CH2.2

Definition [Topology]: Suppose X is a set. Set T is a topogoly on X iff (if and only if) T is a collection of subsets of X s.t. (such that)

(1) $\phi \in T$ (2) $X \in T$ (3) $(U \in T) \wedge (V \in T) \rightarrow U \cap V \in T$

(4) For $\{U_\alpha\}_{\alpha \in \lambda}$ is any collection of subsets of T, $\bigcup_{\alpha \in \lambda} U_\alpha \in T$.

Remark: λ may have any cardinality, even uncountable.

Comment: "Topology" is essentially the "fundamental shape" of the set X, the most specifically meaning written later.

Example:

1. $\{\phi, X\}$ is a topology, in particular it's a "discrete topology"
2. $T = P(X)$, the Power Set of X, or the union of all subsets of X. "indiscrete".
3. Say X is a metric space, T = the set of all open sets in X.

Definition [Topological Space]: A topological Space is a pair (X, T), where X is a set and T is a topology of X. Often denoted "Space" unless otherwise noticed.

Remark: A set X can have multiple topological space. Suppose T, T' are both topologies of a set X and $T \neq T'$. Then (X, T) and (X, T') are different topological space.

Definition [Open Set]: A set $U \subset X$ is called an open set in (X, T) iff $U \in T$.

Comment: The meaning for having topology on X is specifying what subset of X is an open set.

Remark: Open set is an element of T, subsets of X.

Corollary 0.1: Arbitrary union of open sets is open, any finite intersection of open sets is open. [as below]

Theorem 2.1: Let $\{U_i\}_{i=1}^{n}$ be a finite collection of open sets in a topological space (X, T), then $\bigcap_{i=1}^{n} U_i$ is open.

Proof:

1. · To begin with $\forall\ i \in \{1, \dots, n\}$, $U_i \in T$. By definition of topology (3), for any $i, j \in \{1, \dots, n\}$ $U_i \cap U_j = U_{ij} \in T$.

2. Repeat the step 2, for $n-1$ times, we can intersection up all the sets in T. And obviously $\bigcap_{i=1}^{n} U_i \in T$.

3. Then by definition of open set, $\bigcap_{i=1}^{n} U_i \in T \to \bigcap_{i=1}^{n} U_i$ is open. $\boxtimes$.

• Why this proof fail for infinite intersection?

In step 2, there is not a finite amount step that we can keep repeating step 1. Thus this proof fails.

Theorem 2.3: A set U is open in a topological space (X, T) iff $\forall x \in U$, $\exists$ open U_x s.t. $x \in U_x \subset U$.

Verbally: $U \in T$, $x \in U$, there is a smaller $U_x \subset U$ that contains x.

Proof:

$\Rightarrow$: Note that in this book, "$\subset$" essentially means "$\subseteq$".

• Define $U_x = U$, then obviously $U_x \subset U$

• Because $U \in T$, $U_x \in T$. Thus U_x is also open.

$\Leftarrow$:

• Suppose $U \subset X$ s.t. $\forall x \in U$, $\exists U_x \in T$ s.t. $x \in U_x \subset U$.

We claim that $\bigcup_{x \in U} U_x = U$, using double inclusion.

• $\supset$: $\forall p \in U$, $\exists U_p$ s.t. $p \in U_p \subset \bigcup_{x \in U} U_x \to p \in \bigcup_{x \in U} U_x$

$\subset$: $\forall q \in \bigcup_{x \in U} U_x$, there has to be an α s.t. $q \in U_\alpha$. Therefore we know that $q \in U_\alpha \subset U \to q \in U$.

↳ By definition of set union.

• Now by the 4th property in definition of a topology, if $\{U_x\}_{x \in U}$ is a collection of subset of T s.t. $U_x \in T$. Then $\bigcup_{x \in U} U_x \in T$

• Hence $U \in T$. And by definition, U is open in the space (U, T). $\boxtimes$

Definition [neighborhood]: an open set containing x is a neighborhood of x.

→ The last theorem could be written as: a set U is open iff every point has a neighborhood that lies within U.

Example: check if the definition of a topological space fulfills the open sets in $\mathbb{R}$.

• Def: A standard topology T_{std} on $\mathbb{R}$ is a subset U of $\mathbb{R}$ belongs to T_{std} iff $\forall p \in U$, $\exists \varepsilon_p > 0$ s.t. $(p - \varepsilon_p, p + \varepsilon_p) \subset U$.

$U \in T_{std}$.

$\mathbb{R}_{std}$ denotes $(\mathbb{R}, T_{std})$

- We may generalize the topology into Euclidean Space $\mathbb{R}^n$. [set of n-tuple $\mathbb{R}$]
  $x = (x_1, \dots, x_n)$ $y = (y_1, \dots, y_n)$ - Euclidean Distance $= \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}}$.
- Using the definition of Euclidean distance, we define open ball in $\mathbb{R}^n$.
  $B(p, \varepsilon) = \{x \mid d(p, x) < \varepsilon\}$. So Now we extend standard topology T_{std} to $\mathbb{R}^n$

→ Suppose $U \subset \mathbb{R}^n$, $U \in T_{std} \longleftrightarrow \forall p \in U, \exists \varepsilon_p > 0$ s.t. $B(p, \varepsilon_p) \subset U$.

