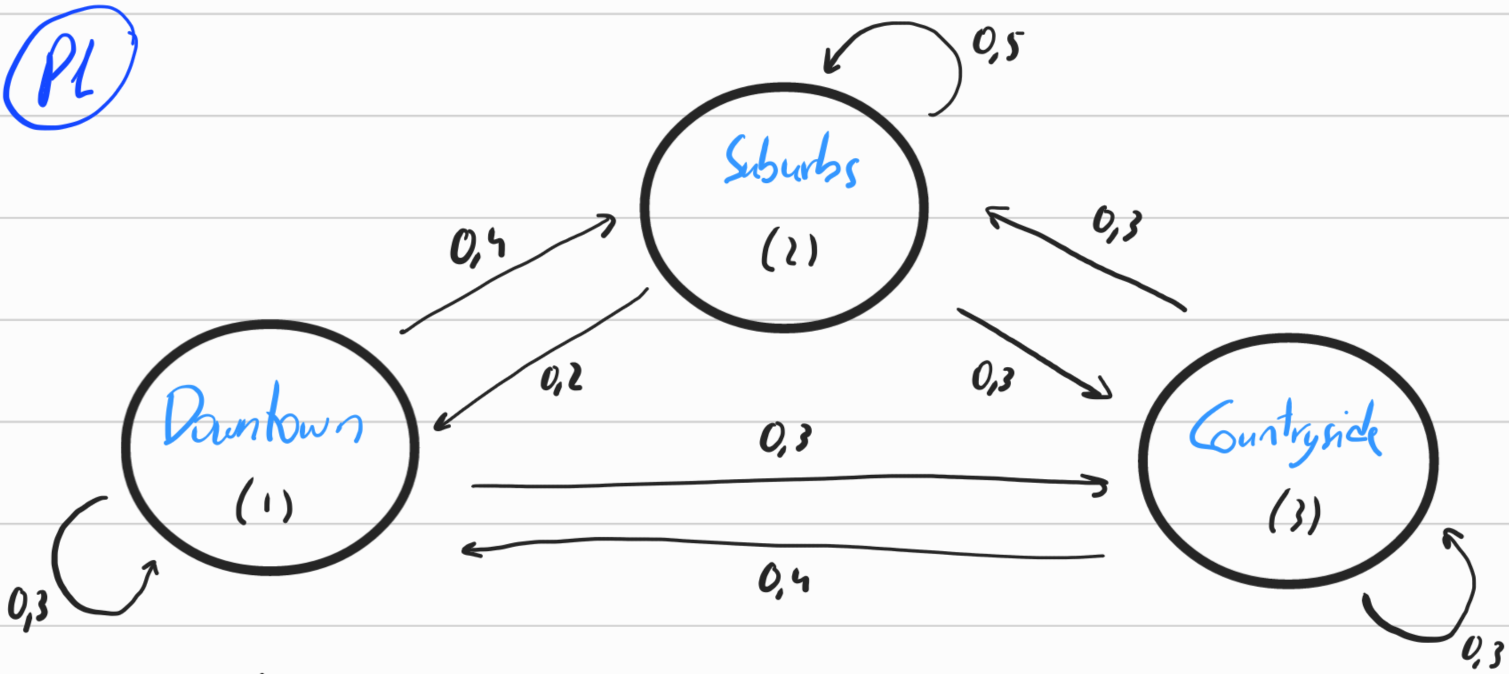
Assignment 2

P2



Probability of transitioning:

Current	To Downtown	To Suburbs	To Countryside
1 Downtown	0,3 $_{11}$	0,4 $_{12}$	0,3 $_{13}$
2 Suburbs	0,2 $_{21}$	0,5 $_{22}$	0,3 $_{23}$
3 Countryside	0,4 $_{31}$	0,3 $_{32}$	0,3 $_{33}$

Transition Matrix for the Markov chain $\Rightarrow$

$$P_{ij} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0,3 & 0,4 & 0,3 \\ 0,2 & 0,5 & 0,3 \\ 0,4 & 0,3 & 0,3 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

(P of transitioning from region i to j)

1. P from Suburbs to Downtown after 2 steps

Transition matrix after 2 steps $\rightarrow$ $P^2_{ij} = P \times P = \begin{pmatrix} 0,29 & 0,41 & 0,3 \\ 0,28 & 0,42 & 0,3 \\ 0,3 & 0,4 & 0,3 \end{pmatrix}$

$$\boxed{P^2_{21} = 0,28}$$

2. Being in Suburbs, P being in Downtown for the 1st time after 2 steps

To Downtown

Suburbs ~~Downtown~~ Downtown

0 1 2

Les 2 possibilitats:

$\text{Suburbs}^{2} \rightarrow \text{Suburbs}^{2} \rightarrow \text{Downtown}^{1}$

