time in the sequence.

- Thus we proved that for any $r \in \mathbb{R}$, r can occur at most finite times. Now take any $U \in T$ s.t. $r \in U$, there can only be finitely many point excluded by U in $\mathbb{R}$. And each point may only occur finitely many times in the sequence.

$\rightarrow$ Thus, $\exists N_r \in \mathbb{N}$ s.t. $\forall n > N_r$, $x_n \neq x$, $x \in \boxed{\{\mathbb{R} \setminus U\}}$ The finite set. and thus $x_n \in U$. Hence $x_n \rightarrow r$.

- The sequence actually converge to every point in $\mathbb{R}$. Contradicts our assumption.

(4) Consider the sequence $x_n = n$. The sequence hits every element in $\mathbb{R}$ at most once, which is finite.

- Using our proof in the orange box, we showed (every r occur only finite time) $\rightarrow$ (the sequence converge to all $r \in \mathbb{R}$).
- Thus (x_n) converge to all $r \in \mathbb{R}$, which is infinitely many. $\square$.

CH3.1

- Definition [Basis]: Let T be a topology on X and let $B \subset T$. Then B is a basis for topology T iff every open set is a union of elements of B. Or

$$(B \text{ is a basis of } T) \longleftrightarrow (\forall U \in T,\ U = \bigcup_i B_i \text{ s.t. } B_i \in B)$$

- Definition [Basis element / Basic open set]: If $b \in B$, b is a basis element of T. or a basic open set of X.

Remark: An empty union is the empty set, so any B will generate an $\emptyset$ by union none of the sets in the Basis.

Comment: The inspiration of developing the concept "basis" is that arbitrary open set union can generate open sets, thus a topological space may have

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Theorem 3.1: Let (X, T) be a topological space, B is a basis of T iff $(B \subset T) \wedge (\forall U \in T, p \in U, \exists V \in B \text{ s.t. } p \in V \subset U)$

Comment: In other word, it means for every open set, every point could be contained by a set in Basis, contained in the open set.

Proof:

$\Rightarrow$: B is a basis $\longrightarrow (B \subset T) \wedge (\forall p \in U \in T, \exists V \in B \text{ s.t. } p \in V \subset U)$.

- By definition of a basis, $\forall U \in T$, $U = \cup_i b_i$, $b_i \in B$.
- The negation of RHS: $(B \not\subset T) \vee (\exists p \in U \in T \text{ s.t. } \boxed{\nexists V \in B \text{ s.t. } p \in V \subset U})$

  $= \forall V \in B$, either $p \notin V$ or $V \not\subset U$.

→ If $p \notin V$, then unioning all the b_i would not include p, thus $V \neq \cup_i b_i$

If $(V \not\subset U) \Leftrightarrow ((V \cap U) \setminus V = P \neq \phi)$, even $p \in V$, the union including V will also have $P \not\subset U$. (P is the part in V but not in U)., thus $V \neq \cup_i b_i$.

$\Leftarrow$: $(B \subset T) \wedge (\forall p \in U \in T, \exists V \in B \text{ s.t. } p \in V \subset U) \rightarrow B$ is a basis.

- Because $\forall U \in T$ and $\forall p \in U$, $\exists V \in B$, $p \in V \subset U$.

  Thus every $p \in U$ can be included by the union $p \in \cup_i b_i$. $\square$.

Example: $B_1 = \{(a,b) \subset \mathbb{R} \mid a, b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$.

Proof:

- For some open set in $\mathbb{R}$, (A,B). For some $p \in (A,B)$, take $\varepsilon < \min\{|A-p|, |B-p|\}$.

  The open ball $N_\varepsilon(p) \subset (A,B)$ and obviously contain p.
- By Theorem 3.1, B_1 is a basis for $\mathbb{R}$.

Two Questions: $\begin{cases} \text{Given a topology} \rightarrow \text{What is its basis (Theorem 3.1)} \\ \text{Given a set} \rightarrow \text{Is it a basis for some topology (Theorem 3.3)} \end{cases}$

Theorem 3.3: Suppose X is a set and B is a collection of subsets of X. B is a basis for some topology on X iff

(1) $\forall p \in X$, $\exists b$ s.t. $p \in b \in B$ and

(2) $\forall U, V \in B$ s.t. $p \in U \cap V$, $\exists W \in B$ s.t. $p \in W \subset (U \cap V)$

Proof:

$\Rightarrow$: (B is a basis of some topology) $\rightarrow$ ((1) $\wedge$ (2))

- $\exists T$ s.t. $\forall U \in T$, $U = \cup_i b_i$ s.t. $b_i \in B$ $\forall i$.
- Since every $p \in U \in T$ can be included by the union $\cup_i b_i$, we know that $\exists i$ s.t. $p \in b_i$, which proves (1)
- For some $b_1, b_2 \in B \subset T$, s.t. $(p \in b_1) \wedge (p \in b_2)$.

  Take $b_1 \cap b_2$. Obviously $p \in b_1 \cap b_2$ and we know $b_1 \cap b_2 \in T$ thus $b_1 \cap b_2 = \cup_i b_i$ where $b_i \in b$. Thus $\exists i$ s.t. $p \in b_i$. And obviously $b_i \subset b_1 \cap b_2$.

- Hence b_i is the W we are looking for.

$\Leftarrow$: $(1) \wedge (2) \rightarrow$ (B is a basis of some topology)

- Define $T = \{$All possible union of $B\}$. WTS T is a topology

$\rightarrow$ For ϕ: the union of non of $b \in B$

$\rightarrow$ X: Since (1), we know $\forall x \in X, x \in b \in B$. Now take all union of every B. it covers every point in X. Thus $X \in T$.

$\rightarrow$ $(U \in T) \wedge (V \in T) \rightarrow U \cap V \in T$.

By our definition $U = \bigcup_{i \in \alpha} B_i$ s.t. $\{B_i\}_{i \in \alpha} \subset B$ $\quad V = \bigcup_{i \in \beta} B_i$ s.t. $\{B_i\}_{i \in \beta} \subset B$.

$\boxed{\text{WTS } U \cap V = (\bigcup_\alpha B_i) \cap (\bigcup_\beta B_i) \in T}$

- We may distribute the intersection: $\boxed{(\bigcup_\alpha B_i) \cap (\bigcup_\beta B_i) = \bigcup_{i \in \alpha, j \in \beta} (B_i \cap B_j)}$

Now by (2), whenever $p \in B_i \cap B_j$, $\exists W \in B$ s.t. $p \in W \subset B_i \cap B_j$. Thus it obviously means that the $B_i \cap B_j$ can be covered by B. Or:

- $\boxed{U_1 \cap V_2 = \bigcup_{i \in \alpha, j \in \beta} (B_i \cup B_j) = \bigcup_{k \in \gamma} U_k}$ where $k \in \gamma \subset B$, which is just $U_1 \cap U_2 \in T$.

$\rightarrow$ $\bigcup_{i \in \alpha} U_i \in T$ where $U_i \in T$

- It's obvious b.c. every U_i is a union of elements in B. Thus any union of U_i in B will also be a union of sets in B. $\boxtimes$.

Comment: Theorem 3.3 Allow us to specify X, B, then easily contruct a topology.