- Below is a proof T_{std} for $\mathbb{R}^n$ is a topology
  (1) In ϕ, there doesn't exist $p \in U$. it fulfill property → $\phi \in T_{std}$
  (2) In $\mathbb{R}^n$, any point $p \in U$ of course have an open ball around it. → $\mathbb{R}^n \in T_{std}$
  (3) If there are two sets $U, V \in T_{std}$, both U and V are open in $\mathbb{R}^n$. In analysis, there is a theorem that finite intersection of open sets are open. → $U \cap V \in T_{std}$.
  (4) If there is $\{U_\alpha\}_{\alpha \in \lambda}$ s.t. $U_\alpha \in T_{std}$ → U_α is open in $\mathbb{R}^n$. In metric space any union of open sets are open → $\bigcup_{\alpha \in \lambda} U_\alpha \in T_{std}$. ⊠

Definition (Finite Complement Topology): Given X, the finite complement topology on X is that:

- A subset U of X is open $(U \in T)$ iff $X - U$ is finite or $U = \phi$.

(1) $U = \phi \in T$ (2) $X - X = \phi$, which is open. Thus $X \in T$

(3) If $U, V \in T$ (1) If both are ϕ, $U \cap V = \phi$ → $U \cap V \in T$.
(2) one is ϕ → $U \cap V = \phi$ → $U \cap V \in T$.
(3) Both aren't ϕ → $X - U$, $X - V$ are both finite.

- Thus $(X - U) \cup (X - V)$ is also finite. We claim $(X - U) \cup (X - V) = X - (U \cap V)$.

If $p \in (X - U) \vee p \in (X - V)$, it mean it's not in either U or V, in other word, it can't occur both in U & V or $U \cap V$. → $p \in X - (U \cap V)$ Same reasoning backward.
It's essentially Demorgan's law.

- $X - (U \cap V)$ is finite → $U \cap V \in T$.

(4) If $\exists \{U_\alpha\}_{\alpha \in \lambda}$ s.t. $\forall \alpha \in \lambda$, $U_\alpha \in T$. Then $\forall \alpha \in \lambda$, $(X - U_\alpha$ is finite$) \vee (U_\alpha = \phi)$. Similarly we use Demorgan's law

- $\bigcap_{\alpha \in \lambda} (X - U_\alpha) = X - \bigcup_{\alpha \in \lambda} U_\alpha$. As $X - U_\alpha$ is finite, infinite intersection is of course finite. Thus $X - \bigcup_{\alpha \in \lambda} U_\alpha$ is finite. → $\bigcup_{\alpha \in \lambda} U_\alpha \in T$.

Definition [Countable Complement Topology]: Replace Finite with countable, proof is the same thing.

CH 2.3

- Definition [Limit Point]: Let (X, T) be a topological space, $A \subset X$, $p \in X$
  p is a limit point of set A if $\forall\, U \in T$ (Open set) s.t. $p \in U$, $(U - \{p\}) \cap A \neq \phi$.
- Comment: p may or may not belong to A.
- Comment: The definition essentially says limit point of a set is a point that every open set containing it intersect with some other points other than the point itself.
- The definition describes the condition in analysis "arbitrarily close" without using the concept of a metric, but only using idea of open set.

Example: Let $X = \mathbb{R}$ and $A = (1,2)$. 0 is a limit point A under indiscrete topology but isn't a limit point under discrete or standard topology.

- This shows under the context of a set may have different topology, exhibits different meaning of being "arbitrarily close."
- Indiscrete topology of $\mathbb{R}$: $\{\phi, \mathbb{R}\}$ $\left\{ \begin{array}{l} \phi \text{ doesn't contain } 0 \\ \mathbb{R} \text{ contain } 0,\ (\mathbb{R} - \{0\}) \cap (1,2) = (1,2) \end{array} \right.$
  Thus it's a limit point under indiscrete $\leftarrow$
- Discrete topology of $\mathbb{R}$: $\{P(\mathbb{R})\}$ $\rightarrow (-\frac{1}{2}, \frac{1}{2}) \in \{P(\mathbb{R})\}$ and $0 \in (-\frac{1}{2}, \frac{1}{2})$.
  But $[(\frac{-1}{2}, \frac{1}{2}) \setminus \{0\}] \cap (1,2) = \phi$ $\rightarrow$ It's not a limit point.
- Standard topology of $\mathbb{R}$: $\{$All open sets in $\mathbb{R}\}$. same example $(-\frac{1}{2}, \frac{1}{2})$ work for it too.

Theorem 2.9. Suppose $p \notin A$ in a topological space (X, T). Then p is not a limit point $\leftrightarrow$ $\exists$ neighborhood U_p s.t. $U \cap A = \phi$.

