CH6.1

Definition [Cover, Open Cover, Subcover]: Let A be a subset of X and let $\mathbb{C} = \{C_\alpha\}_{\alpha\in\lambda}$ be a collection of subsets of X.

1) $\mathbb{C}$ is a cover of A iff $A \subset \bigcup_{\alpha\in\lambda} C_\alpha$
2) $\mathbb{C}$ is an open cover of A iff every C_α is open.
3) A subcover $\mathbb{C}'$ of C of A is a subcollection of C that still covers A.

Definition [Compact]: A space X is compact iff every open cover of X has a finite subcover.

Theorem 6.1: Let X be a finite topological space. Then X is compact.

Proof:

- Let $X = \{x_1, \dots, x_n\}$.
- For any open cover of X say $\{C_\alpha\}_{\alpha\in\lambda}$. For each $j \in \{1, \dots, n\}$, we may find $i_j \in \lambda$ s.t. $x_j \in C_{i_j}$.
- Hence it's a finite subcover. ☒.

Comment: A Compact set can establish some property that a complex set may have.

Theorem 6.2: Let C be a compact subset of $\mathbb{R}_{std}$. Then C has a maximum point. $\exists\, m \in C$ s.t. $\forall\, x \in C$, $x \leq m$.

Proof:

- Let a set U be compact in $\mathbb{R}_{std}$. Then for every open cover, there exists a finite subcover.
- Using this conclusion, we may easily see that U is bounded from above.

→ Let's consider the open cover for U defined as $\{(-\infty, n)\}_{n\in\mathbb{R}}$. It's an open cover for U because it covers the entire $\mathbb{R}$. However, it's compact so only finite n can also cover the set.

- Hence there exists an n s.t. $\forall\, u \in U$, $u < n$. Thus n is the upper bound. We may pick $M = \inf\{n\}$ and we claim it's in U. Suppose it's not.

→ Consider another open cover $U_m = (-\infty, M - \frac{1}{m})_{n\in\mathbb{Z}}\}$. It covers the entire U because it contains all points below u, but not u itself, which is what we assumed.

- But then this set also has a finite subcover. Therefore $\exists\, m_k$ s.t. $\forall\, u \in U$ $u < \frac{1}{m_k} < M$. Thus $\frac{1}{m_k}$ is an upper bound less than n. But it contradicts the fact that n is the least upper bound. $\boxtimes$.

Theorem 6.3: If X is a compact space, then every infinite subset of X has a limit point.

Proof: Using Contrapositive statement.

- Suppose and infinite set $U \subset X$ it doesn't have a limit point. Then for every $u \subset U$, $\exists$ open V s.t. $U \cap V \setminus \{x\} = \phi$.
- Then if we find this set V for every $u \in U$. Then we found an open cover for U. It's obviously not having a finite subcover b.c. even if we remove one set, there's one point in U not being covered. $\boxtimes$.

Corollary 6.4: If X is compact and E is subset of X that has no limit point, then E is finite. (It's just the contrapositive argument of theorem 6.3).

Definition [Finite intersection Property]: A collection of sets has the finite intersection property iff every finite subcollection has a non-empty intersection.

Comment: This definition allows an alternative characterization of compactness.

Theorem 6.5: A space X is compact iff every closed set collection with finite intersection property has non-empty intersection.

Proof:

- Given a collection α of subsets of X, let $C = \{X - A \mid A \in \alpha\}$. Then

(a) A is a collection of open subsets iff C is a collection of closed.

(b) A covers X iff $\bigcap C$ is empty.

(c) $\{A_1, \dots, A_n\}$ covers X iff $\bigcap_{i=1}^{n} C_i$ is empty.

- X is compact, meaning that given any collection A of open subset of X. If A covers X, then some finite subcollection of A also covers X.
- Take its contrapositive: if no finite subcollection of A covers X, then A does not cover X.

