MTH 391W Portfolio Management Paper

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December 2017

1 Introduction

This paper encapsulates the independent study done in conjunction to MTH 210: Introduction to Financial Mathematics. The structure of this paper is based on Chapter 3: Portfolio Management of the book *Mathematics for Finance: An Introduction to Financial Engineering, 2nd Edition* by Marek Capiński and Tomasz Zastawniak. The content of this chapter is rephrased in a concise manner, with more details and explanations supplemented to the proofs for the reader.

In this paper, we consider investment with risks. Such investment may have losses in contrast to risk-free investment.

2 Basic Definitions

Definition 2.1. *S(t) is the value of one share of stock at time t*

Definition 2.2. *V(t) is the value of a portfolio at time t*

Definition 2.3. $K(t) = \frac{S(t)-S(0)}{S(0)}$ *is the return of a stock from time* 0 *to* t*.*

Definition 2.4. $K_V(t) = \frac{V(t)-V(0)}{V(0)}$ *is the return of a portfolio from time* 0 *to* t*.*

3 Risk

The return on a risky investment is a random variable. The uncertainty can be understood as the spread of possible returns around some reference point. The usual reference value is the expected return $E[K]$. The possible spread of returns can be measured by standard deviation $\sigma_K = \sqrt{\mathrm{Var}(K)} = \sqrt{\mathrm{E}\left[(X - \mathrm{E}[K])^2\right]}$.

This notion of measurement describes two aspect of risk:

1. the distances between possible values and the reference point

2. the probabilities of reaching these values

In some circumstances, variance and standard deviation of the logarithmic return are also used as measures of risk.

4 Two Risky Securities

Consider risk and expected return in the simple situation of a portfolio with just two risky securities. Let x_1 and x_2 be the number of shares of stock 1 and 2 in a portfolio, respectively. If short selling is allowed, then x_1, x_2 may be negative.

The weights are defined by

$$w_1 = \frac{x_1 S_1(0)}{V(0)}, \quad w_2 = \frac{x_2 S_2(0)}{V(0)}$$

Here w_k represents the percentage of the initial portfolio value invested in security number k. Note that the weights always sum up to 1:

$$w_1 + w_2 = \frac{x_1 S_1(0) + x_2 S_2(0)}{V(0)} = \frac{V(0)}{V(0)} = 1 \tag{1}$$

If short selling is allowed, then one of the weights may be negative and the other one greater than 1. If no short selling is allowed, then $w_1, w_2 \in [0, 1]$. .

Proposition 4.1. *The return K_V on a portfolio consisting of two securities is the weighted average*

$$K_V = w_1 K_1 + w_2 K_2$$

where w_1 and w_2 are the weights and K_1 and K_2 the returns on the two components.

Proof. Suppose that the portfolio consists of x_1 shares of security 1 and x_2 shares of security 2. Then the initial and final values of the portfolio are

$$\begin{aligned}
V(0) &= x_1 S_1(0) + x_2 S_2(0) \\
V(1) &= x_1 S_1(0)(1 + K_1) + x_2 S_2(0)(1 + K_2) \\
&= V(0) \frac{x_1 S_1(0)}{V(0)} (1 + K_1) + V(0) \frac{x_2 S_2(0)}{V(0)} (1 + K_2) \\
&= V(0) \left[\frac{x_1 S_1(0)}{V(0)} (1 + K_1) + \frac{x_2 S_2(0)}{V(0)} (1 + K_2) \right] \\
&= V(0) \left[w_1(1 + K_1) + w_2(1 + K_2) \right]
\end{aligned}$$

Thus the return on the portfolio is

$$\begin{aligned}
K_V &= \frac{V(1) - V(0)}{V(0)} \\
&= \frac{V(0) \left[w_1(1 + K_1) + w_2(1 + K_2) - 1 \right]}{V(0)} \\
&= \frac{\cancel{V(0)} \left[w_1 K_1 + w_2 K_2 + \cancel{(w_1 + w_2)} - \cancel{1} \right]}{\cancel{V(0)}} \\
&= w_1 K_1 + w_2 K_2
\end{aligned}$$

□

4.1 Risk and Expected Return on a Portfolio

The expected return on a portfolio consisting of two securities can be expressed in terms of the weights and the respective expected returns, by linearity of expectation.

Theorem 4.1.

$$E(K_V) = w_1 E(K_1) + w_2 E(K_2)$$

Theorem 4.2. *The variance of the return on a portfolio is given by*

$$\operatorname{Var}(K_V) = w_1^2 \operatorname{Var}(K_1) + w_2^2 \operatorname{Var}(K_2) + 2 w_1 w_2 \operatorname{Cov}(K_1, K_2)$$

Proof.

$$K_V = w_1 K_1 + w_2 K_2$$

$$\begin{aligned}
\operatorname{Var}(K_V) &= E(K_V^2) - E(K_V)^2 \\
&= E[(w_1K_1 + w_2K_2)^2] - (E[w_1K_1 + w_2K_2])^2 \\
&= w_1^2 E(K_1^2) + w_2^2 E(K_2^2) + 2w_1w_2E(K_1K_2) - w_1^2E(K_1)^2 - w_2^2E(K_2)^2 - 2w_1w_2E(K_1)E(K_2) \\
&= w_1^2[E(K_1^2) - E(K_1)^2] + w_2^2[E(K_2^2) - E(K_2)^2] + 2w_1w_2[E(K_1K_2) - E(K_1)E(K_2)] \\
&= w_1^2 \operatorname{Var}(K_1) + w_2^2 \operatorname{Var}(K_2) + 2w_1w_2 \operatorname{Cov}(K_1, K_2)
\end{aligned}$$

□

To reduce the complexity of notations, we introduce the following notation for the expectation and variance of a portfolio and its components:

$$\mu_V = E(K_V), \quad \sigma_V = \sqrt{\operatorname{Var}(K_V)}$$

$$\mu_1 = E(K_1), \quad \sigma_1 = \sqrt{\operatorname{Var}(K_1)}$$

$$\mu_2 = E(K_2), \quad \sigma_2 = \sqrt{\operatorname{Var}(K_2)}$$

The correlation coefficient is then

$$\rho_{12} = \frac{\operatorname{Cov}(K_1, K_2)}{\sigma_1\sigma_2}.$$

Now Theorem 4.1 and 4.2 give

$$\mu_V = w_1\mu_1 + w_2\mu_2 \tag{2}$$

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \tag{3}$$

Proposition 4.2. *If short sales are not allowed, then the variance σ_V^2 of a portfolio cannot exceed the greater of the variances σ_1^2 and σ_2^2 of the components.*

$$\sigma_V^2 \le max\{\sigma_1^2, \sigma_2^2\}$$

Proof. Assume that $\sigma_1^2 \le \sigma_2^2$. If short sales are not allowed, then $w_1, w_2 \ge 0$ and

$$w_1\sigma_1 + w_2\sigma_2 \le (w_1 + w_2)\sigma_2 = \sigma_2.$$

Since the correlation coefficient satisfies $-1 \le \rho_{12} \le 1$, it follows that

$$\begin{aligned}
\sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \\
&\le w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 \\
&= (w_1\sigma_1 + w_2\sigma_2)^2 \le \sigma_2^2
\end{aligned}$$

If $\sigma_1^2 \ge \sigma_2^2$, the proof is analogous. □

Proposition 4.3. *If $\rho_{12} = 1$ and $\sigma_1 \ne \sigma_2$, then $\sigma_V = 0$ if and only if*

$$w_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}, \quad w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2} \tag{4}$$

(Short sales are necessary, since either w_1 and w_2 is negative.)