Proof :

$\Rightarrow$: p is not a limit point, thus we negate the definition of limit point:

- Given topological space (X, T) $A \subset X$, $p \in X$. $\exists\, U_p \in T$ s.t. $p \in U$ where $(U_p - \{p\}) \cap A = \phi$.
- Obviously, since $\{p\} \cap A = \phi \wedge ((U_p - \{p\}) \cap A = \phi) \rightarrow [\{p\} \cup (U_p - \{p\})] \cap A = \phi$.
  Thus $U_p \cap A = \phi$

$\Leftarrow$: Since $\exists\, U_p$ s.t. $U_p \cap A = \phi$, then of course $(U_p - \{p\}) \cap A = \phi$ b.c. $(U_p - \{p\}) \subset U_p$. $\boxtimes$

Definition [Isolated Point]: Given topological space (X, T), $A \subset X$, $p \in X$. If $(p \in A) \wedge$

(p isn't a limit point), p is an isolated point of A.

This theorem is not suitable for last theorem since $p \notin A$ in last theorem.

Corollary: p is an isolated point of A in topological space $(\mathcal{X}, T)$ $\rightarrow$ $\exists U \in T$ (open set) s.t. $U \cap A = \{p\}$

Proof: We attempt to prove by contrapositive

- Suppose $\forall U \in T$, $U \cap A \neq \{p\}$ $\rightarrow$ then either $U \cap A = \phi$ or $U \cap A = \{p\} \cup \{\text{other}\}$.

  1. $U \cap A = \phi$ $\rightarrow$ if $\boxed{p \in A}$ $p \notin U$ $\forall U \in T$. Then it's not meaningful, as $\mathcal{X} \in T$, then $p \notin \mathcal{X}$. $\rightarrow$ $\boxed{p \notin A \subset \mathcal{X}.}$ contradiction.

     $\rightarrow$ If $p \in U$ but $p \notin A$ $\rightarrow$ negate condition for being an isolate point.

     Thus 1. cannot happen!

  2. Thus for every $U \in T$, $U \cap A = \{p\} \cup \{\text{other point(s)}\}$ $\rightarrow$ $(U \cap A) \setminus \{p\} \neq \phi$.

  - Then it fulfills definition of a limit point, it's not an isolated point. ☒.

Definition [Closure]: Assume topological space $(\mathcal{X}, T)$ and $A \subset \mathcal{X}$. Closure of A in $\mathcal{X}$

Denoted $\bar{A}$, $Cl(A)$, $Cl_{\mathcal{X}}(A)$ = [set A] $\cup$ [all its limit points]

Definition [Closed Set]: Assume topological space $(\mathcal{X}, T)$ and $A \subset \mathcal{X}$. The subset A is closed iff $\bar{A} = A$. (If A contains all its limit point).

Examples: Closed Sets in different topologies

1. In Discrete topology: $T = P(\mathcal{X})$

   Suppose topological space $(\mathcal{X}, T)$, and for a set $A \subset \mathcal{X}$.

   discuss $p \in A$ and $p \notin A$

   - And analyze every point p s.t. $p \notin A$, since $\{p\} \in P(\mathcal{X}) = T$. And obviously $\{p\} \cap A = \phi$. p cann't be a limit point.
   - And $\forall q \in A$, $\{q\} \cap A \setminus \{q\} = \phi \cap A = \phi$ $\rightarrow$ q isn't a limit point.

- Thus no point is a limit point in discrete topology. And trivially, A contains all its limit point, or just ϕ. · Every set in discrete topology is closed!

2. In Indiscrete topology: $T = \{\phi, \mathcal{X}\}$

   - $\forall p \in \mathcal{X}$, $p \notin \phi$ and $\boxed{p \in \mathcal{X}}$ Only set has p. As long as $A \neq \{p\}$, $A \cap \mathcal{X} = A$ and $A - \{p\} \neq \phi$
   - Thus p has to be A's limit point. As long as $A \neq \mathcal{X}$, $\exists q \in \mathcal{X} - A$, $q \notin A$.

Thus the only closed set in $\mathcal{X}$ is $\mathcal{X}$ itself in indiscrete topology.

3. In Finite Complement topology:

Recall, $U \in T$ is open $\leftrightarrow$ ($\mathcal{X} - U$ is finite) $\vee$ ($U = \phi$). For some $A \subset \mathcal{X}$

- Limit Point analysis: $\forall x \in \mathcal{X}$ and $\forall x \in U \in T$. Because $x \notin U = \phi$, $x - U$ is finite.
- Suppose that A is finite, we can just have $\boxed{U = \mathcal{X} - A \cup \{x\}}$ its complement is going to be $A - \{x\}$, is finite. Thus $x \in U \in T$. Finite set has no limit point, thus is closed.
- A is infinite, and its complement is finite. Then $\nexists U \in T$ s.t. $U \cap A \setminus \{x\} = \phi$ since only chance it happen is U is finite

  infinite. Then $\forall U \cap A \setminus \{x\} = \phi$, $U \notin T$ as its complement is infinite. All points are limit points
- Since there must be points out of the set except for $\mathcal{X}$, None are closed! ☒

Theorem 2.13. For any topological space $(\mathcal{X}, T)$ and $A \subset \mathcal{X}$, the set $\bar{A}$ is closed. Or just $\bar{\bar{A}} = \bar{A}$.

