CH 8.1

Definition [Connectedness]: Let X be a topological space. Then X is connected iff X is not union of two disjoint non-empty open sets.

Definition [Separated]: Let X be a topological space. Subsets A, B in X are separated iff $\bar{A} \cap B = A \cap \bar{B} = \phi$. The notation $X = A | B$ means $X = A \cup B$ and A & B are separated.

Theorem 8.1: The following are equivalent

(1) X is connected

(2) There is no continuous function $f: X \to \mathbb{R}_{std}$ s.t. $f(X) = \{0,1\}$

(3) X is not union of 2 disjoint non-empty separated set.

(4) X is not union of 2 disjoint non-empty closed set.

(5) The only subsets of X that are both closed and open is ϕ and X.

(6) For every pair of points p and q and every open cover $\{U_\alpha\}_{\alpha \in \lambda}$ of X there exist a finite of U_α's, $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots, U_{\alpha_n}\}$ s.t. $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$ and for each $i < n$, $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \phi$. (Chain connectedness)

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(1) $\to$ (2): Contrapositive

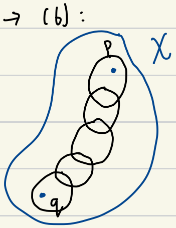
- If there is a continuous map that $f(X) = \{0,1\}$ The set has standard topology.
- Thus $\{0\}, \{1\}$ are closed. (Since in $\mathbb{R}_{std}$, singleton are closed).
  Thus $\{0\}, \{1\}$ are open under subspace topology of $\{0,1\}$.
- $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are 2 disjoint open sets in X. Hence X isn't connected.

Then (2) immediately implies (4), which immediately implies (3), because preimage of a closed set ($\{0\}, \{1\}$) is closed under continuous mapping.

(2) $\to$ (5):

- It's also very obvious, as here $\{0\}, \{1\}$ are both open and closed. Hence their preimages are both open and closed.

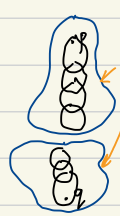
(5) $\to$ (6):

- 

← This is what's meant by (6)

Converse of (6) →

($\exists p, q$ s.t. $\exists \{U_\alpha\}_{\alpha \in \lambda}$ s.t. $\forall$ $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$ $\exists i$ s.t. $U_{\alpha_i} \cap U_{\alpha_n} = \phi$.

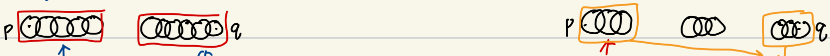


two open + closed sets

· First we fetch every chain and its corresponding i.
We union all open set that's before ith open cover, including the ith
then union all open set that's after ith open cover.

→ We claim the 2 unions are the 2 open & closed sets.

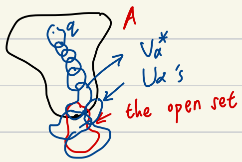
· They are of course open, and we need to prove they are closed.
(Note that WLOG, we assume there is only one "breaking point" in each chain. If there's multiple, we can just union the sets are connected to U_{α_1} & U_{α_n}) Like the following picture



One breaking point union them up separated. 2 breaking points union them up separately.

· For every limit point of A, say a. Every open set containing a intersects A. Since it's a a subset of the entire X. It can be covered by sets in $\{U_\alpha\}_{\alpha\in\lambda}$.

· Find the U_α^* that contains a. By construction, it has a chain of open sets connected to q.



· If the U_α^* is in a failed connecting chain between p and q, then it will be unioned into A, containing a.

· Hence $a \in A$. As needed. A is closed. As needed.

(6) → (1):

· Suppose there are 2 disjoint open sets A & B s.t. $A \cup B = X$.
Suppose X has a basis $\mathcal{B} = \{U_\alpha\}_{\alpha\in\xi}$

· Then make basic open cover for A and B, namely $\{U_\alpha\}_{\alpha\in\lambda}$, $\{U_\alpha\}_{\alpha\in t}$
where λ and t are 2 disjoint sets. And $\lambda, t \subset \xi$.
Since it's basic open, we additionally let $\bigcup_{\alpha\in\lambda} U_\alpha = A$, $\bigcup_{\alpha\in t} U_\alpha = B$.

· Together $\{U_\alpha\}_{\alpha\in\lambda\cup t}$ forms an open cover.

