Introduction to Fourier Series:
Orthogonal Polynomials and Boundary Value Problems
Project Save the fish

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Abstract

This study determines different temperature profiles $T(\phi)$ on the outer surface of a hemispherical pond such that it matches certain temperature constraints at equilibrium. This study is then separated into two parts mainly. First, we calculate $T(\phi)$ given the temperature profile in the form of $T_{10}\cos\phi$; second we come up with our own temperature profile $T(\phi) = \phi^{10} + 8$ that satisfies the given constraints.

1 Introduction

A hemispherical fish pond is covered with ice in the winter. Its bottom provides sustainable heat source given by a temperature profile of $u(a,\phi) = T(\phi)$. However owls can still catch fish in the pond since the fish are forced to stay near the surface because the bottom temperature of the pond is quite high. We therefore adjust the temperature profile at the bottom of the pond so that the fish can stay closer to the bottom of the pond and away from the surface, reducing the loss of fish being eaten by owls.

2 Problem overview

2.1 Problem statement

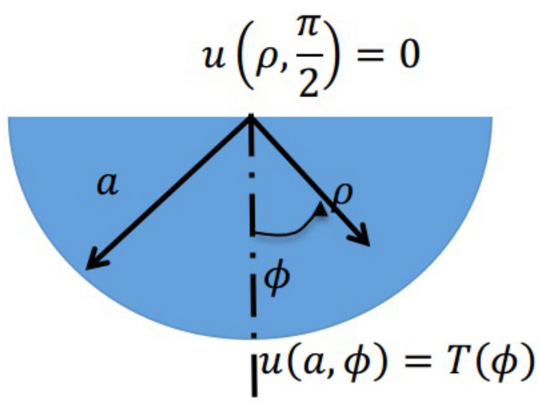


Figure 1: Fish in pond with ice-covered surface

The governing partial differential equation for this problem is:

$$\nabla^2 u(\rho,\phi) = 0$$

In axisymmetric spherical coordinate, the Laplacian of spherical coordinate is simplified to

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial u}{\partial \phi}) - \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}(\sin\phi \frac{\partial u}{\partial \phi}) = 0$$

where

$$0 < \rho < a$$

$$0 < \phi < \frac{\pi}{2}$$

Boundary conditions:

$$u(\rho, \frac{\pi}{2}) = 0$$

$$u(a, \phi) = T(\phi)$$

$$u(a, 0) = 8$$

Finiteness conditions:

$$|u(\rho, 0)| < \infty$$

$$|u(0, \phi)| < \infty$$

2.2 Specific parameters

- radius $a = 5m$
- thermal conductivity $k_0 = 0.6W/(m\,^\circ\mathrm{C})$
- the pond is covered with ice $u(r, \frac{\pi}{2}) = 0$
- The controller at the bottom of the pond is set at temperature $T(0) = T_0 = 8$
- the controllers will only add heat $T(\phi) > 0$

3 Methods and solutions

3.1 Separation of variables

Assume $u(\rho, \phi) = F(\rho)G(\phi)$ and substitute it into $\nabla^2 u$

$$\frac{1}{F}\frac{d}{d\rho}(\rho^2 \frac{dF}{d\rho}) = \frac{1}{G\sin\phi}\frac{d}{d\phi}(\sin\phi \frac{dG}{d\phi}) = \lambda$$

We then use separation variable λ to obtain two to ordinary differential equations (ODE's):

$$\frac{d}{d\phi}(\sin\phi \frac{dG}{d\phi}) + \lambda \sin\phi G = 0$$

$$\frac{d}{d\rho}(\rho^2 \frac{dF}{d\rho}) - \lambda F = 0$$

3.1.1 ODE #1:

$$\frac{d}{d\phi}(\sin\phi\frac{dG}{d\phi})+\lambda\sin\phi G=0$$

Finiteness conditions:

$$|G(0)|<\infty$$
$$|G(\frac{\pi}{2})|<\infty$$

By changing variable

$$x=\cos\phi$$
$$\frac{d}{d\phi}=\frac{dx}{d\phi}\frac{d}{dx}=-\sin\phi\frac{d}{dx}$$

