Università
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Faculty
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Image and Video Processing (Spring 2023)

Assignment 3: Frequency Domain Processing

Apr 28, 2023

1 Theoretical exercise [2 points]

Show that if $f(x, y) = g(x) \cdot g(y)$, then $F(u, v) = G(u) \cdot G(v)$, where capital letters denote Fourier transforms of the original functions.

2 Theoretical exercise [2 points]

Figure 1 shows a spatial domain signal $f(x)$. Compute Fourier transform $G(\mu)$ of function $g(x) = f(x) + f(-x)$. Consider solving this exercise in the following steps:

- Plot function $g(x)$.
- Realized that $g(x)$ can be expressed as a sum of two simpler functions for which we already know the Fourier transform. What are these functions?
- Use the fact that the Fourier Transform is a linear operation to express $G(\mu)$ as a sum of Fourier Transforms of the simpler functions.
- To compute the final expression for the $G(\mu)$ you can use the fact that the Fourier transform of a box function $h(x)$ is a since function, $H(\mu) = AW\text{sinc}(\mu W)$, where A is the amplitude of the box function and W is the width of the box.

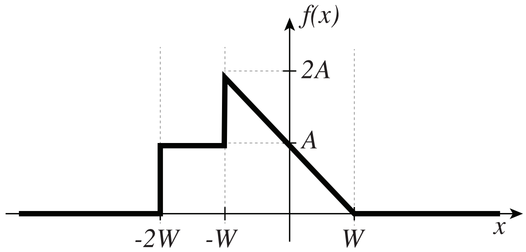

Figure 1: The spatial domain function $f(x)$ for Exercise 2.

3 Theoretical exercise [3 points]

Reducing the size of an image can lead to aliasing unless a low-pass filter is applied before. Consider a task of reducing the size of an image by a factor of c. What is the smallest standard deviation of a Gaussian filter that you would apply in the spatial domain to the original image to avoid aliasing artifacts? To solve this task:

- Perform the derivation on 1D signal.

- Use the fact that the Fourier transform of the Gaussian function is also a Gaussian function. More precisely, filtering a signal in a spatial domain using Gaussian filter with standard deviation σ_s is equivalent to a point-wise multiplication of the Fourier transform of the signal with Gaussian with standard deviation $\sigma_f = \frac{1}{2\sigma_s \pi}$. See practical exercises for experimental analysis.
- Assume that the frequency attenuation of 75% is a good approximation of a cut-off for Gaussian filter. In other words, assume that the spatial frequencies which are attenuated by a filter with a factor smaller than 0.75 do not cause aliasing.

BONUS (1 point): Show that the Fourier transform of a 1D Gaussian filter is also a Gaussian.

4 Theoretical exercise [2 points]

Figure 2 shows the magnitude of the Fourier spectrum of a spatial signal $f(x)$, which is two boxes centered around the frequency $\pm\omega_0$. Derive the formula for function $f(x)$ assuming arbitrary boxes' width (W) and height (A).
HINT: $f(x)$ can be expressed as a product of a sinc with another function. Also, recall the two parts of the convolution theorem.

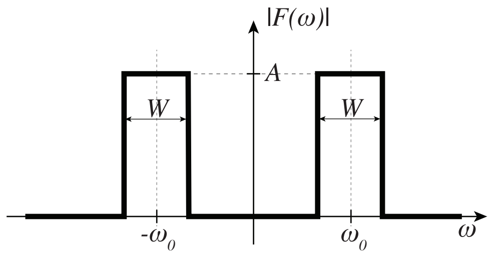

Figure 2: Magnitude of the Fourier spectrum of a spatial signal $f(x)$ from exercise 4.

