CH5.1

Definition [Dense] A is dense in X if and only if $\bar{A} = X$.

Comment: We may think of a dense subset A of a subspace X as a set that permeates the space. Every point in X can approach arbitrarily closely by points of A.

Theorem 5.1: A is dense in X iff every non-empty open set of X containing a point of A.

Proof:

$\Leftarrow$:

- Suppose there is some non-empty open set $A' \subset X$.

$\rightarrow$ We know for a set A, $\bar{A} = X$, meaning either any point $x \in A \subset X$ is a limit point of A or is a point in X.

- If A' doesn't contain any point in A, then it only contains a set of limit point.
- However by definition of limit points, every open set containing it must contain some other points in A. Contradiction.

$\Rightarrow$:

- If A isn't dense, $\bar{A} \neq X$. Since obviously $\bar{A} \subset X$, $\exists x \in X$ s.t. $x \notin \bar{A}$.
- By construction x is neither a limit point nor a point of A.
  Hence x is an isolated point.

$\rightarrow$ $\exists U \subset T$ s.t. it doesn't contain any point in A. $\boxtimes$.

Definition [Separable]: A topological space X is separable iff X has a countable dense set.

Remark: It's not related to the separation property we defined, and isn't related to concept of separated sets defined in the previous chapter.

Theorem 5.2: $\mathbb{R}$ std is separable.

Proof:

- Claim: $\mathbb{Q}$ is countable, we prove it's also dense.
- For every point in $\mathbb{R}$, we may write it as $i_0.i_1 i_2 i_3 \ldots$ $i_0 \in \mathbb{Z}$, $i_1 \in \{0, \ldots, 9\}$

We may approach it with the rational sequence (a_n) s.t. $a_n = \sum_{j=0}^{n} \frac{c_j}{10^j}$

- Hence every point is a limit point of $\mathbb{Q}$, $\overline{\mathbb{Q}} = \mathbb{R}_{std}$. $\mathbb{Q}$ is dense. $\boxtimes$.
- In contrast, the discrete topology is not separable.

Proof: For any countable set $D = \{x_1, x_2, x_3, \dots\}$. The closure of D is always D itself. Therefore we know that $\bar{D} \neq \mathbb{R}$ unless $D = \mathbb{R}$, but $\mathbb{R}$ is uncountable. $\boxtimes$.

Comment: It's tempting to think if one space is separable, then its subspaces should also be separate. However, it's not the case.

- The example is the same example that we used to prove the fact that that $\mathbb{R}_{LL} \times \mathbb{R}_{LL}$ is not a normal space.

→ It's easy to see how $\mathbb{Q}$ & $\mathbb{Q} \times \mathbb{Q}$ is dense under $\mathbb{R}_{LL}$

- However, the negative sloping line subspace inherits the discrete topology, which is not separable under any uncountable set it's put on. $\boxtimes$.

Theorem 5.5: If X and Y are separable spaces, $X \times Y$ is separable

Proof:

- If both X and Y are separable, then $\exists\, U \in X$ that's countable and dense $\exists\, V \in Y$ that's countable and dense.
- Take $U \times V$, we prove it's both countable and dense.

Countable:

→ Since U & V are both countable, they both have the same cardinality as $\mathbb{N}$, therefore we may enumerate U & V as

- $U = \{u_0, u_1, u_2, u_3, \dots\}$
- $V = \{v_0, v_1, v_2, v_3, \dots\}$

→ And now we may use element from $\mathbb{N}$ to match 1 by 1, by the same proof how we prove $\mathbb{Q}$ is countable:

$V \backslash U$	u_0	u_1	u_2	u_3
u_0	(u_0, u_0)	(u_0, u_1)	(u_0, u_2)	(u_0, u_3)
u_1	(u_1, u_0)	(u_1, u_1)	(u_1, u_2)	(u_1, u_3)
u_2	(u_2, u_0)	(u_2, u_1)	(u_2, u_2)	(u_2, u_3)

- Taking this line matching the sets of 2-tuple together with each natural number.
- Thus $U \times V$ is countable.

Dense :

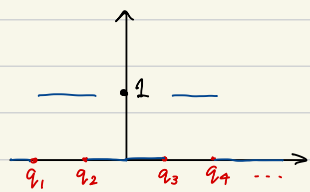
→ For every open set in $X \times Y$, because the fact that U and V are both dense, $\exists\, x \in U$ and $y \in V$ s.t. (x, y) inside the open set.