We obtain Legendre's differential equation

$$\frac{d}{dx}((1-x^2)\frac{d\bar{G}}{dx})+\lambda\bar{G}=0$$

where

$$|\bar{G}(1)|<\infty$$
$$\bar{G}(0)=0$$

Therefore the solution of ODE #1 is in Legendre polynomials

$$G_n(\phi)=P_n(\cos\phi)$$

where

$$\lambda_n=n(n+1)\qquad n=0,1,2,3...$$

3.1.2 ODE #2:

$$\frac{d}{d\rho}(\rho^2\frac{dF}{d\rho})-\lambda F=0$$
$$2\rho\frac{dF}{d\rho}+\rho^2\frac{d^2F}{d\rho^2}-\lambda F=0$$

Finiteness conditions:

$$|F(\rho)|<\infty$$
$$|F(0)|<\infty$$

We let $F(\phi)=c\rho^s$, and simplify the above differential equation to obtain

$$2\rho(c\rho^s)'+\rho(c\rho^s)''-\lambda c\rho^s=0$$
$$c\rho^s(s^2+s-\lambda)=0$$

Therefore

$$s^2+s-\lambda=0$$
$$s^2+s-n(n+1)=0$$
$$s_1=n$$
$$s_2=-(n+1)$$

Therefore the solution of ODE #2 is

$$F(\rho)=A_n\rho^n+B_n\rho^{-(n+1)}$$

While by finiteness condition $F(0)<\infty$
Therefore

$$B_n=0$$
$$F(\rho)=C_n\rho^n$$

Where

$$n=0,1,2,3...$$

3.1.3 Solution of the equilibrium temperature distribution

Therefore by superposition of the sum of the product of the solutions of ODE #1 and ODE #2, we obtain the solution of the equilibrium temperature distribution

$$u(\rho,\phi) = F(\rho)G(\phi) = \sum_{i=0}^{\infty} A_n \rho^n P_n(\cos\phi)$$

where

$$n = 1, 3, 5...$$

Note that the odd sequence of n applied on Legendre's polynomials is determined by the given specific parameter $u(r, \frac{\pi}{2}) = 0$

Therefore

$$u(a,\phi) = \sum_{i=0}^{\infty} C_n a^n P_n(\cos\phi) = T_n(\phi)$$

By orthogonality we have:

$$\int_0^{\frac{\pi}{2}} P_n(\cos\phi)P_m(\cos\phi)\sin\phi d\phi = \begin{cases} \frac{1}{2n+1} & m = n \\ 0 & m \neq n \end{cases}$$

Note that because the domain is from 0 to $\frac{\pi}{2}$ for this axisymmetric problem, we have $\frac{1}{2n+1}$ by orthogonality when $m = n$.

Therefore by multiplying $P_m(\cos\phi)$ and integrating over the domain we have:

$$\int_0^{\frac{\pi}{2}} T_n(\phi)P_m(\cos\phi)\sin\phi\,\mathrm{d}\phi = \sum_{i=0}^{\infty} C_n a^n \int_0^{\frac{\pi}{2}} P_n(\cos\phi)P_m(\cos\phi)\sin\phi\,\mathrm{d}\phi = A_m a^m \frac{1}{2m+1}$$

Therefore we can determine the coefficient:

$$A_m = \frac{\int_0^{\frac{\pi}{2}} T_m(\phi)P_m(\cos\phi)\sin\phi\,\mathrm{d}\phi}{a^m \frac{1}{2m+1}}$$

where

$$m = n = 1, 3, 5...$$

4 Temperature profiles under different constraints

4.1 Temperature profile $T_1(\phi) = T_{10}\cos\phi$

To determine the temperature profile from the general equilibrium temperature distribution solution

$$u(\rho,\phi) = \sum_{i=0}^{\infty} C_n \rho^n P_n(\cos\phi)$$

we match terms with the given temperature profile $u(a,\phi) = T_1(\phi) = T_{10}\cos\phi$

$$u(a,\phi) = \sum_{i=0}^{\infty} C_n a^n P_n(\cos\phi) = T_{10}\cos\phi$$

This implies

$$u(a,\phi) = C_1 a^1 P_1(\cos\phi) = C_1 a\cos\phi = T_{10}\cos\phi$$

while all the other terms in the summation are 0, i.e. $C_{n\neq1} = 0$, i.e. $C_n = 0$, $n = 3, 5, 7....$

