Control Theory

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目录

1 Introduction

After obtaining the mathematical model of a system, several methods can be employed to analyse its dynamic and steady-state performance. Classical control theory offers various techniques to analyse the performance of linear control systems, including the time domain analysis method, root locus method, and frequency domain analysis method. Although different methods possess distinct characteristics and applications, the time domain analysis method directly analyses the system in the time domain, providing intuitive and precise results that convey comprehensive information on the system's time response.

2 Method 1: Time Response

By querying the Laplace transform and inverse Laplace transform table, it is much easier than carrying out convolution, therefore, there are 4 steps to find the time response of a system to a typical input signal.

1. Input signal in s domain $X(s)$
2. System transfer function in s domain $H(s)$
3. The response in s domain is $X(s)H(s)$
4. Inverse Laplace transform to find response in time domain

3 Typical Input Signal

所谓典型输入信号，是指根据系统常遇到的输入信号形式，在数学描述上加以理想化的一些基本输入函数。控制系统中常用的典型输入信号有：单位阶跃函数、单位斜坡（速度）函数、单位加速度（抛物线）函数、单位脉冲函数和正弦函数.

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$

4 Stability Analysis

impulse response $\lim_{t\to\infty} c(t) = 0$

$$C(s) = \frac{M(s)}{D(s)} = \sum_{i=1}^{n} \frac{A_i}{s - s_i} = \frac{K\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{q}(s - s_j)\prod_{k=1}^{r}\left(s^2 + 2\zeta_k\omega_k s + \omega_k^2\right)} \tag{1}$$

$q + 2r = n$ assume $0 < \zeta_k < 1$

$$C(s) = \sum_{j=1}^{q} \frac{A_j}{s - s_j} + \sum_{k=1}^{r} \frac{B_k s + C_k}{s^2 + 2\zeta_k\omega_k s + \omega_k^2} \tag{2}$$

where $A_j = \lim_{s\to s_j}(s - s_j)\, C(s); \quad j = 1, 2 \ldots q$

$$c(t) = \sum_{j=1}^{q} A_j \mathrm{e}^{s_j t} + \sum_{k=1}^{r} B_k \mathrm{e}^{-\zeta_k\omega_k t} \cos\left(\omega_k\sqrt{1-\zeta_k^2}\right) t + \tag{3}$$

$$\sum_{k=1}^{r} \frac{C_k - B_k\zeta_k\omega_k}{\omega_k\sqrt{1-\zeta_k^2}} \mathrm{e}^{-\zeta_k\omega_k t} \sin\left(\omega_k\sqrt{1-\zeta_k^2}\right) t \tag{4}$$

It can be seen that the necessary and sufficient condition for the stability of a linear system is that all roots of the characteristic equation of the closed-loop system have negative real parts: in other words, the poles of the closed-loop transfer function are located in the left half plane of s. 线性系统稳定的充分必要条件是：系统特征方程的根（即系统的闭环传递函数的极点）全部为负实数或具有负实部的共轭复数。（就是根全都在左半平面。这里要注意：左侧就是左侧，哪怕在虚轴上都不可以)

4.1 Routh-Hurwitz Stability Criteria

s^n	a_0	a_2	a_4	a_6	...
s^{n-1}	a_1	a_3	a_5	a_7	...
s^{n-2}	$c_{13} = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = \frac{a_1a_2 - a_0a_3}{a_1}$	$c_{23} = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}$	$c_{33} = \frac{a_1a_6 - a_0a_7}{a_1}$	c_{43}	...
s^{n-3}	$c_{14} = \frac{-1}{c_{13}}\begin{vmatrix} a_1 & a_3 \\ c_{13} & c_{23} \end{vmatrix} = \frac{c_{13}a_3 - a_1c_{23}}{c_{13}}$	$c_{24} = -\frac{1}{c_{13}}\begin{vmatrix} a_1 & a_5 \\ c_{13} & c_{33} \end{vmatrix}$	$c_{34} = \frac{c_{13}a_7 - a_1c_{43}}{c_{13}}$	c_{44}	...
s^{n-4}	$c_{15} = -\frac{1}{c_{14}}\begin{vmatrix} c_{13} & c_{23} \\ c_{14} & c_{24} \end{vmatrix} = \frac{c_{14}c_{23} - c_{13}c_{24}}{c_{14}}$	$c_{25} = \frac{c_{14}c_{33} - c_{13}c_{34}}{c_{14}}$	$c_{35} = \frac{c_{14}c_{43} - c_{13}c_{44}}{c_{14}}$	c_{45}	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^2	$c_{1,n-1}$	$c_{2,n-1}$			
s^1	$c_{1,n}$				
s^0	$c_{1,n+1}$				

1. Case 1: zero in the first column

   (某行的第一列项为零，而其余各项不为零，或不全为零) replace the zero with a small number ϵ and continue.

2. Case 2: zero row

   当劳斯表中出现全零行时，可用全零行上面一行的系数构造一个辅助方程 F(s)=0 (Auxiliary equation), 并将辅助方程对复变量 s 求导，用所得导数方程的系数取代全零行的元，便可按劳斯稳定判据的要求继续运算下去，直到得出完整的劳斯计算表。辅助方程的次数通常为偶数，它表明数值相同但符号相反的根数。所有那些数值相同但符号相异的根，均可由辅助方程求得。辅助方程的根也是判断系统稳定性的重要依据。

   (a) 虚轴上重复的根（不稳定）

   (b) 多个虚轴上不同的根（Marginally stable)

5 Dynamic Performance

5.1 Dynamic Performance

描述稳定的系统在单位阶跃函数作用下，动态过程随时间 t 的变化状况的指标。通常，用 T_r 或者 T_p 评价系统的响应速度；用%OS 评价系统的阻尼程度；T_s 是同时反映响应速度和阻尼程度的综合性指标。

1. Rising Time T_r

   指响应从终值 10% 上升到终值 90% 所需的时间；对于有振荡的系统，亦可定义为响应从零第一次上升到终值所需的时间。上升时间是系统响应速度的一种度量。上升时间越短，响应速度越快。

2. Peak Time T_p

   指响应超过其终值到达第一个峰值所需的时间。

3. Settling Time T_s

   The time for the response to reach and stay within 2% of the final value.

4. Percentage Overshoot %OS

   指响应的最大偏离量 $c(T_p)$ 与终值 $c(\infty)$ 的差与终值 $c(\infty)$ 比的百分数

$$\%OS = \frac{c\left(t_p\right) - c(\infty)}{c(\infty)} \times 100\% \tag{5}$$

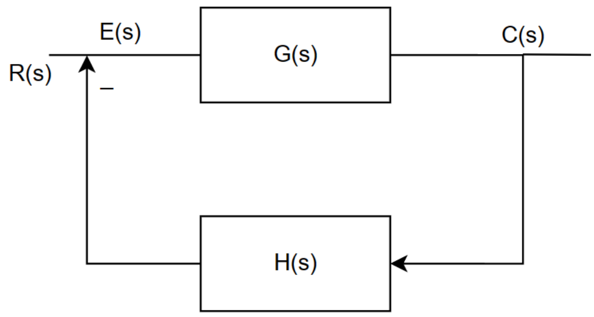


图 1: 误差

6 Steady State Performance (Related to Error)

稳态误差是描述系统稳态性能的一种性能指标，通常在阶跃函数、斜坡函数或加速度函数作用下进行测定或计算。若时间趋于无穷时，系统的输出量不等于输入量或输入量的确定函数，则系统存在稳态误差。稳态误差是系统控制精度或抗扰动能力的一种度量。

6.1 Steady State Error

According to Figure 1:

$$E(s) = R(s) - H(s)C(s) \tag{6}$$

$$E(s)G(s) = C(s) \tag{7}$$

substitute equation (7) to equation (6), we can get the error transfer function:

$$\Phi_e(s) = \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} \tag{8}$$

error signal $e(t)$ include two parts: transient error $e_{ts}(t)$ and steady state error $e_{ss}(t)$, we assume the system is stable, therefore, when $t \to \infty$ we must have $e_{ts}(t) \to 0$. we have

$$\lim_{x \to \infty} x(t) = \lim_{s \to 0} X(s) \tag{9}$$

therefore:

$$e_{ss}(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)} \tag{10}$$

6.2 System Type

To be more precisely, the open-loop transfer function can be written as:

$$G(s)H(s) = \frac{K\prod_{i=1}^{m}(\tau_i s+1)}{s^{\nu}\prod_{j=1}^{n-v}(T_j s+1)} \tag{11}$$

K is the open loop gain.

let $F(s) = \frac{K}{s^v}G_0(s)H_0(s)$ where

$$G_0(s)H_0(s) = \prod_{i=1}^{m}(\tau_i s+1) \,/\, \prod_{j=1}^{n-v}(T_j s+1) \tag{12}$$

we can get:

$$\lim_{s\to 0} G_0(s)H_0(s) = 1 \tag{13}$$

therefore, the steady state error can be rewritten as:

$$e_{ss}(\infty) = \frac{\lim_{s\to 0}\left[s^{v+1}R(s)\right]}{K + \lim_{s\to 0} s^v} \tag{14}$$

分母中的因子 s^v 表明开环传递函数中含有个积分单元，工程上按照 v 的值分别称系统为 0 型，1 型，2 型...

6.3 Error Constant

As shown in equation (14), for a certain input signal, the steady state error only depends on the open-loop transfer function. When we apply three typical inputs to the system, e_{ss}:

1. unit step $R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s\to 0}\frac{1}{1+G(s)H(s)} \tag{15}$$

2. unit ramp $R_(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0}\frac{1}{sG(s)H(s)} \tag{16}$$

3. unit acceleration $R_(s) = \frac{1}{s^3}$

$$e_{ss} = \lim_{s\to 0}\frac{1}{s^2G(s)H(s)} \tag{17}$$

To simplify the equation, use G_{open} to substitute $G(s)H(s)$.

define:

Position constant:

$$K_p = \lim_{s \to 0} G_{\text{open}}(s) \tag{18}$$

Velocity constant;

$$K_v = \lim_{s \to 0} sG_{\text{open}}(s) \tag{19}$$

Acceleration constant:

$$K_a = \lim_{s \to 0} s^2 G_{\text{open}}(s) \tag{20}$$

substitute equation (18),(19),(20) into equation (15),(16),(17) we can get:

$$e_{ss} = 1/(1+K_p) \tag{21}$$

$$e_{ss} = 1/K_v \tag{22}$$

$$e_{ss} = 1/K_a \tag{23}$$

According to the principle of linear superposition, each input component can act on the system separately, and then the steady-state error component can be superimposed to obtain:

$$e_{ss}(\infty) = \frac{R_0}{1+K_p} + \frac{R_1}{K_v} + \frac{R_2}{K_a} \tag{24}$$

6.4 Relationship between system type and error constant

substitute equation (11)(13) into (18)(19)(20), we can get

$$K_p = \lim_{s \to 0} K/s^v; \quad K_v = \lim_{s \to 0} K/s^{v-1}; \quad K_a = \lim_{s \to 0} K/s^{v-2} \tag{25}$$

type	error constant			step $r(t) = R \cdot 1(t)$	ramp $r(t) = Rt$	acceleration $r(t) = \frac{Rt^2}{2}$
	K_p	K_v	K_a	$\mathrm{e_s} = \frac{R}{1+K_p}$	$\mathrm{e}_s = \frac{R}{K_v}$	$\mathrm{e}_s = \frac{R}{K_a}$
0	K	0	0	$\frac{R}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{R}{K}$	∞
2	∞	∞	K	0	0	$\frac{R}{K}$
3	∞	∞	∞	0	0	0

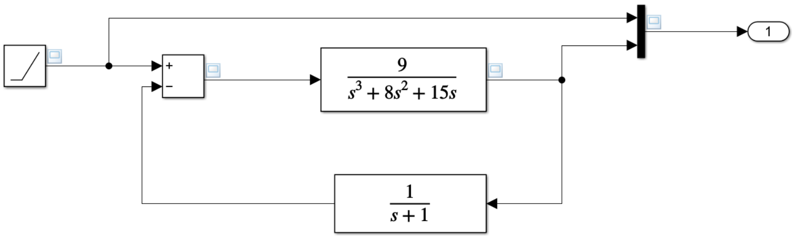


图 2: 二类系统对斜坡信号输入的 MatLab 模拟

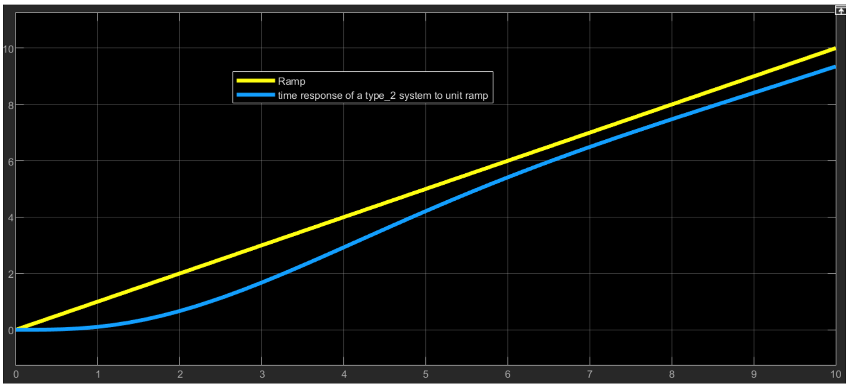


图 3: 二类系统的响应和原输入信号的对比

用 0 型系统跟踪恒速变化的信号时，它的输出量的速度总是赶不上输入信号的速度，以致差距愈来愈大，1 型系统则能以同样速度跟踪恒速变化的信号，但有一定的静态误差，以致输出量总比输入信号“落后”一个固定的量，输入信号变化的速度愈大，落后的量也愈大。0 型和 1 型系统都不能跟踪恒加速度信号，而 2 型系统能跟踪恒加速度信号，但有静态误差，换句话说，它的输出量能与输入信号以同一加速度和同一速度变化，但总是“落后”一个固定的量 e_{ss}。