$\text{Suburbs}_{2} \rightarrow \text{Countryside}_{3} \rightarrow \text{Downtown}_{1}$

$$\boxed{P(1^{st}\text{ time Downtown in 2 Steps}) = P_{22} \times P_{21} + P_{23} \times P_{31} = 0,5 \cdot 0,2 + 0,3 \cdot 0,4 = 0,22}$$

4. Stationary distribution

$$\pi \cdot P = \pi \rightarrow (\pi \cdot P)^T = \pi^T$$

eigenvalues:

eigenvectors:

$$|P^T - \lambda I| = \begin{vmatrix} 0,3-\lambda & 0,2 & 0,4 \\ 0,4 & 0,5-\lambda & 0,3 \\ 0,3 & 0,3 & 0,3-\lambda \end{vmatrix} = 0 \rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 0 \\ \lambda_3 = 0,1 \end{matrix} \xrightarrow{(P^T - \lambda I) \cdot v = 0} v_1 = \begin{pmatrix} 0,96 \\ 1,37 \\ 1 \end{pmatrix} \; v_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \; v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\pi = \frac{1}{\text{sum}(v_i)} \cdot v_i^T \; ; \quad \pi[1] + \pi[2] + \pi[3] = 1$$

$$\boxed{\pi = \frac{1}{0,96 + 1,37 + 1}(0,96,\ 1,37,\ 1) = (0,29,\ 0,41,\ 0,3)}$$

This means, in the long run:

29% of time on Downtown

41% of time on Suburbs

30% of time on Countryside

5. Expected Steps to reach Downtown starting from Suburbs

Expected number of steps = hitting time = $E[i]$

$$E_i = 1 + \sum_j P_{ij} \cdot E_j$$

$E_i = 0$ if i is the target state

$\hookrightarrow E_1 = 0$

$$\left\{\begin{array}{l} E_1 = 0 \\ E_2 = 1 + (\cancel{P_{21} \cdot E_1}^{0} + P_{22} \cdot E_2 + P_{23} \cdot E_3) = 1 + (0{,}5 \cdot E_2 + 0{,}3 \cdot E_3) \\ E_3 = 1 + (\cancel{P_{31} \cdot E_1}^{0} + P_{32} \cdot E_2 + P_{33} \cdot E_3) = 1 + (0{,}3 \cdot E_2 + 0{,}3 \cdot E_3) \end{array}\right\} \rightarrow \begin{array}{l} \boxed{E_2 = 3{,}85} \\ E_3 = 3{,}08 \end{array}$$

P2

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$$

$$L = \prod_{i=1}^{n} f(x_i; \mu, \sigma) \rightarrow -\ln L = -\prod_{i=1}^{n} \ln f(x_i; \mu, \sigma)$$

$$\ln f(x_i; \mu, \sigma) = \ln\left(\frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)\right) = \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \ln \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) =$$

$$= -\ln(\sigma\sqrt{2\pi}) + \frac{-(x-\mu)^2}{2\sigma^2} \cdot \underbrace{\ln e}_{1} = -\ln(\sigma\sqrt{2\pi}) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$-\ln L = -\left(-n \cdot \ln(\sigma\sqrt{2\pi}) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right) = n \cdot \ln(\sigma\sqrt{2\pi}) + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

P3 Derive the MLE for n IID samples for a RV with PDF: $f(x;\lambda) = \frac{1}{24} \cdot \lambda^5 \cdot x^4 \cdot exp(-\lambda x)$ where $\lambda > 0$, $x > 0$

$$\lambda_{MLE} \rightarrow \left.\frac{\partial L}{\partial \lambda}\right]_{\lambda_{MLE}} = 0 \rightarrow \left.\frac{\partial \log L}{\partial \lambda}\right]_{\lambda_{MLE}} = 0$$

$$L = \prod_{i=1}^{n} \frac{1}{24} \cdot \lambda^5 \cdot x_i^4 \cdot exp(-\lambda \cdot x_i) = \left(\frac{1}{24}\right)^n \cdot \lambda^{5n} \left(\prod_{i=1}^{n} x_i^4\right) \cdot exp\left(-\lambda \cdot \sum_{i=1}^{n} x_i\right)$$

$$\log L = n \cdot \log\left(\frac{1}{24}\right) + 5n \cdot \log\lambda + 4 \cdot \sum_{i=1}^{n} \log x_i - \lambda \cdot \sum_{i=1}^{n} x_i$$

$$\frac{\partial \log L}{\partial \lambda} = 0 + 5n \cdot \frac{1}{\lambda} + 0 - \sum_{i=1}^{n} x_i = \frac{5n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\left.\frac{\partial \log L}{\partial \lambda}\right]_{\lambda_{MLE}} = 0 = \frac{5n}{\lambda_{MLE}} - \sum_{i=1}^{n} x_i \longrightarrow \boxed{\lambda_{MLE} = \frac{5n}{\sum_{i=1}^{n} x_i}}$$