We also know $T(0) = T_0 = 8\,^\circ\mathrm{C}$

Therefore

$$C_1 a = T_{10} = 8$$
$$C_1 = \frac{8}{a}$$
$$T_1(\phi) = T_{10}\cos\phi = 8\cos\phi$$

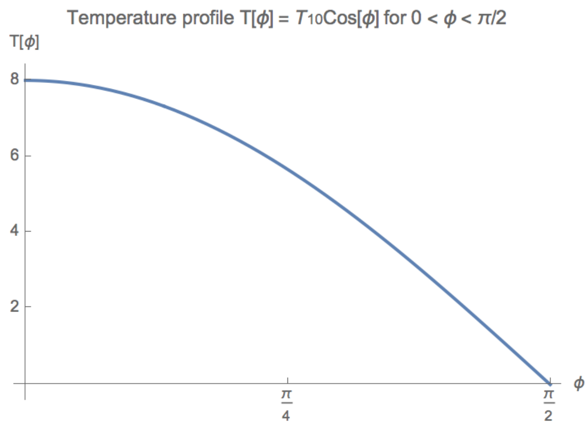


Figure 2: Plot of the first applied temperature profile with ϕ in radians and $T(\phi)$ in °C

Since we know the coefficient $C_1 = \frac{8}{a}$, we know equilibrium temperature distribution

$$u(\rho,\phi) = \frac{8}{a}\rho\cos\phi$$

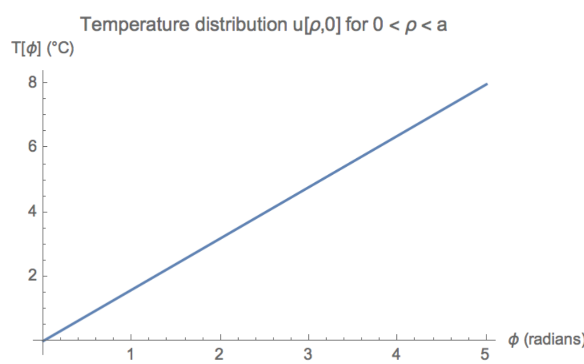


Figure 3: Plot of $u_1(\rho,0)$ for $0<\rho<a$

4.1.1 Calculation of the heat flow through surface $\rho = a$ to the pond

The heat flow into the pond through the surface $\rho = a$ is equal to

$$K_0 \iint_S \nabla u \cdot \underline{n}\big|_{\rho=a}\, dS$$

Where

$$\nabla u = \frac{\partial u}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial u}{\partial \phi}\hat{\phi}$$

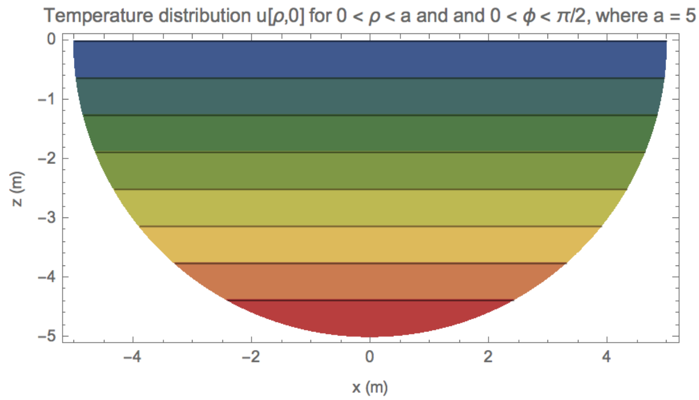


Figure 4: Contour plot of $u_1(\rho, 0)$ for $0 < \rho < a$ and $0 < \phi < \frac{\pi}{2}$

$$\nabla u \cdot \underline{n} = \nabla u \cdot \hat{\rho} = \frac{\partial u}{\partial \rho} = \frac{8}{5} \cos \phi$$

Therefore

$$\iint_S \nabla u \cdot \underline{n}\big|_{\rho=a} dS = k_0 \int_{2\pi}^{0} \int_{\frac{\pi}{2}}^{0} \frac{8}{5} \cos \phi \rho^2 \sin \phi \, d\phi \, d\theta = k_0 \frac{8}{5} a^2 \pi$$

4.2 Customized temperature profile

For this customized temperature profile, we come up with $T_2(\phi) = \phi^{10} + 8$ The reason behind this choice of equation will be evident once the temperature profile is shown.