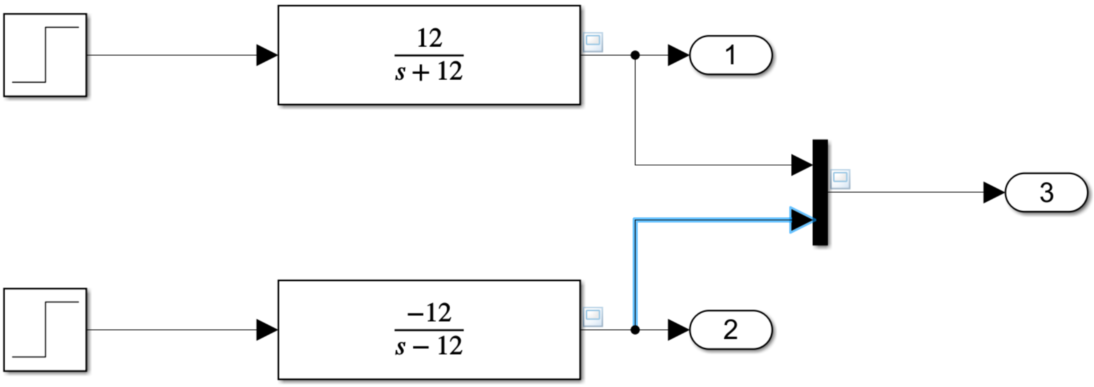


图 4: (一阶系统) 在 Matlab 的 Simulink 模块中演示 a 对系统响应的影响

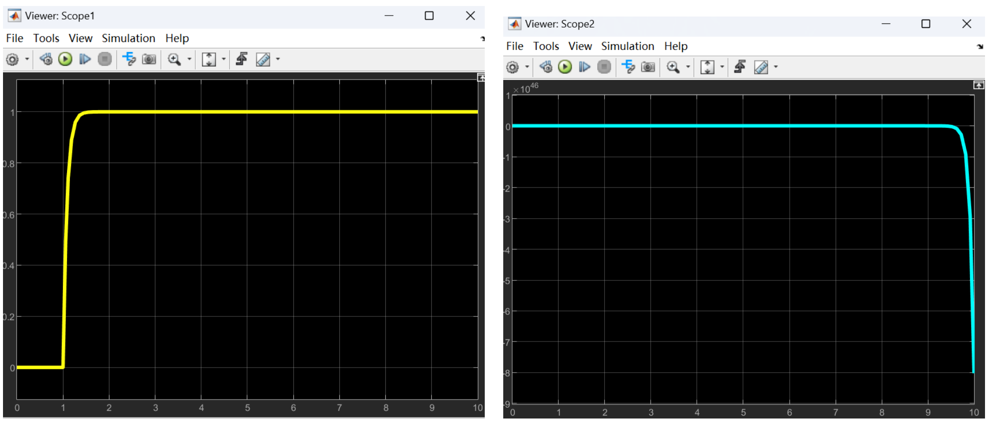
7 1st Order System

(still use step response as input) Transfer function:

$$G(s) = \frac{a}{s+a} \tag{26}$$

$$\mathcal{L}^{-1}\frac{a}{s(s+a)} = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+a}\right) = 1 - e^{-at} \tag{27}$$

we can notice that, when $a > 0$ the time response is convergent, while when $a < 0$ the step response is divergent.



(a) time response with a>0

(b) time response with a<0

图 5: time response

7.1 Dynamic Performance for 1st Order system

the time response,

$$c(t) = 1 - e^{-at} = 1 - e^{-\frac{t}{T}} \tag{28}$$

define $T = \frac{1}{a}$ to be the time constant for the response. when $t = T$ the response of the system $c(T) = 1 - e^{-1} = 0.63$ which is 63% of the final value.

1. Rising Time T_r

$$T_r = t_{c=0.9} - t_{c=0.1} \tag{29}$$

$$1 - e^{-at} = A \tag{30}$$

$$t = -\frac{ln(1-A)}{a} \tag{31}$$

$$T_r = t_{A_1=0.9} - t_{A_2=0.1} = \frac{2.31}{a} - \frac{0.11}{a} \tag{32}$$

2. Settling Time T_s There is not damping in the output waveform, therefore

$$T_s = t_{c=1-2\%\times 1} \tag{33}$$

$$T_s = -\frac{ln(1-(1-2\% \times 1)}{a} = \frac{3.9}{a} \cong \frac{4}{a} = \frac{4}{T} \tag{34}$$

8 2nd Order System

Assume the general system

$$G(s) = \frac{b}{s^2 + as + b} \tag{35}$$

1. Explanation for Over damped Responses ($\sigma > 0$)

$$C(s) = \frac{b}{s(s+\sigma_1)(s+\sigma_2)} = \frac{A}{s} + \frac{B}{(s+\sigma_1)} + \frac{C}{(s+\sigma_2)} \to A + Be^{-\sigma_1 t} + Ce^{-\sigma_2 t} \tag{36}$$

$$A = \frac{b}{(s+\sigma_1)(s+\sigma_2)}|_{s\to 0} B = \frac{b}{s(s+\sigma_2)}|_{s\to -\sigma_1} C = \frac{b}{s(s+\sigma_1)}|_{s\to -\sigma_2} \tag{37}$$

(mentioned before, if any pole appears at the right hand side complex plane, the time response will not be convergent, therefore, only consider the circumstances where poles appear at LHP)

2. Explanation for Under damped Responses

$$C(s) = \frac{b}{s(s+\sigma_d+j\omega_d)(s+\sigma_d-j\omega_d)} = \frac{A}{s} + \frac{B}{(s+\sigma_d+j\omega_d)} + \frac{C}{(s+\sigma_d-j\omega_d)} \tag{38}$$

Apply inverse Laplace transform

$$= K_1 + K_2 e^{-(\sigma_d+j\omega_d)t} + K_3 e^{-(\sigma_d-j\omega_d)t} \tag{39}$$

$$= K_1 + e^{-\sigma_d t}(K_2 e^{-(j\omega_d)t} + K_3 e^{j\omega_d t}) \tag{40}$$

$$= K_1 + e^{t}[K_2(cos\omega_d t - jsin\omega_d t) + K_3(cos\omega_d t + jsin\omega_d t)] \tag{41}$$

$$= K_1 + e^{-\sigma_d t}(K_4 cos\omega_d t + K_5 sin\omega_d t) \tag{42}$$

$$= K_1 + K_6 e^{-\sigma_d t} cos(\omega_d t - \phi) \tag{43}$$

3. Explanation for Un damped Responses

$$C(s) = \frac{b}{s(s+j\omega_d)(s-j\omega_d)} \tag{44}$$

Apply inverse Laplace transform

$$= K_1 + K_2 e^{-j\omega_d t} + K_3 e^{j\omega_d t} = K_1 + K_2(cos\omega_d t - sin\omega_d t) + K_3(cos\omega_d t + isin\omega_d t) \tag{45}$$

$$= K_1 + (K_4 cos\omega_d t + K_5 sin\omega_d t) \tag{46}$$

$$= K_1 + K_6 cos(\omega_d t - \phi) \tag{47}$$

4. Explanation for Critical damped Responses

$$C(s) = \frac{b}{s(s+\sigma_d)^2} = \frac{A}{s} + \frac{B}{(s+\sigma_d)} + \frac{C}{(s+\sigma_d)^2} \tag{48}$$

apply inverse Laplace transform

$$= K_1 + K_2 e^{-\sigma_d} + K_3 t e^{-\sigma_d} \tag{49}$$

response type	poles	time response	
Over damped Responses	2 real at -σ_1 and -σ_2	$c(t) = K_1e^{-\sigma_1 t} + K_2e^{-\sigma_2 t}$	Stable
Under damped Responses	2 complex at $-\sigma_d \pm j\omega_d$	$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$	Stable
Un damped Responses	2 imaginary at $\pm j\omega_d$	$c(t) = A\cos(\omega_d t - \phi)$	Unstable
Critical damped Responses	2 real at -σ_d	$c(t) = K_1e^{-\sigma_d t} + K_2te^{-\sigma_d t}$	Stable

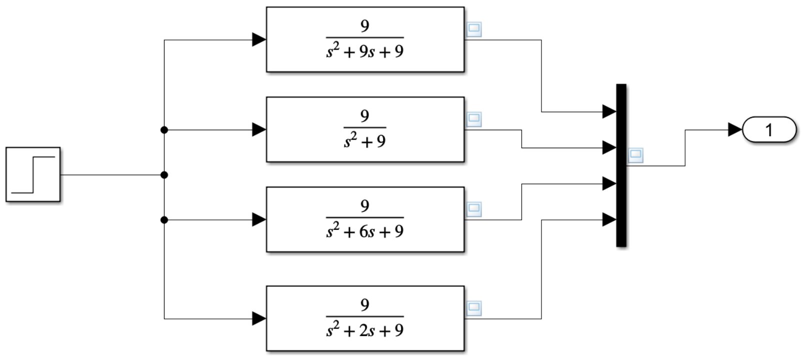


图 6: 不同阻尼情况

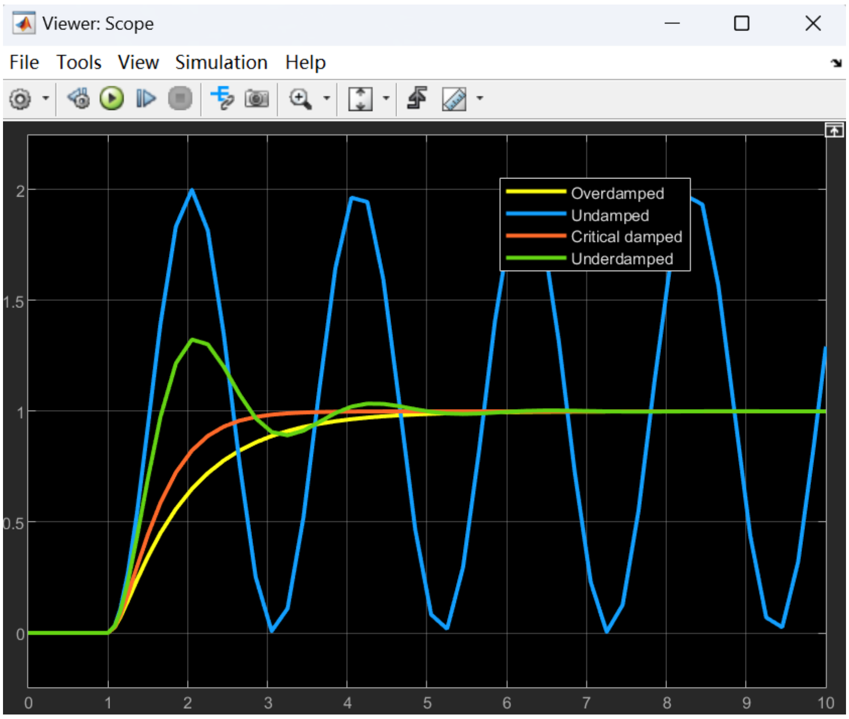


图 7: 对比不同阻尼效果

ζ	poles	Response Type	Stability
$\zeta < 0$	两个正实部根	divergent	Unstable
$\zeta = 0$	纯虚根	Un Damped	Unstable
$0 < \zeta < 1$	负实部共轭复根	Under Damped	Stable
$\zeta = 1$	两个相等负实根	Critical Damped	Stable
$\zeta > 1$	两个不相等负实根	Over Damped	Stable

