PHY 235W Classical Mechanics Term Paper A Brief Study on Spacecraft Dynamics in Circular Orbit

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Abstract

The solution of most spacecraft dynamics and control problems requires a consideration of gravitational forces and moments. When a body is in a uniform gravitational field, its center of mass becomes the center of gravity and the gravitational torque about its center of mass is zero. The gravitational field is not uniform over a body in space, however, and a gravitational torque exists about the body's center of mass. This effect was first considered by D'Alembert and Euler in 1749. Later, in 1780, Lagrange used it to explain why the moon always has the same face toward the Earth. [1] In this paper, a brief study on the motion of a rigid body in circular orbit is conducted along with a derivation of the equations of motion.

1 Introduction

Space has always been a fascinating attraction to us in the past a few thousand years. From observing the phases of the Moon and discoveries of new galaxies, to monitoring and tracking the motion of smaller celestial objects like satellites, spacecrafts, and even space debris [2], our space exploration has never stopped with ambitious goals one after another. From the first artificial Earth satellite in space in 1957 [3], to the International Space Station in 1999, one core technology that ties all of those feats with is the ever maturing technologies of spacecraft dynamics and control.

Orbital mechanics, also called flight mechanics, is the study of the motions of artificial satellites and space vehicles moving under the influence of forces such as gravity, atmospheric drag, thrust, etc. [4] To date, there are more than 1305 operational satellites orbiting around the Earth along with more than 500,000 pieces of space debris ranging from 1 to 10 centimeters to much larger magnitude. [5] Near Earth orbit is so polluted with junk that the International Space Station is often moved to avoid impact with dangerous chunks of space debris. Many of these floatsams are created through collisions, and it is necessary to question whether space travel in the future would be too risky if we get too much junk orbiting the planet. We might seal ourselves inside a shield of shrieking metal moving at 29,000 km/hour. [6]

To better understand the motion of the objects orbiting around the Earth, and their corresponding orbital stability and characteristics, a simple model of the motion of a rigid body in a circular orbit, with certain constraints and assumptions, is developed in this paper with detailed derivation steps divided into sections for the reader, as well as simpler and compatible notation with what we used in PHY 235W.

2 Theory and Derivation

2.1 Rigid body in a circular orbit

Spacecraft in space can be considered as a rigid body with finite volume, mass, and shape that remains unchanged during the observation. The spacecraft can roll, pitch, and yaw freely, as illustrated in Figure 1: Spacecraft attitude control: roll, pitch, and yaw. To study the dynamics of a spacecraft, say in Medium Earth Orbit, we can consider a simple rigid body in a circular orbit. Let's imagine the following picture.

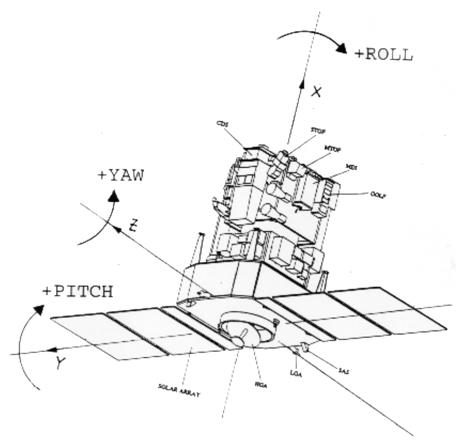

Figure 1: Spacecraft attitude control: roll, pitch, and yaw [7]

A local vertical and local horizontal (LVLH) reference frame A with its origin at the center of mass of an orbiting spacecraft, has a set of basis vectors $\{\vec{a_1},\ \vec{a_2},\ \vec{a_3}\}$, with $\vec{a_1}$ along the orbit direction, $\vec{a_2}$ perpendicular to the orbit plane, and $\vec{a_3}$ toward the Earth, as illustrated in Figure 2: Rigid body in a circular orbit.

The angular velocity of A with respect to the fixed (inertial) reference frame N is

$$\vec{w}^{A/N} = -n\vec{a_2} \tag{1}$$

where $n = \sqrt{\frac{\mu}{R_c^3}}$ is the constant orbit rate and μ is the gravitational parameter of the Earth.