- It's equivalent as saying given any C of closed sets, if every finite intersection of C is non-empty then the intersection of all C is nonempty.
- The other side takes the exact same path backward $\square$

(An exception is a nest sequence of sets): $C_1 \supset C_2 \supset \dots \supset C_n \supset C_{n+1} \supset \dots$ of closed sets in a compact space X. If each of sets C_n is nonempty, $\bigcap^n C_n$ is nonempty. Then $\cap C_n$ is nonempty.)

Theorem 6.6: X is compact iff $\forall U \in T_X$ and any collection of closed sets $\{K_\alpha\}_{\alpha \in A}$ s.t. $\bigcap_{\alpha \in A} K_\alpha \subset U$, $\exists$ a finite number K's whole intersection lies in U.

Proof:

$\Rightarrow$:

- For any closed sets s.t. $\bigcap_\alpha K_\alpha \subset U$. $U^c \subset \bigcup_\alpha K_\alpha^c$, meaning a closed set is covered by a collection of open sets.
- Since X is compact, $\exists$ a finite collection $\{\alpha_1, \dots, \alpha_n\} \subset \alpha$ s.t. $\bigcup_{i=1}^n K_{\alpha_i}^c$ also covers U^c. Now take complement again $U \supset \bigcap_{i=1}^n K_{\alpha_i}$.

$\Leftarrow$: The exact same steps backward. $\square$.

Theorem 6.8: Let A be a closed subspace of a compact space. Then A is compact.

Proof:

- Let Y be a closed subspace of a compact space X.
- Suppose we have a covering of Y by open set in X. We may form an open covering B of X by letting $B = A \cup \{X - Y\}$
- Some finite subcollection of B covers X. If this subcollection contain $X-Y$, we may remove it, since it doesn't intersect Y anyway. If not, then we are done.
- Now the subcollection covers Y. $\square$

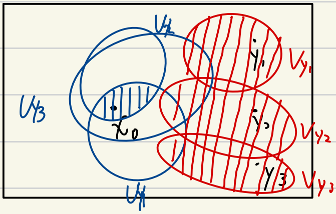
Theorem 6.9: Every compact subspace of a Hausdorff space is closed.

Proof:

- Let Y be a subspace of X, which Hausdorff. We prove that $X-Y$ is open, therefore Y is closed.

· For every $x \in X-Y$ and for every $y \in Y$. Since X is Hausdorff, we may pick $U_y \in T$ s.t. $x \in U$, $V_y \in T$ s.t. $y \in V$. and $U \cap V = \phi$.

→ Obviously, $\bigcup_{y \in V} V_y$ is an open cover of Y. Since Y is compact, $\exists \{y_1, \dots, y_n\}$ s.t. $\bigcup_{i=1}^{n} V_{y_i}$ covers the space Y.



· Now the set $\bigcap_{i=1}^{n} U_{y_i}$ is an open set disjoint from the union of V_{y_i}, which is also containing Y. Thus we proved $X-Y$ is open.

· Thus Y is closed.

Example: The upper proof is a very important criterion that makes metric space the way it is in analysis. For some topological space that's not Hausdorff. compact set may not be closed.

· E.g. The indiscrete topology doesn't have this property

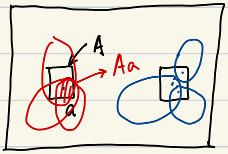
Every singleton set is not a closed set because it's complement is not open, but it certainly has a finite subcover for every open cover

Since the amount of open set in the topology is 1. It's the only possible finite subcover. ⊠.

Theorem 6.12 Every compact Hausdorff space is normal.

Proof:

· Suppose a compact space X and there are 2 closed set A, B.



· For every point a in A, then for every point b in B, b.c. it's Hausdorff space we can find an open set V_{ab} s.t. $a \in V_{ab}$ and V_{ba} s.t. $b \in V_{ba}$

· We found an open cover for B if we union over all B.