表 3: ζ 对二阶系统的极点以及响应的影响

Assume the general system

$$\Phi(s) = \frac{b}{s^2 + as + b} \tag{50}$$

the generalised second order system open loop transfer function is

$$\Phi(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + (1 - \zeta^2)\,\omega_n^2} \tag{51}$$

therefore, $b = \omega_n^2$ ω_n 是自然频率（无阻尼振荡频率）the complex pole of the general system have real part $= \sigma = \frac{-a}{2}$ the magnitude of this value is the exponential decay frequency.

define

$$\zeta = \frac{|\sigma|}{\omega_n} = \frac{\frac{a}{2}}{\omega_n} \tag{52}$$

the closed loop poles is the solution of the characteristics function:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{53}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{54}$$

二阶系统的性质取决于 ζ 的大小。

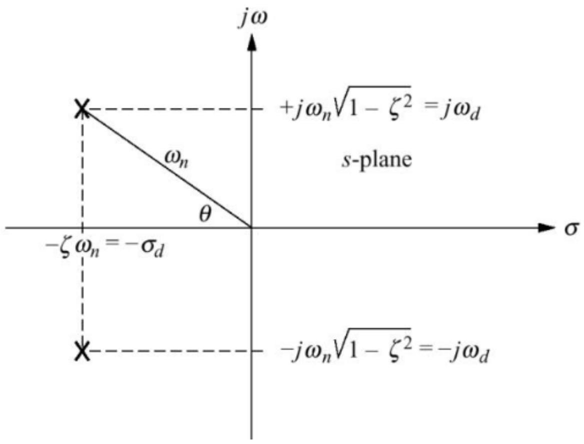


图 8: 欠阻尼二阶系统的极点图

8.1 Dynamic Performance of 2nd Order System

(Use Under damped system as an example)

In the table 2, two poles of the under-damped system are $-\sigma_d \pm j\omega_d$, where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. ($\sigma$ 称为衰减系数，ω_d 叫做阻尼振荡频率)

the time response of the system to the unit step signal input is:

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} \tag{55}$$

Apply inverse Laplace transform:

$$c(t) = 1 - \mathrm{e}^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] \tag{56}$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right) \tag{57}$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\sin\left(\omega_d t + \theta\right) \tag{58}$$

where $\theta = \arctan\left(\sqrt{1-\zeta^2}/\zeta\right)$

suggests that: Under damped second order system is made up of 2 components steady state component is 1 and the transient component is the damped sinusoidal oscillation term (with frequency ω_d) From Figure 6 we can deduce that:

$$\zeta = \cos\theta \tag{59}$$

8.1.1 Rising Time

(定义是响应函数从 0.1 到 0.9 所需要的时间，这里用响应函数从 0 到 1 所需要的时间近似，只要目的是展示响应速度和 ω_d 的关系)

$$c(T_r) = 1 \rightarrow \frac{1}{\sqrt{1-\zeta^2}} \mathrm{e}^{-\zeta\omega_n t_r} \sin\left(\omega_d t_r + \beta\right) = 0 \tag{60}$$

we can get:

$$t_r = \frac{\pi - \beta}{\omega_d} \tag{61}$$

when the damping ratio ζ is a constant, the response speed of the system is proportional to ω_n.

8.1.2 Peak Time

The peak time is the transient time when response value reaches the peak value. Therefore, the derivative of $c(t)$ is zero.

$$\zeta\omega_n \mathrm{e}^{-\zeta\omega_n t_p} \sin\left(\omega_d t_p + \theta\right) - \omega_d \mathrm{e}^{-\zeta\omega_n t_p} \cos\left(\omega_d t_p + \theta\right) = 0 \tag{62}$$

$$\tan\left(\omega_d t_p + \theta\right) = \frac{\sqrt{1-\zeta^2}}{\zeta} = tan\theta \tag{63}$$

we can get, $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \cdots$, therefore, $\omega_d = \pi$. peak time can be written as:

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{64}$$

8.1.3 % OS

$$\%OS = \frac{c\left(t_p\right) - c(\infty)}{c(\infty)} \times 100\% \tag{65}$$

$$c\left(t_p\right) = 1 - \frac{1}{\sqrt{1-\zeta^2}} \mathrm{e}^{-\pi\zeta/\sqrt{1-\zeta^2}} \sin(\pi + \theta) \tag{66}$$

substitute $c\left(t_p\right)$, $\sin(\pi+\theta) = -\sqrt{1-\zeta^2}$ and $c(\infty) = 1$ into equation (65). We can get:

$$\%OS = \mathrm{e}^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% \tag{67}$$

equation (67) suggests that $\%OS$ is only a function of the damping ratio and has nothing to do with the natural frequency ω_n

8.1.4 Settling Time

use Δ to indicate the error between transient response value with steady state value:

$$\Delta = \left| \frac{\mathrm{e}^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \beta\right) \right| \leq \frac{\mathrm{e}^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \tag{68}$$

As mentioned before, the settling time is the transient time for the response to reach and stay within 2% of the final value. Therefore, $\Delta = 0.02$ we can get

$$t_s = \frac{4.4}{\zeta\omega_n} = \frac{4.4}{\sigma} \tag{69}$$

9 Method 2: Root Locus

9.1 Introduction

For an unstable system, we are trying to move the roots to the left half complex plane or make roots have negative real parts, if we just utilize proportion feedback, no matter what gain we choose, the position of roots will never change. We need to figure out where to add poles and zeros in the feedback path. So how poles and zeros affect root locus?

9.1.1 system with unity feedback

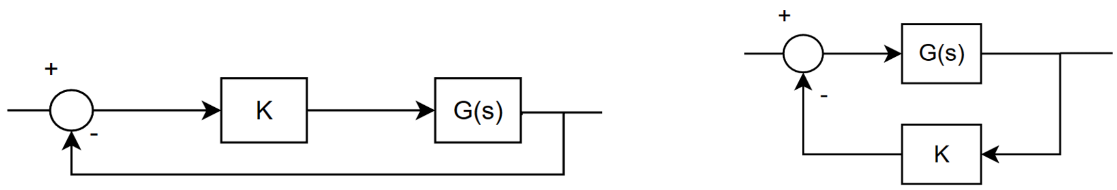


Figure 9. system with unity feedback

These two methods produce the same root locus therefore the same stability of the system (when it comes to root locus, we only care about the denominator)

$$\frac{s^2+s+1}{s^3+4s^2+ks+1} \tag{70}$$

For a closed-loop transfer function like (70), the denominator is not in the standard format, we could rewrite the denominator as:

$$\frac{s^3+4s^2+1}{s^3+4s^2+1}+\frac{Ks}{s^3+4s^2+1} \tag{71}$$

Therefore, the characteristic equation is still $1+KG(s)=0$, where $G(s)=\frac{s}{s^3+4s^2+1}$

9.1.2 system with feedback transfer function $H\left(s\right)$

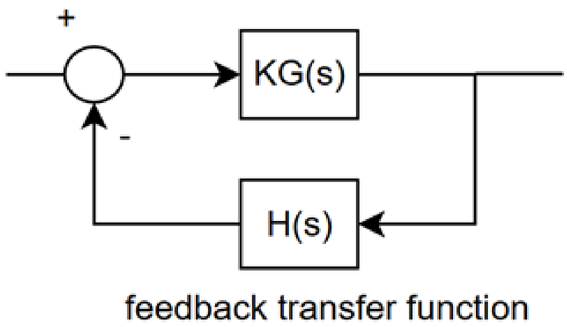


Figure 10. system with feedback transfer function $H\left(s\right)$

The open-loop transfer function is $KG\left(s\right)H\left(s\right)$, as mentioned before, variations in K do not affect the location of any pole of this function.

The closed-loop transfer function $T\left(s\right)$ is:

$$\frac{KG\left(s\right)}{1+KG\left(s\right)H\left(s\right)} \tag{72}$$

We can notice that the poles of $T\left(s\right)$ change with K.
Let us assume: $G\left(s\right)=\frac{N_G(s)}{D_G(s)}$ $H\left(s\right)=\frac{N_H(s)}{D_H(s)}$ Substitute to equation (3), it can be rewritten as:

$$T\left(s\right)=\frac{KN_G\left(s\right)D_H\left(s\right)}{D_G\left(s\right)D_H\left(s\right)+KN_G\left(s\right)N_H\left(s\right)} \tag{73}$$

Since the system's transient response and stability are dependent upon the poles of T(s), we have no knowledge of the system's performance unless we factor the denominator for specific values of K. The root locus will be used to give us a vivid picture of the poles of T(s) as K varies.

9.2 Vector

Any complex number $\sigma+j\ \omega$ can be described in polar form with magnitude M and angle θ. As $M\ \angle\theta$. Let us first assume a function $F\left(s\right)=s+a$, we can conclude that s+a is a complex number and can be represented by a vector drawn from the zeros of the function to the point s. Then we take poles into account, let us assume a function

$$F\left(s\right)=\frac{\prod_{i=1}^{m}\left(s+z_i\right)}{\prod_{j=1}^{n}\left(s+p_j\right)}=\frac{\Pi\ \text{numerator's complex factors}}{\Pi\ \text{denominator's complex factors}} \tag{74}$$

we can get: (by applying Euler formula)

$$\theta \ = \sum \ zero\ angles \ - \sum \ pole\ angles \tag{75}$$

$$= \sum_{i=1}^{m} \angle \left(s + z_i\right) - \sum_{j=1}^{n} \angle \left(s + p_j\right) \tag{76}$$

we can get the magnitude M:

$$M = \frac{\Pi \text{ zero lengths}}{\Pi \text{ pole lengths}} = \frac{\prod_{i=1}^{m} \left|\left(s + z_i\right)\right|}{\prod_{j=1}^{n} \left|\left(s + p_j\right)\right|} \tag{77}$$

9.3 Relationship between closed-loop poles zeros and open-loop poles zeros

In general, the open-loop transfer function with numerator order m and denominator order n can be expressed as:

$$G\left(s\right) H\left(s\right) = \frac{K \prod_{i=1}^{m} \left(\tau_i s + 1\right)}{s^v \prod_{j=1}^{n-v} \left(T_j s + 1\right)} \tag{78}$$

K is the open loop gain; τ_i and $T_j$are time constant; s determine the type of the system.

$$G\left(s\right) = \frac{K_G \left(\tau_1 s + 1\right) \left(\tau_2^2 s^2 + 2\zeta_1 \tau_2 s + 1\right) \cdots}{s^v \left(T_1 s + 1\right) \left(T_2^2 s^2 + 2\zeta_2 T_2 s + 1\right) \cdots} = K_G^* \frac{\prod_{i=1}^{f} \left(s - z_i\right)}{\prod_{i=1}^{q} \left(s - p_i\right)} \tag{79}$$

where K_G is the forward path gain and K_G^* is forward path root locus gain. And the feedback transfer function can be written as:

$$H\left(s\right) = K_H^* \frac{\prod_{j=1}^{l} \left(s - z_j\right)}{\prod_{j=1}^{h} \left(s - p_j\right)} \tag{80}$$

The open-loop transfer function can be written as:

$$G\left(s\right) H\left(s\right) = K^* \frac{\prod_{i=\mathbf{1}}^{f} \left(s - z_i\right) \prod_{j=\mathbf{1}}^{l} \left(s - z_j\right)}{\prod_{i=\mathbf{1}}^{q} \left(s - p_i\right) \prod_{j=\mathbf{1}}^{h} \left(s - p_j\right)} \tag{81}$$

Closed-loop transfer function:

$$\Phi\left(s\right) = \frac{K_G^* \prod_{i=1}^{f} \left(s - z_i\right) \prod_{j=1}^{h} \left(s - p_j\right)}{\prod_{i=1}^{n} \left(s - p_i\right) + K^* \prod_{j=1}^{m} \left(s - z_j\right)} \tag{82}$$

In order to simplify the calculation, we ignore the influence of $K_G; K_G^*; K_H^*$ The forward path transfer function:

$$G(s) = \frac{N_1(s)}{D_1(s)} = \frac{\prod_{i=1}^{n}(s - z_{1i})}{\prod_{i=1}^{n}(s - p_{1i})} \tag{83}$$

The feedback path transfer function:

$$H(s) = \frac{N_2(s)}{D_2(s)} = \frac{\prod_{i=1}^{n}(s - z_{2i})}{\prod_{i=1}^{n}(s - p_{2i})} \tag{84}$$

Closed-loop transfer function:

$$\Phi(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{N_1(s)}{D_1(s)}}{1 + \frac{N_1(s)}{D_1(s)}\frac{N_2(s)}{D_2(s)}} = \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) + N_1(s)N_2(s)} \tag{85}$$

$$= \frac{\prod_{i=1}^{n}(s - z_{1i})\prod_{i=1}^{n}(s - p_{2i})}{\prod_{i=1}^{n}(s - p_{1i})\prod_{i=1}^{n}(s - p_{2i}) + \prod_{i=1}^{n}(s - z_{1i})\prod_{i=1}^{n}(s - z_{2i})} \tag{86}$$