Definition [Lower Limit Topology]: The lower Limit topology is the all possible union of the set $[a,b) = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. It's denoted as $\mathbb{R}_{LL}$ or $\mathbb{R}_{bad}$.

Theorem 3.5: Every open set in $\mathbb{R}_{std}$ is an open set in $\mathbb{R}_{LL}$, but not backward.

Comment: $\mathbb{R}_{LL}$ has more open sets than standard topology.

Proof:

- For an open set (a,b) in $\mathbb{R}_{std}$: $(a,b) = \bigcup_{a_i \in (a,b)} [a_i, b)$, thus (a,b) can be constructed by $[a_i, b)$, the set in space $(T_{LL}, \mathbb{R})$.
- For an open set $[a,b)$ in $\mathbb{R}_{LL}$, and we attempt to describe it as $\bigcup (x_i, y_i)$ where (x,y) is the open set in $\mathbb{R}_{std}$. The issue happens at left end point.

$\rightarrow$ Suppose $[a,b) = \bigcup (x_i, y_i)$, $\exists j$ s.t. $a \in (x_j, y_j)$, which means $x_j < a < y_j$. By density of real number, $\exists d \in \mathbb{R}$ s.t. $x_j < d < a$. But then $d \notin [a,b)$, Hence $\bigcup (x_i, y_i) \supset (x_j, y_j) \not\supset (a,b)$, they are not equal. Contradiction. $\boxtimes$.

Definition [Coarse / Fine]: Suppose $\mathcal{T}, \mathcal{T}'$ are two topologies on the same underlying set X. If $\mathcal{T} \subset \mathcal{T}'$ then $\mathcal{T}'$ is finer than $\mathcal{T}$. Else we say its coarser.

- We say strictly coarser & finer if $\mathcal{T}' \neq \mathcal{T}$.

Example: two topologies on $\mathbb{R}$ that neither are finer than the other.

- The countable complement topology & standard topology.

→ $(a,b) \in \mathcal{T}_{std}$, but not open in $\mathcal{T}_{\text{finite complement}}$

→ $\mathbb{R} \setminus \{q \in \mathbb{Q} : q \in (0,1)\} \in \mathcal{T}_{\text{finite complement}}$, but we can't write it in terms of union of open sets in standard topology.

Definition [Double Headed Snake Topology / $\mathbb{R}_{+00}$] : the set consisting of $\mathbb{R}_+$ together with two points $\{0', 0''\}$. Put a topology on it generates the basis consisting of all intervals in $\mathbb{R}_+$ (a,b) or $(0,b) \cup \{0'\}$ or $(0,b) \cup \{0''\}$ for $a,b \in \mathbb{R}_+$.

- Why this is actually a basis

1. If $x > 0$, $x \in (\frac{x}{2}, \frac{3}{2}x) \in \mathcal{T}_+$
2. If $x = 0'$, $x \in (0,1) \cup \{0'\}$
3. If $x = 0''$, $x \in (0,1) \cup \{0''\}$

- Thus for every case, there is at least one basis containing x.

- The intersection property:

→ If B_1 and B_2 are two basis elements and $x \in B_1 \cap B_2$. By theorem 3.3, we want to find a Basis $B_3 \subset B_1 \cap B_2$ s.t. $x \in B_3$. Again we split the cases.

1. If $x > 0$, then B_1 and B_2 contain open interval in $\mathbb{R}_+$ containing x. The intersection is basically containg x.
2. If $x = 0'$ or $0''$ then B_1 and B_2 both contain $(0,b) \cup \{0'\}$ or $\{0''\}$ now take the smaller b from $(0,b)$ set and the intersection contains $0'$ and $0''$.

Property of $\mathbb{R}_{+00}$: every point is a closed set but it's impossible to find disjoint open sets U & V s.t. $0' \in U$ and $0'' \in V$.

- To show every singleton is closed, we show its complement is open.

1. For some $x > 0$, $\mathbb{R}_{+00} - \{x\} = (0,x) \cup (x,\infty) \cup \{0', 0''\}$
2. For some $x < 0$, $\mathbb{R}_{+00} - \{0'\} = (0,\infty) \cup \{0''\} = [(0,b) \cup \{0''\}] \cup (b,\infty)$.

Same thing for $\{0''\}$.

Definition [Sticky Bubble Topology]: Let H_{bub} be the upper half-plane $\{(x,y) : x, y \in \mathbb{R}, y \geq 0\}$ with a topology whose basis consist of

(1) All balls $B((x,y), r)$ where $0 < r \leq y$ and

(2) All sets $B((x,y),r) \cup \{(x,0)\}$, where $r = y > 0$.

- Some phenomenons in the H_{bub} topology.

1. The closure of the set of rational number on the x-axis.

- For every point other than $\{(q,0) : q \in \mathbb{Q}\}$, or the set $\{\text{Upper half plane}\} \setminus \{(q,0), q \in \mathbb{Q}\}$

  - The point not on x-axis, there's an open ball around it to separate from x-axis.
  - The point $\{(i,0) : i \in \mathbb{I}\}$ was included by $B((i,y),r) \cup \{(i,0)\}$

→ Hence every point in $\{\text{Upper half plane}\} \setminus \{(q,0) : q \in \mathbb{Q}\}$ is an interior point of the set. Thus the set is open, hence $\{(q,0) : q \in \mathbb{Q}\}$ is closed. It's closure is itself. ☒

2. Close set on x-axis in H_{bub} topology.

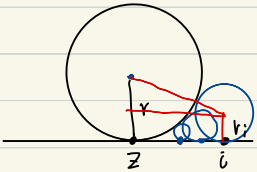
- For any subset of x-axis, we may repeat same set as (1) to prove its complement is open by proving every point in its complement is an interior point. Thus every subset of the x-axis is closed.

3. Let A be a countable subset on x-axis and z is a point not in A.

- $\exists\, U, V \in H_{bub}$ s.t. $U \cap V = \phi$ while $(A \subset U) \wedge (z \in V)$.

Proof:

- Choose a sticky bubble around z, now pick any $r > 0$, $V = B_r(z,r) \cup \{(z,0)\}$
- Now for every point of A, say $A_i = \{a_1, a_2, \dots\}$



- $\boxed{(r - r_i)^2 = (r + r_i)^2 - (i - z)^2}$

Pick r_i that fulfills this limit and thus we know that the

- $[B_{r_i}(a_i, r_i) \cup \{(a_i, 0)\}] \cap [B_r(z,r) \cup \{(z,0)\}] = \phi$

Now we may do induction on i to put the intersection of all i unioned together.

→ Note that the countable union of balls allow us to union all the balls one by one, thus is necessary, we have no idea what happen if union is uncountable.

4. If we repeat the steps on the last step, we may easily see that is A and B are two countable disjoint sets, there are open sets U & V s.t. $A \subset U$ and $B \subset V$

→ Because every point in B can have a open set that aviods A, then union all of them.

CH3.2

Definition [Subbasis]: Let (X, T) be a topological space and S be a collection of subset of X. S is a subbasis for T iff the collection B of all finite intersection of sets in S is a basis for T.

- An element of S is a subbasis element or a subbasic open set

Corollary: A basis for topology is also a subbasis of topology.