· Because X is compact, subspace B is also compact, there $\exists$ a finite subcover.

$B \subset \bigcup_{b \in B} V_{ab} \rightarrow B \subset \bigcup_{b \in \{b_1, \dots, b_n\}} V_{ab} = B_a$ Obviously

· And obviously $A \in \bigcup_{b \in \{b_1, \dots, b_n\}} V_{ab}. = A_a$ $B_a = A_a$

→ Now if we union over A, $\bigcup_{a \in A} A_a$ forms an open cover of A. By definition of compact, there exists a finite subcover s.t. $A \in \bigcup_{a \in \{a_1, \dots, a_n\}} A_a$. Note the $\bigcup A_a$

is a finite union of finite union, hence is finite.

- Then take the union $\bigcap_{a\in\{a_1,\dots,a_n\}} B_a$ is a finite union of open set, which is still open. As all the B_a contains B, then so is $\bigcap B_a$ and additionally

$\underbrace{(\bigcap B_a)}_{\text{Open set containing } B} \cap \underbrace{(\bigcup A_a)}_{\text{Open set containing } A} = \phi.$ As needed. ☒.

Theorem 6.13 : Let B be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets in B has a finite subcover.

Proof :

$\Longrightarrow$:

- If X is compact, and suppose a set $A \subset X$ has an open cover $\{A_a\}_{a\in\lambda}$. For every $a \in \lambda$, there exists a set $B_a \subset B$ s.t. $\bigcup_{B' \in B_a} B' = A_a$.
- Thus $\{A_a\} = \bigcup_{a\in\lambda} B_a \supset A$. It's a group of basic open set. → also form an open cover
- Therefore we know that $\bigcup B_a$ has a finite subcover, consist of basic open set.

$\Longleftarrow$:

- If every basic open cover has a finite subcover.
- Now suppose we have an open cover, we can write it as a union of basic open set and it has a finite basic open cover.
- A basic finite subcover is a finite subcover. ☒.

CH6.2 Some general result from analysis in $\mathbb{R}_{std}$ or $\mathbb{R}^n_{std}$

Theorem 6.14 : For any $a \le b$ in $\mathbb{R}$ $[a, b]$ is compact.

Theorem 6.15 (Heine Borel Theorem): Let $A \subset \mathbb{R}_{std}$. A is compact iff A is closed and bounded.

Theorem 6.17 : Every compact subset C of $\mathbb{R}$ contains a maximum in the set C. In other word, $\exists\, m \in C$ s.t. $\forall x \in C$ $x \le m$.

CH6.3

Theorem 6.18 (Tube lemma) : Let $X \times Y$ be a product space with Y compact. If N is an open set of $X \times Y$ containing $x_0 \times Y$, then there is some open set W in X

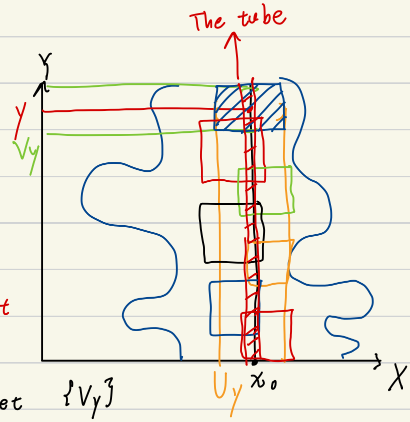
containing x_0 s.t. N contains $W \times Y$.

Proof :

- For every $y \in Y$ and thus the point $(x_0, y) \in X \times Y$, we can find $U_y \in X$ & $V_y \in Y$ s.t.

→ $x_0 \in U_y$, $y \in V_y$ and $U_y \times V_y \subset$.

(Because N is an open set in $X \times Y$, hence every point is an interior point, thus we may find an open in N that contains it).



- Because we may find such a V_y for every y, the set $\{V_y\}$ is indeed a an open cover for Y.