9.4 根轨迹的定义以及相关参数

As mentioned before, K^* can only control the position of closed loop poles. The closed-loop characteristic equation Is:

$$1 + G(s)H(s) = 0 \tag{87}$$

Substitute equation (81) into (87), we can get:

$$K^* \frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} = -1 \tag{88}$$

Any point that satisfies equation (87) and (88) is a point on the root locus. We know that -1 is represented in polar form as

$$1\angle(2k+1)\,180^\circ k = 0, \pm 1, \pm 2, \pm 3... \tag{89}$$

Combine equation (76) and (88) together, we can get:

$$\sum_{j=1}^{m}\angle(s - z_j) - \sum_{i=1}^{n}\angle(s - p_i) = (2k+1)\pi \tag{90}$$

From equation (88) we also can get:

$$K^* = \frac{\prod_{i=1}^{n} |s - p_i|}{\prod_{j=1}^{m} |s - z_j|} \tag{91}$$

$$K^* = \frac{1}{|G(s)\,H(s)|} = \frac{1}{M} = \frac{\Pi \text{ pole lengths}}{\Pi \text{ zero lengths}} \tag{92}$$

9.5 画根轨迹

9.5.1 Starting and ending point

We define the starting point of root locus to be point with root locus gain $K^* = 0$ and ending point to be point with $K^* = \infty$. According to equation (82) the characteristics equation for the closed-loop transfer function:

$$\prod_{i=1}^{n} (s - p_i) + K^* \prod_{j=1}^{m} (s - z_j) \;=\; 0 \tag{93}$$

When $K^* = 0$ we get

$$\prod_{i=1}^{n} (s - p_i) \;=\; 0 \tag{94}$$

therefore, $s = p_i$. We can deduce that, the root locus starts from the pole of open-loop transfer function. When $K^* = \infty$ we get

$$\frac{1}{K^*} \prod_{i=1}^{n} (s - p_i) + \prod_{j=1}^{m} (s - z_j) = 0 \tag{95}$$

therefore, $s = z_j$. We can deduce that, the root locus ends at the zero of open-loop transfer function.

9.5.2 Branches

The number of branches of the root locus equals the number of closed-loop poles. And the root locus is symmetrical about the real axis. (考虑一下定义以及共轭特性)

9.5.3 角度，渐近线，交点

There exists a straightforward approach to address the given inquiry. In general, the quantity of open-loop poles (n) is greater than the number of open-loop zeros (m). For instance, if there are four poles and one zero, it is reasonable to assume that there are three missing zeros. Consequently, the entire plane must be divided into three equivalent parts (120each). (The number of open-loop zeros and open-loop poles is equal in the sense of treating infinity as infinite zeros). Let us prove:

$$(s - z_1)(s - z_2)(s - z_3)\ldots\ldots(s - z_m) \tag{96}$$

$$= s^m - (z_1 + z_2 + z_3 + \ldots.. + z_m)\, s^{m-1} - (z_1 + z_2 + z_3 + \ldots.. + z_{m-1})\, s^{m-2} + \ldots \tag{97}$$

$$G(s)\,H(s) = K^* \frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} = K^* \frac{s^m + (-\sum_{j=1}^{m} z_j)s^{m-1} + \cdots + (-\sum_{j=1}^{2} z_j)s + b_m}{s^n + (-\sum_{i=1}^{m} p_i)s^{n-1} + \cdots + (-\sum_{i=1}^{2} p_i)s + a_n} \tag{98}$$

$$G(s)\,H(s) = K^* \frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} = K^* \frac{s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{99}$$

where:

$$b_1 = -\sum_{j=1}^{m} z_j \ a_1 = -\sum_{i=1}^{n} p_i \tag{100}$$

We know asymptotes refers to the root locus with $s \to \infty$, as s becomes big enough, equation (99) can be approximated as:

$$G(s)\,H(s) = \frac{K^*}{s^{n-m} + (a_1 - b_1)\, s^{n-m-1}} \tag{101}$$

As mentioned before, all the points on the root locus must satisfy equation (87)

Therefore:

$$s\left(1 + \frac{a_1 - b_1}{s}\right)^{\frac{1}{n-m}} = (-K^*)^{\frac{1}{n-m}} \tag{102}$$

Apply binomial theorem (103) on equation (102)

$$(1 + x)^a = \sum_{k=0}^{\infty} C_{\alpha}^{k} x^k = \sum_{k=0}^{\infty} \frac{(\alpha)\, k}{k!} x^k \tag{103}$$

$$\left(1 + \frac{a_1 - b_1}{s}\right)^{\frac{1}{n-m}} = 1 + \frac{a_1 - b_1}{(n-m)\, s} + \frac{1}{2!}\frac{1}{n-m}\left(\frac{1}{n-m} - 1\right)\left(\frac{a_1 - b_1}{s}\right)^2 + \cdots \tag{104}$$

When s becomes big enough, equation (104) can be approximated as:

$$\left(1+\frac{a_1-b_1}{s}\right)^{\frac{1}{n-m}}=1+\frac{a_1-b_1}{(n-m)\,s} \tag{105}$$

Substitute equation (105) into (102), we can get

$$s\left[1+\frac{a_1-b_1}{(n-m)\,s}\right]=(-K^*)^{\frac{1}{n-m}} \tag{106}$$

$$\left(\sigma+\frac{a_1-b_1}{n-m}\right)+\mathrm{j}\omega=(-K^*)^{\frac{1}{n-m}} \tag{107}$$

we know

$$-1=cos\frac{(2k+1)\,\pi}{n-m}+\mathrm{j}sin\frac{(2k+1)}{n-m} \tag{108}$$

Therefore, equation (106) can be written as:

$$\left(\sigma+\frac{a_1-b_1}{n-m}\right)+\mathrm{j}\omega=K^{*\frac{1}{n-m}}\left[\cos\frac{(2k+1)\,\pi}{n-m}+\mathrm{j}\sin\frac{(2k+1)\,\pi}{n-m}\right] \tag{109}$$

$$\sigma+\frac{a_1-b_1}{n-m}=\sqrt[n-m]{K^*}\cos\frac{(2k+1)\,\pi}{n-m} \tag{110}$$

$$\omega=\sqrt[n-m]{K^*}\sin\frac{(2k+1)\,\pi}{n-m} \tag{111}$$

Thus:

$$\sqrt[n-m]{K^*}=\frac{\omega}{\sin\varphi_a}=\frac{\sigma-\sigma_a}{\cos\varphi_a} \tag{112}$$

And:

$$\varphi_a=\frac{(2k+1)\,\pi}{n-m};k=0,1,\cdots,n-m-1 \tag{113}$$

$$\sigma_a=-\left(\frac{a_1-b_1}{n-m}\right)=\frac{\sum_{i=1}^{n}p_i-\sum_{j=1}^{m}z_j}{n-m} \tag{114}$$

9.5.4 闭环重根（分离点）

If the root locus lies between two adjacent open-loop poles on the real axis, one of which may be an infinite pole, then there is at least one separation point between these two poles; likewise, if the root locus lies on the real axis between two Between adjacent open-loop zeros, one of which may be an infinite zero, there is also at least one separation point between these two zeros.

According to equation (82) the characteristics equation for the closed-loop transfer function:

$$D\left(s\right)=\prod_{i=1}^{n}\left(s-p_{i}\right)+K^{*}\prod_{j=1}^{m}\left(s-z_{j}\right)=0 \tag{115}$$

Root locus encounter, indicating that the closed-loop characteristic equation has multiple roots. $f\left(x\right)$ 是 x 的多项式，$f\left(a\right)=f'\left(a\right)=0$ 且 $f''\left(a\right)\neq 0$ 那么 $f\left(a\right)$ 有二重重根。

We already have:

$$D\left(s\right)=\prod_{i=1}^{n}\left(s-p_{i}\right)+K^{*}\prod_{j=1}^{m}\left(s-z_{j}\right)=0 \tag{116}$$

$$\prod_{i=1}^{n}\left(s-p_{i}\right)=-K^{*}\prod_{j=1}^{m}\left(s-z_{j}\right) \tag{117}$$

$$D'\left(s\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left[\prod_{i=1}^{n}\left(s-p_{i}\right)+K^{*}\prod_{j=1}^{m}\left(s-z_{j}\right)\right]=0 \tag{118}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\prod_{i=1}^{n}\left(s-p_{i}\right)=-K^{*}\frac{\mathrm{d}}{\mathrm{d}s}\prod_{j=1}^{m}\left(s-z_{j}\right) \tag{119}$$

let equation (119) over (117), we get:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}s}\prod_{i=1}^{n}\left(s-p_{i}\right)}{\prod_{i=1}^{n}\left(s-p_{i}\right)}=\frac{\frac{\mathrm{d}}{\mathrm{d}s}\prod_{j=1}^{m}\left(s-z_{j}\right)}{\prod_{j=1}^{m}\left(s-z_{j}\right)} \tag{120}$$

Let

$$y\left(s\right)=\prod_{i=1}^{n}\left(s-p_{i}\right) \tag{121}$$

then

$$lny=ln\left(s-p_{1}\right)+ln\left(s-p_{2}\right)+\ldots \tag{122}$$

Derivation on both sides:

$$\frac{1}{y}y'=\frac{1}{s-p_{1}}+\frac{1}{s-p_{2}}+\ldots \tag{123}$$

then

$$y\prime \;=\; y(\frac{1}{s-p_{1}}+\frac{1}{s-p_{2}}+\ldots) \tag{124}$$

We know that:

$$\ln\prod_{i=1}^{n}\left(s-p_{i}\right)=\prod_{i=1}^{n}\ln\left(s-p_{i}\right),\ln\prod_{j=1}^{m}\left(s-z_{j}\right)=\prod_{j=1}^{m}\ln\left(s-z_{j}\right) \tag{125}$$

Thus:

$$\frac{\mathrm{d}\ln\prod_{i=1}^{n}(s-p_i)}{\mathrm{d}s}=\frac{\mathrm{d}\ln\prod_{j=1}^{m}(s-z_j)}{\mathrm{d}s} \tag{126}$$

$$\sum_{i=1}^{n}\frac{\mathrm{d}\ln(s-p_i)}{\mathrm{d}s}=\sum_{j=1}^{m}\frac{\mathrm{d}\ln(s-z_j)}{\mathrm{d}s} \tag{127}$$

Finally, we can get:

$$\sum_{i=1}^{n}\frac{1}{s-p_i}=\sum_{j=1}^{m}\frac{1}{s-z_j} \tag{128}$$

10 Design Via Root Locus

10.1 Proportional controller

A Proportional controller is essentially an amplifier with adjustable gain. During signal conversion, the P controller only changes the gain (magnitude) of the signal without affecting its phase. In cascade compensation, increasing the controller gain K can improve the open-loop gain of the system and reduce the steady-state error of the system, thereby improving the control accuracy of the system.

10.2 integral controller

Systems that feed the integral of the error to the plant are called integral control systems. The output signal of the controller $m(t)$ is proportional to the integral of its input signal $e(t)$. I controller can increase system type, which is conducive to the improvement of the steady-state performance, but I controller will add an open-loop pole located at the original point, so that the signal produces a phase angle lag of 90°, which will influenced transient response of the system.

$$m(t)=K_i\int_0^t e(t)\mathrm{d}t \tag{129}$$

10.3 Derivative controller

systems that feed the derivative of the error to the plant are called derivative control systems. It should be pointed out that the differential controller only acts on the dynamic process, but has no effect on the steady-state process. Additionally, it is very sensitive to system noise, therefore,a single D controller should not be used alone in series with the plant in any case.

$$m(t) = K_d \frac{\mathrm{d}e(t)}{\mathrm{d}t} \tag{130}$$

10.4 Lead Lag compensator

If we differentiate $sin(t)$ we get $cos(t)$, we can deduce that differentiation (s) add positive phase to the output. Similarly, we can get Integration ($\frac{1}{s}$) add positive phase to the output.

1. we expect a lead compensator to add positive phase to the input, so that the output could lead before the input.

2. we expect a lag compensator to add negative phase to the input, so that the output could lag behind the input.

The relationship between Lead and Lag compensator are shown in table 4.

For example, the LEAD compensator is

$$G_c(s) = \frac{s+1}{s+2} \tag{131}$$

and the plant is

$$P(s) = \frac{(s+1.2)(s+2.2)}{(s+3.2)(s+4.2)} \tag{132}$$

the Bode plot simulated by Matlab is shown in 图 11.5.1.

the LAG compensator is

$$G_c(s) = \frac{s+0.111}{s+0.01} \tag{133}$$

and the plant is

$$P(s) = \frac{1}{(s+1)(s+2)(s+10)} \tag{134}$$

the Bode plot simulated by Matlab is shown in 图 11.5.2.

just by moving either the pole or zero closer to the origin could produce phase lead and phase lag.