If $\rho_{12} = -1$, then $\sigma_V = 0$ if and only if

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2} \tag{5}$$

(No short sales are necessary, since both w_1 and w_2 are positive.)

Proof. Let $\rho_{12} = 1$. Then equation (3) takes the form

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2 = (w_1\sigma_1 + w_2\sigma_2)^2$$

and $\sigma_V^2 = 0$ if and only if $w_1\sigma_1 + w_2\sigma_2 = 0$.

This implies equations (4) because $w_1 + w_2 = 1$.

Now let $\rho_{12} = -1$. Then equation (3) takes the form

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\rho\sigma_1\sigma_2 = (w_1\sigma_1 - w_2\sigma_2)^2$$

and $\sigma_V^2 = 0$ if and only if $w_1\sigma_1 - w_2\sigma_2 = 0$.

This implies equation (5) because $w_1 + w_2 = 1$. □

Each portfolio can be represented by a point with coordinates σ_V and μ_V on the σ, μ plane. Figure 1 shows two typical lines representing portfolios with $\rho_{12} = -1$ (left) and $\rho_{12} = 1$ (right). The bold segments correspond to portfolios without short selling.

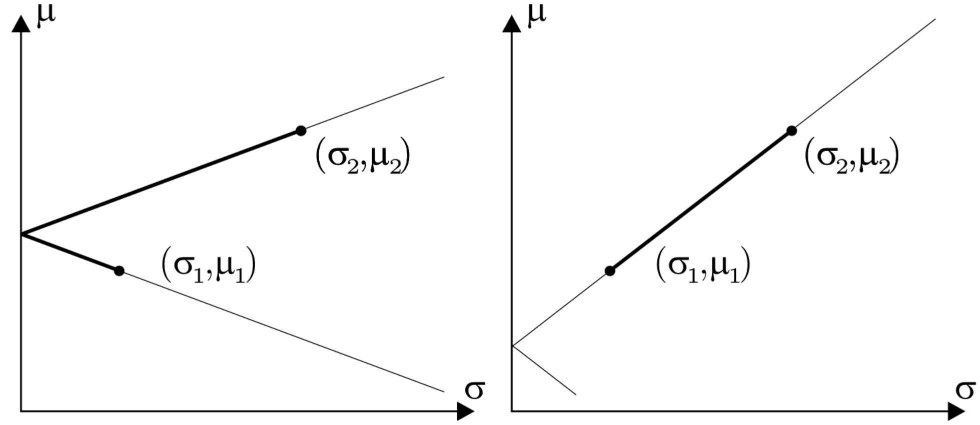

Figure 1: Typical portfolio lines with $\rho_{12} = -1$ (left) and $\rho_{12} = 1$ (right) [1]

If $\rho = -1$, by Proposition (4.3), we have $\sigma_V = |w_1\sigma_1 - w_2\sigma_2|$. In addition, $w_1 + w_2 = 1$ by equation (1) and $\mu_V = w_1\mu_1 + w_2\mu_2$ by equation (2). We can choose $s = w_1$ as a parameter. Then $1 - s = w_2$ and

$$\sigma_V = |s\sigma_1 - (1-s)\sigma_2|, \quad \mu_V = s\mu_1 + (1-s)\mu_2.$$

These parametric equations describe the line in Figure 1 with the bold line segment between (σ_1, μ_1) and (σ_2, μ_2).

If $\rho = 1$, by Proposition (4.3), we have $\sigma_V = |w_1\sigma_1 + w_2\sigma_2|$. Similarly, we can choose $s = w_1$ as a parameter and obtain

$$\sigma_V = |s\sigma_1 + (1-s)\sigma_2|, \quad \mu_V = s\mu_1 + (1-s)\mu_2.$$

These parametric equations describe the straight bold line segment between (σ_1, μ_1) and (σ_2, μ_2) in Figure 1.

To find a portfolio with minimum risk for any given ρ_{12} such that $-1 < \rho_{12} < 1$, we take $s = w_2$ as a parameter. Then equations (2) and (3) take the form

$$\mu_V = (1-s)\mu_1 + s\mu_2 \tag{6}$$
$$\sigma_V^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2. \tag{7}$$

Now we see μ_V as a function of s is a straight line and σ_V^2 is a quadratic function of s with a positive coefficient at s^2. Note that when $0 \le s \le 1$, there is no short selling.

Theorem 4.3. *For $-1 < \rho_{12} < 1$ the portfolio with minimum variance is attained at $s = s_0$, where*

$$s_0 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

If short sales are not allowed, then the smallest variance is attained at

$$s_{min} = \begin{cases} 0 & \text{if} \quad s_0 < 0 \\ s_0 & \text{if} \quad 0 \le s_0 \le 1 \\ 1 & \text{if} \quad 1 < s_0 \end{cases} \tag{8}$$

Proof. We compute the first derivative of σ_V^2 with respect to s and equate it to 0:

$$\begin{aligned} -2(1-s)\sigma_1^2 + 2s\sigma_2^2 + 2(1-s)\rho_{12}\sigma_1\sigma_2 - 2s\rho_{12}\sigma_1\sigma_2 &= 0 \\ -\cancel{2}\sigma_1^2 + \cancel{2}s\sigma_1^2 + \cancel{2}s\sigma_2^2 + \cancel{2}\rho_{12}\sigma_1\sigma_2 - \cancel{2}s\rho\sigma_1\sigma_2 - \cancel{2}s\rho_{12}\sigma_1\sigma_2 &= 0 \\ s(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2) - \sigma_1^2 + \rho_{12}\sigma_1\sigma_2 &= 0 \\ s &= \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} \end{aligned}$$

The above solution for s gives s_0.
While the second derivative is positive,

$$2\sigma_1^2 + 2\sigma_2^2 - 4\rho_{12}\sigma_1\sigma_2 > 2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \ge 0$$

which shows that there is minimum at s_0. It is a global minimum because σ_V^2 is a quadratic function of s.

If short sales are not allowed ($0 \le s \le 1$), we need to find the minimum for $0 \le s \le 1$.
If $s_0 < 0$, then the minimum is at 0.
If $0 \le s_0 \le 1$, then the minimum is at s_0.
If $s_0 > 1$, then the minimum is at 1.
The above cases are illustrated in Figure 2. The bold parts of the curve correspond to portfolios with no short selling.

