COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2021

Assignment 2

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Revised: February 2, 2021

Assignment 2 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 2 as two files, `Assignment_2_`*`YourMacID`*`.tex` and `Assignment_2_`*`YourMacID`*`.pdf`, to the Assignment 2 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_2_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_2.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_2_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_2_YourMacID
```

This assignment is due **Sunday, February 7, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_2_`*`YourMacID`*`.tex` and `Assignment_2_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 7.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Problems

1. **[10 points]**

   Let $\mathsf{SimpleTree}$ be the inductive set defined by the following constructors:

   a. $\mathsf{Leaf} : \mathbb{N} \to \mathsf{SimpleTree}$.
   b. $\mathsf{Branch1} : \mathsf{SimpleTree} \to \mathsf{SimpleTree}$.
   c. $\mathsf{Branch2} : \mathsf{SimpleTree} \times \mathsf{SimpleTree} \to \mathsf{SimpleTree}$.

   The function $\mathsf{leaves} : \mathsf{SimpleTree} \to \mathbb{N}$ is defined by recursion and pattern matching as:

   a. $\mathsf{leaves}(\mathsf{Leaf}(n)) = 1$.
   b. $\mathsf{leaves}(\mathsf{Branch1}(t)) = \mathsf{leaves}(t)$.
   c. $\mathsf{leaves}(\mathsf{Branch2}(t_1, t_2)) = \mathsf{leaves}(t_1) + \mathsf{leaves}(t_2)$.

   The function $\mathsf{branches} : \mathsf{SimpleTree} \to \mathbb{N}$ is defined by recursion and pattern matching as:

   a. $\mathsf{branches}(\mathsf{Leaf}(n)) = 0$.
   b. $\mathsf{branches}(\mathsf{Branch1}(t)) = 1 + \mathsf{branches}(t)$.
   c. $\mathsf{branches}(\mathsf{Branch2}(t_1, t_2)) = 1 + \mathsf{branches}(t_1) + \mathsf{branches}(t_2)$.

   Prove that, for all $t \in \mathsf{SimpleTree}$,

   $$\mathsf{leaves}(t) \leq \mathsf{branches}(t) + 1.$$

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   Proof. Let $P(t) \equiv \mathsf{leaves}(t) \leq \mathsf{branches}(t) + 1$. We will prove $P(t)$ for all $t \in \mathsf{SimpleTree}$ by strong induction.

   Base case: $t = \mathsf{Leaf}(n)$. We must show $P(\mathsf{Leaf}(n))$.

   $$\begin{aligned}
   &\mathrm{P}(\mathsf{Leaf}(\mathrm{n})) \\
   &\equiv \mathsf{leaves}(\mathsf{Leaf}(n)) \leq \mathsf{branches}(\mathsf{Leaf}(n)) + 1 \\
   &\equiv 1 \leq \mathsf{branches}(\mathsf{Leaf}(n)) + 1 && \langle \text{definition of } \mathsf{leaves} \rangle \\
   &\equiv 1 \leq 0 + 1 && \langle \text{definition of } \mathsf{branches} \rangle \\
   &\equiv 1 \leq 1 && \langle \text{arithmetic} \rangle \\
   &\equiv \mathit{True}
   \end{aligned}$$

   So $P(\mathsf{Leaf}(n))$ holds.

Induction step: *Case 1*: We assume $P(t)$ holds for $t \in \mathsf{SimpleTree}$, prove $P(\mathsf{Branch1}(t))$

$$\begin{aligned}
&\mathsf{leaves}(\mathsf{Branch1}(t)) && \\
&= \mathsf{leaves}(t) && \langle \text{definition of } \mathsf{leaves} \rangle \\
&\leq \mathsf{branches}(t) + 1 && \langle \text{induction hypothesis} \rangle \\
&\leq \mathsf{branches}(t) + 1 + 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{branches}(\mathsf{Branch1}(t)) + 1 && \langle \text{definition of } \mathsf{branches} \rangle
\end{aligned}$$

So $P(\mathsf{Branch1}(t))$ holds.

Case 2: We assume $P(t_1)$ and $P(t_2)$ holds for t_1,t_2 $\in \mathsf{SimpleTree}$, prove $P(\mathsf{Branch2}(t_1, t_2))$

$$\begin{aligned}
&\mathsf{leaves}(\mathsf{Branch2}(t_1, t_2)) && \\
&= \mathsf{leaves}(t_1) + \mathsf{leaves}(t_2) && \langle \text{definition of } \mathsf{leaves} \rangle \\
&\leq \mathsf{branches}(t_1) + 1 + \mathsf{branches}(t_2) + 1 && \langle \text{induction hypothesis} \rangle \\
&= \mathsf{branches}(\mathsf{Branch2}(t_1, t_2)) + 1 && \langle \text{definition of } \mathsf{branches} \rangle
\end{aligned}$$

So $P(\mathsf{Branch2}(t_1, t_2))$ holds.