名称	表达式	条件
Lead	$K\frac{s+\omega_z}{s+\omega_p}$	1. $\omega_p \neq 0$ 2. $\omega_z < \omega_p$ 3. zero is closer to the origin than the pole 4. 1 pole and 1 zero
Lag	$K\frac{s+\omega_z}{s+\omega_p}$	1. $\omega_p \neq 0$ 2. $\omega_z > \omega_p$ 3. pole is closer to the origin than the zero 4. 1 pole and 1 zero

表 4: Lead and Lag compensator

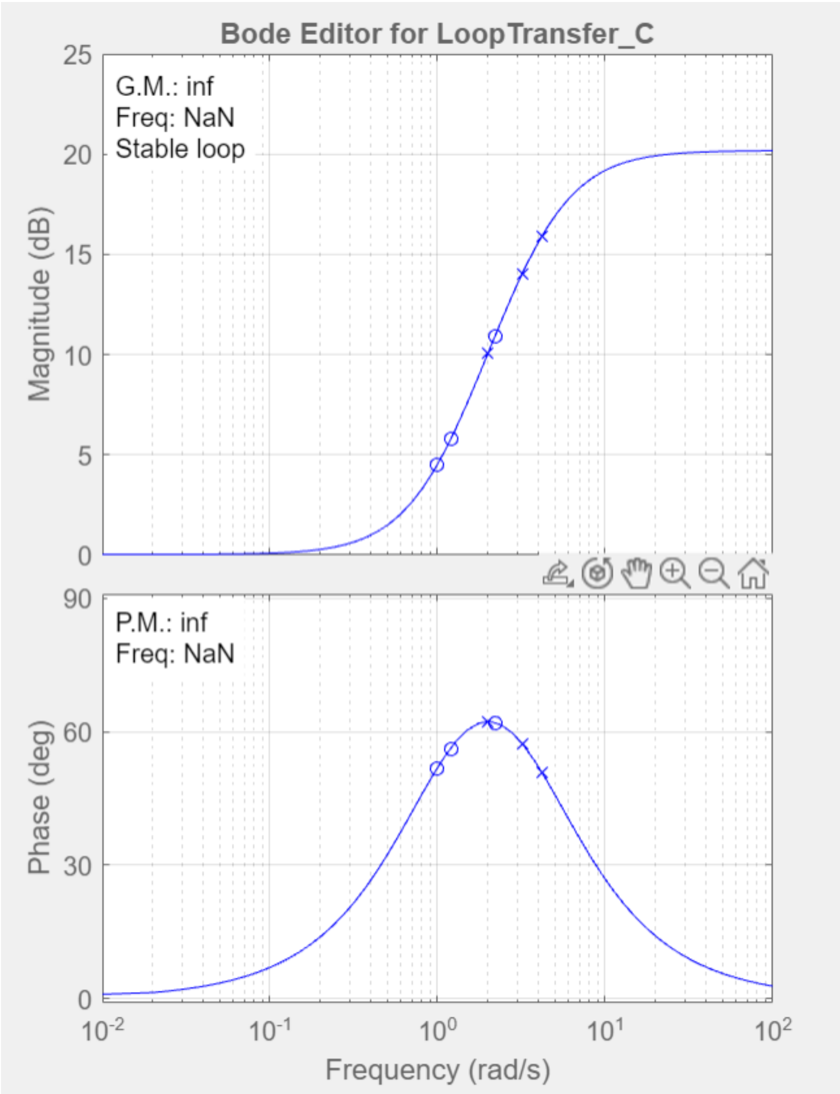


图 11.5.1. Bode plot for system with Lead compensator

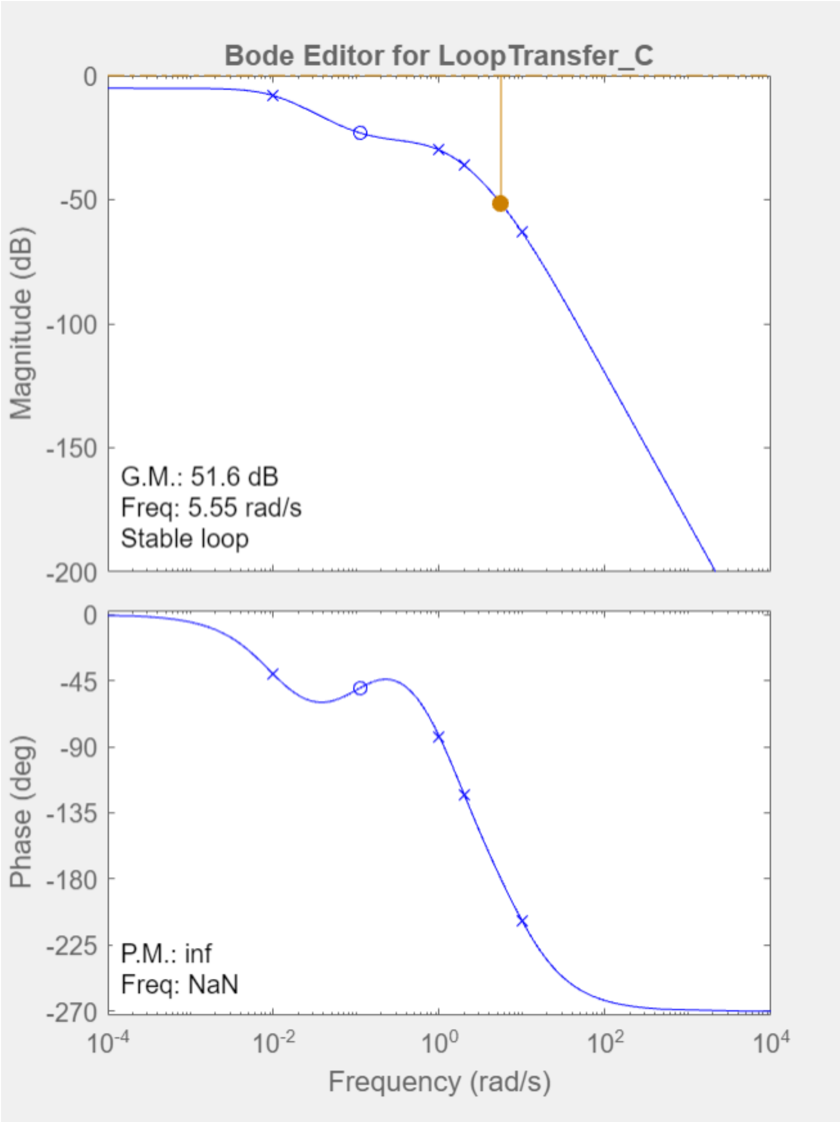


图 11.5.2. Bode plot for system with Lag compensator

10.5 Improving Steady State Error via Cascading Compensation

10.5.1 PI compensator

During cascading compensation, the PI controller adds an open-loop pole at the origin, as well as an open-loop zero at the left half of the s. The pole located at the origin can improve the type of the system to eliminate or reduce the steady-state error of the system and improve the steady-state performance of the system; The added negative real zero point is used to reduce the damping degree of the system and alleviate the adverse effects of the PI controller pole on the stability and dynamic process of the system (reduce the effect on transient response). The root locus is the position of closed-loop poles, as shown in 图 9.(b) if we add an integral controller, the original closed-loop pole no longer on the root locus, after adding a open loop zero, the original closed-loop poles will remain the same points on the compensated root locus.