The constant orbital rate n can be obtained by equating the gravity and centripetal force on the rigid body. Note that $\frac{GMm}{R_c^2} = \frac{\mu m}{R_c^2} = mw^2 R_c$. Therefore $w^2 R_c^3 = \mu$ and $w = \sqrt{\frac{\mu}{R_c^3}} = n$.

The angular velocity of the body-fixed reference frame B with basis vector $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$ is then given by

$$\vec{w}^{B/N} = \vec{w}^{B/A} + \vec{w}^{A/N} = -n\vec{a_2} \tag{2}$$

where $\vec{w}^{B/A}$ is the angular velocity of reference frame B relative to LVLH reference frame A.

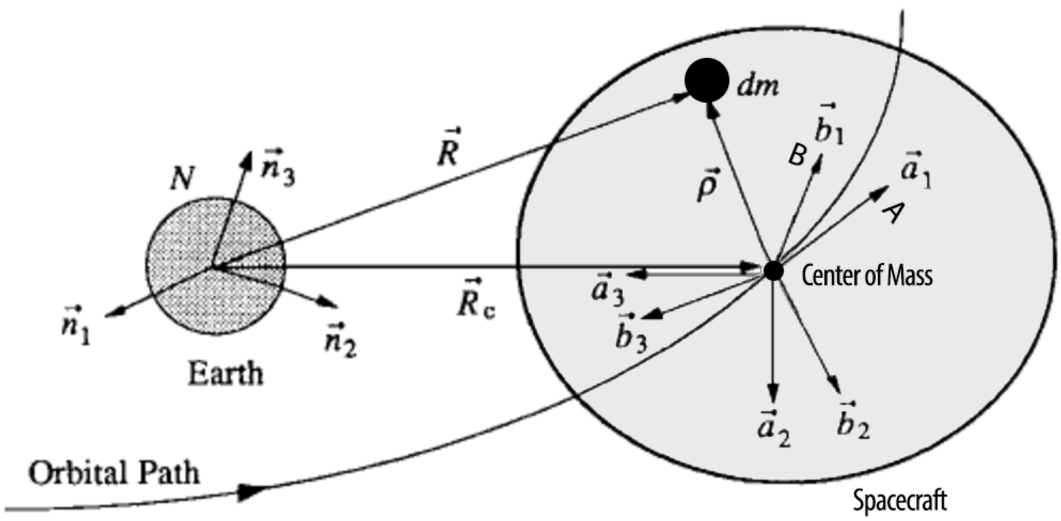

Figure 2: Rigid body in a circular orbit [1]

The orientation of the body-fixed reference frame B with respect to the LVLH reference frame A is in general described by the direction matrix $C = C^{B/A}$ such that

$$\begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ \vec{b_3} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \end{bmatrix} \tag{3}$$

or

$$\begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ \vec{b_3} \end{bmatrix} \tag{4}$$

2.2 Torque in gravitational field

The Earth's gravitational force acting on each small mass element dm of the rigid body is given by

$$d\vec{f} = -\frac{\mu dm\vec{R}}{|\vec{R}|^3} = -\frac{\mu dm(\vec{R_c} + \vec{\rho})}{|\vec{R} + \vec{\rho}|^3} \tag{5}$$

where μ is the gravitational parameter of the Earth, $\vec{R}$ and $\vec{\rho}$ are the position vectors of dm from the Earth's center and the spacecraft's mass center, respectively, and $\vec{R_c}$ is the position vector of the spacecraft's mass center from the Earth's center, as illustrated in Figure 2: Rigid body in a circular orbit.