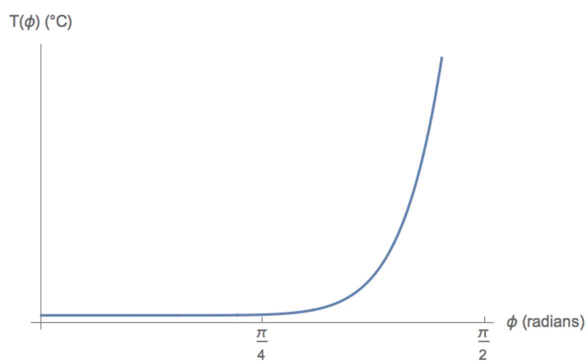


Figure 5: Plot of the second applied temperature profile

In order to find C_n of the temperature profile equation, orthogonality must be used. Orthogonality of Legendre polynomials, for an axisymmetric problem is:

$$\int_0^{\frac{\pi}{2}} P_n(\cos \phi) P_m(\cos \phi) \sin \phi d\phi = \begin{cases} \frac{1}{2n+1} & m = n \\ 0 & m \neq n \end{cases}$$

Note that because the domain is from 0 to $\frac{\pi}{2}$ for this axisymmetric problem, we have $\frac{1}{2n+1}$ from orthogonality when $m = n$.

Since

$$u(a,\phi) = T_2(\phi) = \phi^{10} + 8 = \sum_{i=0}^{\infty} C_n a^n P_n(\cos\phi)$$

By multiplying $P_m(\cos\phi)$ and integrating over the domain we have

$$\int_0^{\frac{\pi}{2}} (\phi^{10} + 8) P_m(\cos\phi) \sin\phi d\phi = \sum_{i=0}^{\infty} \int_0^{\frac{\pi}{2}} C_n a^n P_n(\cos\phi) P_m(\cos\phi) \sin\phi d\phi$$

By orthogonality we have

$$C_n = \frac{(2n+1)}{a^n} \int_0^{\frac{\pi}{2}} \phi^{10} + 8 P_n(\cos\phi) \sin\phi d\phi$$

Therefore

$$u(\rho,\phi) = \sum_{i=0}^{\infty} \left[\frac{(2n+1)}{a^n} \int_0^{\frac{\pi}{2}} (\phi^{10} + 8) P_n(\cos\phi) \sin\phi d\phi \right] \rho^n P_n(\cos\phi)$$

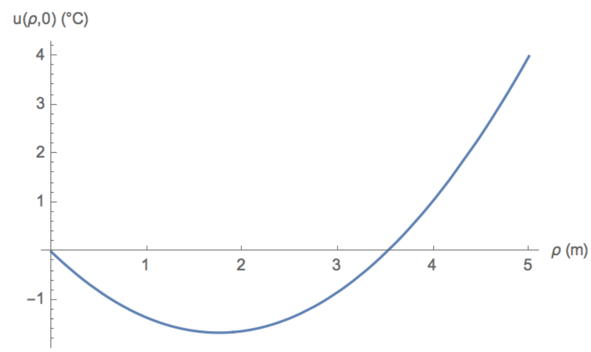


Figure 6: Plot of $u_2(\rho, 0)$ for $0 < \rho < a$

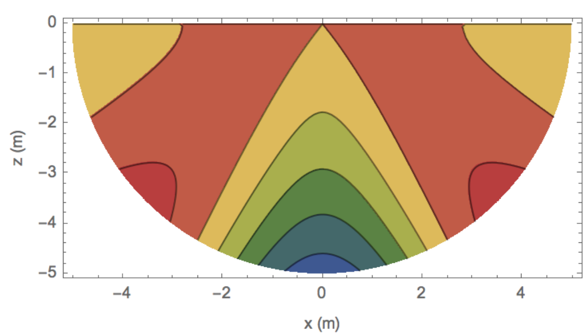


Figure 7: Contour plot of $u_2(\rho, 0)$ for $0 < \rho < a$ and $0 < \phi < \frac{\pi}{2}$