→ Every set in the is an intersection of the set with itself, thus it's the set of all finite intersection.

Comment: Subbasis is a way to specify topologies in a more condensed form

Basic Logic is that subbasis is a "finite intersection topology" of a basis.

Example: $\mathbb{R}_{std}$ has subbasis S consisting of rays $\{x \mid x < a \text{ for some } a \in \mathbb{R}\}$ and $\{x \mid a < x \text{ for some } a \in \mathbb{R}\}$

Proof: For any open set in $\mathbb{R}_{std}$, (a, b) can be the intersection of the set $\{x \mid x < b\} \cap \{x \mid a < x\}$

- Thus the subbasis generates the sets in standard topology.

Theorem 3.15: Let (X, T) be a topological space, and let S be a collection of subsets of X. Then S is a subbasis for T iff

(1) $S \subset T$

(2) $\forall\, U \in T$ and $p \in U$, $\exists$ a finite collection $\{V_i\}_{i=1}^{n}$ s.t. $V_i \in S$ and $p \in \bigcap_{i=1}^{n} V_i \subset U$

Proof:

$\Rightarrow$: S is a subbasis $\rightarrow$ (1) $\wedge$ (2)

(1) Since for every finite collection is in a basis for a topology. Pick the one set intersection, by definition it's in the basis of the topology, thus the topology.

(2) Let $U \in T$ and $p \in U$.

- T is generated by a union of finite intersection of set in S.
- Thus for p, p has to be in one of the finite intersections. Thus we may write it as $p \in \bigcap_{i=1}^{n} V_i \subset U$.

$\Leftarrow$: (1) $\wedge$ (2) $\rightarrow$ S is a subbasis

- To prove S is a subbasis of T, we are essentially proving that the set of S's finite intersection can form a basis, which generates a topology.
- Since for every open set $U \in T$, every p is in a finite intersection of set in S, and the intersection of set in S is a subset of U,
  thus U can be written as the union of such finite intersections.
- As the set of all union of all finite intersection can generate any set of topology it fulfills the definition of basis.

$\rightarrow$ Thus we know that S is a subbasis.

Theorem 3.16 Suppose S is a subset of X. Then S is a subbasis for some topology on X iff every point of X is in some element of S.

In other word, S is a subbasis on X iff S covers X.

Proof:

$\Rightarrow$: S is a subbasis $\longrightarrow$ $\forall p \in X$, $\exists P \in S$ s.t. $p \in P$.

- If S is a subbasis, S is a collection of subsets whose finite intersections form a basis for topology T on X.
- If $\exists p \in X$ s.t. $\forall P \in S$ s.t. $p \notin P$. (The negation of the argument), then any finite intersection of P wouldn't contain p.

$\rightarrow$ If S is a subbasis, the $B = \{\bigcup_{i=1}^{n} V_n\}$ s.t. $V_n \in S$ and $n \in \mathbb{Z}$ forms a basis.

$\rightarrow$ But B doesn't form a basis since by theorem 3.1 (6), for some $V \in T$ s.t. $p \in U$, U can't be written as union of B. Thus B is not a basis.

$\Leftarrow$: $\forall p \in X$, $\exists P \in S$ s.t. $p \in P$ $\rightarrow$ S is a subbasis.

- Let $B = \{\bigcup_{i=1}^{n} V_n\}$ s.t. $V_n \in S$, $T = \{\bigcup_i B_i\}$ for any collection of B_i.
- Now verify T is a topology
  1. $\phi \in T$ 2. $X \in T$ (since $\forall x \in X$, $\exists p \in S$ s.t. $x \in p$).
  3. $\{U_\alpha\} \in T$, U_α is a union of set in B, so is $\bigcup_\alpha U_\alpha$ $\rightarrow$ $\bigcup_\alpha U_\alpha \in T$.
  4. Since B is constructed from finite intersection of sets in S, thus intersection of union of sets in B can be expressed as intersection of union of S. $\boxtimes$

CH 3.3

Definition [Order Topology]: Let X be a set totally ordered by "$\le$". Let $\mathcal{B}$ be the collection of all subsets of X of any of the following for.

$\{x \in X \mid x < a\}$ or $\{x \in X \mid a < x\}$ or $\{x \in X \mid a < x < b\}$

for $a, b \in X$, $\mathcal{B}$ is a basis for T, called order topology on X.

Corollary: Let X be a set totally ordered by $\le$. Let S be collection of sets in the form $\{x \in X \mid x < a\}$ or $\{x \in X \mid a < x\}$ for $a \in X$, S forms a subbasis for the ordered topology on X. The proof is obvious.

Remark: Order topology on $\mathbb{R}$ with "$\le$" relationship is the standard topology on $\mathbb{R}$. T_{std}.

Proof: We know the basis $\{(a,b) : a, b \in \mathbb{R}\}$ constructs the standard topology And for each (a,b), could be written as finite intersection of the subbasis in the corollary above.

- In addition, any finite intersection of the subbasis described above is either in the form

  (a,b) or $(-\infty, b)$ or (a, ∞) which are open in the sense of T_{std}.

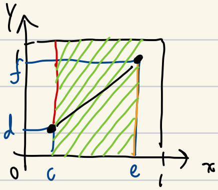
Definition [Cartesian Product]: Given sets A and B, their product (or Cartesian product) $A \times B$ is the set of all ordered pairs (a,b) s.t. $a \in A$, $b \in B$.

Definition [Dictionary Order / Lexicographic Order]: If A, B are totally ordered by "$\le_A$" and "$\le_B$", then Dictionary (Lexicographic) order $\le$ on $A \times B$ is specified by defining $(a_1, b_1) < (a_2, b_2)$ if

$$[a_1 <_A a_2] \text{ or } [a_1 = a_2] \wedge [b_1 <_B b_2]$$

Example: The square $[0,1] \times [0,1]$ with lexicographic order and its associated ordered topology is called the lexicographically order square

- A Picture of open sets in lexicographically ordered square.



- Open sets look like $\{(a,b) \in X \times X \mid (c,d) < (a,b) < (e,f)\}$

$\rightarrow$ $c < a < e$ or $(a = c \wedge d < b)$ or $(a = e \wedge b < f)$

$\rightarrow$ The red, green blue part is an open set in lexigraphically ordered sphere.

Example: Closure in lexigraphically ordered square.

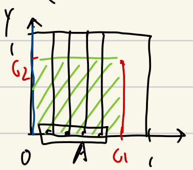
• Remark: What's a limit point in lexigraphical order: for every open set include the point p, U_p, $U_p \cap A \setminus \{p\} \neq \phi$.

E.g. $A = \{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$ → find the closure

→ Claim: $(0,0)$ is a limit point of the set. To prove it, pick any neighborhood or interval around $(0,0)$ or $(c_1,d_1) < (0,0) < (c_2,d_2)$ and we shall prove the interval intersects A.