Therefore, $P(t)$ holds for all $t \in \mathsf{SimpleTree}$ by strong induction. $\square$

2. **[10 points]**

Let BinNum be the inductive set defined by the following constructors:

$\mathsf{Zero} : \mathsf{BinNum}.$

$\mathsf{One} : \mathsf{BinNum}.$

$\mathsf{JoinZero} : \mathsf{BinNum} \to \mathsf{BinNum}.$

$\mathsf{JoinOne} : \mathsf{BinNum} \to \mathsf{BinNum}.$

The members of BinNum represent binary numerals like 1011 and 010. Zero represents 0; One represents 1; and if u represents U, then $\mathsf{JoinZero}(u)$ represents U0 and $\mathsf{JoinOne}(u)$ represents U1. For example,

$$\mathsf{JoinOne}(\mathsf{JoinZero}(\mathsf{JoinOne}(\mathsf{One})))$$

represents the binary number 1101.

The function

$$\mathsf{len} : \mathsf{BinNum} \to \mathbb{N}$$

maps a member of BinNum to its length. len is defined by the following equations using recursion and pattern matching:

$$\mathsf{len}(\mathsf{Zero}) = 1.$$
$$\mathsf{len}(\mathsf{One}) = 1.$$
$$\mathsf{len}(\mathsf{JoinZero}(u)) = \mathsf{len}(u) + 1.$$
$$\mathsf{len}(\mathsf{JoinOne}(u)) = \mathsf{len}(u) + 1.$$

The function

$$\mathsf{val} : \mathsf{BinNum} \to \mathbb{N}$$

maps a member of BinNum to the value of the binary numeral it represents. For example,

$$\mathsf{val}(\mathsf{JoinOne}(\mathsf{JoinZero}(\mathsf{JoinOne}(\mathsf{One})))) = (1101)_2 = 13.$$

val is defined by the following equations using recursion and pattern matching:

$$\mathsf{val}(\mathsf{Zero}) = 0.$$
$$\mathsf{val}(\mathsf{One}) = 1.$$
$$\mathsf{val}(\mathsf{JoinZero}(u) = 2 * \mathsf{val}(u).$$
$$\mathsf{val}(\mathsf{JoinOne}(u) = (2 * \mathsf{val}(u)) + 1.$$

The function

$$\mathsf{add} : \mathsf{BinNum} \times \mathsf{BinNum} \to \mathsf{BinNum}$$

is intended to implement addition on members of BinNum. It is defined by the following equations using recursion and pattern matching:

$$\mathsf{add}(u, \mathsf{Zero}) = u.$$
$$\mathsf{add}(\mathsf{Zero}, u) = u.$$
$$\mathsf{add}(\mathsf{One}, \mathsf{One}) = \mathsf{JoinZero}(\mathsf{One}).$$
$$\mathsf{add}(\mathsf{JoinZero}(u), \mathsf{One}) = \mathsf{JoinOne}(u).$$
$$\mathsf{add}(\mathsf{One}, \mathsf{JoinZero}(u)) = \mathsf{JoinOne}(u).$$
$$\mathsf{add}(\mathsf{JoinOne}(u), \mathsf{One}) = \mathsf{JoinZero}(\mathsf{add}(u, \mathsf{One}).$$
$$\mathsf{add}(\mathsf{One}, \mathsf{JoinOne}(u)) = \mathsf{JoinZero}(\mathsf{add}(u, \mathsf{One}).$$
$$\mathsf{add}(\mathsf{JoinZero}(u), \mathsf{JoinZero}(v)) = \mathsf{JoinZero}(\mathsf{add}(u, v).$$
$$\mathsf{add}(\mathsf{JoinOne}(u), \mathsf{JoinZero}(v)) = \mathsf{JoinOne}(\mathsf{add}(u, v).$$
$$\mathsf{add}(\mathsf{JoinZero}(u), \mathsf{JoinOne}(v)) = \mathsf{JoinOne}(\mathsf{add}(u, v).$$
$$\mathsf{add}(\mathsf{JoinOne}(u), \mathsf{JoinOne}(v)) = \mathsf{JoinZero}(\mathsf{add}(\mathsf{add}(u, v), \mathsf{One})).$$

Notice that the algorithm behind the definition is essentially the same algorithm that children learn to add numbers represented as decimal numerals. The last equation is a bit complicated because it involves a carry of 1.