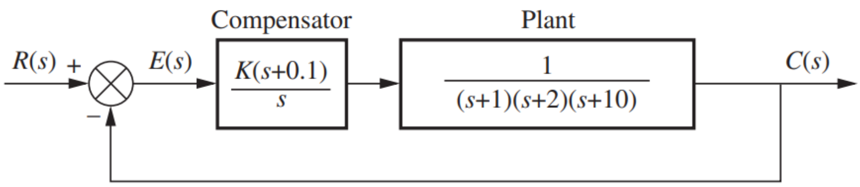


图 8. system with PI controller

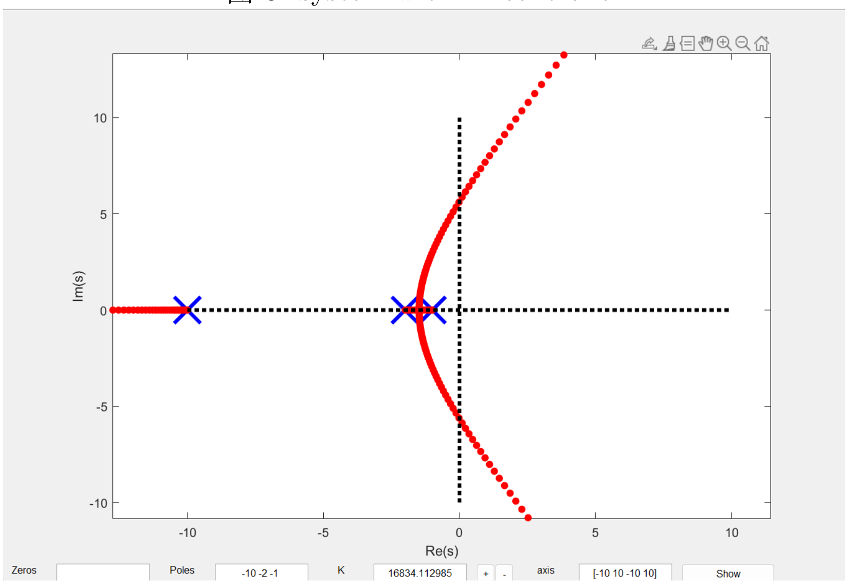


图 9. original system

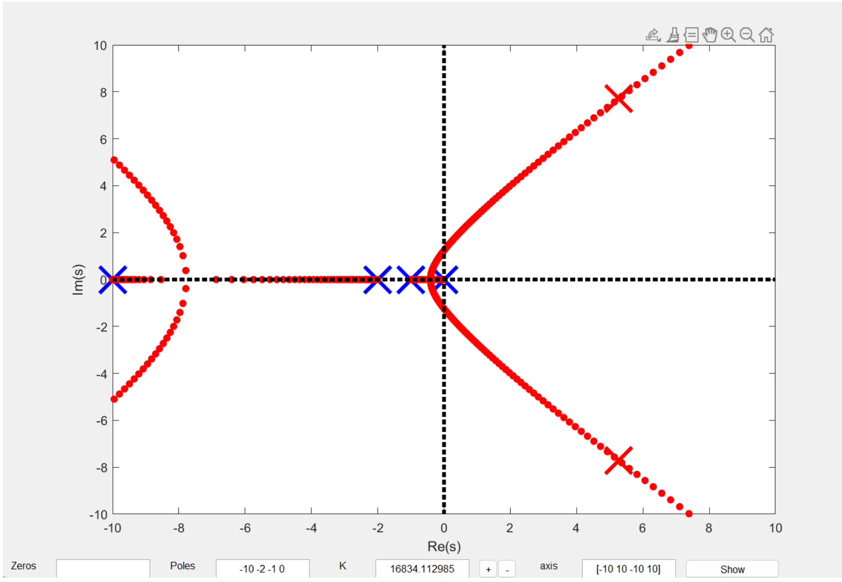


图 10. system with I controller

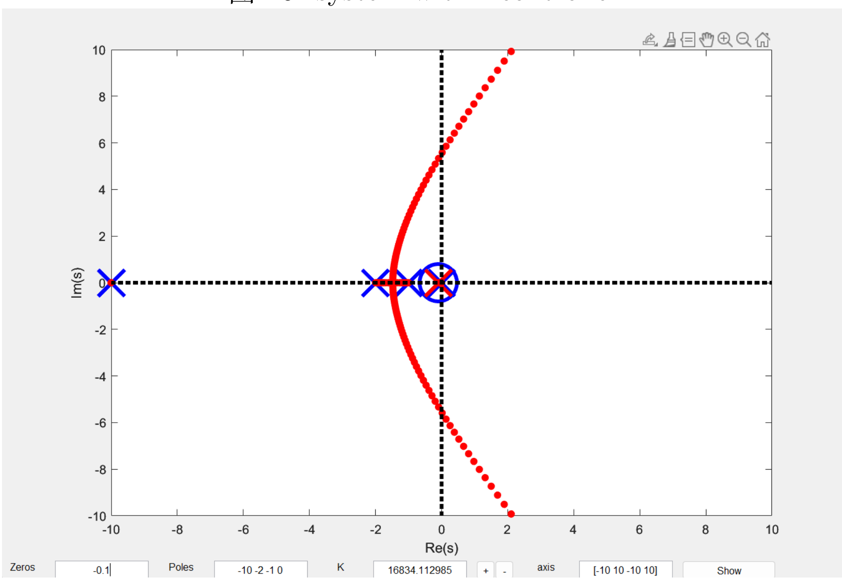


图 11. system with PI controller

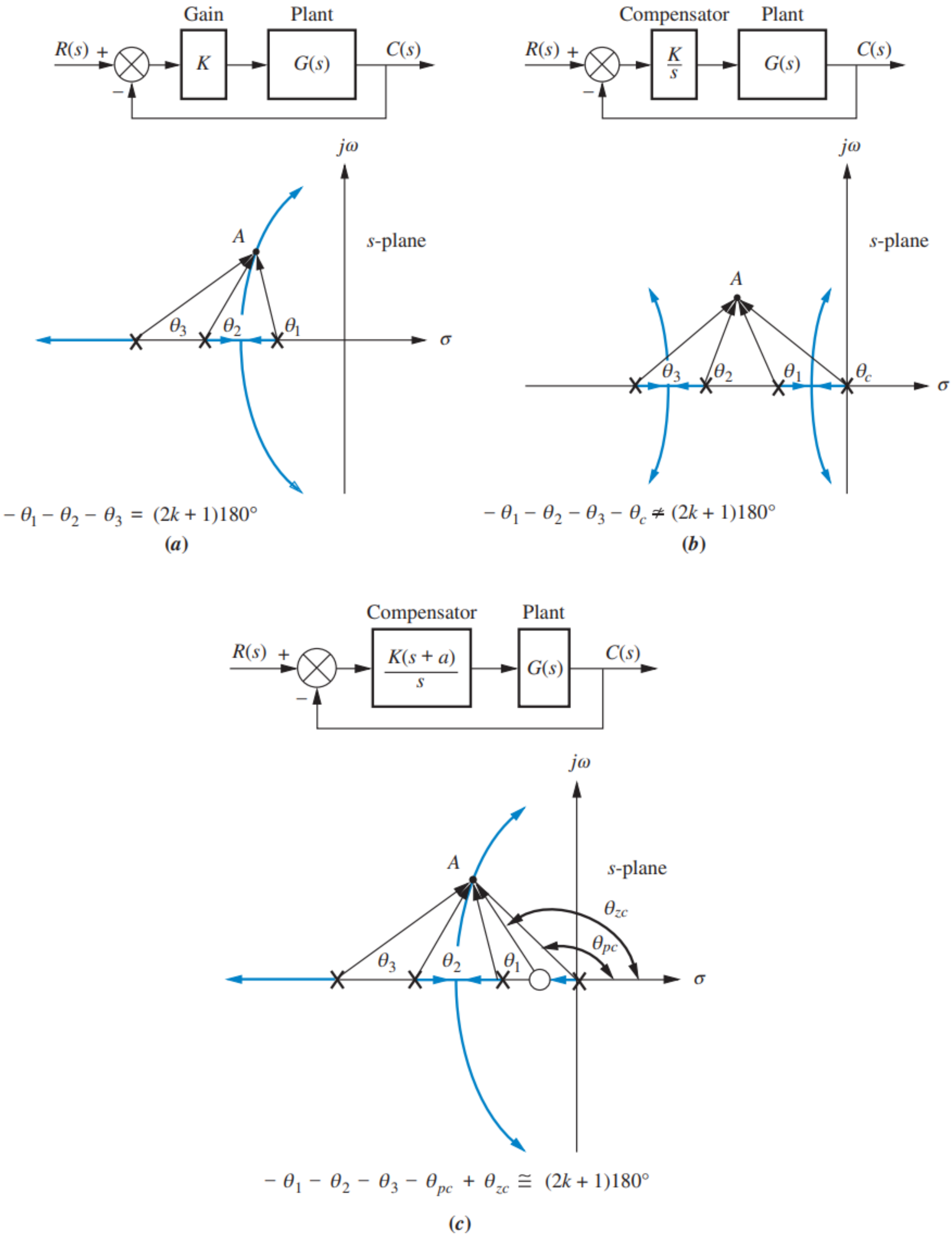
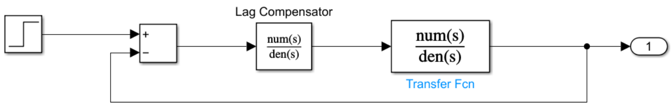


图 9: System with PI controller

10.5.2 Lag compensator

Lag and Lead compensator have same transfer function with different coefficient. As shown in 图 11.5.2. lag compensator add negative phase to the system. Lag compensator essentially drag the asymptotes slightly further into the right half plane. The main advantage of lag compensator is: lag compensator reduces $e_s s$. As shown in equation (10),

$$e_{ss}(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} \tag{135}$$



Lag compensator $G_c(s) = \frac{s+z}{s+p}$, the transfer function of the plant can be reduced to $\frac{G_n(s)}{G_d(s)}$, therefore the compensated e_{ssc} can be written as:

$$e_{ssc}(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+\frac{G_n(s)}{G_d(s)}\frac{s+z}{s+p}} \tag{136}$$

for a unit step as input, equation (136) can be reduced to:

$$e_{ssc}(\infty) = \frac{G_d(0)p}{G_d(0)p + G_n(0)z} \tag{137}$$

we could obtain that:

$$\frac{z}{p} = \frac{G_d(0) - e_{ssc}G_d(0)}{e_{ssc}G_n(0)} \tag{138}$$

we can adjust the steady state error by changing the ratio between the zero and the pole of the lag compensator. However, unlike ideal PI controller, error is improved but not driven to zero.

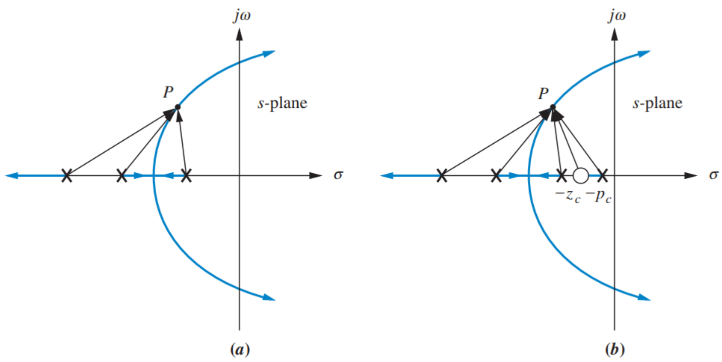


图 10.6.3. system with PI controller

10.6 Improving Transient Response via Cascading Compensation

10.6.1 PD compensator

By definition, the relationship between the input $e(t)$ and the output $m(t)$ of the PD compensator is:

$$m(t) = K_p e(t) + K_p \tau \frac{\mathrm{d}e(t)}{\mathrm{d}t} \tag{139}$$

the general form of the transfer function of PD compensator can be written as:$G_c(s) = s + z_c$

The main advantage of PD compensator is: speed up the original system. As mentioned in chapter 8.1.4, the settling time is an general indicator to show the respond speed of the system and the damping ratio.

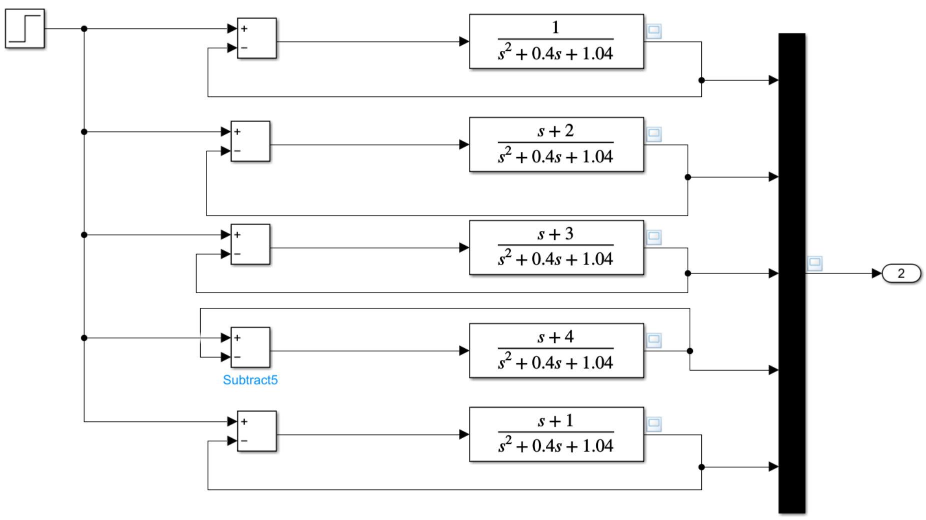
$$T_s = \frac{4}{\zeta \omega_n} \tag{140}$$

when operating with the same damping ratio (same θ), there will be a cross point A of the root locus and the radial line ($\angle\theta$), as shown in 图 8, we can figure out ω_n at A, the larger the ω_n, the smaller the settling time, the faster the system's respond pace.

Assume an original system with 2 poles: -0.2+j,-0.2-j from equation (114) we can get the centroid of the root locus is:

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{-0.2 + j - 0.2 - j}{2} = -0.2 \tag{141}$$

after we adding a zero$-0.1, -2, -3, -4$, the centroid of the root locus becomes$-0.5, -2.4, -3.4, -4.4$ correspondingly. compare the time response in each case:



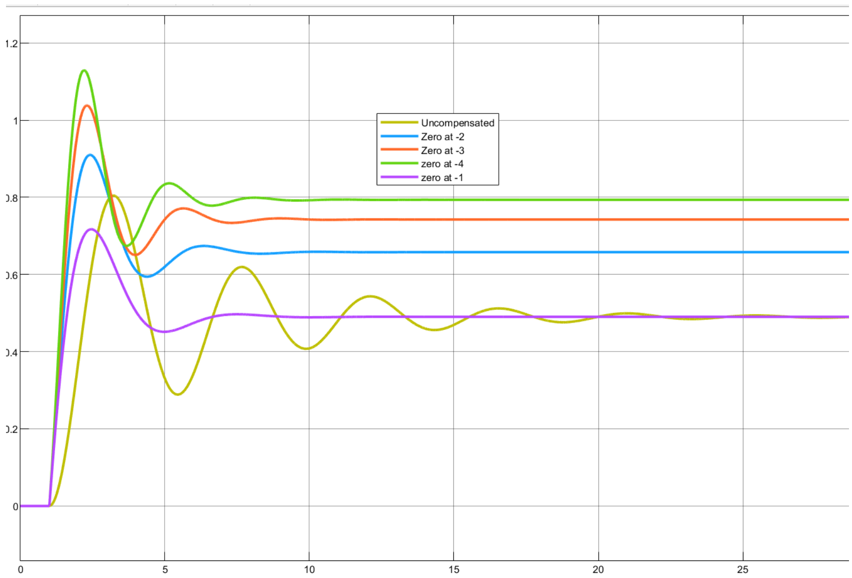
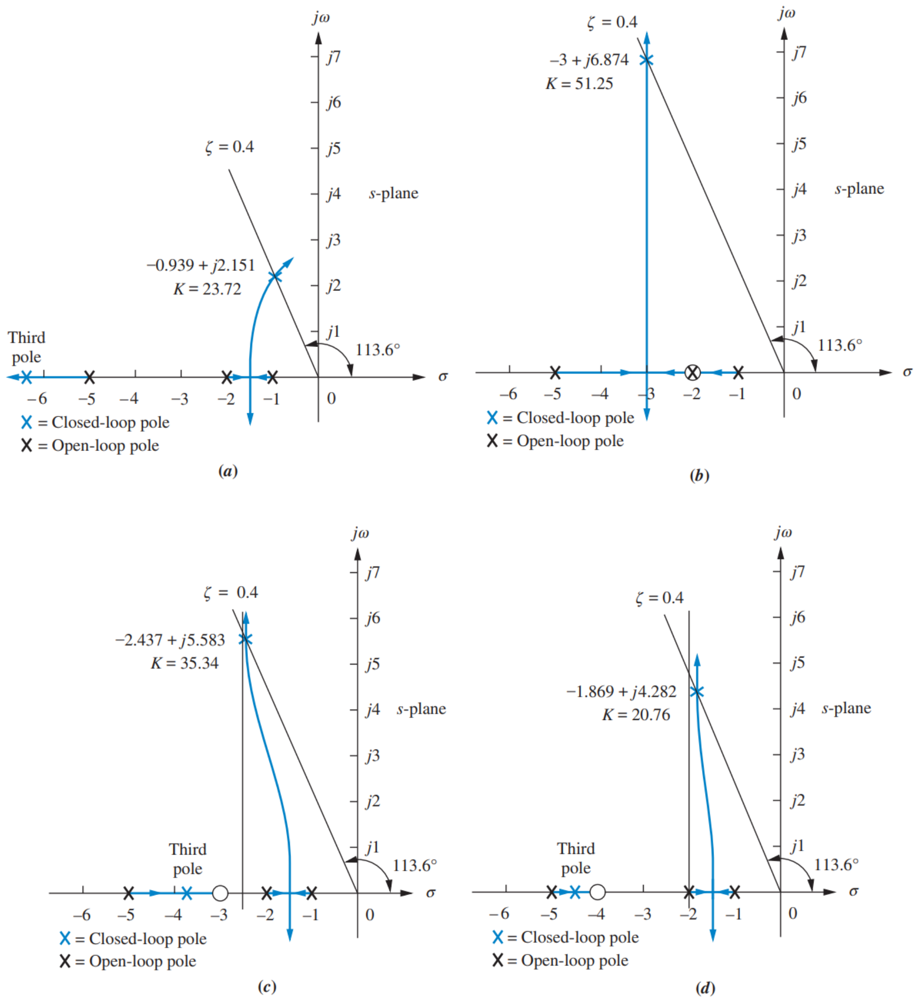