• The neighborhood has the set of points such that

$$(x>0 \text{ or } (x=0) \wedge (y>0)) \wedge (x < c_2 \text{ or } (x=c_1) \wedge (y<c_2))$$



- The set contains all point of x-axis that's smaller than c_1, thus the sequence will eventually enter it. For any c_1
- If point is not on x-axis, and x-value $= \frac{1}{n}$ for some n, then its neighborhood will intersect $(\frac{1}{n}, 0)$

→ The closure is $\{(\frac{1}{n}, y) : y \in [0,1], n \in \mathbb{N}\} \cup \{0,0\}$. ☒

CH3.4

Definition [Subspace]: If (X, T) is a topological space, $Y \subset X$. The collection

$$T_y = \{U \mid U = V \cap Y \text{ for some } V \in T\}$$

is the topology on y called subspace topology, or relative topology on Y inherited from X. (Y, T) is a topological subspace of X. If $U \in T_y$, U is open in Y.

Theorem 3.25 Let (X, T) be a topological space and $Y \subset X$. The the collection of sets T_y is in fact a topology on Y.

Proof:

1. $\phi \in T_y$ obviously
2. We claim $\bigcup U = Y$ where $U = V \cap Y$ for some $V \in T$.
   - Obviously, For every U, $U \subset Y$, thus union of U has to be a subset of Y. Thus we know that $\bigcup U \subset Y$
   - For every $y \in Y$, $\exists U \in \bigcup_i U_i$ s.t. $y \in U \in \bigcup_i U_i$, as needed.
3. If $A \in T_y$ and $B \in T_y$, then $A \cup B \in T_y$.

- By definition, $A = \bigcup_{i\in\alpha} U_i$, $B = \bigcup_{i\in\beta} U_i$ where $\{U_i : i\in\alpha\}$ and $\{U_i : i\in\alpha\}$ are in the set $\{U : (U = V\cap Y) \wedge (V \in T_x)\}$
- Thus $A\cap B = (\bigcup_{i\in\alpha} U_i) \cap (\bigcup_{i\in\beta} U_i)$ → We may distribute the intersection into the union $= \bigcup_{i\in\alpha, j\in\beta} (U_i \cap U_j)$

→ We know that $\forall i, \exists V_i$ s.t. $U_i = V_i \cap Y$, same thing with j.

Hence $(U_i \cap U_j) = (V_i \cap Y)\cap(V_j \cap Y) = (V_i \cap V_j)\cap Y$

→ Since T_x is a topology, $V_i\cap V_j \in T_x$ thus $(U_i \cap U_j) \in T_y$.

4. For any $\{U_\alpha\}$ s.t. $U_\alpha \in T_y$, then $\bigcup_\alpha U_\alpha \in T_y$.

- Similar to step 3, $\bigcup_\alpha U_\alpha = \bigcup_\alpha (V_\alpha \cap Y) = (\bigcup_\alpha V_\alpha)\cap Y$. Hence $\bigcup_\alpha V_\alpha \in T_y$.

It's obviously in x's topological space, ⊠.

Example: The set $Y = [0,1)$ as a subset of $\mathbb{R}_{std}$, examine the open & closed sets.

- For example the set $[\frac{1}{2}, 1)$

→ Is it open? If it's open, then $[\frac{1}{2},1) \in T_Y$, thus $[\frac{1}{2},1) = V\cap Y$ where $V \in T_{std}$. However, any set V that fulfills the relation, such as $[\frac{1}{2}, 2)$, does not open under $\mathbb{R}$std.

→ Is it closed? Examine if its complement in Y is open (as by theorem 3.25, T_Y also generates a topology)

- The complement of Y in $[0,1)$ is given by $[0,\frac{1}{2})$, it's an open set in Y since $\exists V = (-1,\frac{1}{2}) \in T_{std}$. Hence $[\frac{1}{2},1)$ is indeed closed in Y.

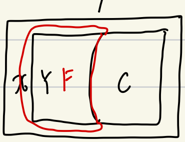
Theorem 3.28: Let (Y, T_y) be a subspace of (X, T). A subset $C\subset Y$ is closed in (Y, T_y) iff $\exists D\subset X$ s.t. D is closed in (X, T) s.t. $C = D\cap Y$.

Proof:

⇒: C is closed in (Y, T_y) → $\exists D$ closed in (X, T) s.t. $C = D\cap Y$.

- If C is closed, then complement of C in Y, $Y - C$ is open in Y
- $Y - C$ is open, then $\exists F\in T$ s.t. $F\cap Y = Y - C$. Now examine the set $X - F$

It obviously a closed set and $\boxed{(X-F)\cap Y = C}$



- Thus we found $D = X - F$ is the set we're looking for.

$\Leftarrow$: $\exists D$ closed in (X,T) s.t. $C = D \cap Y$ $\rightarrow$ C is closed.

- Since D is closed. $X - D$ is open, thus $G = (X-D) \cap Y$ is open in Y.
- Hence $Y - (X-D) \cap Y = Y - [(X \cap Y) - (D \cap Y)] = Y - [Y - C] = C$ is closed. $\boxtimes$.

Corollary 3.29: Let (Y, T_Y) be a subspace of (X, T). A subset $C \subset Y$ is closed in (Y, T_Y) iff $Cl_X(C) \cap Y = C$.

$\rightarrow$ The proof is basically repeating the proof of Theorem 2.13.

Theorem 3.30 Let (X, T) be a topological space, and let (Y, T_Y) be a subspace, if B is a basis for T_X, then $B_Y = \{b \cap Y \mid b \in B\}$ is a basis for T_Y.

Proof:

- Since B is a basis for T, then for any set $V \subset X$ s.t. $V \in T_X$. V can be written as union of sets in B.

$\rightarrow$ Now for any set of $U \in T_Y$, $\exists V \in T_X$ s.t. $V \cap Y = U$. Since $V = \{$union of sets in $B\}$ and we know that $U = \{$union of the set intersection with $B\}$.

- Hence we know that the topology T_Y could be generated by the set of $b \cap Y$. Thus $b \cap Y$ is a basis for Y. $\boxtimes$.

CH 3.5

Definition [Projection Function]: Let X and Y be topologies, the projection function $\pi_X : X \times Y \rightarrow X$ $\pi_Y : X \times Y \rightarrow Y$ are defined by $\pi_X(x,y) = x$ and $\pi_Y(x,y) = y$.

Definition [Product topology]: The topology whoses basis is all sets of the form $U \times V$ where U is an open set in X and V is an open set in Y.

Remark: The $A \times B$ was defined as Cartesian product of the two sets. The set of all pairs of (a,b) s.t. $a \in A$ and $b \in B$.

- Below is a verification that $U \times V$ is a basis of topology

Proof:

- We are going to verify it satisfies theorem 3.3, which states:

(1) Each point of $X \times Y$ is in some element of $U \times V$.

$\rightarrow$ For some (x,y) where $x \in X$ and $y \in Y$, since the set U & V are open set in X & Y, the set and Y themselves are also open by definition of topology. Thus $(x,y) \in X \times Y \in \{U \times V : U \in T_X, V \in T_Y\}$

(2) If M & N are sets in $\{U \times V : U \in T_X, V \in T_Y\}$ and $p \in M \cap N$, $\exists W \in B$ s.t. $p \in W \subset (M \cap N)$. call the point (a,b) in product topology

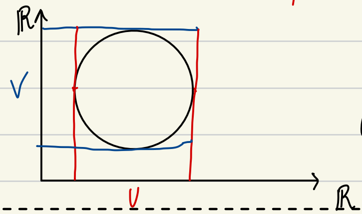
→ By definition $M = U_M \times V_M$ s.t. $U_M \in T_X$, $V_M \in T_Y$ and $a \in U_M$, $b \in V_M$.