Lemma 1. For all $u, v \in \mathsf{BinNum}$,

$$\mathsf{len}(\mathsf{add}(u, v)) \leq \mathsf{len}(u) + \mathsf{len}(v).$$

Theorem 1. For all $u, v \in \mathsf{BinNum}$,

$$\mathsf{val}(\mathsf{add}(u, v)) = \mathsf{val}(u) + \mathsf{val}(v).$$

Theorem 1 states that add correctly implements addition on the members of BinNum.

Prove Theorem 1 assuming Lemma 1. (You are not required to prove Lemma 1.) Hint: Use strong induction with $P(n) \equiv \mathsf{val}(\mathsf{add}(u, v)) = \mathsf{val}(u) + \mathsf{val}(v)$ for all $u, v \in \mathsf{BinNum}$ such that $n = \mathsf{len}(u) + \mathsf{len}(v)$.

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In order to prove Theorem 1 better, we introduce Lemma 2 and Lemma 3. **Lemma 2**. For all $u, v \in \mathsf{BinNum}$

$\mathsf{len}(u) + \mathsf{len}(v) \leq \mathsf{len}(\mathsf{JoinOne}(u)) + \mathsf{len}(\mathsf{JoinOne}(v))$

Lemma 3. For all $u, v \in \mathsf{BinNum}$

$\mathsf{len}(\mathsf{add}(u, v)) + \mathsf{len}(\mathsf{JoinOne}(\mathsf{Zero})) \leq \mathsf{len}(\mathsf{JoinOne}(u)) + \mathsf{len}(\mathsf{JoinOne}(v))$

Proof. :

Lemma 2:

$$\begin{aligned}
&\mathsf{len}(u) + \mathsf{len}(v) \\
&\leq \mathsf{len}(u) + 1 + \mathsf{len}(v) + 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{len}(\mathsf{JoinOne}(u)) + \mathsf{len}(\mathsf{JoinOne}(v)) && \langle \text{definition of } \mathsf{len} \rangle
\end{aligned}$$

Lemma 3:

$$\begin{aligned}
&\mathsf{len}(\mathsf{add}(u, v)) + \mathsf{len}(\mathsf{JoinOne}(\mathsf{Zero})) && \langle \text{LHS} \rangle \\
&\leq \mathsf{len}(u) + \mathsf{len}(v) + \mathsf{len}(\mathsf{JoinOne}(\mathsf{Zero})) && \langle \text{Lemma 1} \rangle \\
&= \mathsf{len}(u) + \mathsf{len}(v) + \mathsf{len}(\mathsf{Zero}) + 1 && \langle \text{definition of } \mathsf{len} \rangle \\
&= \mathsf{len}(u) + \mathsf{len}(v) + 1 + 1 && \langle \text{definition of } \mathsf{len} \rangle \\
&= \mathsf{len}(\mathsf{JoinOne}(u)) + \mathsf{len}(\mathsf{JoinOne}(v)) && \langle \text{definition of } \mathsf{len} \rangle
\end{aligned}$$

We now prove **Thereom 1**

Proof: We assume $P(a)$ for all a < n, and prove $P(n)$ by strong induction.

Case 1: $v = \mathsf{Zero}$

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(u, \mathsf{Zero})) && \langle \text{LHS} \rangle \\
&= \mathsf{val}(u) && \langle \text{definition of } \mathsf{add} \rangle \\
&= \mathsf{val}(u) + 0 && \langle \text{arithmetic} \rangle \\
&= \mathsf{val}(u) + \mathsf{val}(\mathsf{Zero}) && \langle \text{definition of } \mathsf{val} \rangle
\end{aligned}$$

Case 2: $u = \mathsf{Zero}$.

