = \phi$, the $U \cap V = \phi$, thus X is normal. $\square$

Theorem 4.11 (Incredible Shrinking Theorem): A topological space X is normal iff for each pair of open sets U, V s.t. $U \cap V = X$, $\exists\, U', V'$ s.t. $\overline{U'} \subset U$ and $\overline{V'} \subset V$ and $U' \cup V' = X$.

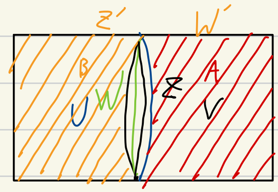
Proof :

$\Rightarrow$: (X is Normal) $\to$ (X is incredible shrinking)

Define $A = X - U$, $B = X - V$, obviously both A and B are closed.

- Note that $U \cup V = X$

$$A \cap B = X - (X - (A \cap B)) = X - (U \cup V) = \phi$$



- Thus we found two disjoint closed set in X.
- Because X is normal, A, B can be separated by 2 open neighborhood, $\exists\, W, Z \in T$ s.t. $A \subset W$, $B \subset Z$ and $W \cap Z = \phi$.

$\to$ Define $U' = X \setminus Z$, $V' = X \setminus W$. And obviousl. $U' \cap V' = \phi$

- For any element $x \in U'$, $x \notin Z$. Since $A \subset Z$, $x \notin A$. $x \in U$.
  $x \in V'$, $x \in V$.

  Thus $V \subset V'$, $U \subset U'$.

$\to$ To Prove $U' \cup V' = X$, we can observe that

$U' \cup V' = (X - Z) \cup (X - W) = X - (W \cap Z)$

Since we know $W \cap Z = \phi$ $U' \cup V' = X - \phi = X$.

$\Leftarrow$:

- For every open sets U and V with $U \cup V = X$, $\exists\, U' \subset U$ and $V' \subset V$ with $U' \cup V' = X$.
- Now let A and B 2 disjoint open sets. Define $U = X - B$ $V = X - A$.
  Then $A \subset U$ and $B \subset V$, obviously $U \cup V = (X - B) \cup (X - A) = X$
- Hence we know that $\exists\, \overline{U'} \subset U$ and $\overline{V'} \subset V$ s.t. $U' \cup V' = X$.

We observe the following :

$\to U' \subset U = X - B \to U' \cap B = \phi$

$\to V' \subset V = X - A \to V' \cap A = \phi$

- As $U' \cup V' = X$, $\overline{U'} \cup \overline{V'} = X$. Since $\overline{U'}$ and $\overline{V'}$ are both closed sets, we know

Hence $\overline{U'}^c \cap \overline{V'}^c = X^c = \phi \to \overline{U'}^c$ & $\overline{V'}^c$ are both open set. We claim these are the sets that fulfill the requirement.

$\to$ Since $\overline{U'} \subset U$, $\overline{U'}^c \supset U^c$. $\to \overline{U'}^c \supset A$ and similarly $\overline{V'}^c = B$

$\to$ Additionally $\left(\overline{U'}^c \cap \overline{V'}^c\right)^c \overset{\text{By definition}}{=} \overline{U'} \cup \overline{V'} = X \quad \to \quad \overline{U'}^c \cap \overline{V'}^c = X^c = \phi$.

Thus disjoint. As needed. ☒.

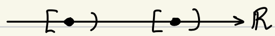
Difference between T_1, T_2, T_3 and T_4

- Finite complement Topology is T_1 but not T_2.

$\to$ Given any two distinct points $x, y \in X \setminus \{y\}$ is open but not y, and vice versa.

$\to$ Any two complement open set must have infinite intersection. Hence it's not a T_2 space.

- Lower limit topology is T_2 but not T_3.

$\to$ Given any two point, we can alway find 2 distinct open sets that separate them (for a metric space). 

$\to$ Given a point and a closed set

Take $[-\infty, 0)$ as an open set

$(0, \infty) = \mathbb{R} - [-\infty, 0)$ is closed. And take $\{0\}$ as a point.

For every open neighbood that contains $\{0\}$ must have intersection with $(\infty\, 0)$.

- Hbub (Sticky bubble topology) is T_3 but not T_4.

$\to$ For every closed set, just like in $\mathbb{R}^2$, every point may have a little "wiggle room". Thus we fit in an open ball that separates the point from open set.

$\to$ Why Hbub is not normal.

  - Every subset L must be closed since every point in L is the complement $\bigcup_{y \neq x} \bigcup_{\varepsilon > 0} D_\varepsilon^y$ of a union $D_\varepsilon^y = D_\varepsilon \cup \{y\}$
  - Thus the rationals and the irrationals are two closed subsets in X and Y but cannot be separated. ☒.

CH4.2

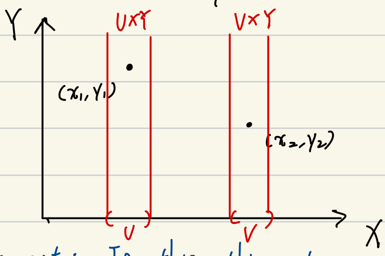
Theorem 4.16: Let X and Y be Hausdorff. Then $X \times Y$ is Haurdorff.

Proof:

- For 2 distint point in $X \times Y$ where both X and Y are T_2. Call them (x_1, y_1), (x_2, y_2). Namely $x_1 \neq x_2$ or $y_1 \neq x_2$.
- WLOG, assume $x_1 \neq x_2$. Since X is T_2, then we may take 2 disjoint open sets

U, V s.t. $x_1 \in U$, $x_1 \in V$. $U \cap V = \phi$.