$N = U_N \times V_N$ $\quad U_N \in T_X$, $V_N \in T_Y$ $\quad a \in U_N$, $b \in V_N$.

→ Since U_M & V_M, U_N & V_N are open sets in a topology, their intersection are all open sets in the topology. Thus we may pick W to be the set $M \cap N$ themselves. Thus $W \subset M \cap N$. $\boxtimes$.

Example: Product Space on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Remark: $U \times V$ are open in $\mathbb{R} \times \mathbb{R}$, but not every open set in $\mathbb{R}^2$ could be written as $U \times V$.



- An open ball is not a product of open sets.
- A closed ball is not a product

(A shown in the figure, there are always parts missed out no matter how you pick U and V).

Corollary: Product of closed sets are closed.

- Let $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ be closed sets in X_1 and X_2. WTS $A_1 \times A_2$ is closed in the product space $X_1 \times X_2$.

→ The complement of $A_1 \times A_2$: $(X_1 \times X_2) - (A_1 \times A_2)$. We may distribute the Cartesian Product as the following:

$$(X_1 \times X_2) - (A_1 \times A_2) = ((X_1 - A_1) \times X_2) \cup (X_1 \times (X_2 - A_2))$$

- Because $X_1 - A_1 \in T_{X_1}$ and $X_2 \in T_{X_2} \to (X_1 - A_1) \times X_2$ is in the basis.

$X_1 \in T_{X_1}$ and $X_2 - A_2 \in T_{X_2} \to (X_2 - A_2) \times X_1$ is in the basis.

Thus both $(X_1 - A_1) \times X_2$ and $X_1 \times (X_2 - A_2)$ are open in the product topology

- Thus $(X_1 \times X_2) - (A_1 \times A_2)$ is open $\to A_1 \times A_2$ is closed. $\boxtimes$

Remark: this corollary is also true for higher order product space (product of multiple topological space).

Theorem 3.35: The product topology on $X \times Y$ is generated by subbasis of inverse images of open sets under the projection function. To be specific, the subbasis is given by $\{\pi_X^{-1}(U) : U \in T_X\} \cup \{\pi_Y^{-1}(U) \mid V \in T_Y\}$

Comment: we want show that product topology on $X \times Y$ can be generated by looking only at the rectangle pieces of the form $U \times Y$ and $X \times V$.

- It's a significant theorem as it shows the projections capture the structure of the product space. It allow us to see how X, Y coordinates behave individually.
- The proof is basically showing the subbasis generate the basis $\{U \times V : U \in T_X, V \in T_Y\}$.

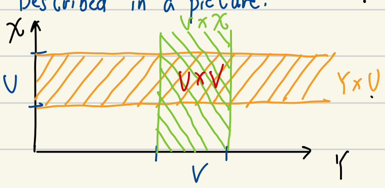
Proof:

$\Rightarrow$: Every basis element $U \times V$ in the product topology can be written as finite intersection of sets in the set $\pi_X^{-1}(U) \cup \pi_Y^{-1}(U)$.

- For every basic element for $U \times V$, we can write it out as $U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$

As $\begin{cases} \pi_X^{-1}(U) = U \times Y \\ \pi_Y^{-1}(V) = X \times V \end{cases}$ which are both in the subbasis described in the theorem.

Described in a picture:



- $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) = U \times V$ is therefore the basis generated by the given subbasis

Since obviously the product topology is generated by all $U \times V$, and each such set can be constructed by the basis generate by the subbasis.

$\Leftarrow$: Every set in the topology generated by the subbasis is open in product topology.

- For a set generated by the subbasis $W = \bigcup \left(\bigcap_{i=1}^{n} \pi_X^{-1}(U_i) \cap \bigcap_{j=1}^{m} \pi_Y^{-1}(V_j) \right)$

$\rightarrow$ $W = \bigcup \left(\bigcap_{i=1}^{n} (U_i \times Y) \cap \bigcap_{j=1}^{m} (X \times V_j) \right) = \bigcup \left(\left(\left[\bigcap_{i=1}^{n} U_i \right] \times Y \right) \cap \left(\left[\bigcap_{j=1}^{m} V_j \right] \times X \right) \right)$

- The finite intersection of U & U in X & Y's topology is also open, hence

$W = \bigcup \left(\boxed{[\cap U_i] \times [\cap V_i]} \right)$ which is one of the basic element of the product topology

$\rightarrow$ Hence the union of sets in the basis is the set in product topology as needed. ☒

Example: The standard topology on $\mathbb{R}$ and product topology on $\mathbb{R} \times \mathbb{R}$ generates the same topology on $\mathbb{R}^2$.

$\rightarrow$ The proof is easy as the basis defined by both topology satisfies theorem 3.3 for every open sets in the other topology.

i.e. Interior point in an open set $\mathbb{R}^2$ can be covered by basic element in $\mathbb{R} \times \mathbb{R}$ & $\mathbb{R}_{std}$. ☒

Definition [Infinite Cartesian Product]: Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of topological spaces, the product $\prod_{\alpha\in\lambda} X_\alpha$ or Cartesian product, is the set of functions that
$\{f:\lambda \to \bigcup_{\alpha\in\lambda} X_\alpha \mid \forall\alpha\in\lambda,\ f(\alpha)\in X_\alpha\}$ and $f(\alpha)$ is the αth coordinate of f. The spaces X_α are the factors of the infinite product.

A point in the product may be thought of as a function that associates to each α an element $f(\alpha)$ of the factor X_α.

Definition [Product topology]: For each β in λ, define $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ by $\pi_\beta(f) = f(\beta)$.
We define product topology on $\prod_{\alpha\in\lambda} X_\alpha$ to be the one generated by the subbasis of sets of subbasis of sets of the form $\pi_\beta^{-1}(U_\beta)$, $U_\beta \in T_{X_\beta}$.

Comment: There are 2 ways to define the produce and under the case of product of 2 topological spaces, in fact finitely many, the subbasis & Carterian method generates the same product topology. But the case of infinite case makes the subbasis generation more natural.

Theorem 3.37 A basis for the product topology $\prod_{\alpha\in\lambda} X_\alpha$ is the collection all sets of the form $\prod_{\alpha\in\lambda} U_\alpha$ where $(U_\alpha \in T_{X_\alpha}\ \forall\alpha\in\lambda) \wedge (U_\alpha = X_\alpha$ for all but finite $\alpha)$.

- The theorem is basically a restatement of the definition of the box topology.
- As we are using here the product topology, we would start from the given subbasis.

Proof:

→ If B is a basis for the product topology that subbasis S generates. Then $\mathcal{B}$ $= \{$all finite intersection of sets in $S\}$

- The finite intersection of sets in S still makes a set in S, as $\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta)$ $= \pi^{-1}(U_\beta \cap V_\beta)$ $\forall U_\beta, V_\beta$ in X_β. By induction, any integer i should work.