· Thus we know that for every open set in $X \times Y$, $\exists$ an element in $U \times V$. Hence $V \times V$ is dense. ☒.

Theorem 5.6 : The space $2^{\mathbb{R}}$ is separable.

· Remark : We may consider $2^{\mathbb{R}}$ as the power set as its elements are essentially "sets" in x that are "1" or "included" & "0" or "excluded".

· Define the set with the open set like the following



$E = \{ n \in \mathbb{N}^+ ,$ define $\{q_1, \dots, q_n\} \in \mathbb{Q}^n : q_1 < \dots < q_n \}$

$e = \forall\, x \in (-\infty, q_1)$, x^{th} coordinate has value 0

$\forall\, x \in (q_{i \in odd}, q_{i \in even})$, x^{th} coordinate has value 1

$\forall\, x \in (q_{i \in even}, q_{i \in odd})$, x^{th} coordinate has value 0.

$\forall\, x \in (q_n, \infty)$, x^{th} coordinate has value 0.

$G = \{$ The set of all E and their corresponding $e. \in 2^X \}$. We claim G is the set.

· That's countable and Dense in E.

→ For any basic open set, there are only finite element that not $\{0, 1\}$. At their coordinates, we may always find an $E = \{q_1, \dots, q_n\}$ that saticfy the limit condition (being either 1 or 0).

· Hence for any basic open set, we can always find a G - point contained in it.

· Thus G is dense in $2^{\mathbb{R}}$, where E, hence G is obviously countable. ☒.

CH5.2

Definition [2^{nd} Countable Space] : A space X is 2^{nd} countable iff it has a countable basis.

Comment : In a 2^{nd} countable space every open set can be built from a countable set. It's defined by Hausdorff in Mengenlehre.

· Examples :

1) The real line $\mathbb{R}$ has a countable basis, the collection of all open interval

(a,b) with rational endpoint. In fact, $\mathbb{R}^n$ & $\mathbb{R}^\omega$ all have a countable basis.

- The $\mathbb{R}^n$'s countable basis is the subsets in $\mathbb{R}^n$ where each coordinate has the open set with rational endpoints.
- The $\mathbb{R}^\omega$'s countable basis: subsets of $\mathbb{R}^\omega$ where each coordinate $= \mathbb{R}$ except for finitely many coordinate with rational endpoints.

Theorem 5.9 Let X be 2^{nd} countable. Then X is separable.

Proof:

- If X is 2^{nd} countable, then there exists a countable basis.
- Pick a basic element B_i, choose an element $b_i \in B_i$, now for any i we may always pick such an element.

$\to$ Thus if we construct the countable set $\{b_i\}$, it's dense in X. Since every open set is a union of basic open sets. It has to contain some $\{b_i\}$. $\square$

Example:

(1) $\mathbb{R}_{LL}$ is separable but not 2^{nd} countable.

- The countable dense subset is $\mathbb{Q}$. Each open set $[a,b)$ must contain a $\mathbb{Q}$.
- Why it doesn't have a countable basis?

  Suppose it has a countable basis $B = \{B_1, B_2, B_3, \dots\}$

$\to$ $\forall x \in \mathbb{R}$, $[x, x+1)$, there must exist subset of B, say B_{k_x} s.t. $x \in B_{k_x} \subseteq [x, x+1)$

- Consider the map $x \to B_{k_x}$. It is injective b.c. $x \neq x' \to B_{k_x} \subseteq [x, x+1)$ and $B_{k_{x'}} \subseteq [x', x'+1)$, $B_{k_x} \neq B_{k_{x'}}$

$\to$ But the set of $x \in \mathbb{R}$ is uncountable. Hence $\mathbb{R}_{LL}$ is not 2^{nd} countable.

(Reason why $[a,b)$, $a, b \in \mathbb{Q}$ doesn't work like $\mathbb{R}_{std}$: consider $[\sqrt{2}, \sqrt{2}+1)$).

(2) H_{bub} is separable but not 2^{nd} countable.

- The countable dense set is (a,b), $b > 0$, $a, b \in \mathbb{Q}$.
- If it has a countable basis $B = \{B_1, B_2, B_3, \dots\}$, we can find a injective map between $\mathbb{R}$ and B as last example, by making each element on the real line with an open set in H_{bub} that's tangential to it. $\square$

Theorem 5.11 Every Uncountable set in a 2^{nd} countable space has a limit point.