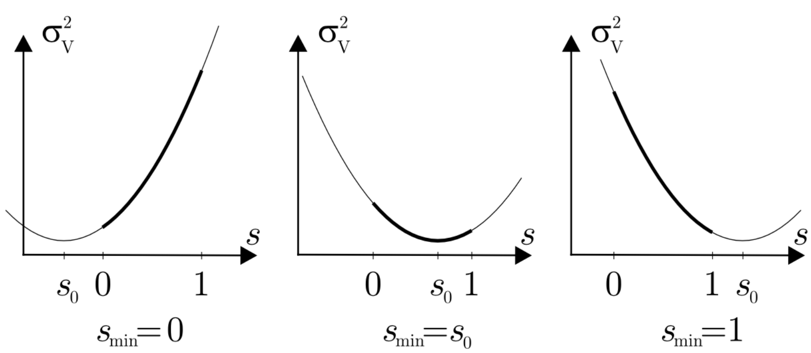

Figure 2: The minimum of σ_V^2 as a function of s [2]

□

The curve on the σ, μ plane defined by the parametric equations (6) and (7) represents all possible portfolios with given $\sigma_1, \sigma_2 > 0$ and $-1 \le \rho_{12} \le 1$. If short selling is allowed, the parameter s can be any real number. If short selling is not allowed ($0 \le s \le 1$), we only obtain a segment of the curve.

As s increases from 0 to 1, the corresponding point (σ_V, μ_V) travels along the curve in the direction from (σ_1, μ_1) to (σ_2, μ_2). Figure 3 shows two examples of such curves, with ρ_{12} close to but greater than -1 (left) and with ρ_{12} close to but smaller than 1 (right). In both cases, the bold line segments correspond to the portfolios without short selling.

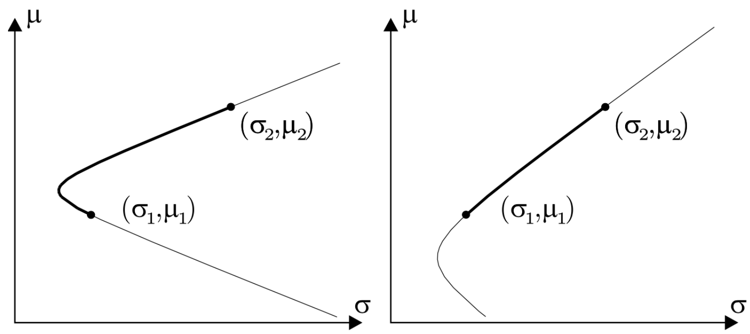

Figure 3: Typical portfolio lines with $-1 < \rho_{12} < 1$ [2]

Corollary 4.1. *Suppose that $\sigma_1 \leq \sigma_2$. The following three cases are possible:*

1. *If $-1 \leq \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then there is a portfolio without short selling such that $\sigma_V < \sigma_1$ (curves 4 and 5 in Figure 4)*
2. *If $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then $\sigma_V > \sigma_1$ for each portfolio (curves 3 in Figure 4)*
3. *If $\frac{\sigma_1}{\sigma_2} < \rho_{12} \leq 1$, then there is a portfolio with short selling such that $\sigma_V < \sigma_1$, but for each portfolio without short selling $\sigma_V \geq \sigma_1$ (curves 1 and 2 in Figure 4)*

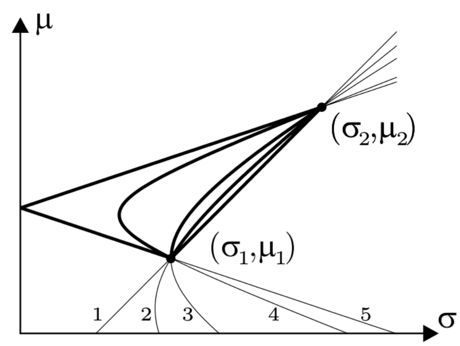

Figure 4: Portfolio lines for various values of ρ_{12} [1]

Proof. By equation (7) and Theorem 4.3, we have

$$\sigma_V^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2$$

$$s_0 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

1) If $-1 \leq \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then

$$[\sigma_1^2 - \rho_{12}\sigma_1\sigma_2] - [\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2] = \rho_{12}\sigma_1\sigma_2 - \sigma_2^2 < \sigma_1^2 - \sigma_2^2 \leq 0,$$

since $\sigma_1 \leq \sigma_2$. Therefore

$$[\sigma_1^2 - \rho_{12}\sigma_1\sigma_2] - [\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2] < 0$$

$$s_0 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} < 1.$$

In addition, the numerator of s_0

$$\sigma_1^2 - \rho_{12}\sigma_1\sigma_2 = \sigma_1\sigma_2(\frac{\sigma_1}{\sigma_2} - \rho_{12}) > 0$$

since $\rho_{12} < \frac{\sigma_1}{\sigma_2}$.

Therefore $0 < s_0 < 1$, which means there is no short selling of security 2 in the portfolio with with minimum variance.

On the other hand, since $\rho_{12} < \frac{\sigma_1}{\sigma_2} \leq 1$ $(\sigma_1 \leq \sigma_2)$, $\rho_{12} < 1$.

$$\begin{aligned}\sigma_V^2 &= (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2 \\ &< (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\sigma_1\sigma_2 = [(1-s)\sigma_1 + s\sigma_2]^2\end{aligned}$$

In addition, since $\sigma_V \geq 0$, $\sigma_2 \geq \sigma_1 \geq 0$, and for the portfolio with minimum variance, $s = s_0$, we have

$$\begin{aligned}\sigma_V &< [(1-s)\sigma_1 + s\sigma_2] \\ \sigma_{Vmin} &< [(1-s_0)\sigma_1 + s_0\sigma_2] \\ &< (1-s_0)\sigma_1 \\ &< \sigma_1\end{aligned}$$

Therefore there is a portfolio without short selling such that $\sigma_V < \sigma_1$.

2) If $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then

$$s_0 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} = \frac{\cancel{\sigma_1^2} - \cancel{\sigma_1^2}}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} = 0.$$

Since

$$\sigma_V^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2$$

We have

$$\begin{aligned}\sigma_{Vmin}^2 &= (1-s_0)^2\sigma_1^2 + s_0^2\sigma_2^2 + 2s_0(1-s_0)\rho_{12}\sigma_1\sigma_2 \\ &= \sigma_1^2\end{aligned}$$

Therefore $\sigma_V \geq \sigma_1$ for every portfolio because σ_1^2 is the minimum variance.

3) If $\frac{\sigma_1}{\sigma_2} < \rho_{12} \leq 1$, since

$$s_0 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

where the numerator

$$\sigma_1^2 - \rho_{12}\sigma_1\sigma_2 \leq \sigma_1^2 - \sigma_1\sigma_2 \leq 0$$

then $s_0 \leq 0$, which means there is short selling of security 2 in the portfolio with minimum variance.