→ The $\pi^{-1}_\beta(U_\beta) = (X_1, X_2, \dots, U_\beta, \dots X_i)$ in specific. That's why $\pi^{-1}(U_\beta)\cap\pi^{-1}(V_\beta) =$ $(X_1,\dots,U_\beta,\dots,X_i)\cap(X_1,X_2,\dots,V_\beta,\dots,X_i) = (X_1,\dots,U_\beta\cap V_\beta,\dots,X_i) = \pi^{-1}(U_\beta\cap V_\beta)$.

- This set ensures the intersection with the same β coordinate also makes an element, so we only need to worry about intersection for different coordinate.
- Elements of B are of the form $B = \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})$ where $i\in\lambda$ and $U_{\beta_i}\in T_{X_{\beta_i}}$

→ Again visualize this intersection:
$\bigcap(X_1,\dots,U_{\beta_i},\dots) = (X_1,\dots,U_{\beta_1},\dots,U_{\beta_n},\dots)$ which is all but finite coordinate has $U_{\beta_i} \neq X_{\beta_i}$. Which what the proof asked for. $\boxtimes$

Definition [Generate Product Notation]: If each X_α factor is the same topological space. Then $\prod_{\alpha \in \lambda} X_\alpha$ is sometimes denoted X^λ. Where λ is the index set of the product or an ordinal number.

- A countable product of copies of $\mathbb{R}$ is denoted by $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{R}^\omega$.

Example: The space $\{0,1\}^A = \prod_{\alpha \in A} \{0,1\}$ is a product of discrete two-point spaces one for element a in A. Element of this topological space is a function $f: A \to \{0,1\}$

- Note for every $\alpha \in A$, the projection function π_α takes the function π_α takes the function f to $f(a)$. (Restate Definition).
- $\{0,1\}$ has discrete topology on the set.
- Consider the subset in this product topology.

$U(a,0) = \pi_a^{-1}(0) = \{ f \in \{0,1\}^A \mid f(a) = 0 \}$

$U(a,1) = \pi_a^{-1}(1) = \{ f \in \{0,1\}^A \mid f(a) = 1 \}$

- A subbasis for this topology $S = \{ U(a,\delta) \mid a \in A, \delta \in \{0,1\} \}$ is a subbasis for the topology on $\{0,1\}^A$.

Visualization: $U(\beta,1) = (\{0,1\}, \ldots, \{1\}, \ldots)$ For the β^{th} coordinate of the product.

- The space can be expressed as 2^A - it's a bijection between power of $\{0,1\}^A$.

Remark: What does it mean by union in a product topology?

- By Theorem 3.37, we know the basis constructed by the subbasis s.t. $\pi_\alpha^{-1}(U_\alpha)$, which is in specific $\prod_\alpha U_\alpha$ $\quad (U_\alpha \subset T_{X_\alpha}) \wedge (U_\alpha = X_\alpha$ for all but finitely many $\alpha)$
- For example, two set in a basis of the set $\mathbb{R}^\omega$ could be given by

$A = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \ldots, \underbrace{(0,1)}_{mth}, \ldots)$ and we make $A \cup B$

$B = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \ldots, \underbrace{(0,1)}_{nth}, \ldots)$

- Note the union is not taking union at every coordinate, which just makes every coordinate $\mathbb{R}$. For example the element $a = (0, 0, \ldots, \underbrace{3}_{mth}, \ldots, \underbrace{3}_{nth}, \ldots)$ is not in the set $A \cup B$

→ Then what is the complement of $A \cup B$? By demorgan's law $A^c \cap B^c$. And obviously $A^c \cap B^c = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \ldots, \underbrace{\mathbb{R} \setminus (0,1)}_{mth}, \ldots, \underbrace{\mathbb{R} \setminus (0,1)}_{nth}, \ldots)$

Because for A^c & B^c, coordinate doesn't matter pick which number.

Example: Let T be the topology on $2^{\mathcal{X}}$ with basis generated by subbasis S. There are several interesting properties.

(1) Every basic open set in $2^{\mathcal{X}}$ are both open and closed.

- By definition, any open set in $2^{\mathcal{X}}$ could be written as the union of sets in the basis.
- The sets in the basis, given by Theorem 3.37, basis of the set $\prod_{\alpha\in\mathcal{X}}\{0,1\}_\alpha$ is $\{\prod_{\alpha\in\mathcal{X}} U_\alpha : (U_\alpha \in T_{\{0,1\}}) \wedge (U_\alpha = \{0,1\}$ for all α but finitely many $\alpha)\}$

Therefore, the basic open sets under product topology can be written as

$$\boxed{\left(\bigcap_{n\in\mathbb{Z}} (\text{Subbasic Element})\right)^c = \left((\text{Basic Element})\right)^c}$$

- These $(\text{basic element})^c = \left(\bigcap_{n\in\mathbb{Z}} (\text{Subbasic Element})\right)^c = \bigcup_{n\in\mathbb{Z}} (\text{Subbasic Element})^c$

→ These subbasic element could be written just flipping $\{0\} \leftrightarrow \{1\}$ and $\{\phi\} \Leftrightarrow \{0,1\}$ at the one coordinates, thus are also subbasic elements

- Since these subbasic elements are open sets, their union are also open sets in the space, thus their complement is closed. ☒.

(2) If a collection of subbasic open sets of $2^{\mathcal{X}}$: (every point in $2^{\mathcal{X}}$ lie in some of subbasic open sets) → (there are two subbasic open sets in the collection s.t. covers 2^X).

- Suppose for the sake of contrapositive, there is no such two sets.

→ We know that for every subbasic open set, there is 1 and only 1 one coordinate s.t. $U_\beta \in T_{X_\beta}$ and $U_\beta \neq X_\beta$.

- Then we may alway pick a point when for every coordinate, pick the point in the complement of the open set if the coordinate $U_\beta \neq X_\beta$. Then the point is not in any of the subbasic open sets. Which completes the proof.

(3). (If a collection of basic open set cover $2^{\mathcal{X}}$) → (A finite collection of open sets in the collection that covers 2^X).

- We can basically do the same proof as above.

The (2) and (3) shows us 1. For a basic open set, we need a finite collection of open set complement to fill the coordinates $U_\beta \neq X_\beta$ (Thm 3.37). 2. For subbasic open set, we are look for a set & its complement to cover the entire 2^X.

- At this point, I want to do a general summary on general (infinite product topology)
- Generation from Subbasis:

$\{\text{Subbasis}\} = \mathcal{S} = \bigcup_{\beta \in \lambda} \mathcal{S}_\beta$, where $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \in T_{X_\beta}\}$

$\{\text{Basis}\} = \{B\} = \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})$ where $U_{\beta_i} \in X_{\beta_i}$ $\forall i = 1, \dots, n$.

- Thus $\vec{x} = (x_\alpha) \in B$ iff it's β_i th coordinate is in U_{β_i}. No restriction on rest of the coordinates. Hence B can also be written as

$$B = \prod_{\alpha \in J} U_\alpha \qquad U_\alpha = X_\alpha \text{ if } \alpha \neq \{\beta_i\}_{i=1}^{n}$$

Definition [Box Topology]: Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces, take a basis for a topology on the product space $\prod_{\alpha \in \lambda} X_\alpha$. The collection of all sets of the form $\prod_{\alpha \in \lambda} U_\alpha$, $U_\alpha \in T_{X_\alpha}$. The topology is called box topology.