→ Now for every $p \in A$, $q \in B$. If there $\exists$ a connected chain $U_{\alpha_1}, \dots, U_{\alpha_n}$ s.t. $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$. Then some $U_{\alpha_1}, \dots, U_{\alpha_i}$, $\{\alpha_1, \dots, \alpha_i\} \subset \lambda$
$U_{\alpha_{i+1}}, \dots, U_{\alpha_n}$, $\{\alpha_{i+1}, \dots, \alpha_n\} \subset t$.

→ Then $U_{\alpha_i} \subset A$, $U_{\alpha_{i+1}} \subset B$ must have an empty intersection. ☒

Theorem 8.3: $\mathbb{R}_{std}$ is connected. (Same proof as in analysis).
Proof:

- If $\mathbb{R}_{std}$ is not connected, there are 2 non-empty disjoint open sets U & V s.t. $\mathbb{R} = U \cup V$ and $U \cap V = \phi$.
- Since $U \cup V$ are open, their complement which is each other, is closed.

Assume $a \in U$, and since U is open, $\exists \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subseteq U$ for some $\varepsilon > 0$.

→ Now consider $S = \{x \in \mathbb{R} \mid [a, x) \subseteq U\}$. S is non-empty, as $(a-\varepsilon, a+\varepsilon)$ is contained in S.

- Let $b = \operatorname{Sup} S$. By the least upper bound property, b exists.

Is b in U? No, because if b is in U, $\exists \delta > 0$ s.t. $(b-\delta, b+\delta) \subset S$.

- Hence b is not the greatest element in S.

Is b in V? No, because if b is in V, $\exists \delta > 0$ s.t. $(b-\delta, b+\delta) \subset V$.

- Hence b is not the least upper bound, $b - \frac{\delta}{2}$ is a smaller upper bound.

→ Contradiction, the union is not the entire $\mathbb{R}$. $\boxtimes$

Theorem 8.4: Let A, B be separated subsets of a space X. If C is a connected subset of $A \cup B$. Then either $C \subset A$ or $C \subset B$.

Proof:

- Because A, B are separated, they are of course disjoint.
- Since $C \subset A \cup B$. $C \cap A$ & $C \cap B$ is a partition of C.

→ We claim that $C \cap A$ & $C \cap B$ are closed under subspace topology. Hence we can say it's not connected by theorem 8.1 (3).

→ To show that $\bar{A} \cap C = A \cap C$ because $\bar{A} \cap B = \phi \rightarrow (\bar{A} \cap C) \cap (B \cap C) = \phi$.

- Because C is covered by $A \cap C$ and $B \cap C$. If none of $\bar{A} \cap C$ is situated in $B \cap C$, $\bar{A} \cap C \subset A \cap C$. Since $\bar{A} \cap C \supset A \cap C$, $\bar{A} \cap C = A \cap C$.

→ Similarly $\bar{B} \cap C = B \cap C$. Thus $A \cap C$ & $B \cap C$ are both closed. Hence can be written as union of 2 disjoint closed sets. It's not connected.

- The only possibility is C is entirely contained in $A \cap C$ or $B \cap C$. Then it's the only set. Then in that case $C \subset A$ or $C \subset B$. $\boxtimes$

Theorem 8.5: Let $\{C_\alpha\}_{\alpha \in \lambda}$ be a collection of connected subsets of X, and let E be another connected subset of X s.t. $\forall \alpha \in \lambda$, $E \cap C_\alpha \neq \phi$. Then $E \cup (\bigcup_{\alpha \in \lambda} C_\alpha)$ is connected.

Proof:

- Suppose $E \cup (\bigcup_{\alpha \in \lambda} C_\alpha)$ is not connected. It can be written as union of 2 disjoint separated sets that are non-empty. Call them A & B (By theorem 8.1).
- Because E is a connect set, by theorem 8.4. $E \subset A$ or $E \subset B$. WLOG, assume $E \subset A$.
- Because B is non-empty, $E \subset A$, and A & B are disjoint. $\exists\, b \in C_{\alpha^*}$ where $\alpha^* \in \lambda$. Also because C_{α^*} is also connected. $C_{\alpha^*} \subset B$ also by theorem 8.4.