图 10.7 the time response of the system with and without PD compensator



是一个示意图，目的是展示 PD 对系统根轨迹的影响

10.6.2 Lead compensator

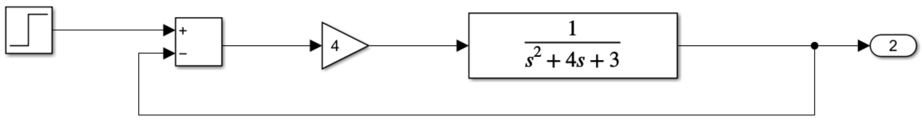


图 11.7.2.1. system without lead compensator

We assume a plant has transfer function:

$$P(s) = \frac{1}{(s+1)(s+3)} \tag{142}$$

the lead compensator:

$$G_c(s = \frac{s+4}{s+5} \tag{143}$$

From equation (114) we can get the centroid of the root locus is:

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = -2 \tag{144}$$

we can get the root locus of the system:

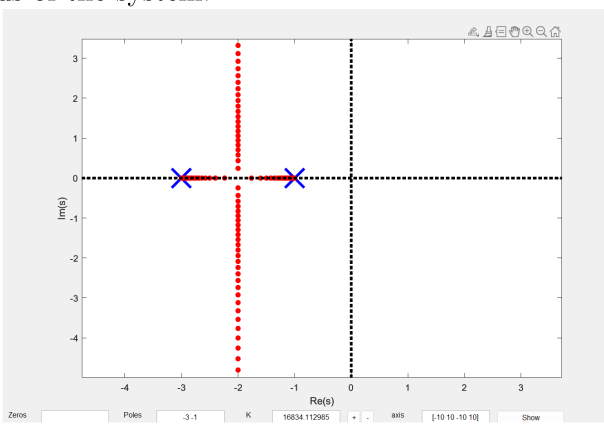


图 11.7.2.2. root locus of system without lead compensator

After adding a zero and a pole (lead compensator):

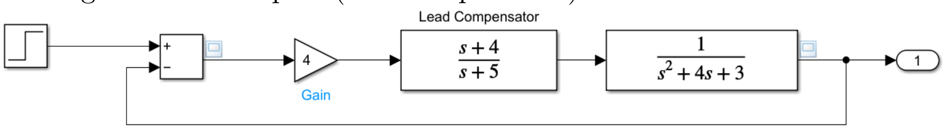


图 11.7.2.3. system with lead compensator

the centroid of the root locus is transferred to :

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{-1-3-5+4}{2} = -2.5 \tag{145}$$

then the root locus:

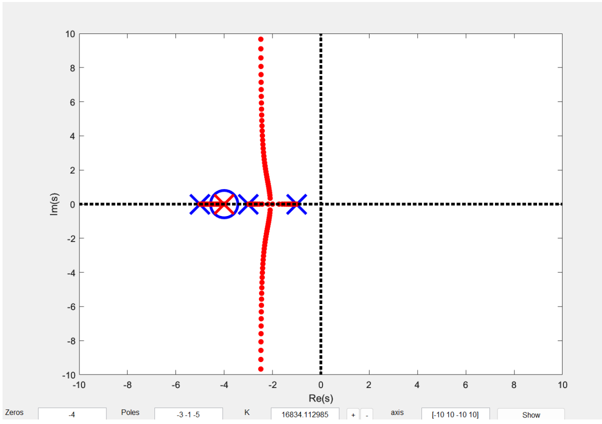


图 11.7.2.4. root locus of system with lead compensator

Add a lead compensator, we have essentially dragged the asymptotes further into the left half plane, add stability to the system by moving the closed loop poles further into the left half plane. (Main advantages of Lead compensator: 1. enhance stability 2. faster T_r rising time)

10.6.3 How to use Lead Lag compensator to meet requirement

A system with single pole (for example: $\frac{1}{s+0.1}$), then the transient response of this system to an impulse signal input is $e^{-0.1t}$, as moving the pole further into the left have plane, the impulse response is going to die off quicker. Assume a system with two poles at -0.1 and -5, the transient response will die off follow the equation $Ae^{-0.1t} + Be^{-5t}$, it is obvious that the total response tend to follow pole -0.1 more than pole -5, -0.1 is considered as a dominant pole. (如果一个极点极为靠近虚轴，我们可以忽略去其他极点的影响).

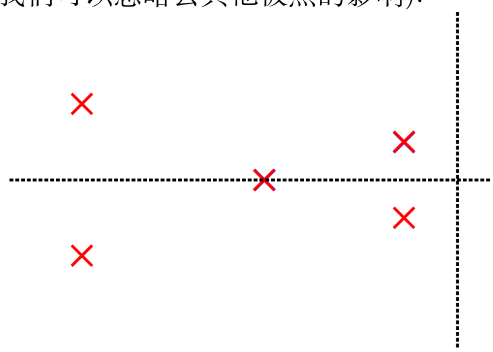


图 E1

在图 E1 中最右面的两个极点和剩余的极点相比，更加靠近虚轴，因此这两个极点被称为主极点，如果两个主极点极为靠近虚轴，我们可以忽略其他极点，认为此系统有着和二阶系统相似（近乎相同）的性质。

If the required poles are to the left of the current uncompensated root locus then we have to add lead compensator to move the root locus to the left; if the required poles are to the right, then we add lag compensator instead.

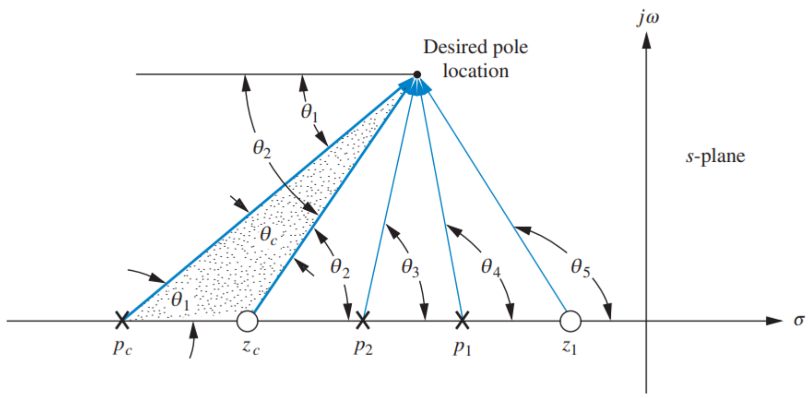


图 11.7.3: use lead compensator to reside desired pole location on the root locus

11 Method 3: Frequency Response

11.1 Bode plot

Pay attention that the frequency response of a system is the steady state response where all transient died out (exponential part become constant $\rightarrow \sigma = 0$) therefore, $s = j\omega$. For a system that has transfer function $H(s) = \frac{2s+1}{s}$ (parallel go through Proportion and integration), substitute $s = j\omega$ into transfer function we can get the steady state frequency response:

$$H(s) = \frac{2s+1}{s} = 2 - \frac{1}{\omega}j \tag{146}$$

the gain equals to $\sqrt{real^2 + imag^2}$ when ω goes from 0 to ∞, the gain goes from 2 to ∞ and phase goes from $-90°$ to 0. In Bode plot, we use gain's decibel and phase to represent the characteristics of the system. Consider the following transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{147}$$

the gain (magnitude) of the system:

$$|G(j\omega)| = \left.\frac{K|(s+z_1)||(s+z_2)|\cdots|(s+z_k)|}{|s^m||(s+p_1)||(s+p_2)|\cdots|(s+p_n)|}\right|_{s\rightarrow j\omega} \tag{148}$$

Converting the magnitude response into decibel (dB):

$$20 \log |G(j\omega)| = 20 \log K + 20 \log |(s+z_1)| + 20 \log |(s+z_2)| + \cdots - 20 \log |s^m| - 20 \log |(s+p_1)| - \cdots |_{s \to j\omega} \tag{149}$$

the phase of the system:

$$\varphi = \tan^{-1} \frac{\text{imag part}}{\text{real part}} \tag{150}$$

11.2 Nyquist Plot

11.2.1 Why we need Nyquist Plot

The general format of closed-loop transfer function is:

$$\frac{G(s)}{1+G(s)H(s)} \tag{151}$$

the stability of a closed-loop transfer function is determined by the poles, which is also the zeros of $1+G(s)H(s)$. When we are dealing with a highly complex system, roots analysis becomes pale,since roots for $G(s)H(s)$ and $1+G(s)H(s)$ differ from each other greatly. 总结来说，目的就是用已知的开环系统来对闭环系统的稳定性进行分析。

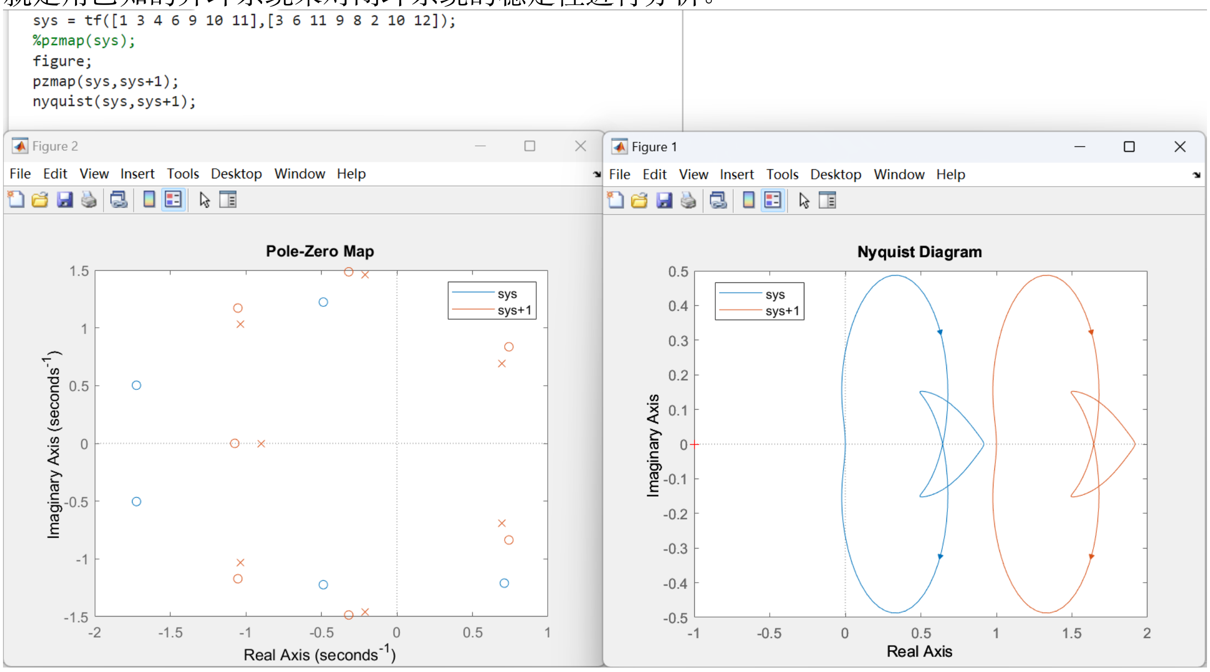


图 11.2.1: 直接进行极点零点分析和 Nyquist 分析的对比

11.2.2 随便说说

Let s be a complex variable, and $F(s)$ be a rational fractional function of s. For any point S on the s plane, through the mapping relationship of the complex variable function $F(s)$, the image about s can be determined on the $F(s)$ plane.

1. Pick the point in s plane.
2. Draw all the phasors. (from poles and zeros to s)
3. Magnitude of the phasor in $F(s)$ plane can be calculated by: multiplying all of the zero phasor magnitudes and dividing by all of the pole phasors magnitudes.
4. Phase of the phasor in $F(s)$ plane can be calculated by: adding all the zero phases and subtracting all the pole phases.