Below are some theorems that moves from finite product $\to$ infinite product.

Theorem 3.43: Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology or both giver the product topology.

Proof:

$\to$ If $\prod_\alpha A_\alpha$ & $\prod_\alpha X_\alpha$ are both box topology:

- For every α, $\exists B_\alpha \subset X_\alpha$ s.t. $A_\alpha = B_\alpha \cap X_\alpha$. Then for the product $\prod_\alpha A_\alpha$, $\boxed{\prod_\alpha X_\alpha \cap \prod_\alpha B_\alpha = \prod_\alpha A_\alpha}$ (As infinite product, each intersection is just intersecting the nth coordinate. As needed.

$\to$ If the product are both product topology:

- For an open set such that is open in product topology, it's a union of its basis. Or $\prod_\alpha A_\alpha = \bigcup_{\beta \in \lambda} (\prod_\alpha V_\alpha)_\beta$ where $\forall \beta \in \lambda$, $V_\alpha = X_\alpha$ except for a finite amount of α.
- We know for every A_α, $\exists B_\alpha \subset X_\alpha$ s.t. $B_\alpha \cap X_\alpha = A_\alpha$. If any of the β, $\prod_\alpha V_\alpha$ is not in the subspace topology, in particular $\exists \alpha_0$ s.t. $\forall B_{\alpha_0} \subset X_{\alpha_0}$ $B_{\alpha_0} \cap V_{\alpha_0} \neq V_{\alpha_0}$

$\to$ Then the union, also contains the set with the given β.

Hence A_{α_0} can't be written as $B_\alpha \cap V_\alpha$ $\to$ not a subspace! $\square$.

Property of product topology: $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.

Proof:

→ Let $Y = \prod_{j=1}^{n-1} X_n$ and $Z = \prod_{j=1}^{n} X_n$, now define $f: Y \times X_n \to Z$ as $f((a_1, \dots, a_{n-1}), a_n) = (a_1, \dots, a_n)$ f is obviously a bijection.

- Thus f^{-1} exist an $f^{-1}: Z \to Y \times X_n$ where $f^{-1}(a_1, \dots, a_n) = ((a_1, \dots, a_{n-1}), a_n)$.

Next we prove both f and f^{-1} are continuous.

→ Because both are finite product, their topologies are the same under both box and product topologies.

- Thus $\prod_{j=1}^{n} B_j$, B_j is a basic open set of X_j, is a basis of Z.

Similarly $\prod_{j=1}^{n-1} B_j$, is a basic open set of X_n.

$f^{-1}(B_1, \dots, B_n) = ((B_1, \dots, B_{n-1}), B_n)$ is open in $Y \times X_n$, f is continuous

→ For the same reasoning backward, f^{-1} is also continuous.

→ $Y \times X_n$ is indeed homeomorphic with Z. $\boxtimes$.

Theorem 3.45: Let $x_1, x_2, \dots$ be a sequence of the points of the product space $\prod X_\alpha$. Show this sequence converge to x iff $\pi_\alpha(x_1), \pi_\alpha(x_2)$ converge to $\pi_\alpha(x)$ for each α.

Remark: It only works in product topology, not box topology.

Proof:

$\Rightarrow$:

- Suppose each $\pi_\alpha(x_1)$, $\pi_\alpha(x_2)$ converge to $\pi_\alpha(x)$. Let U be a neighborhood on x in $\prod_\alpha X_\alpha$. $\exists\, V = \prod_\alpha B_\alpha$ contained in U that contains x.

(Because for every point in X_α, it's covered by the set of all basic element).

- For each α designate $K_\alpha \in \mathbb{N}$ s.t. $n \geq K_\alpha \to \pi_\alpha(\vec{x_n}) \in B_\alpha$.

(By definition of limit of a sequence, given that $\prod_\alpha B_\alpha$ is indeed an open set)

- Note that for each B_α, either $B_\alpha = X_\alpha$ or B_α is some open set in K_α.

→ If it's that $B_\alpha = X_\alpha$, then $K_\alpha = 1$, since every index in the sequence has to be in the set.

→ If it's that $B_\alpha \in T_{X_\alpha}$, there is always such K_α for every index, since the amount of such coordinate is finite. We can directly take the supreme over the entire finite set. Let $K = \sup\{K_\alpha : K_\alpha > 1\}$, therefore when $n \geq K$, then $x_{n\beta} \in B_\beta$ for every n and β. Then $(x_{n\beta})$ converges to x

(Note that this proof doesn't work in the case of box topology, as the supremum of an infinite set might not be finite).

$\Leftarrow$: (This direction also work in box topology).

- Suppose (x_n) converge to x.
- Let $U' \in \prod_\alpha X_\alpha$ s.t. $\pi_\alpha(x) \in U'$. There is some open set in X_α contained in U' that $\pi_\alpha(x)$. Call the set B'_α.

  (It is because we know that a sequence converge in X_α to $\pi_\alpha(x) \Leftrightarrow \pi_\alpha(x)$ is a limit point in this coordinate, and the sentence above denotes the definition of limit point in X_α).
- By the definition of product topology, suppose there is a basis element $\prod_\beta B''_\beta$ where $\begin{cases} B''_\beta = B'_\alpha & \text{for a finite set of coordinate} \\ B''_\beta = X_\alpha & \text{for the rest of the coordinate.} \end{cases}$
- Then by definition $\exists K' \in \mathbb{N}$ s.t. $\pi_\alpha(x_n) \in B''_\alpha = B'$ for the finite set of coordinate.

$\rightarrow$ It suggest we can always find a basic element that the sequence $(x_\beta)_n$ eventually enter, since basic open set is also an open set, it fulfills the definition that this sequence $(x_\beta)_n$ converge to the point x. $\square$

Comment: Last theorem is actually very "sneaky", it only certify the convergence of sequence with the convergence in every coordinate $\alpha \in \lambda$. It has not mentioned anything about the point being a limit point in product topology.

In fact, being a limit point in the product topology doesn't mean there is a sequence have to converge to it, below is an example:

- Let $A = \{(x_A) : x_A = 1$ only for finitely may α, 0 otherwie$\}$.

We claim the point $x = (1)$ is a limit point of A.

Proof:

- Take any basic open set that contains x

$U = \prod_{\alpha \in \mathbb{R}} U_\alpha$ is open $\{0,1\}$ for all but finitely many $U_\alpha = \{0,1\}$

Obviously $A \in U$

$S = \{\alpha : U_\alpha \neq \{0,1\}\}$ (S is a finite set) for $\alpha \in S$, $U_\alpha = \{1\}$.

- Now take $y = (y_\alpha)$ such that $\begin{cases} y_\alpha = 1 & \alpha \in S \\ y_\alpha = 0 & \alpha \notin S \end{cases}$
- Then obviously $\vec{y} \in A$ & $\vec{y} = U$

- This part is proving that $U \cap A \setminus \{x = (1)\} \neq \phi$
- Hence x is a limit point.

→ However, there is no such sequence where the sequence converge to $\vec{x}$.

Let $\vec{y_i} = (y_{i\alpha})$. The set of indices where $\vec{y_i}$ has 1 is countable.

