Option Pricing Exact Solutions, I

Goals

- Understanding basic option pricing theory
- Setting up algorithms for pricing formulae
- Learning to map algorithms to code
- Getting a working solution; improving and extending the solution

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Prerequisites/Checklist

- C++ hands-on knowledge
- Some experience with option pricing theory, maths and background
- Take it step-by-step (in general)
- Revise previous slide shows if your C++ knowledge is a bit rusty

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Topics

- Plain option pricing, one factor
- American perpetual options
- Affine models (CIR, Vasicek)

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European Options

- One-factor calls and puts
- Exact formulae for price (and sensitivities (greeks)) known
- These formulae expressed in terms of the normal cumulative distribution function
- (We provide home-grown version; a more standard solution is Boost statistics library)
- We show formulae for call options (puts are published)

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Call Option

$$C = Se^{(b-r)T}N(d_1) - Ke^{-rT}N(d_2)$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$$

$$d_1 = \frac{ln(S/K) + (b+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(S/K) + (b-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

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Call Sensitvities

$$\Delta_C \equiv \frac{\partial C}{\partial S} = e^{(b-r)T}\, N(d_1)$$

$$\Gamma_C \equiv \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_C}{\partial S} = \frac{Kn(d_1)e^{(b-r)T}}{S\sigma\sqrt{T}}$$

$$Vega_C \equiv \frac{\partial C}{\partial \sigma} = S\sqrt{T}e^{(b-r)T}\, n(d_1)$$

$$\Theta_C \equiv -\frac{\partial C}{\partial T} = -\frac{S\sigma e^{(b-r)T}n(d_1)}{2\sqrt{T}} - (b-r)Se^{(b-r)T}N(d_1) - rKe^{-rT}N(d_2)$$

Remarks

- Use the formulae to price options on stocks (+- dividends), indexes, futures and currencies
- We can use it for different purposes (e.g. determining the accuracy of numerical methods)
- The formula can be extend to n-factor problems (but an explicit formula may not be forthcoming!)
- There are assumptions underlying the Black-Scholes model

Early Exercise Features

- American options can be exercised at any time up to the expiry date
- No exact solution known, although ‘exact approximations’ exist (Barone-Adesi-Whaley)
- In general we need to employ numerical methods
- For the infinite time to expiration case we do have an exact formula (American perpetual option)

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Remarks

- In general, the solution V(S) is independent of time (depends only on the level of the underlying S)
- The option value can never go below the early-exercise payoff $\boxed{V \geq \max(K - S, 0)}$ (for a put option)
- Behaviour is described by the solution of an ordinary differential equation
- Can be solved exactly (and of course, numerically)
- Can use the price to test other schemes when the expiry time approaches infinity

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Perpetual Call

- McKean (1965) and Merton (1973)

$$y_1 = \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$c = \frac{K}{y_1 - 1}\left(\frac{y_1 - 1}{y_1}\frac{S}{K}\right)^{y_1}$$

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Perpetual Put

$$y_2 = \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$p = \frac{K}{1 - y_2}\left(\frac{y_2 - 1}{y_2}\frac{S}{K}\right)^{y_2}$$

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Affine Interest Rate Models

- We also wish to model interest rate derivatives
- Many models used (Vasicek, CIR, Hull-White,...)
- Fundamental to this is to model the short rate and term structure of interest rates
- This usually takes the form of a stochastic differential equation (SDE)

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SDE Model

$$dr = \kappa\left(\theta - r\right)dt + \sigma r^{\beta} dw$$

$r = r(t) =$ level of short rate at time t.
$dW =$ increment of a Wiener process.
$\theta =$ long-term level of r.
$K =$ speed of mean reversion.
$\sigma =$ volatility of the short rate.
$\beta = 0$ for Vasicek model, $\frac{1}{2}$ for CIR model.

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Affine Models

- Provide bond price based on a given SDE

$$\boxed{P(t,s) = A(t,s)e^{-rB(t,s)}}$$

- The parameters A(t) and B(t) will be different for each model
- Most models are mean-reverting