Proof:

- Suppose $A \subset \mathcal{X}$ and $\bar{A} \subset \mathcal{X}$ is its closure. If x is a limit point of $\bar{A}$, then $x \in \bar{A}$.
- x is a limit point of $\bar{A}$, by definition, $\forall x \in U \in T$, $(U \cap \bar{A}) - \{x\} \neq \phi$

  Because $\bar{A} = A \cup \{\text{limit point of } A\}$,

  Either $U \cap A - \{x\} \neq \phi$ or $U \cap (\text{limit points of } A) - \{x\} \neq \phi$.
- $U \cap A - \{x\} \neq \phi$, then the set U intersects A, $\forall U$. Thus x is a limit point of A.
- $U \cap (\text{limit points of } A) - \{x\} \neq \phi$, meaning $\exists p \in \{\text{limit point of } x\}$ within the intersection

  $\rightarrow \forall U \in T$ s.t. $p \in U$, $\exists q \in A$ s.t. $q \in (U \cap A) - \{p\}$. (Definition of limit point).

  Thus $\forall U \in T$, $\exists q \in U$ s.t. $q \in U \cap (\text{limit point of } A) - \{x\}$.
- By Green + Blue part $\rightarrow \exists q \in (U \cap \bar{A}) - \{x\}$ s.t. $q \in A$. Thus $x \in \bar{A}$, a limit point. ☒

Theorem 2.14 $(\mathcal{X}, T)$ is a topological space. The set A is closed iff $\mathcal{X} - A$ is open.

Proof:

$\Rightarrow$: A is closed $\rightarrow$ $\mathcal{X} - A$ is open.

- By definition: A is closed $\leftrightarrow$ A contains all its limit points, take contrapositive statement: If x is not a limit point $\rightarrow$ $x \notin A$ (*)
- Now for $x \in \mathcal{X} - A$, WTS $\exists U \in \mathcal{T}$ s.t. $U \cap A = \phi$.
- Because $x \notin A$, x is not a limit point of A (By the 1st step), $\exists$ a neighborhood $V \in \mathcal{T}$ s.t. $U \cap A = \phi$ $\leftrightarrow$ $U \subset \mathcal{X} - A$
- This fulfills the definition of being open.

(By Theorem 2.3): The basic logic is that you give me an x, I give you a U_x s.t. $x \in U_x \subset A$. The U_x in this case is the neighborhood.

$\Leftarrow$: $\mathcal{X} - A$ is open $\rightarrow$ A is closed.

- $\forall x \in \mathcal{X} - A$, $\exists U \in \mathcal{T}$ s.t. $x \in U \subset \mathcal{X} - A$
- In other word, $\forall x \in A$, $\boxed{\exists U \in \mathcal{T} \text{ s.t. } U \cap A \setminus \{x\} = \phi}$ It is exactly the contrapositive statement of A being closed as mentioned. (*). Thus A is closed ⊠

Theorem 2.15 $(\mathcal{X}, \mathcal{T})$ is a topological space, U is an open set and A is a closed subset of $\mathcal{X}$ $\rightarrow$ $\underbrace{(U - A \text{ is open})}_{\mathrm{I}} \wedge \underbrace{(A - U \text{ is closed})}_{\mathrm{II}}$

Proof:

- Part (I)
  - For some $x \in U - A$, obviously $x \in U - A \subset \mathcal{X} - A$.
  - By (theorem 2.14), A is closed $\leftrightarrow$ $\mathcal{X} - A$ is open. By (theorem 2.3) $\mathcal{X} - A$ is open $\leftrightarrow$ $\exists V \in \mathcal{T}$ s.t. $x \in V \subset \mathcal{X} - A$.
  - Apply (theorem 2.3) again, $x \in U - A \subset U$ $\leftrightarrow$ $\exists W \in \mathcal{T}$ s.t. $x \in W \subset U$

  Now because $V \in \mathcal{T}$, $W \in \mathcal{T}$. Apply (theorem 2.1) $V \cap W$, a finite intersection of open sets $V \cap W \in \mathcal{T}$. Let $(V \cap W) = V'$
  - As $V \subset \mathcal{X} - A$, $W \subset U$, $V \cap W \subset U \cap (\mathcal{X} - A) = V \cap W \subset U - A$
  - Thus $\forall x \in U - A$, $\exists V' \in \mathcal{T}$ s.t. $x \in V' \subset U - A$
- Part (II). It's same proof as part I, just that we prove $x \in \mathcal{X} - (A - U)$ is open. It's the part out of A and inside U ⊠.

Theorem 2.16 $(\mathcal{X}, \mathcal{T})$ is a topological space, then
(i) ϕ is closed (ii) $\mathcal{X}$ is closed (iii) Union of finitely closed set is closed

(iv) $\{A_\alpha\}_{\alpha\in\lambda}$ is any collection of closed sets in $(\mathcal{X}, T)$ $\bigcap_{\alpha\in\lambda} A_\alpha$ is closed.

Proof:

(i) By theorem 2.14, $\mathcal{X} - \phi = \mathcal{X} \in T$ is open (ii) $\mathcal{X} - \mathcal{X} = \phi \in T$ is open.