Since $\rho_{12} < 1$, we also have

$$\begin{aligned}\sigma_V^2 &= (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2 \\ &< (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2s(1-s)\sigma_1\sigma_2 = [(1-s)\sigma_1 + s\sigma_2]^2\end{aligned}$$

In addition, since $\sigma_V \geq 0$, $\sigma_2 \geq \sigma_1 \geq 0$, and for the portfolio with minimum variance, $s = s_0$, we have

$$\begin{aligned}
\sigma_V &< [(1-s)\sigma_1 + s\sigma_2] \\
\sigma_{Vmin} &< [(1-s_0)\sigma_1 + s_0\sigma_2] \\
&< [(1-s_0)\sigma_1 + s_0\sigma_1] \\
&< \sigma_1
\end{aligned}$$

Therefore there is a portfolio with short selling such that $\sigma_V \leq \sigma_1$.
On the other hand, since $\sigma_V \geq 0$, $\sigma_1 \leq \sigma_2$ and $\frac{\sigma_1}{\sigma_2} < \rho_{12}$, we have

$$\begin{aligned}
\sigma_V^2 \geq \sigma_{Vmin}^2 &= (1-s_0)^2\sigma_1^2 + s_0^2\sigma_2^2 + 2s_0(1-s_0)\rho_{12}\sigma_1\sigma_2 \\
\sigma_V \geq \sigma_{Vmin} &= \sqrt{(1-s_0)^2\sigma_1^2 + s_0^2\sigma_2^2 + 2s_0(1-s_0)\rho_{12}\sigma_1\sigma_2} \\
&> \sqrt{(1-s_0)^2\sigma_1^2 + s_0^2\sigma_1^2 + 2s_0(1-s_0)\sigma_1^2} = \sqrt{(1-\cancel{2s_0}+\cancel{s_0^2+s_0^2}+\cancel{2s_0}-\cancel{2s_0^2})^2}\sigma_1 = \sigma_1
\end{aligned}$$

Therefore for $s \geq s_0$, $\sigma_V > \sigma_1$ for every portfolio without short selling. □

Proposition 4.4. *Consider a portfolio consisting of a risky security with expected return μ_1 and standard deviation $\sigma_1 > 0$, and a risky-free security with return r_F and standard deviation zero, the standard deviation σ_V of the portfolio depends on the weight w_1 of the risky security as follows:*

$$\sigma_V = |w_1|\sigma_1$$

Proof. Let $\sigma_1 > 0$ and $\sigma_2 = 0$. Then Equation (3) reduces to $\sigma_V^2 = w_1^2\sigma_1^2$. Therefore $\sigma_V = |w_1|\sigma_1$.

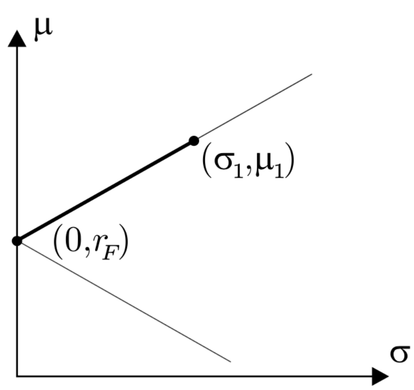

Figure 5: Portfolio line for one risky and one risk-free security [2]

□

5 Several Securities

5.1 Risk and Expected Return on a Portfolio

A portfolio constructed from n different securities can be described in terms of their weights

$$w_i = \frac{x_i S_i(0)}{V(0)}, \quad i = 1, ..., n,$$

where x_i is the numeber of shares of type i in the portfolio, $S_i(0)$ is the initial price of security i, and $V(0)$ is the amount initially invested in the portfolio.

We arrange the weights as a one-row matrix

$$\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & ... & w_n \end{bmatrix}$$

Since the weights add up to one, we have

$$1 = \boldsymbol{u}\boldsymbol{w}^T \tag{9}$$

where

$$\boldsymbol{u} = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}$$

is a one-row matrix with all n entries equal to 1, $\boldsymbol{w}^T$ is a one-column matrix, the transpose of $\boldsymbol{w}$.

Definition 5.1. *A portfolio with weights $\boldsymbol{w}$ satisfying equation (9) is called an* ***attainable portfolio****. The set of all attainable portfolios is called the* ***attainable set****.*

Suppose that the returns on the securities are $K_1, .., K_n$. The expected returns $\mu_i = E(K_i)$ for $i = 1, .., n$ will also be arranged into a one-row matrix

$$\boldsymbol{m} = \begin{bmatrix} \mu_1 & \mu_2 & ... & \mu_n. \end{bmatrix}$$

The covariance between returns will be denoted by $C_{ij} = \mathrm{Cov}(K_i, K_j)$. They are the entries of the $n \times n$ *covariance matrix.*

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

The covariance matrix is symmetric and non-negative definite. The diagonal elements are simply the variance of returns, $C_{ii} = \mathrm{Var}(K_i)$. In addition, we assume that $\det \boldsymbol{C} \neq 0$ so that $\boldsymbol{C}$ has an inverse $\boldsymbol{C}^{-1}$.

Proposition 5.1. *The expected return $\mu_V = E(K_V)$ and variance $\sigma_V^2 = \mathrm{Var}(K_V)$ of a portfolio with weights $\boldsymbol{w}$ are given by*

$$\mu_V = \boldsymbol{m}\boldsymbol{w}^T \tag{10}$$

$$\sigma_V^2 = \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T. \tag{11}$$

Proof.

$$\mu_V = E(K_V) = E(\sum_{i=1}^n w_i K_i) = \sum_{i=1}^n w_i \mu_i = \boldsymbol{m}\boldsymbol{w}^T$$

$$\begin{aligned} \sigma_V^2 &= \mathrm{Var}(K_V) = \mathrm{Var}(\sum_{i=1}^n w_i K_i) \\ &= \mathrm{Cov}(\sum_{i=1}^n w_i K_i, \sum_{i=1}^n w_j K_j) = \sum_{i,j=1}^n w_i w_j c_{ij} \\ &= \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T \end{aligned}$$

□

5.2 Minimum Variance Portfolio

Definition 5.2. *The portfolio with the smallest variance in the attainable set is called the **minimum variance portfolio**.*

Proposition 5.2. *The minimum variance portfolio has weights*

$$\boldsymbol{w} = \frac{\boldsymbol{u}\boldsymbol{C}^{-1}}{\boldsymbol{u}\boldsymbol{C}^{-\mathbf{1}}\boldsymbol{u}^T}$$

provided that the denominator is non-zero.

Proof. We need to find the minimum of equation (11) subject to the constraint of equation (9). Using the method Lagrange multiplier, we have

$$F(\boldsymbol{w}, \lambda) = \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T - \lambda \boldsymbol{u}\boldsymbol{w}^T,$$

where λ is a Lagrange multiplier.

By setting $\nabla_w F(\boldsymbol{w}, \lambda) = 0$, we obtain $2\boldsymbol{w}\boldsymbol{C} - \lambda \boldsymbol{u} = 0$. Therefore

$$\boldsymbol{w} = \frac{\lambda}{2}\boldsymbol{u}\boldsymbol{C}^{-1}, \tag{12}$$

which is a necessary condition for a minimum. Substituting this into equation (9) we have

$$1 = \boldsymbol{u}\boldsymbol{w}^T = \boldsymbol{u}\left[\frac{\lambda}{2}\boldsymbol{C}^{-\mathbf{1}}\boldsymbol{u}^T\right] = \frac{\lambda}{2}\boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T$$

where we use the fact that $\boldsymbol{C}^{-1}$ is a symmetric matrix because $\boldsymbol{C}$ is. Therefore

$$\frac{\lambda}{2} = \frac{1}{\boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T}.$$

By substitute this result into equation (12), we have the asserted formula. □

5.3 Minimum Variance Portfolios with Prescribed Expected Return

Consider the set of all attainable portfolios having a prescribed expected return. We now study the problem of finding the portfolios in this set having minimal variance.