- Now take $\pi_X^{-1}(U) = U \times Y$ and $\pi_X^{-1}(V) = V \times Y$, which are two subbasic open sets that distinctively contain (x_1, y_1) and (x_2, y_2)



- In addition $U \times Y \cap V \times Y$ is obviously ϕ, as needed.
- Hence $X \times Y$ is also Hausdorff. ☒.

Comment: Is this thing true for infinite product topology? Yes!

- Let $\vec{x}, \vec{y} \in \prod_\alpha X_\alpha$, where $\vec{x} \neq \vec{y}$. Then clearly $\exists\, i \in \alpha$ s.t. $x_i \neq y_i$

  Since X_i is T_2, $\exists\, U_i \in T_{X_i}$ s.t. $x_i \in U_i$

  $\exists\, V_i \in T_{X_i}$ s.t. $y_i \in V_i$

- Now take the inverse projection of $\pi_{X_i}(U_i)$ and $\pi_{X_i}(V_i)$, are the 2 disjoint open sets that respectively contains $\vec{x}$ and $\vec{y}$. Hence $\prod_\alpha X_\alpha$ is also Hausdorff.
- In fact, the proof works for either product & box topology. ☒.

Theorem 4.17: Let X and Y be regular, then $X \times Y$ is regular.

Proof:

- Obviously under the case of finite product, product topology is the same as the box topology. Suppose a point in product space, (x, y) s.t. $x \in X$ and $y \in Y$.
- And a closed set in $X \times Y$.

Lemma: $U \times V$, $U \subset X$, $V \subset Y$, is closed $\longleftrightarrow$ U is closed in X, V is closed in Y.

Proof:

$\Rightarrow$:

- If either U or V is not closed. WLOG, assume it's V.
- Since it's not closed, it does not contain some of it's limit points, say p. Then pick a point in $X \times Y$, say (p, y), where $y \in Y$.

$\rightarrow$ Take any neighborhood of the point $U' \times V'$, by definition $X' \cap X = \phi$ thus we know that (p, y) is a limit point, but it's not in U, it can't be in

$U \times V$ hence $U \times V$ is not closed.

$\Leftarrow$: If both U, V are closed, then $(U \times V)^c$ should be open, we prove it.

- We know that $\pi_X^{-1}(U^c) = U^c \times Y$ $\quad \pi_Y^{-1}(V^c) = X \times V^c$ are two subbasic open sets. $\underline{(U^c \times Y) \cup (X \times V^c)}$ $= ((U^c \times Y)^c \cap (X \times V^c)^c)^c$
$= ((U \times Y) \cap (X \times V))^c = (U \times V)^c$.
- Hence obviously this set is union of 2 open set, is open. Thus $U \times V$ is closed. ⊠.

Note that this proof is also true for infinite product, as any union of open sets is still open by definition.

- Thus we know, the set $U \times V \subset X \times Y$ is closed $\leftrightarrow$ U is closed & V is closed.
- Since X is regular, if for the set x and U
  We may find open sets $U_1 \cap U_2 = \phi$ s.t. $x \in U_1$ and $U \subset U_2$
  $V_1 \cap V_2 = \phi$ $\quad y \in V_1$ $\quad V \subset V_2$

$\rightarrow$ Thus we know that $U \times V \subset U_2 \times V_2$ $\quad (x, y) \in U_1 \times V_1$.

- $X \times Y$ is regular if both X and Y are both regular. ⊠.

Remark: However, this theorem is not true for normal space.

- A classical counterexample is the set $\mathbb{R}_{LL} \times \mathbb{R}_{LL}$
- Below is a brief description that $\mathbb{R}_{LL}$ is normal.

Proof:

- We know that theorem 4.9. X is normal iff $\forall$ closed set A in X and open set U containing A, $\exists V \in T_X$ s.t. $A \subset V$ and $\bar{V} \subset U$.

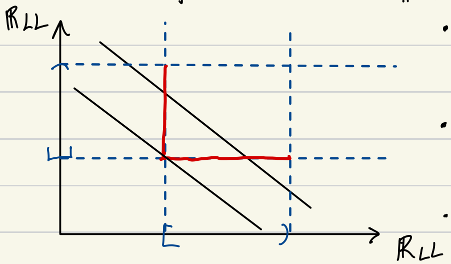
$\rightarrow$ If A and B are 2 two disjoint closed set in lowest limit topology
As A, B are closed, their complements are closed.

- In lower limit topology, $\forall x \in A$, we may find a basic element $[x, x+\varepsilon)$ that contains x and does not intersect B. Same thing for each $y \in B$.

$\rightarrow$ Construction of disjoint open set

- For each point $a \in A$, pick an interval $U_a = [a, r_a)$ s.t. $U_a \cap B = \phi$
  $V_b = [b, r_b)$ s.t. $V_b \cap A = \phi$.
- The set $U = \bigcup_{a \in A} U_a$ and $V = \bigcup_{b \in B} V_b$ form disjoint open sets of A & B. ⊠.

- However for the set $\mathbb{R}_{LL} \times \mathbb{R}_{LL}$, where both set has lower limit topology.



- Now for any of the negative sloping line, having subspace topology.
- By definition it the set $U \times V$ is open, then the intersection between $U \times V$ has to be open.
- However, the intersecting line segment is either the form of •——• or •

→ Which are not open in $\mathbb{R}_{LL}$. Hence it inherits the discrete topology, which is not normal. $\boxtimes$.