Proof:

- We use a contrapositive argument: suppose a set S in a 2^{nd} countable space X doesn't have a limit point, then it's a countable set.
- By definition of 2^{nd} countable space, it has a countable basis. We call it $\mathcal{B} = \{B_1, B_2, \dots\}$

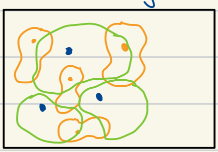
→ As the set doesn't have a limit point:

$\neg(\exists x \in X$ s.t. $\forall U \in T_x$ s.t. $x \in U$, $\exists x' \neq x$ s.t. $x' \in S$, $x' \in U)$

$\Leftrightarrow \forall x \in X$ s.t. $\exists U \in T_x$ s.t. $x \in U$, $\forall x' \neq x$ s.t. $x' \in S$, $x' \notin U)$.

(1st line, S has one limit point, 2nd line, S has 0 limit point).

- S



- It means for every point in $x \in S$, we may find a open set that contains x but excludes every other points in S.
- We know this open set $= \bigcup_{i \in I} B_i$, $B_i \in \mathcal{B}$.

→ Thus for some $i_x \in I$, $x \in B_{i_x}$, obviously B_i disjoint from $S \setminus \{x\}$

→ Due to the disjointness, we may designate a map $f: x \to B_{i_x}$. It's obviously injective. (As no two $x \in S$ can share a same B_{i_x}).

- If S is an uncountable set, then there are uncountably many B_{i_x} that are distinct, which contradicts the assumption that $\mathcal{B}$ is countable. ☒.

Theorem 5.13. If X and Y are 2^{nd} Countable, then the product space $X \times Y$ is 2^{nd} Countable.

Proof:

- If X & Y both exists a countable basis

$$\begin{cases} X\text{'s basis}: \{X_1, X_2, X_3, \dots\} \\ Y\text{'s basis}: \{Y_1, Y_2, Y_3, \dots\} \end{cases}$$

→ We claim the set $\{X_i \times Y_j : i, j \in \mathbb{Z}^+\}$ is a basis. Since for any open set takes the form $U \times V$ where $U \subset X$ and $V \subset Y$.

- We may built the construction by 1. Take the $\alpha \subset \mathbb{Z}^+$ s.t. $\bigcup_{i \in \alpha} X_i = U$ and $\beta \subset \mathbb{Z}^+$ $\bigcup_{i \in \beta} X_i = V$. Then $\bigcup_{i \in \beta, j \in \alpha} X_i \times Y_j = U$. ☒.

CH5.3

Definition [Neighborhood Basis]: Let p a point in a space X. A collection of open set $\{U_\alpha\}_{\alpha \in \lambda}$ in X is a neighborhood basis for p iff
(i) Each U_α contains p and
(ii) Every open set containing p contains some U_α.

Definition [1^{st} Countable Space]: A topological space is 1^{st} countable iff every point in X has a countable neighborhood basis.

Theorem 5.14: Let X be 2^{nd} Countable Space. Then X is 1^{st} countable.

Proof: The theorem is trivial.

(Comment: 2^{nd} Countability refers to countability of basis for whole topology. 1^{st} countable refers to a local version of the basis).

Theorem 5.15: If X is a topological space $p \in X$, and p has a countable neighborhood basis, then p has a countable neighborhood basis.

Proof:

- Suppose X has a countable neighborhood basis name it $\{U_1, U_2, U_3, \dots\}$
  Let $\{V_n : V_n = \bigcap_{i=1}^{n} U_i\}$ and we claim it's a countable neighborhood basis.
- First, of course it's countable and nested by construction.
- For every open neighborhood that contains p, it can be written as a union of U. Suppose some V_j is within the union, the set $\bigcap_{i=1}^{j} U_i \subset U_j$ is also within the union.

$\rightarrow$ Thus the set $\{V_n\}$ is a countable nested neighborhood basis. ⊠.

Example:

(1) $\mathbb{R}_{LL}$ is 1^{st} Countable:

- For every p, we may define the set of neighborhood of p $[p, p+\frac{1}{n})$.
  Now for any open neighborhood of p, we may always find a $[p, p+\frac{1}{n})$ contained in the neighborhood.