→ Because Y is compact, we may find $\{y_1, \ldots, y_n\} \subset Y$ s.t. $V_{y_1} \cup \ldots \cup V_{y_n} \supset Y$.

- For these y_i's, their corresponding U_{y_i} all contains x_0.

Define $W = U_{y_1} \cap \ldots \cap U_{y_n}$ (a finite intersection of open sets)

→ We want to show that $x_0 \in W$ and $W \times Y \subset N$.

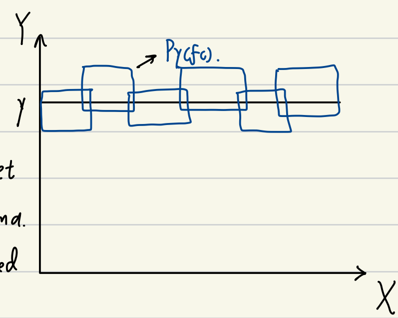
- Since $x_0 \in U_{y_i}$ $\forall i \in \{1, \ldots, n\}$, $x_0 \in W = U_{y_1} \cap \cdots \cap U_{y_n}$.

And additionally for any $y \in V_y$, $\exists i \in \{1, \ldots, n\}$ s.t. $i \in V_{y_i}$. Hence we know that $(x, y) \in U_y \times V_y \subset N$. $\square$.

Theorem 6.19 : If X & Y are both compact, then $X \times Y$ is also compact.

Proof :

- Suppose we have an open cover $\{P_\alpha\}_{\alpha \in \lambda}$ for $X \times Y$. WTS it has a finite subcover.
- Because the open cover covers every (x, y), we fix a y and define the set $\{P_{\alpha y} = U_{\alpha y} \times V_{\alpha y}\}$ be the set in $\{P_\alpha\}$ that covers $y \times X$.



- By Unioning all of the $P_{\alpha y}$'s, we have an open set that contains $y \times X$. Which allow us to use tube Lemma.
- $\exists W_y \subset Y$ s.t. $W_y \times X \in \bigcup P_{\alpha y}$. And since we picked y generally, we may find W_y for any y.

→ Thus the set $\{W_y \times X\}_{y \in Y}$ forms an open cover of $X \times Y$ as $\{W_y\}$ forms an open cover for Y. Hence $\exists$ finite subcover of Y from W_y. Which cover $X \times Y$. $\square$.

Some Supplements before we start next theorem: 3 equivalent Axioms

- Definition: A set X is partially ordered by "$\leq$" iff for any elements x, y and z in X.
  (1) $x \leq x$
  (2) if $x \leq y$ and $y \leq z \rightarrow x \leq z$
  (3) $x \leq y$ and $y \leq x \rightarrow y = x$
  A partially ordered set is also called a "poset"
- Definition: A poset is totally ordered iff it's partially ordered and every 2 elements are comparable.
- Definition: A set is well-ordered iff it's totally ordered any every non-empty subset has a least element.
- Definition: Let P be a poset with relation $\leq$, let A be a subset of P. An element b in P is an upper bound of A iff $\forall a \in A$, $a \leq b$.

1. Zorn's Lemma. Let X be a partially ordered set in which every totally ordered subset has an upper bound. Then X has a maximal element.
2. Axiom of choice. Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a set of non-empty sets. Then there is a function $f: \lambda \rightarrow \bigcup_{\alpha \in \lambda} A_\alpha$ s.t. $\forall \alpha \in \lambda$ $f(\alpha) \in A_\alpha$. 选择公理
   (We can create a new set from a collection of sets in a certain way. We choose one element from each set or construct a set that contains one element from each of the sets in a given set of non-empty sets).
3. Every set can be well ordered, every set can be put in one-to-one correspondence with a well-ordered sets.

Theorem 6.21 (Alexander Subbasis Theorem): Let S be a subbasis for a space X. Then X is compact iff every subbasic open cover has a finite subcover.