4.2.1 Calculation of the heat flow through surface $\rho = a$ to the pond

Heat flow is calculated in the same way as the previous one. We first calculate

$$\nabla u \cdot \underline{n} = \nabla u \cdot \hat{\rho} = \frac{\partial u}{\partial \rho} = \sum_{i=1}^{\infty} n C_n \rho^{(n-1)} P_n(\cos\phi)$$

Therefore the heat flow into the pond through the surface $\rho = a$ is equal to

$$K_0 \int \int_S \nabla u \cdot \underline{n}\big|_{\rho=a} dS = 4.93K_0$$

5 Discussion

For the first solution, since the temperature at the boundary is a function of Cosine, we notice that it have the same form as the first 0th degree Legendre polynomial. From there, we deduced that the C_n where $n = 2, 3, 4...$ has to equal to zero. As Figure 3 shows, the temperature at the center, i.e., $\phi = 0$ is a linear function; therefore the contour plot of the temperature profile increases linearly as illustrated by Figure 4. For sanity check, we performed a heat flow derivation over the boundaries. The heat flow in equals the heat flow out, therefore our answer is indeed correct.

For the second solution, we choose a temperature profile $\phi^{10} + 8$ because we want a rapid heat generation as ϕ gets larger while keeping the temperature at the bottom, i.e., boundary condition satisfied. In order to calculate C_n we use the orthogonality property of the Legendre polynomial to deduce C_n. Once the coefficient C_n is deduced, we proceed to plot our temperature profile. As shown in Figure 7, as ϕ gets larger, the pond gets heated rapidly up until about Φ is about $\frac{\pi}{3}$ where the temperature profile starts to lower due to the influence of the frozen surface. For sanity check, we performed a heat flow derivation over the boundaries. The heat flow in equals the heat flow out, therefore our answer is indeed correct.

6 Short summary

Using the method of separation of variable, we first derive the general solution of equilibrium temperature distribution in an axisymmetric pond. An overview of the derivation and the general solution of the equilibrium temperature distribution are included at the beginning of study. The derivation also includes plots of the temperature profile within the pond due to the heat applied to the surface of the pond. We also calculate the net heat flow as a sanity check, i.e.$K_0 \int \int_S \nabla u \cdot \underline{n}\big|_{\rho=a} dS$, to ensure that our equilibrium temperature distribution equation is correct.

The problem is then split into two parts with two different heat application on the boundary of the pond. We first matched terms of our first solution with the given temperature profile in the form of $T_{10} \cos \phi$ to determine the coefficient. When determining the second temperature profile, we use the orthogonality property of the Legendre polynomials to calculate the unknown coefficient from the equilibrium temperature distribution function. Once we know the coefficient for each solution, we can obtain two temperature profiles under the given constraints.

Acknowledgements

This study couldn't be completed without the help from Professor Sheryl Gracewski at the Department of Mechanical Engineering at University of Rochester. Some parts of study were advised by fellow MTH 281 students Devanjith Fonseka and Shuchen Wu. The format of work has been adapted from the template of the International Conference on Mathematical Methods in Science and Engineering with the support of LaTeX.

References

[1] R. Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, 2012.