The gravity-gradient torque M about the spacecraft's mass center is then expressed as

$$\vec{M} = \int \rho \times d\vec{f} = -\mu \int \frac{\rho \times \vec{R_c}}{|\vec{R_c} + \vec{\rho}|^3} dm \tag{6}$$

And we could use the following approximation for the denominator $|\vec{R_c} + \vec{\rho}|^{-3}$:

$$\begin{aligned} |\vec{R_c} + \vec{\rho}|^{-3} &= R_c^{-3} \left\{ \frac{|\vec{R_c} + \vec{\rho}|}{R_c} \right\}^{-3} \\ &= R_c^{-3} \left\{ \frac{(\vec{R_c} + \vec{\rho})^2}{R_c^2} \right\}^{-\frac{3}{2}} \\ &= R_c^{-3} \left\{ 1 + \frac{2(\vec{R_c} \cdot \vec{\rho})}{R_c^2} + \frac{\rho^2}{R_c^2} \right\}^{-\frac{3}{2}} \\ &= R_c^{-3} \left\{ 1 - \frac{3(\vec{R_c} \cdot \vec{\rho})}{R_c^2} + \text{higher-order terms from Taylor expansion} \right\} \end{aligned} \tag{7}$$

where $R_c = |\vec{R_c}|$ and $\rho = |\vec{\rho}|$.

Now we substitute equation (7) to equation (6) and neglect the higher-order terms, the gravity-gradient torque can be written as

$$
\begin{aligned}
\vec{M} &= -\mu \int \big(\vec{\rho} \times \vec{R}_c\big) \left\{ R_c^{-3} \left[1 - \frac{3(\vec{R}_c \cdot \vec{\rho})}{R_c^2} \right] \right\} dm \\
&= -\mu \int \frac{\big(\vec{\rho} \times \vec{R}_c\big)}{R_c^3} \left[1 - \frac{3(\vec{R}_c \cdot \vec{\rho})}{R_c^2} \right] dm \\
&= -\mu \int \frac{\big(\vec{\rho} \times \vec{R}_c\big)}{R_c^3} dm + \mu \int \frac{3(\vec{R}_c \cdot \vec{\rho})\big(\vec{\rho} \times \vec{R}_c\big)}{R_c^5} dm \\
&= \frac{-\mu}{R_c^3} \left[\int \vec{\rho} dm \right] \times \vec{R}_c + \frac{3\mu}{R_c^5} \int (\vec{R}_c \cdot \vec{\rho})\big(\vec{\rho} \times \vec{R}_c\big) dm \\
&= \frac{3\mu}{R_c^5} \int (\vec{R}_c \cdot \vec{\rho})(\vec{\rho} \times \vec{R}_c) dm
\end{aligned} \tag{8}
$$

where $\int \vec{\rho} dm = 0$ by the choice of the center of mass. The ρ vectors, which originate from the center of mass, will all cancel when we integrate with the infinitesimal mass dm of the rigid body.

Equation (8) could be further simplified as the follows:

$$
\begin{aligned}
\vec{M} &= -\frac{3\mu}{R_c^5} \vec{R}_c \times \int \vec{\rho}(\vec{\rho} \cdot \vec{R}_c) dm \\
&= -\frac{3\mu}{R_c^5} \vec{R}_c \times \int \vec{\rho}\vec{\rho} dm \cdot \vec{R}_c \\
&= -\frac{3\mu}{R_c^5} \vec{R}_c \times \left\{ \int \left[\rho^2 \hat{J} - (\rho^2 \hat{J} - \vec{\rho}\vec{\rho}) \right] dm \cdot \vec{R}_c \right\} \\
&= -\frac{3\mu}{R_c^5} \vec{R}_c \times \left\{ \left[\int \rho^2 \hat{J} dm \right] - \int \big(\rho^2 \hat{J} - \vec{\rho}\vec{\rho}\big) dm \right\} \cdot \vec{R}_c \\
&= -\frac{3\mu}{R_c^5} \vec{R}_c \times \left\{ \left[\int \rho^2 \hat{J} dm \right] - \hat{I} \right\} \cdot \vec{R}_c \\
&= -\frac{3\mu}{R_c^5} \vec{R}_c \times \left[\int \rho^2 \hat{J} dm \right] \cdot \vec{R}_c + \frac{3\mu}{R_c^5} \vec{R}_c \times \hat{I} \cdot \vec{R}_c \\
&= -\frac{3\mu}{R_c^5} \vec{R}_c \times \vec{R}_c \cdot \left[\int \rho^2 \hat{J} dm \right] + \frac{3\mu}{R_c^5} \vec{R}_c \times \hat{I} \cdot \vec{R}_c \\
&= \frac{3\mu}{R_c^5} \vec{R}_c \times \hat{I} \cdot \vec{R}_c
\end{aligned} \tag{9}
$$