Proof:

$\Rightarrow$: Obviously True

$\Leftarrow$:

Use contrapositive: if X is not compact, $\exists \{S_\alpha\}_{\alpha \in \lambda}$ s.t. $X \subset \bigcup_{\alpha \in \lambda} S_\alpha$ and X doesn't have a finite subcover.

- If X is not compact: $\exists U = \{U_\alpha\}_{\alpha \in \lambda}$ be a basic open cover of X that doesn't have a finite basic open cover. (Equivalent to that of open cover by theorem 6.12)

- Now consider a collection of open cover that $\bar{\mathcal{U}}$ s.t.

  (i) $\mathcal{U} \subset \bar{\mathcal{U}}$ (ii) $\bar{\mathcal{U}}$ has no finite cover

- The set $\bar{\mathcal{U}}$ is indeed a partially ordered set and each must has an upper bound

  The "upper bound" is essentially the union over all open set in $\bar{\mathcal{U}}$.

→ Then by Zorn's theorem: there's a maximal element $\bar{\mathcal{U}}^*$ that satisfy (i) & (ii).

- We claim $S \cap \bar{\mathcal{U}}^*$ is an open cover for X. Suppose not, $\exists\, x \in X$ that's not in $S \cap \bar{\mathcal{U}}^*$. $\exists\, S_1, \dots, S_n$ s.t. $x \in S_1 \cap \dots \cap S_2$ (By property of basic open set).

→ $\forall\, i \in \{1, \dots, n\}$ S_i can't be in $\bar{\mathcal{U}}^*$.

- Thus for the set $S_1 \cup \bar{\mathcal{U}}^*$, a set greater than the maximal open cover, has a finite subcover. Call it $U_{11} \cup \dots \cup U_{1n(1)} \cup S_1 \supset X$. Repeat the steps for $1, \dots, k$

  We got k finite subcover in the form $U_{i1} \cup \dots \cup U_{in(i)} \cup S_i$

- Take union of all the intersection $\bigcap_{j=1}^{n} \{U_{ji} \,\&\, S_i\}$. Each open cover contains X, hence the intersection contains X and the intersection is finite.

→ Additionally, it doesn't contain any $S_i \in \{S_1, \dots, S_n\}$ because each $\{U_{ji} \,\&\, S_i\}$ only contain one each. We found a finite subcover for $\bar{\mathcal{U}}^*$, contradiction. ◻.

Theorem 6.23 (Tychonoff's theorem): Any product of compact space is compact.

Proof:

- Suppose we have a product space: $\prod_{\alpha \in \lambda} X_\alpha$ where X_α is compact for every α.
- Suppose S is a subbasic open cover. Name it $S = \{S_\beta\}_{\beta \in \lambda}$

  Let $\{S_\alpha\}$ be the subset of S that's not having X_α at the α coordinate.

- We claim $\exists\, \alpha_0$ s.t. $\{S_{\alpha_0}\}$ cover X. Because if not, then $\exists\, x_\alpha$ s.t. none of its coordinate thats covered by any α, which make S not an open cover.

→ Now S_{α_0} covers X, as we know, and we know that X_α is compact, hence the set has a finite subcover. Because element in $\{S_{\alpha_0}\}$ is open and not X_α only at X_{α_0} thus at α_0 th coordinate.

- Thus we found a finite subbasic open subcover from any subbasic open cover. By Alexander's subbasis theorem, the space is compact. ◻.

Tangent: A result of Axiom of choice & Zorn's Lemma.

Theorem 6.0 : Every Vector Space has a basis.

Proof:

· In general, we want to show if L is a linear independent subset of X, $\exists B$ being a basis of X s.t. $L \subset X$.

· Let A be the set of linearly independent subsets of X containing L. Then A is partially ordered by inclusion.

→ For every chain $C \subseteq A$, define $\hat{C} = \bigcup C$ - it's clearly an upper bound.