→ Thus $A \supset E$, $B \supset C_{\alpha^*}$. Since A, B are disjoint, E and C_{α^*} are disjoint. Thus $\exists\, \alpha^* \subset \lambda$ s.t. $E \cap C_{\alpha^*} = \emptyset$, as needed. $\square$.

Theorem 8.6: Let C be a connected subset of the topological space X. If D is a subset of X s.t. $C \subset D \subset \bar{C}$, then D is connected.

Proof:

- If D is not connected, it can be written as 2 disjoint reparated sets by theorem 8.1. Call them A and B.
- Because $C \subset D \subset A \cup B$. By theorem 8.4 $C \subset A$ or $C \subset B$. Assume $C \subset A$. Since $D \subset \bar{C}$, B consist of C's limit points.
- However then $\bar{A} = \bar{C}$ (since $\bar{\bar{C}} = \bar{C}$, theorem 2.13) it must intersect B, Contradict the fact that A & B are separated. $\square$

Example: The closure of topologist's sine curve in $\mathbb{R}^2_{std}$ is connected.

- Recall Topologist's sine curve is defined as $S = \{(x, \sin\frac{1}{x}) \mid x \in (0,1)\}$

Proof:

- As $x \to 0$, $\sin\frac{1}{x}$ oscillate between $(-1, 1)$. Hence the vertical segment $\{0\} \times [-1, 1]$ is in the closure. As mentioned before.
- As $x \to 0$, $(1, \sin(1))$ is also in the closure.

In all $\bar{S} = S \cup (\{0\} \times [-1,1] \cup \{(1, \sin(1))\})$

- $S \cup \{(1, \sin(1))\}$ is of course connected, since it's a continuous function.
- Hence $\bar{S} \subset \bar{S}$ is connected by theorem 8.6. ☐

Theorem 8.8 Let X be a connected space, C is a connected subset of X and $X - C = A | B$. Then $A \cup C$ & $B \cup C$ are each connected.

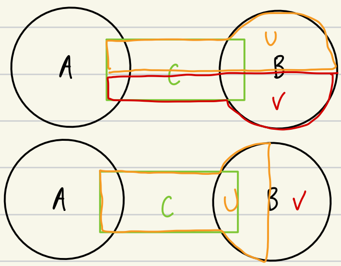
Proof:

- If one of $A \cup C$ or $B \cup C$ is not connected. WLOG, assume $B \cup C$ is not connected

  WTS X or C is not connected

- By theorem 8.1, $B \cup C$ can be written as 2 disjoint separated sets.

  Call them U and V. $B \cup C = U | V$.



- If X is connected, $C \cap U$ and $C \cap V$ are obviously 2 separated sets under subspace topology
- And their union is obviously C. C is not connected.
- If C is connected, $B \cap U$ and $B \cap V$ are 2 separated sets under subspace toplogy and their union is B. ☐

Theorem 8.9: Let $f: X \to Y$ be a continuous, surjective function. If X is connected then Y is connected.

Proof:

- Prove using contrapositive, suppose Y can written as 2 disjoint open sets. U, V
- Since f is continuous $f^{-1}(U)$ & $f^{-1}(V)$ are open under topology of X.

  Since $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X$ - since f is surjective.

  because U & V are disjoint, $f^{-1}(U)$ & $f^{-1}(V)$ are disjoint:

→ Therefore X is written as 2 disjoint open sets, hence is not connected. ☐

Theorem 8.10 (Intermediate Value theorem): Let $f: \mathbb{R}_{std} \to \mathbb{R}_{std}$ be a continuous map. If $a, b \in \mathbb{R}$ and r is a point of $\mathbb{R}$ s.t. $f(a) < r < f(b)$, then $\exists c \in (a,b)$ s.t. $f(c) = r$.

Proof:

Before we prove the theorem we prove a Lemma that in $\mathbb{R}$ connected sets are interval.

- Here, interval is defined as a C where every 2 points $a, b \in C$ $\forall x \in (a, b)$ $x \in C$. Then C is an interval.

$\Rightarrow$:

- If there is a set is not connected under $\mathbb{R}_{std}$. It can be written as 2 disjoint open sets. U & V
- Each open set can be written as union of basic open sets, which look like (a_i, b_i). Since the open set is disjoint, every pair of basic open are disjoint.