where $\hat{I} = \int (\rho^2 \hat{J} - \vec{\rho}\vec{\rho}) dm$ is the inertia dyadic, and $\hat{J}$ is the unit dyadic, defined as $\hat{J} \cdot v = v \cdot \hat{J}$ for any vector v, which means $\vec{R}_c \times \hat{J} \cdot \vec{R}_c = \vec{R}_c \times \vec{R}_c = 0$.

We can then further transform the the gravity-gradient torque we obtained from equation (9) to the following vector/dyadic form

$$
\begin{aligned}
\vec{M} &= \frac{3\mu}{R_c^5}\vec{R_c} \times \hat{I} \cdot \vec{R_c} \\
&= 3\frac{\mu}{R_c^3}\frac{\vec{R_c}}{R_c} \times \hat{I} \cdot \frac{\vec{R_c}}{R_c} \\
&= 3n^2\vec{a_3} \times \hat{I} \cdot \vec{a_3}
\end{aligned} \tag{10}
$$

where $n = \sqrt{\mu/R_c^3}$ is the constant orbital rate, derived from equation (1), and $\vec{R_c}/R_c$ is in $\hat{a}_3$ direction, as illustrated in Figure 2: Rigid body in a circular orbit.

The above remarkable approximation and simplification of equations (7), (8), and (9) are referred from *Space Vehicle Dynamics and Control (2nd Edition)* by Bong Wie. To better understand how inertia dyadic and unit dyadic work, I have supplemented many necessary derivation steps, which the reference not mentioned or skipped, for the reader.

2.3 Reference frame undergoing rotation

We know that the rotational equation of motion of a rigid body in a circular orbit is in terms of torque M and angular momentum $\vec{L} = \hat{I} \cdot \vec{w}^{B/N}$ has the following form

$$
\vec{M} = \left\{\frac{d\vec{L}}{dt}\right\}_{N(inertial)} = \left\{\frac{d\vec{L}}{dt}\right\}_{B} + \vec{w}^{B/N} \times \vec{L} \tag{11}
$$

which could also be written as the following equated by the result of equation (10)

$$
\vec{M} = \hat{I} \cdot \dot{\vec{w}} + \vec{w} \times (\hat{I} \cdot \vec{w}) = 3n^2\vec{a_3} \times \hat{I} \cdot \vec{a_3} \tag{12}
$$

where $\vec{w} = \vec{w}^{B/N}$

Because $\vec{w}$ and $\vec{a_3}$ can be expressed in terms of basis vectors of the body-fixed reference frame B as

$$
\vec{w} = w_1\vec{b_1} + w_2\vec{b_2} + w_3\vec{b_3} \tag{13}
$$

$$
\vec{a_3} = C_{13}\vec{b_1} + C_{23}\vec{b_2} + C_{33}\vec{b_3} \tag{14}
$$

and the inertia dyadic is related to inertia matrix as

$$
\hat{I} = \sum_i^3 \sum_j^3 I_{ij}\vec{b_i}\vec{b_j} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{15}
$$

substitute equations (13), (14), and (15) into Equation (12) in matrix form, we obtain

$$\begin{aligned}\vec{M} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} \dot{w}_1 \vec{b_1} \\ \dot{w}_2 \vec{b_2} \\ \dot{w}_3 \vec{b_3} \end{bmatrix} + \\ \begin{bmatrix} 0 & -w_3 b_3 & w_2 b_2 \\ w_3 b_3 & 0 & w_1 b_1 \\ -w_2 b_2 & w_1 b_1 & 0 \end{bmatrix} \left[\left(\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \begin{bmatrix} w_1 \vec{b_1} \\ w_2 \vec{b_2} \\ w_3 \vec{b_3} \end{bmatrix} \right] \\ = 3n^2 \begin{bmatrix} 0 & -C_{33} b_3 & C_{23} b_2 \\ C_{33} b_3 & 0 & -C_{13} b_1 \\ -C_{23} b_2 & C_{13} b_1 & 0 \end{bmatrix} \left(\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \begin{bmatrix} C_{13} \vec{b_1} \\ C_{23} \vec{b_2} \\ C_{33} \vec{b_3} \end{bmatrix}\end{aligned} \tag{16}$$