Proposition 5.3. *For each $\mu_V \in \mathbb{R}$, the minimum variance portfolios with expected return μ_V are precisely the portfolios with weights*

$$\boldsymbol{w} = \frac{\lambda_1}{2}\boldsymbol{u}\boldsymbol{C}^{-1} + \frac{\lambda_2}{2}\boldsymbol{m}\boldsymbol{C}^{-1}$$

where λ_1 and λ_2 are solutions of the following system of linear equations:

$$\frac{1}{2}M\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}\mu_V\\ 1\end{bmatrix}, \quad M = \begin{bmatrix}\boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{m}^T & \boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{m}^T\\ \boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{u}^T & \boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T\end{bmatrix}.$$

Proof. Here we need to find the minimum of equation (11) subject to two constraints equation (9) and equation (10). We take

$$G(\boldsymbol{w}, \lambda_1, \lambda_2) = \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T - \lambda_1\boldsymbol{u}\boldsymbol{w}^T - \lambda_2\boldsymbol{m}\boldsymbol{w}^T$$

where λ_1 and λ_2 are Lagrange multipliers.

By setting $\nabla_w G(\boldsymbol{w}, \lambda_1, \lambda_2) = 0$, and we obtain $2\boldsymbol{w}\boldsymbol{C} - \lambda_1\boldsymbol{u} - \lambda_2\boldsymbol{m} = 0$, which implies that

$$\boldsymbol{w} = \frac{\lambda_1}{2}\boldsymbol{u}\boldsymbol{C}^{-1} + \frac{\lambda_2}{2}\boldsymbol{m}\boldsymbol{C}^{-1} \tag{13}$$

Substituting this into the constraints equation (9) and equation (10), we obtain the system of linear equations

$$1 = \boldsymbol{u}\left[\frac{\lambda_1}{2}\boldsymbol{C}^{-1}\boldsymbol{u}^T + \frac{\lambda_2}{2}\boldsymbol{C}^{-1}\boldsymbol{m}^T\right] = \frac{\lambda_1}{2}\boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T + \frac{\lambda_2}{2}\boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{m}^T$$
$$\mu_V = \boldsymbol{m}\left[\frac{\lambda_1}{2}\boldsymbol{C}^{-1}\boldsymbol{u}^T + \frac{\lambda_2}{2}\boldsymbol{C}^{-1}\boldsymbol{m}^T\right] = \frac{\lambda_1}{2}\boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{u}^T + \frac{\lambda_2}{2}\boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{m}^T.$$

This is precisely the system in the statement of the proposition. □

5.4 Minimum Variance Line

Consider the matrix

$$M = \begin{bmatrix} \boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{m}^T & \boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{m}^T \\ \boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{u}^T & \boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T \end{bmatrix}$$

appearing in Proposition 5.3. For the rest of this study, we assume M is invertible. Proposition 5.3 immediately gives

Proposition 5.4. *For each $\mu_V \in \mathbb{R}$, there is a unique minimum variance portfolio with expected return μ_V, and it has weights*

$$\boldsymbol{w} = \frac{\lambda_1}{2}\boldsymbol{u}\boldsymbol{C}^{-1} + \frac{\lambda_2}{2}\boldsymbol{m}\boldsymbol{C}^{-1}$$

where

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2M^{-1}\begin{bmatrix} \mu_V \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} \boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{m}^T & \boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{m}^T \\ \boldsymbol{m}\boldsymbol{C}^{-1}\boldsymbol{u}^T & \boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T \end{bmatrix}$$

Note the weights depend linearly on μ_V.

Definition 5.3. *For each $\mu \in \mathbb{R}$, let V_μ be the unique minimum variance portfolio with expected return μ. Define the set*

$$L = \{V_\mu : \mu \in \mathbb{R}\}.$$

*The set L is called the **minimum variance line**.*

The next theorem says that the set of weights of portfolios in L is a line in $\mathbb{R}^n$. This explains why L is called the minimum variance line.

Theorem 5.1. *Let $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ be the weights of any two different portfolios V_1, V_2 in L with different expected returns $\mu_{V1} \neq \mu_{V2}$. The weights $\boldsymbol{w}$ of each portfolio V in L can be obtained as a linear combination of the weights of these two different portfolios:*

$$\boldsymbol{w} = \alpha\boldsymbol{w}_1 + (1-\alpha)\boldsymbol{w}_2$$

for some $\alpha \in \mathbb{R}$.

Proof. For each portfolio V in L, given any two different portfolios V_1, V_2 in L with different expected returns $\mu_{V1} \neq \mu_{V2}$, we can find α such that

$$\mu_V = \alpha\mu_{V1} + (1-\alpha)\mu_{V2}$$

Since V_1 and V_2 belong to L, by Proposition 5.3 they satisfy the following linear forms

$$\boldsymbol{w_1} = \mu_{V_1}\boldsymbol{a} + \boldsymbol{b}, \quad \boldsymbol{w_2} = \mu_{V_2}\boldsymbol{a} + \boldsymbol{b}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are matrices.

We can then obtain

$$\begin{aligned}
\boldsymbol{w} &= \mu_V \boldsymbol{a} + \boldsymbol{b} = [\alpha\mu_{V1} + (1-\alpha)\mu_{V2}]\boldsymbol{a} + \boldsymbol{b} = \alpha\mu_{V1}\boldsymbol{a} + (1-\alpha)\mu_{V2}\boldsymbol{a} + \boldsymbol{b} \\
&= \alpha\mu_{V1}\boldsymbol{a} + \alpha\boldsymbol{b} + (1-\alpha)\mu_{V2}\boldsymbol{a} + \boldsymbol{b} - \alpha\boldsymbol{b} \\
&= \alpha(\mu_{V1}\boldsymbol{a} + \boldsymbol{b}) + (1-\alpha)\mu_{V2}\boldsymbol{a} + (1-\alpha)\boldsymbol{b} \\
&= \alpha(\mu_{V1}\boldsymbol{a} + \boldsymbol{b}) + (1-\alpha)(\mu_{V2}\boldsymbol{a} + \boldsymbol{b}) \\
&= \alpha\boldsymbol{w_1} + (1-\alpha)\boldsymbol{w_2}
\end{aligned}$$

□

5.5 Efficient Frontier

Given the choice between two securities a rational investor will, if possible, choose the security with higher expected return and lower risk.