But why $\hat{C} \in A$? Let $V := \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq \hat{C}$ be a finite vector collection.

$\exists C_1, \dots, C_n \in C$ s.t. $\vec{v}_i \in C_i \quad \forall\ 1 < i < n$. Since C is a chain

· Then $\exists C_1, \dots, C_n$ s.t. $\vec{v}_1 \in C_1, \dots, \vec{v}_n \in C_n$

Because C is a chain, $\exists k$ w/ $1 \le k \le n$ s.t. $C_k = \bigcup_{i=1}^{n} C_i$ and thus $V \subseteq C_k$

· Therefore V is linearly independent. $\hat{C}$ is an element of A.

→ By Zorn's Lemma, a poset s.t. every well-ordered set has a maximal element, A has a maximal element M. By definition of A, we show M should be a basis of X.

· Assume not, $\exists \vec{x} \in A - \{\text{Span of } M\}$. It's linearly independent to other $\vec{v}$'s.

$\exists a_1, \dots, a_n, a_{n+1} \in \mathbb{R}$ s.t.

$$a_1\vec{v}_1 + \dots + a_n\vec{v}_n - a_{n+1}\vec{x} = 0.$$

$$\begin{cases} \text{If } a_{n+1} = 0 \rightarrow a_1 \,\&\, a_n = 0 \text{ since } \vec{v}_1, \dots, \vec{v}_n \text{ are LI. } M \text{ is not maximal.} \\ \text{If } a_{n+1} \neq 0 \rightarrow x = \frac{a_1}{a_{n+1}}\vec{v}_1 + \dots + \frac{a_n}{a_{n+1}} x_n \longrightarrow x \in \text{span}(M) \text{ Contradiction.} \end{cases}$$

→ Hence M has to be the basis of X.

Interesting remark: Compactness and Hausdorff are in perfect tension.

· As we have finer topology: There's more open set, therefore we may enclose small regions with many open sets it's not likely to be compact.
(For example, every infinity sets in indiscrete topology is not compact, but every set in indiscrete is compact, because it only has one open set).

· But when there's more open set, we may use open sets to sketch the boundary of any set, thus it's more likely to be Hausdorff.

- Hence when a set is more likely to be compact, it's likely to be Hausdorff.

In fact, if a set is Hausdorff and compact:

(i) Any finer topology is not compact and;

(ii) Any coarser topology is not Hausdorff.

Proof:

- Suppose T_1 on X is compact and Hausdorff, let T_2 be finer and T_3 be coarser.
- By theorem 6.9 Any compact set under Hausdorff topology is closed. Thus if a

I set is not closed under Hausdorff topology, then it's not compact.

- Since T_2 is a finer topology, there is a set U^* that's open in T_2 but is not open in T_2.

→ Since U^{*c} is not closed in T_1, we know there is an open cover without a finite subcover call it $\{V_\alpha\}_{\alpha \in \lambda}$. Now add U^* to the collection $\{U^*\} \cup \{V_\alpha\}_{\alpha \in \lambda}$ is now an open cover in T_2 (U^* is open in T_2)

- If T_2 is compact, there is a finite subcover for X. It might or might not contain U^*. If yes, remove it and the rest of the finite subcover is a part of $\{V_\alpha\}$ that covers U^{*c}, which is still valid in T_1.

II

- If T_3 is a coarser topology than T_1, $\exists U^\#$ that open in T_1 but not in T_3. We may again invert theorem 6.9: If a set is not closed but is compact in a topology, then the topology is not Hausdorff.

→ The set we claim it is $U^{\# c}$ in T_3. To show this consider any open cover of $U^{\# c}$ in T_3. It does not contain $U^\#$ since it's not open anymore. The rest of the open covers are also open in T_1 thus have a finite subcover something to T_3.

- Thus $U^{\# c}$ is a closed but compact subset in T_3. Hence by contrapositive of 6.9. T_3 cannot be Hausdorff. ☒.