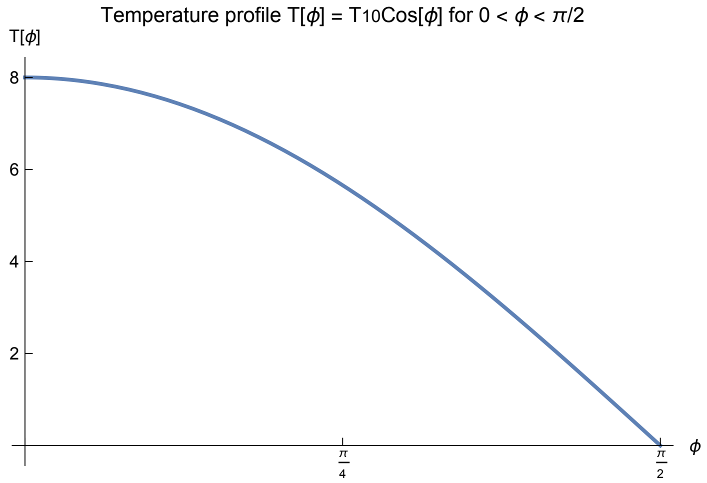
Appendix

Mathematica Code

The first temperature profile

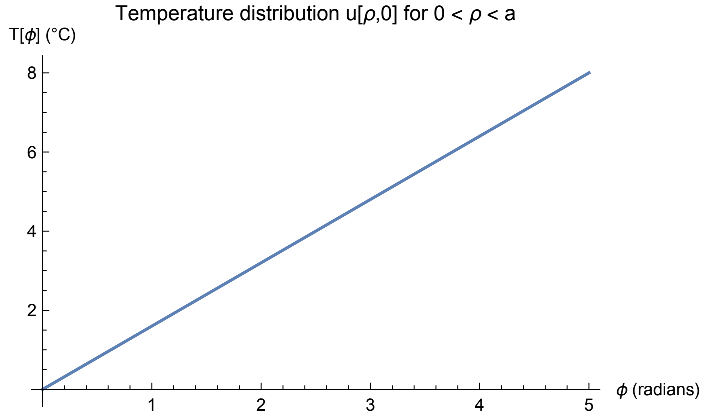
In[1]:=
```
T[ϕ_] = 8 Cos[ϕ];
Plot[T[ϕ], {ϕ, 0, π / 2}, AxesLabel → {"ϕ", "T[ϕ]"}, PlotRange → All,
 PlotLabel → "Temperature profile T[ϕ] = T10Cos[ϕ] for 0 < ϕ < π/2",
 AxesLabel → {"ϕ (radians)", "T(ϕ) (°C)"},
 PlotStyle → Thick, Ticks → {{0, π/4, π/2}, {0, 2, 4, 6, 8}}]
```

Out[2]=



In[3]:=
```
a = 5;
u[ρ_, ϕ_] = (8 ρ Cos[ϕ]) / a;
Plot[{u[ρ, 0]}, {ρ, 0, 5} , AxesLabel → {"ϕ (radians)", "T[ϕ] (°C)"},
 PlotRange → All, PlotLabel → "Temperature distribution u[ρ,0] for 0 < ρ < a",
 AxesLabel → {"ϕ (radians)", "T(ϕ)°C"}]
```

Out[5]=

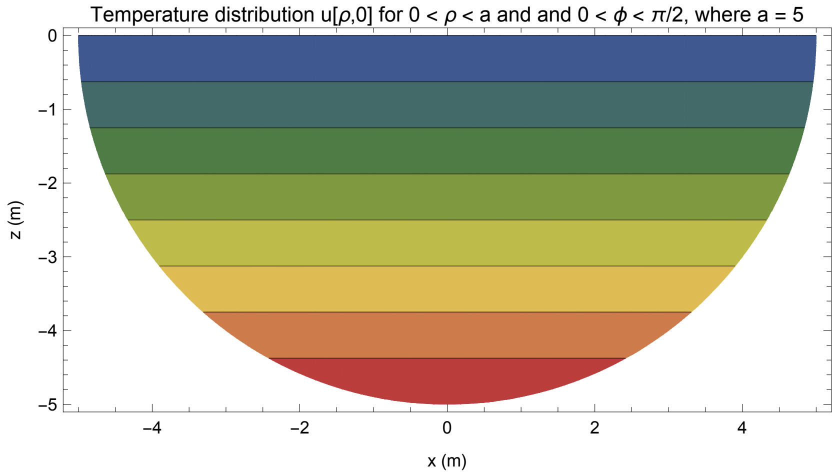


In[6]:=

The first contour plot

In[7]:=
```
ContourPlot[u[ρ, ϕ] /. {ρ → -√(x² + z²), ϕ → ArcTan[z, x]}, {x, -a, a},
  {z, -a, 0}, PlotRange → All, PlotLabel → "Temperature distribution
     u[ρ,0] for 0 < ρ < a and and 0 < ϕ < π/2, where a = 5",
  RegionFunction → (#1^2 + #2^2 < a² &), AspectRatio → 0.5,
  ColorFunction -> "DarkRainbow",
  FrameLabel → {"x (m)", "z (m)"}, ContourLabels → False]
```