Definition 5.4. *We say that a security with expected return μ_1 and standard deviation σ_1 dominates another security with expected return μ_2 and deviation σ_2 whenever*

$$\mu_1 \geq \mu_2 \quad \textit{and} \quad \sigma_1 \leq \sigma_2$$

Definition 5.5. *A portfolio is called efficient if there is no other portfolio, except itself, that dominates it. The set of efficient portfolios among all attainable portfolios is called the* ***efficient frontier****.*

We assume that every rational investor will choose an efficient portfolio, always preferring a dominating portfolio to a dominated one. However, different investors may select different portfolios on the efficient frontier, depending on their individual preferences.

5.6 Market Portfolio

We now assume that a risk-free security is available in addition to n risky securities. The return on the risk-free security will be denoted by r_F. The standard deviation for the risk-free security is zero.

Consider a portfolio consisting of the risk-free security and a specified risky security (possibly a portfolio of risky securities) with expected return μ_1 and standard deviation $\sigma_1 > 0$. By Proposition 4.4, all such portfolios form a V-shaped curve consisting of two broken half-lines meeting at the point $(0, r_F)$ on the σ, μ plane, as illustrated in Figure 6.

Therefore by taking portfolios consisting of the risk-free security and a security with σ_1, μ_1 anywhere in the attainable set (represented by shaded area in Figure 6), we can construct any portfolio between the solid two half-lines meeting at the point $(0, r_F)$. The area between these two solid half-lines is now a new set of attainable portfolios which include the risk-free security.

The efficient frontier of this new set of portfolios, which may include the risk-free security, is the bold half-line passing through $(0, r_F)$ and tangent to the hyperbola which corresponds to the minimum variance line constructed from risky securities. According to our assumptions, every rational investor will select his or her portfolio on this bold half-line.

Definition 5.6. *The bold-half line described above is called the* ***capital market line (CML)****.*

Note that the above argument works as long as the risk-free return r_F is not too high, so the upper half-line is tangent to a bullet (i.e. an attainable portfolio). (If r_F is too high, then the tangency is not possible.)

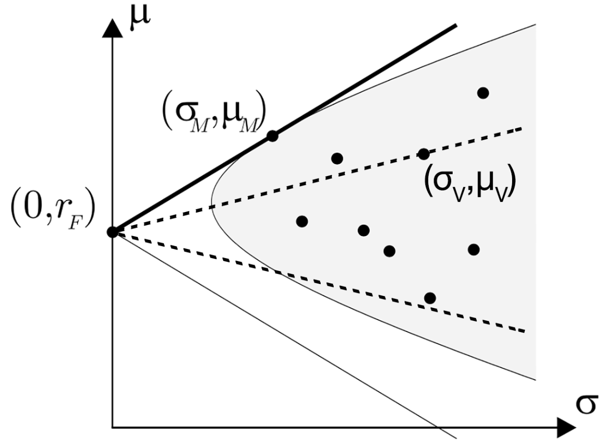

Figure 6: Efficient frontier for portfolios with a risk-free security [1]

Definition 5.7. *The portfolio M corresponding to the tangency point (σ_M, μ_M) is called the market portfolio.*

To compute the weights in the market portfolio, we observe that the straight line passing through $(0, r_F)$ and (σ_M, μ_M) has the steepest slope among all lines through $(0, r_F)$ and (σ_M, μ_M) for any portfolio V.

Proposition 5.5. *The weights $\boldsymbol{w}$ of the market portfolio in the attainable set constructed from risky assets satisfy the condition*

$$\gamma \boldsymbol{w}\boldsymbol{C} = \boldsymbol{m} - \delta \boldsymbol{u} \tag{14}$$

for some real numbers $\gamma > 0$ and δ.

Proof. The slope of the line passing through $(0, r_F)$ and (σ_V, μ_V) for any portfolio V in the feasible set constructed from risky assets can be written as

$$\frac{\mu_V - r_F}{\sigma_V} = \frac{\boldsymbol{m}\boldsymbol{w}^T - r_F}{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}}.$$

where $\boldsymbol{w}$ are the weights in portfolio V.

The maximum is to be taken over all weights $\boldsymbol{w}$ subject to the constraint $\boldsymbol{u}\boldsymbol{w}^T = 1$. Using the Lagrange multiplier method, we have

$$F(\boldsymbol{w}, \lambda) = \frac{\boldsymbol{m}\boldsymbol{w}^T - r_F}{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}} - \lambda \boldsymbol{u}\boldsymbol{w}^T$$

where λ is a Lagrange multiplier.

By setting $\nabla_w F(\boldsymbol{w}, \lambda) = 0$, we have

$$\nabla_w F(\boldsymbol{w}, \lambda) = \frac{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}\boldsymbol{m} - \frac{\boldsymbol{w}\boldsymbol{m}^T - r_F}{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}}\boldsymbol{w}\boldsymbol{C}}{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T} - \lambda \boldsymbol{u} = 0$$

$$\frac{\boldsymbol{m} - \frac{\boldsymbol{w}\boldsymbol{m}^T - r_F}{(\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T})^2}\boldsymbol{w}\boldsymbol{C}}{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}} - \lambda \boldsymbol{u} = 0$$

$$\frac{\boldsymbol{m} - \frac{\boldsymbol{w}\boldsymbol{m}^T - r_F}{(\sigma_V)^2}\boldsymbol{w}\boldsymbol{C}}{\sigma_V} - \lambda \boldsymbol{u} = 0.$$

Since $\sigma_V = \sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}$, this gives

$$\frac{\mu_V - r_F}{\sigma_V^2}\boldsymbol{w}\boldsymbol{C} = \boldsymbol{m} - \lambda\sigma_V \boldsymbol{u}$$

Evidently, to obtain (14) we should take $\gamma = \frac{\mu_V - r_F}{\sigma_V^2}$ and $\delta = \lambda\sigma_V$.

Because the tangent line has positive slope, we have $\mu_V > r_F$, which implies $\gamma > 0$. To show that δ actually exists, it remains to find λ.

Multiplying by $\boldsymbol{w}^T$ on the right and using that $\boldsymbol{u}\boldsymbol{w}^T = 1$, $\boldsymbol{m}\boldsymbol{w}^T = \mu_V$, and $\sigma_V = \sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}$, we find

$$\begin{aligned}
\boldsymbol{m}\boldsymbol{w}^T - \lambda\sigma_V\boldsymbol{u}\boldsymbol{w}^T &= \frac{\mu_V - r_F}{\cancel{\sigma_V^2}}\cancel{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T} \\
\mu_V - \lambda\sigma_V &= \mu_V - r_F \\
\cancel{\mu_V} - \lambda\sigma_V &= \cancel{\mu_V} - r_F \\
\lambda &= \frac{r_F}{\sigma_V}
\end{aligned}$$

Thus $\delta = \lambda\sigma_V = r_F$. □

The capital market line (CML) joining the risk-free security $(0, r_F)$ and the market portfolio (σ_M, μ_M) is described by

$$\mu = r_F + \frac{\mu_M - r_F}{\sigma_M}\sigma, \quad \sigma \geq 0 \tag{15}$$

Definition 5.8. *For a portfolio on the capital market line with risk σ, the term*

$$\frac{\mu_M - r_F}{\sigma_M}\sigma$$

is called the ***risk premium****.*

The risk-free premium $\frac{\mu_M - r_F}{\sigma_M}\sigma$ is an additional return above the risk-free level. It provides compensation for exposure to risk.