(iii), (iv) Take $\mathcal{X} - A_\alpha$ and take union / intersection. ◻.

Theorem 2.20 For any set A in topological space, the closure of A equals the intersection of all closed sets containing A.

- Or $\bar{A} = \bigcap_{B\supset A,\, B\in C} B$

Proof: We prove using double inclusion

$\subseteq$: To prove $\bar{A} \subseteq \bigcap B$, we prove $\bar{A} \subset B$ $\forall B$.

- Because B contains all its limit point, it of course contain all A's limit point. To be specific. $\forall x$ s.t. $\forall x \in U \in T$, $U \cap A \setminus \{x\} \neq \phi$

  Because $\rightarrow U \cap A \setminus \{x\} \subset U \cap B \setminus \{x\} \neq \phi$. Thus it fulfills the definition of a limit point of B.

- Since $B \supset A$ and $B \supset$ (all limit point of A), $B \supset \bar{A}$. Thus $\bar{A} \subset \bigcap B$

$\supseteq$: For some point that's in all B s.t. $B \supset A$ and B is closed.

- Suppose $x \notin \bar{A}$, $\exists\, x \in U \in T$ s.t. $U \cap A = \phi$
- Because U is open, by theorem 2.13 $\mathcal{X} \setminus U$ is closed and $A \subset \mathcal{X} \setminus U$. And since $x \in$ [every closed set containing A] (in other word, $\mathcal{X} \setminus A$ is one of the B's), we know $x \in \mathcal{X} \setminus U$. $\rightarrow$ Contradiction as $x \in U$. ◻

Comment: The basic Logic is that $x \notin \bar{A} \rightarrow \exists$ closed superset of A that exclude x.

E.g. Subsets of $\mathbb{R}$ and their closure under different topology.

	$(0,1)$	$(\infty, 0]$	$[0,1]$
Discrete	$(0,1)$	$(-\infty, 0]$	$[0,1]$
Indiscrete	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Finite Complement	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Standard	$[0,1]$	$(-\infty, 0]$	$[0,1]$

Theorem 2.22 Let $A, B \subset \mathcal{X}$. Then: (1) $A \subset B \rightarrow \bar{A} \subset \bar{B}$ (2) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

Proof I: A proof using limit point

(1) Suppose $x \in \bar{A}$, if $x \in A$, then $x \in B \subset \bar{B}$, we are done.

If $x \notin A$, then $\forall x \in U \in T$, $U \cap A \setminus \{x\} \neq \phi$

Thus $U \cap A \setminus \{x\} \subset U \cap B \setminus \{x\} \neq \phi$. Thus

$x \in$ [The set of all limit point of B] $\to$ $x \in \bar{B}$.

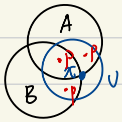
(2) 1. $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$

- Suppose $x \in \overline{A \cup B}$. If $x \in A \cup B$, then $(x \in A) \vee (x \in B) \to (x \in A \subset \bar{A}) \vee (x \in B \subset \bar{B})$

$\to (x \in \bar{A}) \vee (x \in \bar{B}) \to x \in \bar{A} \cup \bar{B}$.

- $x \notin A \cup B$. Suppose for $x \notin \bar{A} \cup \bar{B}$, by Demorgan's law For the sake of contrapositive

$(x \notin \bar{A}) \wedge (x \notin \bar{B}) \leftarrow (x \in X - (A \cup B) \leftrightarrow x \in (X - A) \cap (X - B))$



- Thus there exists $U_A, U_B \in T$ s.t. $x \in U_A, U_B$, $U_A \cap A = \phi$, $U_B \cap B = \phi$.
- Take $U_A \cap U_B$, obvious it's not empty, at least they both have x.

$\to$ By definition of topology, $(U_A \in T) \wedge (U_B \in T)$, $(U_A \cap U_B) \in T$

Now because $(U_A \cap A) = \phi \wedge (U_B \cap B) = \phi$

$\to$ 1. $(U_A \cap U_B) \subset U_A \cap A = \phi$ 2. $(U_A \cap U_B) \subset U_B \cap B = \phi$ $\to$ $(U_A \cap U_B) \cap (A \cup B) = \phi$

- Thus we found $\forall x \in \bar{A} \cup \bar{B}$, $\exists U = (U_A \cap U_B) \in T$ s.t. $U \cap (A \cup B) = \phi$
- Therefore by definition, $x \notin \overline{A \cup B}$

2. $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$

- Suppose $x \in \bar{A} \cup \bar{B}$, meaning either $x \in \bar{A}$ or $x \in \bar{B}$. WLOG, assume $x \in \bar{A}$.
- If $x \in A$, $x \in (A \cup B) \subset \overline{(A \cup B)}$ obviously

If $x \notin A$, $\forall U \in T$ s.t. $x \in U$, $U \cap A \setminus \{x\} \neq \phi$

and obviously $[U \cap A] \subset [U \cap (A \cup B)] \setminus \{x\} \neq \phi$. Thus, it fulfills the definition of $x \in \overline{(A \cup B)}$. Thus $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. ⊠

Proof II: Using theorem 2.20

$(U \in T, V \in T, W \in T,)$

(2)· By theorem 2.20

1. $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$

- Since $A \subset A \cup B$. $\bar{A} \subset \overline{A \cup B}$ (Below is a proof). For some $x \in \bar{A}$

1. If $x \in A$, $x \in A \subset A \cup B \subset \overline{A \cup B}$ and we're done
2. If $x \notin A$, $\forall U \in T$ s.t. $x \in U$, $U \cap A \setminus \{x\} \neq \phi$.