Note that the cross product in matrix form is conducted as the following

$$\vec{a} \times \vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{17}$$

We cancel basis vectors $\vec{b}$ on both sides of the equation and get

$$\begin{aligned}\vec{M} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ = 3n^2 \begin{bmatrix} 0 & -C_{33} & C_{23} \\ C_{33} & 0 & -C_{13} \\ -C_{23} & C_{13} & 0 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} C_{13} \\ C_{23} \\ C_{33} \end{bmatrix}\end{aligned} \tag{18}$$

2.4 Euler's equations of motion for the rigid body

Now, we would like to see how the rigid body moves under the influence of the gravity-gradient torque. To describe the orientation of the body-fixed reference frame B with respect to the LVLH reference frame A in terms of three Euler angles θ_i (i = 1, 2, 3), consider the rotational sequence of $C_1(\theta_1) \leftarrow C_2(\theta_2) \leftarrow C_3(\theta_3)$ to B from A. For this sequence we have

$$\begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ \vec{b_3} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \end{bmatrix}$$
$$= \begin{bmatrix} c\theta_2 c\theta_3 & c\theta_2 s\theta_3 & -s\theta_2 \\ s\theta_1 s\theta_2 c\theta_3 - c\theta_1 s\theta_3 & s\theta_1 s\theta_2 s\theta_3 + c\theta_1 c\theta_3 & s\theta_1 c\theta_2 \\ c\theta_1 s\theta_2 c\theta_3 - c\theta_1 s\theta_3 & c\theta_1 s\theta_2 s\theta_3 - s\theta_1 c\theta_3 & c\theta_1 c\theta_2 \end{bmatrix} \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \end{bmatrix} \tag{19}$$

where we use the shorthands $c\theta_i = \cos\theta_i$ and $s\theta_i = \sin\theta_i$.

Also, for the sequence sequence of $C_1(\theta_1) \leftarrow C_2(\theta_2) \leftarrow C_3(\theta_3)$, the angular velocity of B relative to A is represented as

$$\vec{w}^{B/A} = w_1' \vec{b_1} + w_2' \vec{b_2} + w_3' \vec{b_3} \tag{20}$$

where

$$\begin{bmatrix} w_1' \\ w_2' \\ w_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta_2 \\ 0 & c\theta_1 & s\theta_1 c\theta_2 \\ 0 & -s\theta_1 & c\theta_1 c\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{21}$$

Now, with

$$\vec{w}^{B/A} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \\ w_3' \end{bmatrix}$$

and

$$\vec{a_2} = C_{12}\vec{b_1} + C_{22}\vec{b_2} + C_{32}\vec{b_3} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c\theta_2 s\theta_3 \\ s\theta_1 s\theta_2 s\theta_3 + c\theta_1 c\theta_3 \\ c\theta_1 s\theta_2 s\theta_3 - s\theta_1 c\theta_3 \end{bmatrix}$$

We represent $\vec{w}^{B/N} = \vec{w}^{B/A} + \vec{w}^{A/N} = \vec{w}^{B/A} - n\vec{a_2}$ in the following matrix form after canceling basis vectors $\vec{b}$ on both sides of the equation

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta_2 \\ 0 & c\theta_1 & s\theta_1 c\theta_2 \\ 0 & -s\theta_1 & c\theta_1 c\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} - n \begin{bmatrix} c\theta_2 s\theta_3 \\ s\theta_1 s\theta_2 s\theta_3 + c\theta_1 c\theta_3 \\ c\theta_1 s\theta_2 s\theta_3 - s\theta_1 c\theta_3 \end{bmatrix} \tag{22}$$