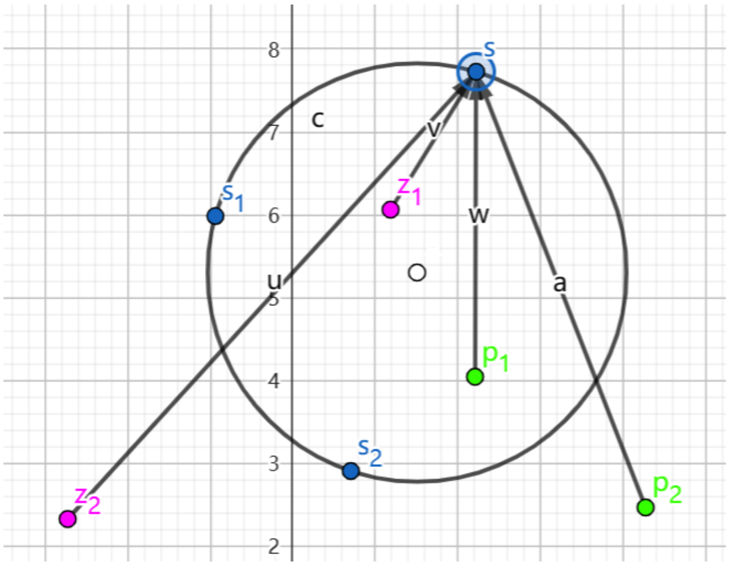


图 11.3.1: 闭合曲线包围的点和未包围的点的对比

Choose a closed curve Γ on the s plane, Γ does not pass through any zero and pole of F(s), let s start from any point A on the closed curve T, move clockwise along the factory circle for one circle, and then return to point A, Correspondingly, a closed curve Γ_F is also formed from the point $F(A)$ to the point $F(A)$ on the $F(s)$ plane.

$$\delta\angle[F(s)] = \oint_{\Gamma} \angle[F(s)]\mathrm{d}s \tag{152}$$

For system with poles and zeros shown in 图 11.3.1, as shown in equation (76)

$$\angle[F(s)] = \angle(s - z_1) + \angle(s - z_2) - \angle(s - p_1) - \angle(s - p_2) \tag{153}$$

therefore,

$$\delta\angle[F(s)] = \delta\angle\left(s-z_1\right) + \delta\angle\left(s-z_2\right) - \delta\angle\left(s-p_1\right) - \delta\angle\left(s-p_2\right) \tag{154}$$

$$\delta\angle\left(s-z_2\right) = \oint_{\Gamma} \angle\left(s-z_2\right) \mathrm{d}s = \int_{\Gamma\widetilde{s_1 s_2}} \angle\left(s-z_2\right) \mathrm{d}s + \int_{\Gamma\widetilde{s_2 s_1}} \angle\left(s-z_2\right) \mathrm{d}s = 0 \tag{155}$$

同理也可以得到 $\delta\angle\left(s-p_2\right) = 0$

$$\delta\angle\left(s-z_1\right) = \delta\angle\left(s-p_1\right) = -2\pi \tag{156}$$

如果 s 沿着 Γ 转一周，很显然 Γ 未被包围的极点零点的相角变化量为 0，而被 Γ 包围的极点和零点的相角变化量为 2π. (矢量顺时针转一圈相角变化是 -2π, 逆时针是 2π).

如果 s 平面上的闭环曲线包围了 $F(S)$ 的一个零点并沿闭合曲线顺时针一周，则 $F(S)$ 映射的闭合曲线绕原点顺时针一周。

$$\theta \sum{}_{i=1}^{m} \angle\left(s+z_i\right) - \sum_{j=1}^{n} \angle\left(s+p_j\right) = -2\pi \tag{157}$$

如果 s 平面上的闭环曲线包围了 $F(S)$ 的一个极点并沿闭合曲线顺时针一周，则 $F(S)$ 映射的闭合曲线绕原点逆时针一周。

$$\theta \sum{}_{i=1}^{m} \angle\left(s+z_i\right) - \sum_{j=1}^{n} \angle\left(s+p_j\right) = 0 - (-2\pi) = 2\pi \tag{158}$$

We can summarize the following formula:

$$R = P - Z \tag{159}$$

R: $F(s)$ 平面上（被映射系统平面）闭合曲线 Γ_F 逆时针包围原点的圈数，$R<0$ 和 $R>0$ 分别表示 Γ_F 顺时针包围和逆时针包围 $F(s)$ 平面的原点.

P: 在 s 平面闭合曲线 Γ 内被包围的极点数

Z: 在 s 平面闭合曲线 Γ 内被包围的零点数

11.2.3 Nyquist Plot

上述论证证实了一个问题，映射可以通过图像含有的圈数来判断极点和零点的差值，回到原先的问题，判断闭环系统的稳定性，其实就是在找 $1+G(s)H(s)$ 的零点，只要零点不出现在右半复平面，我们就可以保证这个系统至少 marginly stable, 现在的问题就是，我们需要用上述方法扫描整个 RHP (right half plane), 即我们需要一个闭合曲线 Γ 包围整个 RHP.

在 9.3 中已经论证过 $1+G(s)H(s)$ 的好处，即可以可以同时包含开环传函和闭环传函的极点，这里用更简单的方法提一嘴：

$$F(s)=1+G(s)H(s)=1+\frac{B(s)}{A(s)}=\frac{A(s)+B(s)}{A(s)} \tag{160}$$

可以看出 $F(s)$ 的零点为闭环传递函数的极点,$F(s)$ 的极点为开环传递函数的极点. 所以公式 (159) 的参数可以重新被定义为：

P: $G(s)H(s)$ 在 RHP 内的极点数（已知数）

Z: $1+G(s)H(s)$ 在右半平面内的极点数

在图 11.2.1 中也展示了 $1+G(s)H(s)$ 的另一个好处，即闭合曲线 Γ_F 包围 $F(s)$ 平面原点的圈数等价于闭合曲线 $\Gamma_G H$ 包围 $F(s)$ 平面点 $(-1,j0)$ 的圈数

11.2.4 Γ 的选择

无非就是两种情况，虚轴上有极点和虚轴上无极点。系统的闭环稳定性取决于系统闭环传递函数极点即 $F(s)$ 的零点的位置，因此当选择 s 平面闭合曲线 Γ 包围 s 平面的 RHP 时，若 $F(s)$ 在 s 右半平面的零点数 $Z=0$, 则闭环系统稳定。考虑到前述闭合曲线 Γ 应不通过 $F(s)$ 的零极点的要求。

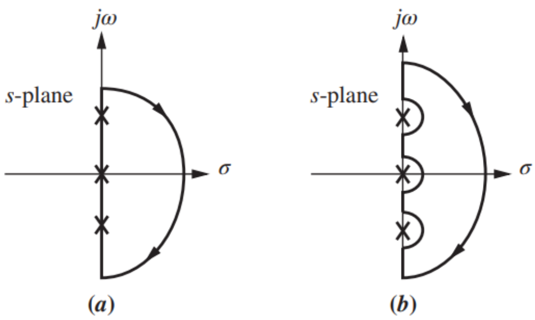


图 11.2.4 闭合曲线 Γ 的选取

因为 $F(s)$ 和 $G(S)H(S)$ 有同样的极点，因此考虑 $G(S)H(S)$ 即可。

1. 当 $G(s)H(s)$ 无虚轴上的极点时

第一部分：从原点开始，正虚轴和第一象限中半径为 ∞ 的 $\frac{1}{4}$ 圆

$$s = j\omega, \omega \in [0, \infty) \quad s = \infty e^{j\theta}, \theta \in [0, +90°] \tag{161}$$

第二部分：第四象限中半径为 ∞ 的 $\frac{1}{4}$ 圆，负虚轴到原点

$$s = j\omega, \omega \in (-\infty, 0] \quad s = \infty e^{j\theta}, \theta \in [0, -90°] \tag{162}$$

2. 当 $G(s)H(s)$ 在虚轴上有极点时

第一部分：开环系统有积分环节

$$s = \epsilon e^{j\theta}, \epsilon \to 0, \theta \in [-90°, 90°] \tag{163}$$

第二部分：开环系统有等幅振荡环节

$$s = \pm \mathrm{j}\omega_n + \epsilon \mathrm{e}^{\mathrm{j}\theta}, \epsilon \to 0, \theta \in [-90°, 90°] \tag{164}$$

11.2.5 已知 Γ 画 Γ_{GH}

当 $G(s)H(s)$ 无虚轴上的极点时举例：画出 $G(s) = \frac{1}{s^2+3s+2}$

1. $\omega = 0$

$$G(s) = \frac{1}{s^2 + 3s + 2}|_{\omega=0} = \frac{1}{2} \tag{165}$$

2. $\omega = \infty$

$$G(s) = \frac{1}{s^2 + 3s + 2}|_{\omega=\infty} = 0 \tag{166}$$

3. find the imaginary intercepts

   将 $s = j\omega$ 代入 $G(s)$

$$G(j\omega) = \frac{1}{(2-\omega^2) + 3\omega j} \frac{(2-\omega^2) - 3\omega j}{(2-\omega^2) - 3\omega j} = \frac{(2-\omega^2) - 3\omega j}{(2-\omega^2) + 9\omega^2} \tag{167}$$

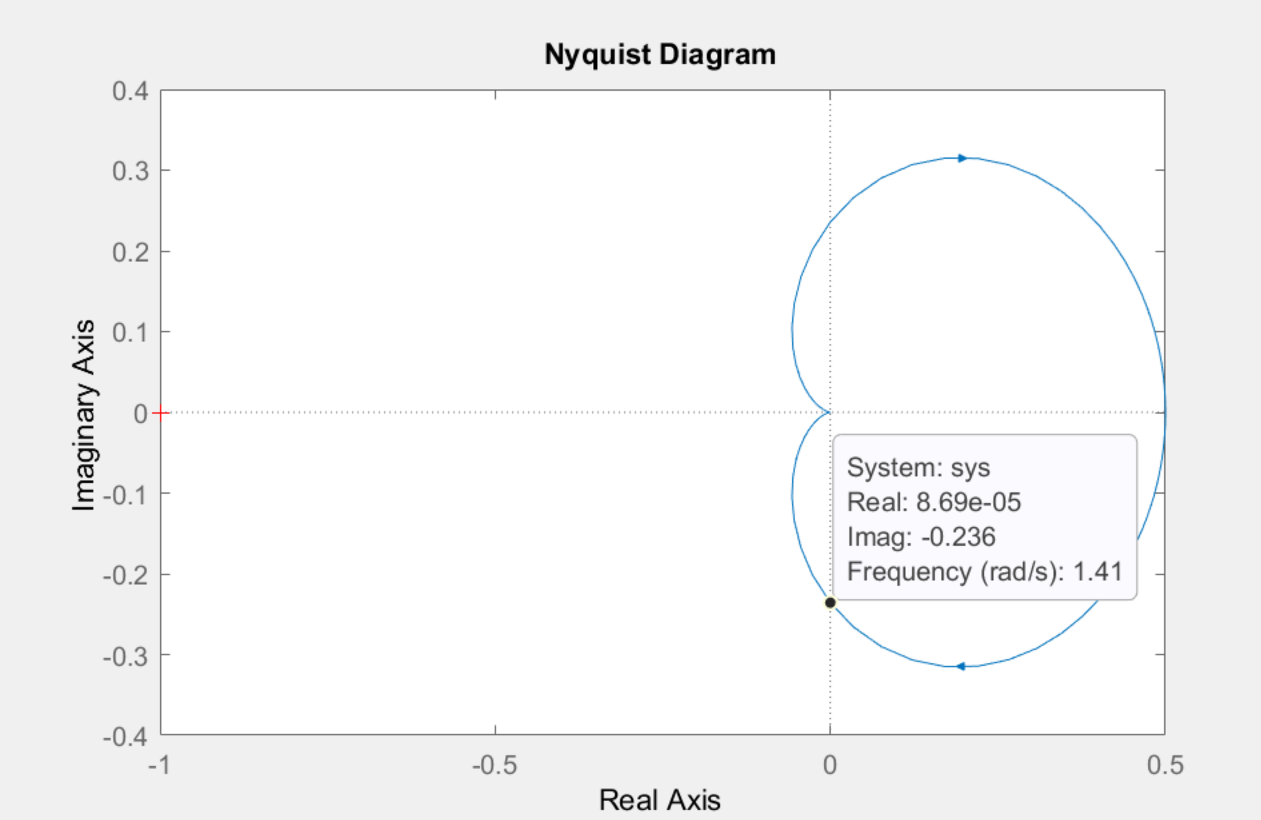


图 10: 一张屑图

the real part $= \frac{(2-\omega^2)}{(2-\omega^2)+9\omega^2}$ the imaginary part $= \frac{-3\omega j}{(2-\omega^2)+9\omega^2}$. 为了找到在虚轴上的纵坐标，让 $G(j\omega)$ 实部为零，可得 $\omega = \sqrt{2}$ 代入虚部，即可求解。

4. find real intercepts 为了找在实轴上的横坐标，让 $G(j\omega)$ 虚部为零，此种情况已经考虑过了。

最终得到 Nyquist Plot 的起点是 1/2，终点是 0，与虚轴的交点是-0.236，因为 Nyquist plot 关于实轴对称，所以可得最终的图像。

剩下还有有积分环节的和有等幅振荡环节的，实在懒得写了, 上面这个屑例子是我抄的，我自己举的例子计算太麻烦了，虽然可以达到和 Matlab 近似的准确度，后续可能会在 GitHub 上发出。

-结-

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