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{Zero}, v)) && \langle \text{LHS} \rangle \\
&= \mathsf{val}(u) && \langle \text{definition of } \mathsf{add} \rangle \\
&= 0 + \mathsf{val}(u) && \langle \text{arithmetic} \rangle \\
&= \mathsf{val}(\mathsf{Zero}) + \mathsf{val}(v) && \langle \text{definition of } \mathsf{val} \rangle
\end{aligned}$$

Case 3: $u = \mathsf{JoinZero}(u\prime)$ and $v = \mathsf{JoinZero}(v\prime)$

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinZero}(u\prime), \mathsf{JoinZero}(v\prime))) && \langle \text{LHS} \rangle \\
&= \mathsf{val}(\mathsf{JoinZero}(\mathsf{add}(u\prime, v\prime))) && \langle \text{definition of } \mathsf{add} \rangle \\
&= 2 * \mathsf{val}((\mathsf{add}(u\prime, v\prime))) && \langle \text{definition of } \mathsf{val} \rangle \\
&= 2 * (\mathsf{val}(u\prime) + \mathsf{val}(v\prime)) && \langle \text{Induction Hypothesis} \rangle \\
&= 2 * (\mathsf{val}(u\prime)) + 2 * (\mathsf{val}(v\prime)) && \langle \text{arithmetic} \rangle \\
&= \mathsf{val}(\mathsf{JoinZero}(u\prime)) + \mathsf{val}(\mathsf{JoinZero}(v\prime)) && \langle \text{definition of } \mathsf{val} \rangle
\end{aligned}$$

Case 4: $u = \mathsf{JoinOne}(u\prime)$, $v = \mathsf{JoinZero}(v\prime)$

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinOne}(u\prime), \mathsf{JoinZero}(v\prime))) && \langle \text{LHS} \rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(\mathsf{add}(u\prime, v\prime))) && \langle \text{definition of } \mathsf{add} \rangle \\
&= 2 * \mathsf{val}((\mathsf{add}(u\prime, v\prime))) + 1 && \langle \text{definition of } \mathsf{val} \rangle \\
&= 2 * (\mathsf{val}(u\prime) + \mathsf{val}(v\prime)) + 1 && \langle \text{Induction Hypothesis} \rangle \\
&= 2 * (\mathsf{val}(u\prime)) + 1 + 2 * (\mathsf{val}(v\prime)) && \langle \text{arithmetic} \rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(u\prime)) + \mathsf{val}(\mathsf{JoinZero}(v\prime)) && \langle \text{definition of } \mathsf{val} \rangle
\end{aligned}$$

Case 5: $u = \mathsf{JoinZero}(u\prime)$, $v = \mathsf{JoinOne}(v\prime)$.

$$
\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinZero}(u\prime), \mathsf{JoinOne}(v\prime))) && \langle\text{LHS}\rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(\mathsf{add}(u\prime, v\prime))) && \langle\text{definition of } \mathsf{add}\rangle \\
&= 2 * \mathsf{val}((\mathsf{add}(u\prime, v\prime))) + 1 && \langle\text{definition of } \mathsf{val}\rangle \\
&= 2 * (\mathsf{val}(u\prime) + \mathsf{val}(v\prime)) + 1 && \langle\text{Induction Hypothesis}\rangle \\
&= 2 * (\mathsf{val}(u\prime)) + 2 * (\mathsf{val}(v\prime)) + 1 && \langle\text{arithmetic}\rangle \\
&= \mathsf{val}(\mathsf{JoinZero}(u\prime)) + \mathsf{val}(\mathsf{JoinOne}(v\prime)) && \langle\text{definition of } \mathsf{val}\rangle
\end{aligned}
$$

Case 6: $u = \mathsf{JoinOne}(u\prime)$, $v = \mathsf{JoinOne}(v\prime)$

$$
\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinOne}(u\prime), \mathsf{JoinOne}(v\prime))) && \langle\text{LHS}\rangle \\
&= \mathsf{val}((\mathsf{JoinZero}(\mathsf{add}(\mathsf{add}(u\prime, v\prime), \mathsf{One}))) \quad = 2 * \mathsf{val}(\mathsf{add}(\mathsf{add}(u\prime, v\prime), \mathsf{One})) && \langle\text{definition of } \mathsf{add}\rangle \\
&= 2 * (\mathsf{val}(\mathsf{add}(u\prime, v\prime)) + \mathsf{val}(\mathsf{One})) && \langle\text{definition of } \mathsf{val}\rangle \\
&= 2 * (\mathsf{val}(u\prime) + \mathsf{val}(v\prime + \mathsf{val}(\mathsf{One})) && \langle\text{Induction Hypothesis}\rangle \\
&= 2 * (\mathsf{val}(u\prime) + \mathsf{val}(v\prime + 1) && \langle\text{definition of } \mathsf{val}\rangle \\
&= (2 * \mathsf{val}(u\prime) + 1) + (2 * \mathsf{val}(v\prime) + 1) && \langle\text{arithmetic}\rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(u\prime)) + \mathsf{val}(\mathsf{JoinOne}(v\prime)) && \langle\text{definition of } \mathsf{val}\rangle
\end{aligned}
$$

So P(n) has been successfully proved by strong induction. $\square$