$\rightarrow$ Hence every end point of basic open set is not included (easily proved). If we pick a point in one basic open set from U and one basic open set from V. There will be a point not included, thus it's not an interval.

$\Leftarrow$:

- If it's not an interval, meaning $\exists\, a, b \in C$ s.t. $\exists\, x \in (a, b)$ $x \in C$ is not included. Call the set S.
- Then call $m = \operatorname{Sup} S$, $n = \operatorname{Inf} S$
- Take $S \cap (x, m+1)$ & $S \cap (n-1, x)$ are 2 open sets under subspace topology of S, and their union obvious contain S, hence S can written as 2 disjoint open sets
- Thus S is not connected $\boxtimes$.

- Because f is a continuous surjective map, and since (a, b) is an interval, or a connected set in $\mathbb{R}$ by the Lemma we proved.
- $f((a, b))$ is also a connected set, hence an interval by theorem 8.9. Now by definition, $f(a), f(b) \in f(X)$ are points in the interval
- Thus $\forall$ $r \in (f(a), f(b))$, $r \in f((a,b))$. Which mean $\exists\, r \in (a, b)$ s.t. $f(c) = v$. $\boxtimes$

Theorem 8.11: For topological spaces X and Y, $X \times Y$ is connected iff each of X and Y are connected.

Proof:

$\Rightarrow$:

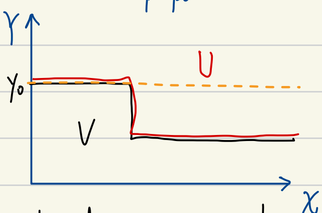
- Suppose one of X & Y is not connected and we WTS $X \times Y$ is not connected.
- WLOG, assume that X is not connected, then it can be written as union of 2 open sets, $X = U \cup V$.
- Obviously $U \times Y$ & $V \times Y$ are 2 open sets that unions to be $X \times Y$. And they are disjoint. Hence $X \times Y$ is not connected.

$\Leftarrow$:

- Suppose $X \times Y$ can be written as union as 2 disjoint open sets.
- We want to show either X or Y isn't connected. Indeed we show Y is connected, then X cannot be connected.

$\rightarrow$ Call the 2 disjoint sets that are unioned to form $X \times Y$, U and V.

A visual preparation :



- It's clear how if Y is connected, U & V can only be written as $(U' \times Y)$ & $(V' \times Y)$ where $U', V' \in \mathcal{T}_X$ and $U' \cup V' = X$.
- Because if not, then there must be a point y_0, like the diagram on the left s.t. $X \times \{y_0\} = U'' \times \{y_0\} \cup V'' \times \{y_0\}$ under the subspace topology.
- However, U'' & V'' has to be open under subspace topology else U & V wouldn't be then, hence Y can't be connected.
- Thus we proved that $X \times Y = (U' \times Y) \cup (V' \times Y)$
  Therefore we know that $X = U' \cup V'$ $\rightarrow$ not connected. $\square$.

Theorem 8.12 : For spaces $\{X_\alpha\}_{\alpha \in \lambda}$, $\prod_{\alpha \in \lambda} X_\alpha$ is connected iff $\forall \alpha \in \lambda$, X_α connected.

Proof :

$\Rightarrow$:

- Some procedure as theorem 8.11. If $\exists \alpha^* \in \lambda$ s.t. $X_\alpha = U \cap V$.
- $U \times \prod_{\alpha \in \lambda \setminus \{\alpha^*\}} X_\alpha \cup V \times \prod_{\alpha \in \lambda \setminus \{\alpha^*\}} X_\alpha = \prod_{\alpha \in \lambda} X_\alpha$

$\rightarrow$ Hence the product space is not connected.

$\Leftarrow$:

1. Suppose $\prod_{\alpha\in\lambda} X_\alpha$ is not connected, then $\prod_{\alpha\in\lambda} X_\alpha = U \cap V$ for some disjoint open U, V in the product topology. Suppose by contradiction, all X_α is connected.

2. Let $\vec{x} \in U$ and $\vec{y} \in V$. Then exists basic open sets B_1, B_2 s.t. $\vec{x} \in B_1 \subset U$ and $\vec{y} \in B_2 \subset V$. Let coordinate of $\vec{x}$ at αth coordinate be x_α.