Let β be an index where no y_i has a 1 or $\forall i,\ y_{i\beta} \neq 1$

- Let $U = \prod_\alpha U_\alpha$ where $U_\alpha = \{1\}$ $\alpha = \beta$
  $U_\alpha = \{0,1\}$ $\alpha \neq \beta$. ☒.

Theorem 3.46: The set $\prod_\alpha U_\alpha$ is closed in product topology is closed if U_α is closed in X_α for every U_α.

- $\prod_\alpha U_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(U_\alpha)$, where $(\pi_\alpha^{-1}(U_\alpha))^c = \pi_\alpha^{-1}(U_\alpha^c)$ is an open subbasic element, thus $\pi_\alpha^{-1}(U_\alpha)$ is closed.
- An infinite intersection of closed set is still closed. ☒.

Comment: It's an unintuitive result, compared to the property of open set.

Example: In the product space $2^{\mathbb{R}}$, what is the closure of all elements of $2^{\mathbb{R}}$ are 0 on every irrational coordinate & 0 or 1 on every rational coordinate?

Proof:

→ Our claim for the answer of the question: $\overline{Z} = Z$, or Z itself is closed.

- Every set in $2^{\mathbb{R}}$'s every dimension is closed, thus an product of closed is closed, by the last theorem we just proved. ☒.

- Different topologies on $\mathbb{R}^\omega$

1. Box Topology
2. Product Topology
3. Uniform Topology

Definition [Uniform Topology]: The topology of $\mathbb{R}^\omega$ constructed from basis $\{\vec{F}(\vec{x}, \varepsilon) : \vec{x} \in \mathbb{R}^\omega, \varepsilon \in \mathbb{R}^+\}$ when $\vec{x} \in (x_1, x_2, \dots)$ and the $E(\vec{x}, \varepsilon)$ is defined as $E(\vec{x}, \delta) = \{(y_1, y_2, \dots) \in \mathbb{R}^\omega : |y_i - x_i| < \delta \ \forall i \in \omega\}$

$F(\vec{x}, \delta) = \bigcup_{0<\delta<\varepsilon} E(\vec{x}, \delta)$

- The coarser & finer comparison:

Box Product $\xrightarrow{\text{finer}}$ Uniform Topology $\xrightarrow{\text{finer}}$ Product topology.

Proof:

- To prove the statement, we show uniform topology is coarser than box topology and finer than product topology.

(1) Product topology.

- Can open sets in product topology expressed as union of open sets in uniform topology?

→ For every basic open set in product topology, $U = \prod_{\alpha \in \omega} U_\alpha$ when $U_\alpha \subset \mathbb{R}$ for a finite set β and $U_\alpha = \mathbb{R}$ $\alpha \in \omega - \beta$

- $\mathbb{R}$ has the standard topology. For every oint in U, say $(x_1, x_2, \dots)$

For every x_i's where $i \in \beta$, $\exists \varepsilon_i > 0$ s.t. $N_\varepsilon(x_i) \subset U_i$

Now take $\varepsilon = \min\{\varepsilon_i \in \mathbb{R}^+ : i \in \beta\} > 0$ (Recall β is finite).

- Then $i \in \beta$ x_i can be covered by $\varepsilon_i(x_i) \subset U_i$

$i \notin \beta$ x_i can also be covered nce $X_i = \mathbb{R}$.

→ $E(\vec{x}, \varepsilon)$ is a subset of U for ever $\alpha \in \omega$. $E(\vec{x}, \varepsilon) \subset U$. Hence every point in a basic open set, hence open set ca be covered by on open set in uniform topology.

→ Conversely, for some basic open set in $E(\vec{x}, \varepsilon) : \vec{x} \in \mathbb{R}^\omega$ and $\varepsilon \in \mathbb{R}^+\}$

It can't be expressed as union of basic open sets in product topology since it has infinite coordinate of $\mathbb{R}$, which $E(\vec{x}, \varepsilon)$ don't.

(2) Box Topology

- For every basic open set U in box topology, written as $\prod_\alpha U_\alpha$, $U_\alpha \in \mathbb{R}_{std}$. $\forall \alpha \in \omega$.
- Now for every $\vec{x} \in U$, each $x_i \in U_i$, $\exists \varepsilon_i > 0$ s.t. $x_i \in N_{\varepsilon_i}(x_i)$. But when take $\inf\{\varepsilon_i, i \in \omega\}$ under box topology. The number might be 0.
- We can't expressed all box topology basic open set as union of basic open set for uniform topology.

· On other hand, for any basic open set U in uniform topology, every coordinate it an open ball $N_{\varepsilon_i}(x_i)$.

→ For a point $\{y_i : |y_i - x_i| < \varepsilon_i < \varepsilon\}$ for a given ε. We may always fit an basic open set of box topology inside containing x_i for every point.

· Hence box topology is strictly finer than uniform topology.

Example: To illustrate the difference, consider $\mathbb{R}^\infty \subseteq \mathbb{R}^\omega$ defined as the sequences that are eventually zero.

$$\mathbb{R}^\infty = \{\vec{x} : \exists (\alpha_i)_{i=1}^{n} \subseteq \omega \quad x_{\alpha_i} \neq 0, \ n \in \mathbb{Z}^+\}$$

· Now consider the closures under the three different topologies.

1) Product Topology: $Cl(\mathbb{R}^\infty) = \mathbb{R}^\omega$

· For any point in $\mathbb{R}^\omega$, or any sequence $\vec{x} = (x_1, x_2, \dots)$ we want to show it's a limit point of $\mathbb{R}^\infty$.

· Let a basic open set in product topology be $\prod_{i=1}^{n} U_i \times \prod_{\omega \setminus \{1,\dots,n\}} \mathbb{R}$

Construct a sequence $\vec{x}' = (x_1', x_2', \dots, x_n', 0, 0, \dots) \leftarrow$ a point in $\mathbb{R}^\infty$.

2) Box topology: $\mathbb{R}^\infty$ is closed. Or $Cl(\mathbb{R}^\infty) = \mathbb{R}^\infty$.

· We may show the complement of $\mathbb{R}^\infty$ is open.

· For every point $\vec{x} = (\mathbb{R}^\infty)^c$, $\vec{x}$ has infinite non-zero term.

For every non-zero term, say x_i, pick U_i in $\mathbb{R}$ that $\{0\} \notin U_i$ but $x_i \in U_i$. It doesn't contain any point in $\mathbb{R}^\infty$.

· Hence, we know that every point in $(\mathbb{R}^\infty)^c$ has an open set in $(\mathbb{R}^\infty)^c$ that contains it.

3) Uniform Topology: A the sequence in $\mathbb{R}^\omega$ that converge to 0. Like $(1, \frac{1}{2}, \frac{1}{3}, \dots)$.
Call the sequence $\mathbb{R}^{s0}$.

· For every point in $\mathbb{R}^{s0}$, take a neighborhood: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$ $|x_n - 0| < \varepsilon$.

· Not take neighborhood in uniform topology. Named $E(\vec{x}, \varepsilon)$. $\forall n > N$, by definition the neighborhood $\mathbb{R} \supset N_\varepsilon(x_i) \ni 0$.

→ All contains 0, except for $n < N$. Which is finite. Hence we know that it contains a point in $\mathbb{R}^\infty$.