We can also obtain an expression of $\dot{\theta}_i$, where $i = 1, 2, 3$ by transforming equation (22)

$$\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\\\dot{\theta}_3\end{bmatrix} = \frac{1}{c\theta_2}\begin{bmatrix}c\theta_2 & s\theta_1 s\theta_2 & c\theta_1 s\theta_2\\0 & c\theta_1 c\theta_2 & -s\theta_1 c\theta_2\\0 & -s\theta_1 & c\theta_2\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} + \frac{n}{c\theta_2}\begin{bmatrix}s\theta_3\\c\theta_2 c\theta_3\\s\theta_2 s\theta_3\end{bmatrix} \tag{23}$$

3 Results and Discussion

Now we can go back to equation (12) and use Euler's equations for rigid body in a force field to substitute the left side, and use the result of equation (18) after matrix multiplication to represent the right side

$$M_1 = I_1\dot{w}_1 = (I_2 - I_3)w_2w_3 = -3n^2(I_2 - I_3)C_{23}C_{23} \tag{24}$$

$$M_2 = I_2\dot{w}_2 = (I_3 - I_1)w_3w_1 = -3n^2(I_3 - I_1)C_{33}C_{13} \tag{25}$$

$$M_3 = I_3\dot{w}_3 = (I_1 - I_2)w_1w_2 = -3n^2(I_1 - I_2)C_{13}C_{23} \tag{26}$$

where C_{13}, C_{23}, and C_{33} can be referred from equation (19).

3.1 Small angle approximation

Furthermore, for small angles approximation, we use $\sin\theta_i \approx \theta_i$ and $\cos\theta_i \approx 1$. Now, equation (22) becomes

$$w_1 = \dot{\theta}_1 - n\theta_3 \tag{27}$$

$$w_2 = \dot{\theta}_2 - n \tag{28}$$

$$w_3 = \dot{\theta}_3 + n\theta_1 \tag{29}$$

Using the above approximations of w_i, we can further calculate the derivatives of w_i and obtain the equations of motion of a rigid body in a circular orbit from the above 3 torque equations, as follows, for roll, pitch, and yaw, respectively:

$$I_1\ddot{\theta}_1 - n(I_1 - I_2 + I_3)\dot{\theta}_3 + 4n^2(I_2 - I_3)\theta_1 = 0 \tag{30}$$

$$I_2\ddot{\theta}_2 + 3n^2(I_1 - I_3)\theta_2 = 0 \tag{31}$$

$$I_3\ddot{\theta}_3 + n(I_1 - I_2 + I_3)\dot{\theta}_1 + n^2(I_2 - I_1)\theta_3 = 0 \tag{32}$$

where θ_1, θ_2, and θ_3 are often called, respectively, the roll, pitch, and yaw attitude angles of the spacecraft relative to the LVLH reference frame A. The equations of the motion can be solved completely once we know the mass distribution of the rigid body (for the calculation of its moment of inertia). Of course, orbital rate n could also be adjusted for Mars, Jupiter, the Sun or other celestial bodies.

For these small angels θ_1, θ_2, θ_3, the body-fixed reference frame B is related to the LVLH frame A by

$$\begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ \vec{b_3} \end{bmatrix} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \end{bmatrix} \tag{33}$$

or

$$\begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \end{bmatrix} \approx \begin{bmatrix} 1 & -\theta_3 & \theta_2 \\ \theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} \begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ \vec{b_3} \end{bmatrix} \tag{34}$$

The angular velocity of B in N for this case of small relative angles of B with respective to A is also given by

$$\vec{w} = \vec{w}^{B/N} = w_1\vec{b_1} + w_2\vec{b_2} + w_3\vec{b_3} \approx (\dot{\theta}_1 - n\theta_3)\vec{b_1} + (\dot{\theta}_2 - n)\vec{b_2} + (\dot{\theta}_3 + n\theta_1)\vec{b_3} \tag{35}$$

In summary, in the above derivation of the equations of motion, we assume that the object in space is a rigid body in a circular orbit, and use Newton's law of universal gravitation accordingly to first calculate the force and torque from the Earth on the rigid body. The techniques used to calculate and simplify the torque on the rigid body include the use of inertia and unit dyadic in the calculation of the gravity-gradient torque and Taylor expansion of the position vector of dm to the Earth's center, where we assume the radius R_c is very large so that the terms with orders more than three in the denominators could be neglected in equation (7). Note that a circular geosynchronous (i.e. geostationary) orbit, in the plane of the Earth's equator has a radius of approximately 42,164 km (26,199 mi). [8] Once we have an expression of the torque, we equate the result using Euler's equations of motion for rigid body rotation, and obtain the full equations of motion. Small angle approximation of the equations of motion is followed by treating $\sin\theta = \theta$ and $\cos\theta = 1$.