6 Capital Asset Pricing Model (CAPM)

6.1 Beta Factor

In investment, is it important to understand how the return K_V on a security or a portfolio will react to trends affecting the whole market.

We can plot the return K_V on a given security or portfolio for each market scenario vs. the return K_M on the market portfolio and compute the line of best fit, as illustrated in Figure 7. The equation of the line of best fit will be

$$K_V = \beta_V K_M + \alpha_V$$

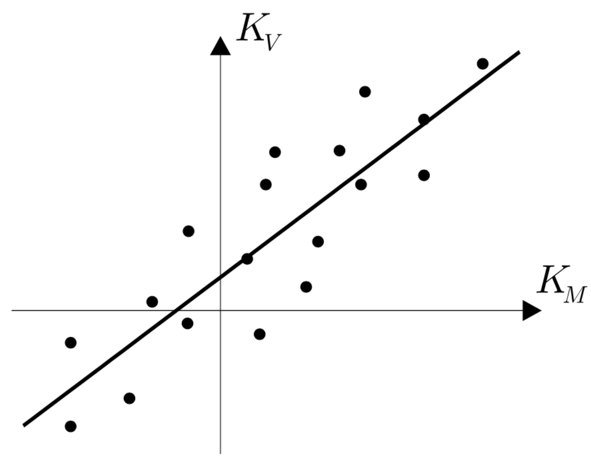

Figure 7: Line of best fit [2]

The values of $\alpha + \beta K_M$ can be viewed as predictions for the return on the given portfolio, for any given β and α.

Definition 6.1. *The difference between the actual return and the predicted return $\epsilon = K_V - (\alpha + \beta K_M)$ is called the* ***residual random variable****.*

The line of best fit requires that

$$\begin{aligned} E(\epsilon^2) &= E(K_V^2 + (\alpha - 2K_V(\alpha + \beta K_M) + \beta K_M^2)) \\ &= E(K_V^2 - 2K_V(\alpha + \beta K_M) + \alpha^2 - 2\alpha\beta K_M + \beta^2 K_M^2) \\ &= E(K_V^2 - 2\beta K_V K_M + \beta^2 K_M^2 + \alpha^2 - 2\alpha K_V - 2\alpha\beta K_M) \\ &= E(K_V^2) - 2\beta E(K_V K_M) + \beta^2 E(K_M^2) + \alpha^2 - 2\alpha E(K_V) + 2\alpha\beta E(K_M) \end{aligned}$$

attains its minimum at $\beta = \beta_V$ and $\alpha = \alpha_V$.
To minimize $E(\epsilon^2)$, we take the partial derivatives with respect to β and α and equate them to zero:

$$\begin{aligned} \frac{\partial E(\epsilon^2)}{\partial \beta} &= 0 \\ -2E(K_V K_M) = 2\beta E(K_M^2) + 2\alpha E(K_M) &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial E(\epsilon^2)}{\partial \alpha} &= 0 \\ 2\alpha - 2E(K_V) + 2\beta E(K_M) &= 0 \end{aligned}$$

We then have a system of linear equations:

$$\begin{aligned} \alpha_V E(K_M) + \beta_V E(K_M^2) &= E(K_V K_M) \\ \alpha_V + \beta_V E(K_M) &= E(K_V) \end{aligned}$$

The solutions for gradient β_V and intercept α_V of the line of best fit are the follows:

$$\beta_V = \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2}, \quad \alpha_V = \mu_V - \beta_V \mu_M$$

where $\mu_V = E(K_V)$, $\mu_M = E(K_M)$ and $\sigma_M^2 = \mathrm{Var}(K_M)$.

Definition 6.2. *We call*

$$\beta_V = \frac{\operatorname{Cov}(K_V, K_M)}{\sigma_M^2}$$

the ***beta factor*** *of the given portfolio or individual security.*

Since $\mu_V = \beta_V \beta_M + \alpha_V$, the return on a security with a positive beta factor tends to increase as the return on the market portfolio increases, while the return on a security with a negative beta factor tends to increase if the return on the market portfolio goes down.

The risk $\sigma_V^2 = \operatorname{Var}(K_V)$ of a security or portfolio can be written as

$$\sigma_V^2 = \operatorname{Var}(\epsilon_V) + \beta_V^2 \sigma_M^2$$

Proof.

$$\begin{aligned}
\sigma_V^2 &= \operatorname{Var}(K_V) \\
&= \operatorname{Var}(\epsilon_V + (\alpha_V + \beta_V K_M)) \\
&= \operatorname{Var}(\epsilon_V) + \operatorname{Var}(\alpha_V + \beta_V K_M) + \operatorname{Cov}(\epsilon_V, \alpha_V + \beta_V K_M) \\
&= \operatorname{Var}(\epsilon_V) + \cancel{\operatorname{Var}(\alpha_V)} + \operatorname{Var}(\beta_V K_M) + \cancel{\operatorname{Cov}(\alpha_V, \beta_V K_M)} + \cancel{\operatorname{Cov}(\epsilon_V, \alpha_V + \beta_V K_M)} \\
&= \operatorname{Var}(\epsilon_V) + \beta_V^2 \operatorname{Var}(K_M) \\
&= \operatorname{Var}(\epsilon_V) + \beta_V^2 \sigma_M^2
\end{aligned}$$

All terms cancelled above are 0, and are proved individually as the follows.

$$\operatorname{Var}(\alpha_V) = \mathrm{E}\left[(\alpha_V - \mathrm{E}(\alpha_V))^2\right] = \mathrm{E}\left[(\alpha_V - \alpha_V)^2\right] = E[0] = 0$$

$$\operatorname{Cov}(\alpha_V, \beta_V \sigma_M) = \mathrm{E}\left[(\alpha_V - \mathrm{E}[\alpha_V])(\beta_V \sigma_M - \mathrm{E}[\beta_V \sigma_M])\right] = \mathrm{E}\left[(\cancel{\alpha_V - \alpha_V})(\beta_V \sigma_M - \mathrm{E}[\beta_V \sigma_M])\right] = E[0] = 0$$

Since $\alpha_V + \beta_V K_M$ is an estimator of K_V, we can write $\alpha_V + \beta_V K_M$ as $\hat{K_V}$ for convenience. Then we have

$$\begin{aligned}
\operatorname{Cov}(\epsilon_V, \alpha_V + \beta_V K_M) &= \operatorname{Cov}(\epsilon_V, \hat{K_V}) \\
&= \operatorname{Cov}((\boldsymbol{I} - \boldsymbol{H}) K_V, \boldsymbol{H} K_V) \\
&= (\boldsymbol{I} - \boldsymbol{H}) \operatorname{Cov}(K_V, K_V) \boldsymbol{H}^T \\
&= (\boldsymbol{I} - \boldsymbol{H}) \sigma_{K_V}^2 \boldsymbol{I} (\boldsymbol{H}^T) \\
&= \sigma_{K_V}^2 (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{H}^T \\
&= \sigma_{K_V}^2 (\boldsymbol{H}^T - \boldsymbol{H}\boldsymbol{H}^T) \\
&= \sigma_{K_V}^2 (\boldsymbol{H} - \boldsymbol{H}) \\
&= 0
\end{aligned}$$

Note that $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T$ is the hat matrix where $\boldsymbol{X} = [K_M]$. The properties $\hat{K_V} = \boldsymbol{H} K_V$, $\boldsymbol{H}^T = \boldsymbol{H}$ and $\boldsymbol{H}\boldsymbol{H}^T = \boldsymbol{H}$ are used [3].