3. Suppose $B_2 = \prod_{\alpha\in\lambda} C_\alpha$ where $C_\alpha = X_\alpha$ for all but finitely α.

4. Thus $\exists\, \vec{y}' \in B_2$, coordinate $(\dots, y_\alpha', \dots)$ s.t. $y_\alpha' = x_\alpha$ for all by finitely many α. Call these coordinates $\{\alpha_1, \dots, \alpha_k\}$

5. Now consider the set $X_{\alpha_1} \times \prod_{\alpha\in\lambda\setminus\{\alpha_1\}}$ (the "line" in X_{α_1} coordinate). Since the line is restricted in all directions except for one. It's homeomorphic to X_{α_1} hence is also connected by assumption.

6. By theorem 8.4. the set $X_{\alpha_1} \setminus \prod_{\alpha\in\lambda\setminus\{\alpha_1\}}$ is either contained in U or V. Because $\vec{x} \in U$, $X_{\alpha_1} \setminus \prod_{\alpha\in\lambda\setminus\{\alpha_1\}} X_\alpha \subset U$

7. Hence we may "zigzag" in finitely many coordinates to find y'. Or:
   - $(y_{\alpha_1}', \dots, x_\alpha, \dots) \in U$ since the point $y' \in X_{\alpha_1} \setminus \prod_{\alpha\in\lambda\setminus\{\alpha_1\}} X_\alpha$
   - Similar as step 6, the "line" in X_{α_2}, $\{y_{\alpha_1}'\} \times X_{\alpha_2} \times \prod_{\alpha\in\lambda\setminus\{\alpha_1,\alpha_2\}} \{x_\alpha\} \in U$ is homeomorphic to X_{α_2} thus is connected. The line is also contained in U.
   - Hence $(y_{\alpha_1}', y_{\alpha_2}', \dots, x_\alpha, \dots) \in U$ and we repeat.

8. We repeat the process for every $(\alpha_1, \dots, \alpha_k)$, then $(y_{\alpha_1}, \dots, y_{\alpha_k}, \dots, x_\alpha, \dots)$ is contain in U. But the point is in B_2 and in B_1, thus $U \cap V \neq \emptyset$, contradiction. ☒.

(Note in the proof, we turn theorem 8.4 "disjoint separated sets" into "disjoint open set, which are essentially equivalent.)

Example: Box product of countably infinitely many copies of $\mathbb{R}_{std}$ is not connected.

Proof:

- The set of all sequences that's eventually zero is closed under box product, by the last example of chapter 3.
- The set of all such sequence is also open. Consider one of such sequence we can always find an open set in box product that contains it and contained in the set. The set is both open and closed ☒.

CH 8.3

Definition [Component & Connected component]: Let X be a space and $p \in X$. The component or connected component of p in X is the union of all connected subsets of X that contain p.

Theorem 8.18 : Each component of X is connected, closed and not contained in any strictly larger connected subset of X.

Proof :

- For some point $p \in X$. We consider the connected component that's constructed around p.
- The component is connected, since it's a union of connected sets with a point in intersect.
  - → In specific, if we want to split the set into 2 disjoint open sets, one has to contain p and one set doesn't, call them U & V
  - → The open set that doesn't contain p, must have at least 1 point and the connected set S that originally contain the point by construction contains p.
  - → Since the set is originally connected, $S \cap V$ can't be open. Thus V is not a union of open set, it's not open, contradiction.
- The component is a closed set :
  - → Suppose that there is a limit point of X that's not contained in the component.
  - → Then the closure of the component is a set strictly containing the component, and it's a connected set by theorem 8.6. It contradicts the fact we are going to prove next.
- Component is the maximal connected set in X that contains itself.
  - → If not, then union the bigger connected set to the component to make a bigger component, since $p \in$ the bigger connected set.
  - → Thus we know that the original component is not actually a component. ☒

Theorem: 8.19 The set of components of a space X is a partition of X.

Proof :

- Recall the definition of a partition P for a topological space X is

1. $\bigcup_{A \in P} A = X$ 2. $\forall A, B \in P$ s.t. $A \neq B$, $A \cap B = \phi$

3. Every $A \in P$ is non-empty.

1. · For any $x \in X$ there exists a component that contains it, since if not, then construct from the definition, and add it to the set.
   · Thus $\bigcup A$ covers X.