Thus $(U \cap A \setminus \{x\}) \subset (U \cap (A \cup B) \setminus \{x\}) \neq \phi$ $\to$ $x \in \overline{A \cup B}$ by definition.

- For the same reason $\bar{B} \subset \overline{A\cup B}$. Thus $\bar{A}\cup\bar{B} \subset \overline{A\cup B}$.

2. $\overline{A\cup B} \subset \bar{A}\cup\bar{B}$
   - Since $A\subset\bar{A}$, $B\subset\bar{B}$, $A\cup B \subset \bar{A}\cup\bar{B}$.
   - Because $(\bar{A}\in C)$, $(\bar{B}\in C)$, $\bar{A}\cup\bar{B}\in C$. By [theorem 2.20], because $\overline{A\cup B}$ is the smallest closed set that contains $A\cup B$. As $\bar{A}\cup\bar{B}$ is another closed set that contains $A\cup B$, $\overline{A\cup B} \subset \bar{A}\cup\bar{B}$.

Extending the Theorem: is $\overline{\bigcup_{\alpha\in\lambda} A_\alpha} = \bigcup_{\alpha\in\lambda} \overline{A_\alpha}$?

- The answer is negative, the following is an counterexample.
  Let $A_n = \{\frac{1}{n}\} \in \mathbb{R}$ for $n\in\mathbb{Z}^+$ under T_{std}.
- Under standard topology, $\{\frac{1}{n}\}$ has no limit point, thus contains all it's limit points $\rightarrow$ $\{\frac{1}{n}\}$ is closed. By theorem 2.13 $\overline{\{\frac{1}{n}\}} = \{\frac{1}{n}\}$
  Thus we know that $\bigcup_{n\in\mathbb{Z}^+}\overline{\{\frac{1}{n}\}} = \{\frac{1}{n} : n\in\mathbb{N}\} = \bigcup_{\alpha\in\lambda} A_\alpha$
- But for closure of $\bigcup_{\alpha\in\lambda} A_\alpha = \bigcup_{n\in\mathbb{N}}\{\frac{1}{n}\}$, there is an additional limit point of $\{0\}$ since $\forall\varepsilon$, $\exists N\in\mathbb{N}$ s.t. $\frac{1}{N}>\varepsilon$, $\forall n>N$, $|\frac{1}{n}-0| < \frac{1}{N} < \varepsilon$.
  Thus for every neighborhood of $\{0\}$, there is always $\{\frac{1}{n}\}$ in the nbhd

$$\overline{\bigcup_{n\in\mathbb{Z}^+}\{\tfrac{1}{n}\}} = \left[\bigcup_{n\in\mathbb{Z}^+}\{\tfrac{1}{n}\}\right]\cup\{0\} \neq \bigcup_{n\in\mathbb{Z}^+}\{\tfrac{1}{n}\} = \bigcup_{n\in\mathbb{Z}^+}\overline{\{\tfrac{1}{n}\}}.$$

$\boxtimes$.

CH 2.4

- Definition [Interior]: An interior of a set A in a topological space X, denoted A° or $\mathrm{Int}(A)$, as $\mathrm{Int}(A) = \bigcup_{V\subset A,\, V\in T} U$
- Definition [Interior Point]: Points of $\mathrm{Int}(A)$ is interior point of A.

Remark: As closure is the smallest closed set that contains A, an interior is the largest open set contained by A.

Theorem 2.26: Let A be a subset of a topological space (X, T), then (p is an interior point) $\Leftrightarrow$ ($\exists U\in T$ s.t. $p\in U\subset A$).

Proof:

$\Rightarrow$: $p\in \mathrm{Int}\, A$, by definition $p\in\bigcup_{V\subset A,\, V\in T} U$. $\exists U_p\in T$ s.t. $p\in U_p$. Since $U_p\subset A$, it is exactly what RHS suggest.

$\Leftarrow$: If $\exists U_p \in T$ s.t. $p \in U_p \subset A$, clearly $U_p \subset \bigcup_{U \subset A, U \in T} U = Int(A) \rightarrow U_p \in Int(A)$. ☒.

Corollary 2.27: $U \in T$ in topological space $(X, T) \longleftrightarrow \forall x \in U, x \in Int(U)$.

Definition: The boundary of A, or Bd(A), $\partial A = \bar{A} \cap \overline{X - A}$

Theorem 2.28: Let A be a subset of a topological space X, $Int(A)$, $Bd(A)$ and $Int(X-A)$ are disjoint and their union is X.