Out[7]=

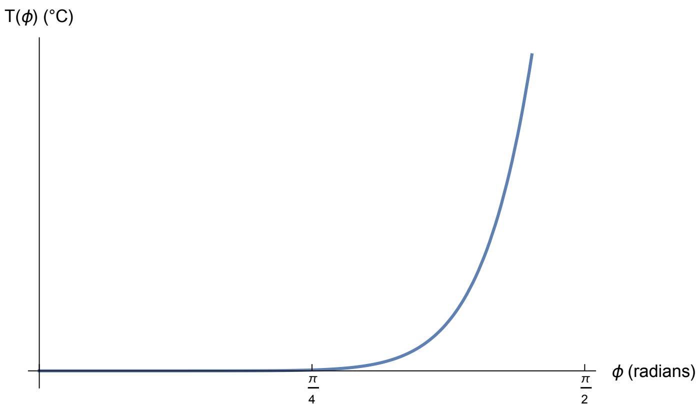


In[8]:=

The second temperature profile

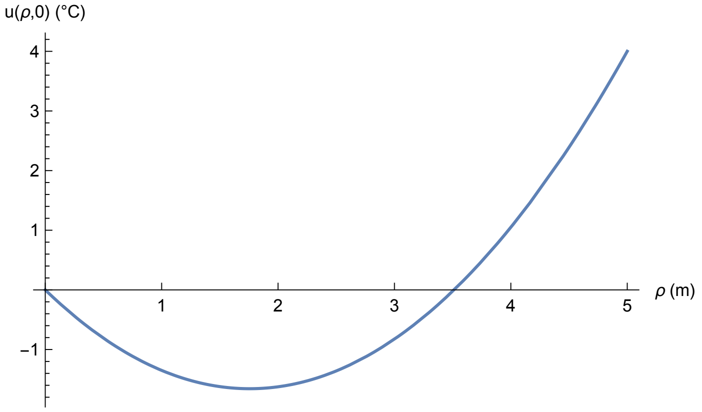
```
T2[ϕbar_] := (ϕbar¹⁰ + 8)
Plot[(ϕbar^10 + 8), {ϕbar, 0, π/2},
 AxesLabel → {"ϕ (radians)", "T(ϕ) (°C)"}, Ticks → {{0, π/4, π/2}}]
```

Out[10]=



In[11]:=
```
a = 5.;
cn[n_] := (2 n + 1)
  NIntegrate[(ϕbar^10 + 8) LegendreP[2 n + 1, Cos[ϕbar]] Sin[ϕbar], {ϕbar, 0, π/2}]
u1PS[ρ_, ϕ_, nMax_] := Sum[cn[n] (ρ/a)^n LegendreP[2 * n + 1, Cos[ϕ]], {n, 1, nMax}]
Plot[{u1PS[ρ, 0, 2]}, {ρ, 0, a},
 PlotRange → All, AxesLabel → {"ρ  (m)", "u(ρ,0)  (°C)"},
 PlotPoints → 10, Ticks → {{0, 1, 2, 3, 4, 5}, Automatic}]
```

Out[14]=



In[15]:=
The Second Contour Plot

In[16]:=
```
ContourPlot[u1PS[ρ, ϕ, 2] /. T0 → 4 /. {ρ → -Sqrt[x^2 + z^2], ϕ → ArcTan[z, x]},
 {x, -a, a}, {z, -a, 0}, RegionFunction → (#1^2 + #2^2 < a^2 &),
 AspectRatio → 0.5, ColorFunction -> "DarkRainbow",
 FrameLabel → {"x  (m)", "z  (m)"}, ContourLabels → False]
```