2. Suppose there are 2 non-disjoint component $C \cap D \neq \phi$.
   · Let $x \in C \cap D$. Since C and D are connected and share the point x, $C \cup D$ is connected.
   · But then $C \cup D$ strictly contain C & D, hence C & D are not actually component.

3. · For every x, $\{x\}$ is connected. The component constructed from x at least contain $\{x\}$, hence is non-empty. $\square$.

CH8.4

Definition [Path] : A path from x to y in space X is a continuous map $f: [0,1] \to X$ s.t. $f(0) = x$ $f(1) = y$. It's also called an arc.

Definition [Arcwise & Path Connected] : Every pair of points in X can be joined by a path in X.

Theorem 8.35 : Every path connected space X is connected.

Proof :

- We show that if a space X is not connected, then it's not path connected.
- Using theorem 8.1 (2), there is a continuous function $f: X \to \mathbb{R}std$ s.t. $f(X) = \{0,1\}$.
- We want to show there doesn't exist a path, or a continuous map $g: [0,1] \to X$ between 2 points in X.

→ Pick 2 point $a, b \in X$. $f(a) = 0$, $f(b) = 1$

→ Now restrict g so that $g(0) = a$, $g(1) = b$

- We know that $f \circ g : [0,1] \to \{0,1\}$ can't be continuous because $[0,1]$ is an interval, it's connected in $\mathbb{R}std$. (A lemma proved in theorem 8.10)

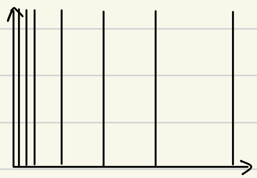
→ Thus by theorem 7.9, we know either f or g is discontinuous. Since f we know is continuous, we know g can't be continuous.

(We restrict a & b so that we ensure it the g wouldn't map to one point, or it will be continuous). ∎

Example :

Definition [Topologist's comb] : $C = \{(x, 0) \mid x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) \mid y \in [0,1]\}$

To visualize, the space looks like



Definition [Flea and comb space] : The flea and comb space is the union of the topologist's comb with the (0,1).

- The flea and comb space is connected but not path connected.

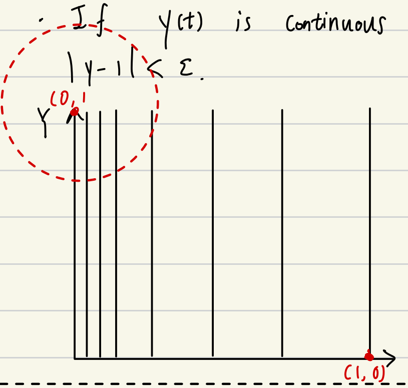
Proof :

- The set is connected is obvious, same reason we proved the topologist's sine curve. The closure of the entire set is the original set union $\{(0,x): x \in (0,1)\}$. And the proof follows using theorem 8.6.
- Next we prove why the flea and comb isn't path connected.
  Assume there is a path $\gamma: [0,1] \to C$ with $\gamma(0) = [0,1]$ and $\gamma(1) = (1,0)$
  $\gamma(t) = (x(t), y(t))$

→ $x(t)$ is continuous $x(0)=0$ $x(1)=1$
  $y(t)$ is continuous $y(0)=1$ $y(1)=0$

→ We want to show that the map can't be continuous. We show $x(t), y(t)$ can't be continuous at the same time.

- If $y(t)$ is continuous at 0, meaning $\forall \varepsilon > 0, \exists \delta$ s.t. $|t-0| < \delta$, then $|y-1| < \varepsilon$.



- We can find a maximum $t \in [0,1]$, where t can change from 0 to some finite number.
- At $t_0 >$ (the maximum time)
  $x(t_0) = \frac{1}{n}$ since it's confined away from the horizontal line.
- Therefore $x(t)$, $t \in [0, t_0]$ can't be connected.

Example: The closure of a topologist's sine curve is connected but not path connected.