- Or $(Int(A) \cap Bd(A) = \phi) \wedge (Bd(A) \cap Int(X-A) = \phi)$
  $(Int(A) \cup Bd(A) \cup Int(X-A))$

Proof:

1. It's obvious that $(A \cap X-A = \phi) \wedge (Int(A) \subset A) \wedge (Int(X-A) \subset X-A)$
   $\Longrightarrow Int(X-A) \cap Int(A) = \phi$.

2. Let $x \in Bd(A) = \bar{A} \cap \overline{X-A}$.
   - First, what kind of x is not on the boundary?

By Definition, it's the x that's either $1 (x \notin X-A) \cup (\exists U \in T$ s.t. $x \in U, U \cap X-A = \phi)$
or $2 (x \notin A) \cup (\exists U \in T$ s.t. $x \in U, U \cap A = \phi)$

- Case 1 is $x \notin \overline{X-A}$. Case 2 is $x \notin \bar{A}$

$\rightarrow$ Note we may also denote the cases as 1. ... $U \subset A$ 2. ... $U \subset X-A$.

In other word, we found a 1. $U \in T$ s.t. $U \subset A \rightarrow x \in U \subset \bigcup_{U \subset A, U \in T} U = Int(A)$.

For same reason 2. $U \in T$ s.t. $U \subset X-A \rightarrow x \in Int(X-A)$

$\rightarrow$ Thus we prove if $x \notin Bd(A)$, $(x \in Int(A)) \vee (x \in Int(X-A))$, which suggest.

- $Bd(A) \cap (Int(A) \cup Int(U-A)) = \phi \rightarrow Bd(A) \cap Int(A) = Bd(A) \cap Int(X-A) = \phi$.

3. $\boxed{\text{The last statement}}$ Also proves that if $x \in Bd(A)$, then it has to be in the rest of two options, suggest that
   $\boxed{Bd(A) \cup Int(A) \cup Int(X-A) = X.}$ ☒.

Examples of Interior and Boundary of set in $\mathbb{R}$.

	Discrete	Indiscrete	Finite - Complement	T std
$\{x\}$	$\{x\}, \phi$	$\phi, \mathbb{R}$	$\phi, \{x\}$	$\phi, \{x\}$
{Finite Set}	F, ϕ	$\phi, \mathbb{R}$	ϕ, F	ϕ, F

{Co Finite Set}	U, ϕ	$\phi, \mathbb{R}$	$U, \mathbb{R}\setminus U$	$\mathbb{R}\setminus U$
(a,b)	$(a,b), \phi$	$\phi, \mathbb{R}$	$\phi, \mathbb{R}$	$(a,b), \{a,b\}$
ϕ	ϕ, ϕ	ϕ, ϕ	ϕ, ϕ	ϕ, ϕ
$\mathbb{R}$	$\mathbb{R}, \phi$	$\mathbb{R}, \phi$	$\mathbb{R}, \phi$	$\mathbb{R}, \phi$.

CH2.5

Definition [Sequence] : A sequence in a topological space X is a function that $f: \mathbb{N} \to X$. The image of i is x_i.

- We usually just write the sequence by listing its images $x_1, x_2, x_3, \ldots$ Or $\boxed{(x_i)_{i\in\mathbb{N}}}$

Definition [Limit of a sequence] : $(x_i)_{i\in\mathbb{N}}$ converge to p (denoted $x_i \to p$) iff $\forall\, U \in T$ s.t. $p \in U$, $\exists N \in \mathbb{N}$ s.t. $(i > N) \to (x_i \in U)$.

- Comment : (x_i) converge to p suggest every neighborhood of p has infinitely many x_i's in the neighborhood.

Theorem 2.30 : Let A be a subset of the topological space (X, T). Let $p \in X$, $(\{x_i\}_{i\in\mathbb{N}} \subset A) \wedge (x_i \to p) \longrightarrow p \in \bar{A}$.

Proof:

- We know that $\forall\, U \in T$ s.t. $p \in U$, $\exists N \in \mathbb{N}$ s.t. $(i > N) \to (x_i \in U)$.
- And that since every $x_i \in A$

  Thus $\forall\, U \in T$ s.t. $p \in U$, $U \cap A \setminus \{p\} \supset \{x_i : i > N\} \neq \phi$. Which fulfills the definition of being a limit point. ☒

Theorem 2.31 : In standard topology on $\mathbb{R}^n$, if p is a limit point of A, then $\exists\, (x_i)_{i\in\mathbb{N}}$ s.t. $x_i \in A$ $\forall\, i \in \mathbb{N}$, $x_i \to p$.

Proof: ($B_r(s)$: open ball with radius r and centered at p).