□

The first term $\operatorname{Var}(\epsilon_V)$ is called the residual variance or diversifiable risk. As for a market portfolio, $\operatorname{Var}(\epsilon_M) = 0$. This part of risk can 'diversified away' by investing in the market portfolio. The second term $\beta_V^2 \sigma_M^2$ is called the systematic or undiversifiable risk. Therefore, the market portfolio involves only systematic risk.

The beta factor β_V to some extent measures systematic risk associated with a security or portfolio. The higher the systematic risk, the higher the return expected by investors as a premium for taking this kind of risk. However, diversifiable risk gives no additional premium, having no effect on the expected return $\mu_V = E[K_V]$. This is because diversifiable risk can be eliminated by investing in many securities, and particularly by investing in the market portfolio.

6.2 Security Market Line

Consider an arbitrary portfolio with weights $\boldsymbol{w}_V$. Denote $\boldsymbol{w}_M$ as the weights in the market portfolio. The market portfolio belongs to the efficient frontier of the attainable set of portfolios consisting of risky securities. Thus, by Proposition 5.5

$$\gamma \boldsymbol{w}_M \boldsymbol{C} = \boldsymbol{m} - \delta \boldsymbol{u}$$
$$\boldsymbol{w}_M \boldsymbol{C} = \frac{\boldsymbol{m} - \delta \boldsymbol{u}}{\gamma}$$

for some numbers $\gamma > 0$ and δ. The beta factor of the portfolio with weights $\boldsymbol{w}_V$ can, therefore, be written as

$$\beta_V = \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2} = \frac{\boldsymbol{w}_M \boldsymbol{C} \boldsymbol{w}_V^T}{\boldsymbol{w}_M \boldsymbol{C} \boldsymbol{w}_M^T} = \frac{\gamma(\boldsymbol{m} - \delta \boldsymbol{u})\boldsymbol{w}_V^T}{\gamma(\boldsymbol{m} - \delta \boldsymbol{u})\boldsymbol{w}_M^T} = \frac{\boldsymbol{m}\boldsymbol{w}_V^T - \delta \boldsymbol{u}\boldsymbol{w}_V^T}{\boldsymbol{m}\boldsymbol{w}_M^T - \delta \boldsymbol{u}\boldsymbol{w}_M^T} = \frac{\mu_V - \delta}{\mu_M - \delta} \tag{16}$$

where $\boldsymbol{m}\boldsymbol{w}_V^T = \mu_V$, $\boldsymbol{m}\boldsymbol{w}_M^T = \mu_M$, $\boldsymbol{u}\boldsymbol{w}_V^T = 1$, and $\boldsymbol{u}\boldsymbol{w}_M^T = 1$ by equations (9) and (10).

To find δ consider the risk-free security, with return r_F and beta factor $\beta_F = 0$. We Substitute β_F and r_F for β_V and μ_V in equation (16) and obtain $r_F - \delta = 0$, which implies $\delta = r_F$ and leads to the following theorem.

Theorem 6.1. *The expected return μ_V on a portfolio (or an individual security) is a linear function of the beta coefficient β_V of the portfolio,*

$$\mu_V = r_F + (\mu_M - r_F)\beta_V \tag{17}$$

Definition 6.3. *The straight line (on the β, δ plane) formed by plotting the expected return against the beta coefficient of any portfolio or individual security is called the **security market line**.*

The security market line is plotted to the right of the capital market line for comparison in Figure 8. Different portfolios and individual securities are indicated by dots in both graphs.

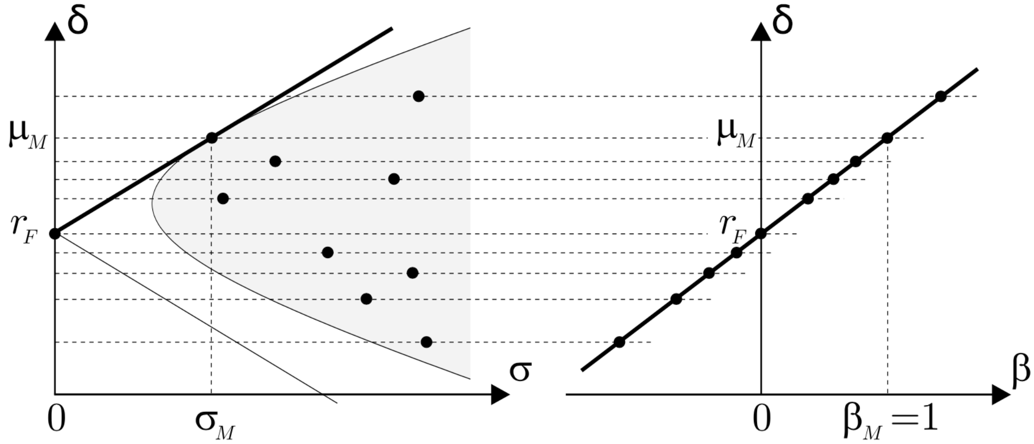

Figure 8: Capital market line and security market line [2]

Similarly as in equation (15) for the capital market line, the term $(\mu_M - r_F)\beta_V$ in equation (17) is the *risk premium*, for taking the systematic risk. However, equation (15) applies only to portfolios on the capital market line, whereas equation (17) applies to all portfolios and individual securities.

The market will remain at equilibrium so as long as the estimates of expected returns and beta factors satisfy equation (17). However, any new information about the security V may affect the estimated expected returns and beta factors. As a result, the new estimated values may not satisfy equation (17) any more.

For example, if

$$\mu_V > r_F + (\mu_M - r_F)\beta_V$$

holds for a particular security. Investors will want to increase their position in this security, which offers a higher expected return than required as compensation for systematic risk. Demand will exceed supply, the price of the security will rise and the expected return will fall.

On the other hand, if the reverse inequality

$$\mu_V < r_F + (\mu_M - r_F)\beta_V$$

holds, investors will want to sell the security. In this case supply will exceed demand, the price will fall and the expected return will rise.

Each of the above two process will continue until the prices and with them the expected returns of all securities settle at a new level, restoring equilibrium. The two inequalities are very important because they tell investors whether any security is underpriced or overpriced, respectively.

Acknowledgement

I would like to thank Professor Kyle Hambrook, for his explanation and patience thtoughout my independent study. It has been a great semester working with him.

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