Proof:

- We connect a path between $(0,0)$ and $(1, \sin(1))$
- $x(t)$ is continuous with $x(0) = 0$ and $x(1) = 1$.
  For $t > 0$, $x(t) > 0$ - since $x(t) \to 0$ when $t \to 0^+$
- For $t > 0$, $\gamma(t)$ lies on S, so $y(t) = \sin\left(\frac{1}{x(t)}\right)$
  $t \to 0^+$, $x(t) \to 0$ $\frac{1}{x(t)} \to \infty$
  which makes $\sin\frac{1}{x}$ oscillate between -1 & 1

→ But $y(t)$ must approach $y(0) = 0$ at $t \to 0^+$. But $\sin\left(\frac{1}{x}\right)$ doesn't approach any which violate requirement $y(t) \to 0$. $\gamma(t)$ can't exist. ☒

Theorem 8.38: The product of path connected space is path connected.

Proof:

- We do this for any product $\prod_{\alpha\in\lambda} X_\alpha$. We suppose every X_α is path connected.

The logic is essentially same as the proof for theorem 8.21.

- For any 2 points, say $\vec{a} = (a_i)_{i\in I}$ and $\vec{b} = (b_i)_{i\in I}$ be 2 points in X.

  For each $i \in I$, X_i is path connected, $\exists\, r_i : [0,1] \to X_i$ s.t.

  $$r_i(0) = a_i \qquad r_i(1) = b_i$$

→ Now construct $\Gamma : [0,1] \to X$ by defining $\Gamma(t) = (r_i(t))_{i\in I}$. $\Gamma(t)$ is the point in X whose ith coordinate is $r_i(t)$.

- Why Γ is continuous? Because using theorem 7.40: $g: Z \to \prod_{\alpha\in\lambda} X_\alpha$ is continuous iff $\pi_\beta \circ g$ is continuous for each $\beta \in \lambda$.

  Here we let $Z = [0,1]$ and $g = \Gamma$

→ $\pi_\beta \circ \Gamma(t) = r_i(t)$ is continuous by our construction thus $\Gamma(t)$ is continuous.

- When $t = 0$: $\Gamma(0) = (r_i(0))_{i\in I} = (a_i)_{i\in I} = \vec{a}$
- When $t = 1$: $\Gamma(1) = (r_i(1))_{i\in I} = (b_i)_{i\in I} = \vec{b}$. As needed. ☒.

Remark: The notion of path connectedness along us to determine what sets are homeomorphic & what is not.

Theorem 8.39: Suppose $f: X \to Y$ is a homeomorphism, the image of a connected component of X under f is a connected component of Y.

- A homeomorphism is a continuous bijection with a continuous inverse.

→ It's of course a surjection. And to be specific, every restriction of f is also a bijection. (*)

- By definition of a connected component: we can construct a connected component from a point p by unioning all the connected subsets of X containing p.

→ For every such connected set $C_\alpha \ni p$, $\alpha \in \lambda$

- Construct the connected component of $f(p)$ using the same method in Y. We claim the connected component is just the connected component in X constructed from p.
- For every connected sets D_β in Y, we may use the $f^{-1}(D_\alpha)$, since it's a continuous bijection. It's a connected set in X, by theorem 8.9, containing p. (As

$D_\beta \ni f(p)$). Thus $\exists \alpha \in \lambda$ s.t. $f(C_\alpha) = D_\beta$

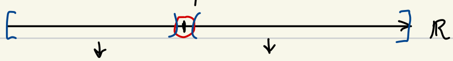
- Hence for every connected set in Y containing $f(p)$ (one of the open set union) it mapped back to a connected set in X.

→ The map applies the same way from X to Y, too. Thus $f(C_\alpha) = D_\beta$ in Y. So the mapping doesn't miss out anything in the process.

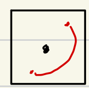
- Hence f(component constructed from p) = (Component constructed from $f(p)$) $\boxtimes$

Example: Using the theorem, we can distinguish homeomorphic sets, for example, $\mathbb{R}$ and $\mathbb{R}^2$.

- If they are homeomorphic, there has to exist a continuous bijection, due to the continuous inverse the connect components "stay together".
- However, if we remove a point from $\mathbb{R}$, it become 2 connected component.



Two connected components



→ If we remove a point from $\mathbb{R}^2$, the part left is still a connected component.

- But connected component can only be homeomorphic to one connected component, not 2. Thus we know that $\mathbb{R}$ & $\mathbb{R}^2$ are not homeomorphic. $\boxtimes$.