- p is a limit point $\leftrightarrow \forall\, U \in T_{std}$ s.t. $p \in U$, $U \cap A \setminus \{x\} \neq \phi$
- $\forall\, k \geq 1$, construct $B_{\frac{1}{k}}(p)$, by the last step $\to B_{\frac{1}{k}}(p) \cap A \neq \phi$

$\to$ Pick any $x_k \in B_{\frac{1}{k}}(p) \cap A$

- Now by construction, $d(x_k, p) < \frac{1}{k}$. And $\forall\, U \in T$, $p \in U$, take $s = \inf_{w \in U} d(w, p)$

Apparently $S > 0$, since if $S = 0$, $\exists\, w = p$, $\forall\, U \in T$ s.t. $p \in U$, $U \not\subset A$.
(We may view that p has to be at the boundary)

- Thus $S > 0$, and by Archimedian Principle we know that $\exists\, k \in \mathbb{N}$ s.t. $\frac{1}{k} < S$. Hence $B_{\frac{1}{k}}(p) \subset S$. Thus these points s.t. $B_{\frac{1}{k}}(p) \cap A \setminus \{p\} \subset S$.

→ This fulfills the definition of sequence convergence. ☒.

Remark: Convergent sequence converge to a unique limit on $(\mathbb{R}, T_{std})$, it may not be the case for other topologies.

Example:

- Consider sequences with finite complement topology in $\mathbb{R}$, what sequences converge and to what value they converge.

Recap: $U \in T$ if $X - U$ is finite or $U = \phi$.

(1) Sequence with no limit: $x_n = (-1)^n$

→ Suppose the sequence has a limit p, then it may at most cover a point in the set $\{-1, 1\}$, as they are the only 2 elements in the sequence.

→ WLOG, assume $p = -1$. $p \in \mathbb{R} \setminus \{1\} \in T$. But for any $n \in \mathbb{N}$, there is always an $x_n = 1$ after it. Thus it doesn't fulfill definition of convergence.

- Similarly with $p \neq -1$ or 1, then we take $U = \mathbb{R} \setminus \{-1, 1\} \in T$.

(2) Sequence with one limit: $x_n = 0$

→ For every $U \in T$ s.t. $0 \in U$, for any $N \in \mathbb{N}$, the rest x_n with $n > N$ are all zero. Thus 0 is obviously a limit.

→ There is no limit point other than 0: for any p other than 0, $p \in \mathbb{R} \setminus \{0\} \in T$. Which is an open set that contains no element in the sequence.

(3) Sequences with two limits: Doesn't exist.

- Suppose for a sequence (x_n) with 2 distinct limits p and q.
- Take $U_q = \mathbb{R} \setminus \{p\} \in T$, $\exists\, N_1 \in \mathbb{N}$ s.t. $n > N_1 \rightarrow x_n \in U_q$. There are at most N_1 time occurrence of p in the sequence. (as $n > N_1 \rightarrow x_n \in X \setminus \{p\}$)
- Take $U_p = \mathbb{R} \setminus \{q\} \in T$, $\exists\, N_2 \in \mathbb{N}$ s.t. $n > N_2 \rightarrow x_n \in U_p$. Similar, at most N_2 occurence of q in the sequence.

Let $U_q = \mathbb{R} \setminus \{r\} \in T$ ...

→ Actually, take any $r \in \mathbb{R}$, using the $\boxed{\text{last two steps}}$ r occur at most finite time in the sequence.

- Thus we proved that for any $r \in \mathbb{R}$, r can occur at most finite times. Now take any $U \in T$ s.t. $r \in U$, there can only be finitely many point excluded by U in $\mathbb{R}$. And each point may only occur finitely many times in the sequence.

$\rightarrow$ Thus, $\exists N_r \in \mathbb{N}$ s.t. $\forall n > N_r$, $x_n \neq x$, $x \in \boxed{\{\mathbb{R} \setminus U\}}$ The finite set. and thus $x_n \in U$. Hence $x_n \rightarrow r$.

- The sequence actually converge to every point in $\mathbb{R}$. Contradicts our assumption.

(4) Consider the sequence $x_n = n$. The sequence hits every element in $\mathbb{R}$ at most once, which is finite.

- Using our proof in the orange box, we showed (every r occur only finite time) $\rightarrow$ (the sequence converge to all $r \in \mathbb{R}$).
- Thus (x_n) converge to all $r \in \mathbb{R}$, which is infinitely many. $\boxtimes$.

CH3.1

- Definition [Basis]: Let T be a topology on X and let $B \subset T$. Then B is a basis for topology T iff every open set is a union of elements of B. Or

$$(B \text{ is a basis of } T) \longleftrightarrow (\forall U \in T,\ U = \cup_i B_i \text{ s.t. } B_i \in B)$$

- Definition [Basis element / Basic open set]: If $b \in B$, b is a basis element of T. or a basic open set of X.

Remark: An empty union is the empty set, so any B will generate an ϕ by union none of the sets in the Basis.

Comment: The inspiration of developing the concept "basis" is that arbitrary open set union can generate open sets, thus a topological space may have

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Theorem 3.1: Let (X, T) be a topological space, B is a basis of T iff $(B \subset T) \wedge (\forall U \in T, p \in U, \exists V \in B \text{ s.t. } p \in V \subset U)$

Comment: In other word, it means for every open set, every point could be contained by a set in Basis, contained in the open set.

Proof:

$\Rightarrow$: B is a basis $\longrightarrow (B \subset T) \wedge (\forall p \in U \in T, \exists V \in B \text{ s.t. } p \in V \subset U)$.