4 Conclusion

In this paper, several key assumptions are made to make the problem easier to solve. We are very lucky to have a relatively straightforward approximation of the gravity-gradient torque in that the Earth's orbit is nearly circular. The eccentricity of the Earth's orbit is on average below 0.02 in the past hundreds of thousands of years. [9] Otherwise our treatment of angular moment and the constant orbital rate will be much more complicated if the orbit has a higher eccentricity. In fact, our gravity-gradient torque would not be that simple too, since in reality any object in space is subject to the superposition of all gravity-gradient torques of every single known celestial body, regardless the spatial distance. As a result, the net gravity-gradient torque on the rigid body in space could be either extremely complicated or relatively simple, dependent on the position of the rigid body.

However, the significance of understanding spacecraft dynamics and orbital mechanics is still immense. Many technologies we used in daily life originally came from the research and development for space exploration. From Light-Emitting Diodes (LEDs), solar panels, to microwaves; from better software and structural analysis to water purification systems [10], we are very lucky today to experience the exponential growth of technology since the industrial revolution in the 18th century.

The root of rigorous study of orbital mechanics can be traced back to the 17th century when a brilliant young Englishman named Isaac Newton (1642 – 1727) put forward his laws of motion and formulated his law of universal gravitation. 176 Years after Newton, in 1903, the first successfully airplane, defying gravity and skepticism of aerodynamics, was built by the Wright Brothers in the United States. [11] Then through 2 World Wars and only 66 years after the great invention of the Wright Brothers, in 1969, the humans landed on the Moon, marking a new age on the history of human civilization, as well as an important step for humans as a species — for the first time ever, life goes beyond Earth.

To conclude this paper, I would like to emphasize the importance of spacecraft dynamics, as well as space exploration in general, and refer to a question asked by Elon Musk and his answer followed.

> *What are the important steps in the evolution of life?*
>
> *Obviously there was the advent of single-celled life. There was differentiation to plants and animals. There was life going from the oceans to land. There was [were] mammals... consciousness. And I would argue also on that scale, should fit 'life goes multi-planetary'. I think if one could make a reasonable argument that something is important enough to fit on the scale of evolution. Then it's... it's important. And maybe worth a little bit of our resources.* [12] — Elon Musk, CEO of SpaceX and Tesla Motors

5 Acknowlegement

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6 Appendix

6.1 Inspiring moments in the research of this paper

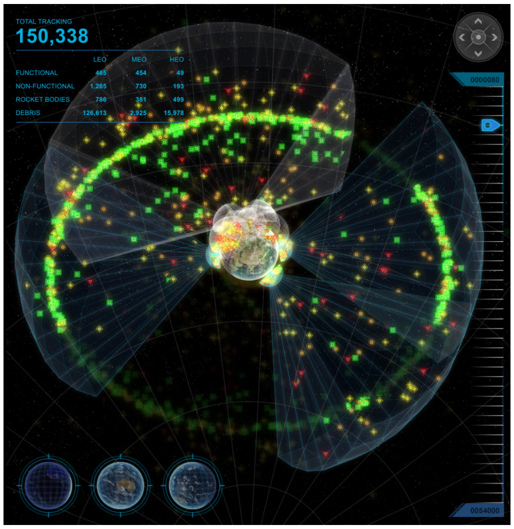

Figure 3: Lockheed Martin's Space Fence Program tracking over 150 thousand pieces of space junk simultaneously [13]

Figure 4: A long-exposure photograph reveals the apparent rotation of the stars around the Earth. (Photograph ©1992 Philip Greenspun.) [14]

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