(2) $2^{\mathbb{R}}$ is not 1^{st} Countable: Conside $\overbrace{(0,0,0,\dots)}^{\mathbb{R}\text{ coord}}$, name it $\vec{0}$ from now on. WTS it doesn't have a nbhd basis.

- Let $U_{\alpha_0} = \underbrace{\{0\}}_{\alpha_0\text{th coordinate.}} \times \prod_{\alpha \neq \alpha_0} \{0,1\}$. Every U_{α_0} contains $\vec{0}$

- Suppose $\{V_\beta\}$ is a local countable basis.
- Obviously $\forall \alpha_0$, $U_{\alpha_0} \supset V_\beta$ for some β. Then $\exists \beta$ s.t. $U_{\alpha_0} \supset V_\beta$ for uncountable many coordinates.

→ Since this mapping $\alpha_0 \to \beta$ is injective. Say the β that α_0 maps to is β_{α_0}, $V_{\beta_{\alpha_0}} \subset U_{\alpha_0}$

- It can't be in any other U_α since it's having 0 on α_0 th coordinate. But any other U_α has $\{0,1\}$ on α_0 th coordinate.

→ Hence because α_0 is uncountable, β also has to be uncountable. ☒.

Theorem 5.18 Suppose x is a limit point of A in a 1^{st} countable space X. Then there is a sequence of points $\{a_i\}_{i\in\mathbb{N}}$ in A that converge to x.

Comment: This property is essentially matching 1^{st} countable space to metric spare we are familiar of.

Proof:

- Suppose A has a limit point in 1^{st} countable space X. Call it p.
- Since X is 1^{st} countable, there is a nested neighborhood basis around p. $\{B_i\}_{i\in\mathbb{Z}^+}$ is the neighborhood basis.
- Since p is a limit point, thus $\forall i \in \mathbb{Z}^+$, $\exists a_i \in A \cap B_i \setminus \{p\}$

→ The sequence converge to p because $\forall$ open set that contains p, there $\exists i \in \mathbb{Z}$. that $B_i \subset$ the open.

- Now since the open sets are nest, $\forall j > i$, B_j is also nested. ☒.

Example: First Countability of $\mathbb{R}^\omega$ under different topologies.

(Recall at end of Chapter 3).

1) Box topology: Not First Countable

- Say the point $\vec{x}$, assume it has a local countable basis. The local countable basis is open in every coordinate.
- The local basis is given by $\{B_n : n \in \mathbb{N}\}$ at $\vec{x}$. Each basis element is written as $B_n = \prod_{k=1}^{\infty} U_{n,k}$

→ Consider the basic element k,k. Let $x_k \in J_k \subsetneq U_{k,k}$

- Now let $W = \prod_{k=1}^{\infty} J_k$. It's an open neighborhood that contains none of the countable basis.

2) Product Topology : 1^{st} Countable

- Still consider the local basis at $\vec{x}$, defined as

$$B_{k,m} = N_{\frac{1}{m}}(x_1) \times N_{\frac{1}{m}}(x_2) \times \dots \times N_{\frac{1}{m}}(x_k) \times \prod_{\omega \setminus \{1,\dots,k\}}^{\infty} \mathbb{R}$$

- $k \in \mathbb{Z}$, $m \in \mathbb{Z}$ hence $Card(B) = Card|\mathbb{Z} \times \mathbb{Z}|$ is countable.
- For every open set that contains x in product topology, $\exists$ basic open set $U = \prod_{n=1}^{\infty} U_n$. V_n is open but not $\mathbb{R}$ all but finitely many coordinate.
- There is a maximum distance a_k s.t. $N_{a_k}(x_k) \subset U_k$.
- Let m be big enough so that $\frac{1}{m} < \inf_k \{a_k : k\text{th coordinate of } U_k \neq \mathbb{R}\}$ Then $B_{k,m}$ is contained in V. Hence the open set.

(Because Product has only finitely many coordinates we may modify - hence we can manual fit every coordinate into the product.)

3) Uniform Topology

- Consider the local basis at $\vec{x} = F(\vec{x}, q)$ where $q \in \mathbb{Q}$ is countable.
- For every open set containing $\vec{x}$, $F(\vec{x}, \varepsilon) = \bigcup_{0<\delta<\varepsilon} E(\vec{x}, \delta)$, we may find $0 < q < \varepsilon$. And it fits in the open set.

$\rightarrow$ It fulfills definition of local countable basis.