5 Theoretical exercise [3 points]

Consider the task of filtering a signal with a spatial domain Gaussian filter $h(x)$ shown in Figure 3 (left). A student, who was not paying attention during the lectures, decided to approximate the filter with a kernel of inappropriate size (s), effectively windowing the original filter to $g(x)$ (Figure 3, right). The new filter has now a different Fourier transform than the original one ($G(\omega) \neq H(\omega)$), but it can be expresses as:

$$G(\omega) = H(\omega) \star X(\omega), \tag{1}$$

Your task is to derive an expression for $X(\omega)$ where s is an additional parameter. To this end, note that $g(x)$ can be expressed as a multiplication of function $h(x)$ with another simple function. Then use convolution theorem (second part) to derive $X(\omega)$.
Use MATLAB to plot an example of $H(\omega)$ and $G(\omega)$ and justify why using a smaller kernel size is a bad idea, i.e., what artifacts it will cause and what will be their source. You can use the MATLAB `conv` function for this.

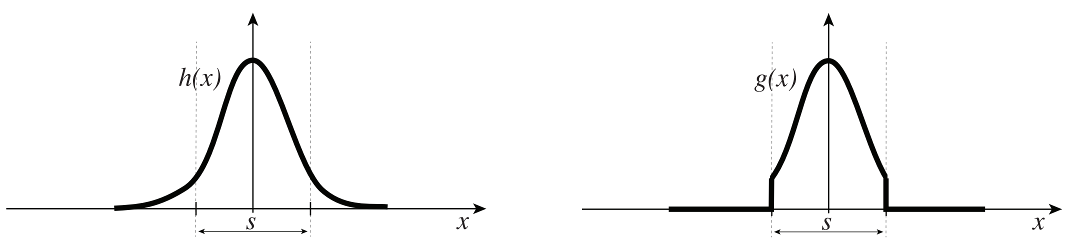

Figure 3: Visualization of the Gaussian filter $h(x)$ and its windowed version $g(x)$ for the exercise 5.

6 Gaussian filtering [4 points]

Implement, in MATLAB, Gaussian filtering both in the spatial and frequency domains and demonstrate that convolving an image with a Gaussian filter with standard deviation σ_s in the spatial domain is equivalent to point-wise multiplication in the frequency domain with Gaussian filter with standard deviation $\sigma_f = \frac{1}{2\sigma_s \pi}$. As a test image for this exercise, use an image similar to the one shown in Figure 4. Use MATLAB to construct such an image with size 1024×1024 pixels. For filtering both spatial and frequency domains assume padding with zero values. In the report, please show examples of filtered images with different pairs of σ_s and σ_f. **BONUS (1 point):** Use MATLAB `tic` and `toc` functions to analyze how the performance of equivalent filtering in spatial and temporal domains depends on the parameter σ_s. In particular, include in your report a plot of the execution time for both domains as a function of σ_s.

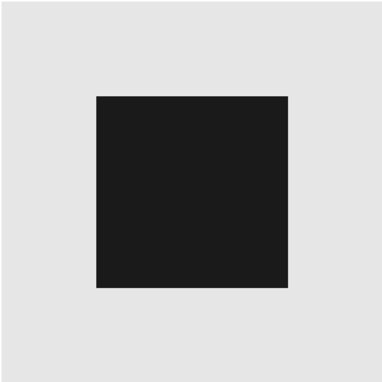

Figure 4: An input image for the exercise 6.

7 Image Restoration [4 points]

Consider a task of removing a repetitive pattern from an image using filtering in the frequency domain. Figure 5 demonstrates an input and the corresponding output of such a procedure. Design and implement a filtering procedure which perform such restoration. Explain your technique, show Fourier plots of all the steps, as well as the final image. Use the input image provided with the assignment.

Input image

Output image

Figure 5: Input and output images from exercise 7.

8 BONUS: Image Interpolation Analysis [2 point]

Suppose an image is to be up-scaled by a factor of two (doubling the size). Assuming no pre-filtering, you can use either nearest-neighbour interpolation, or linear interpolation to up-scale the image. What will be the ratio between the Fourier transforms of the linear interpolation up-scaled image, and the nearest-neighbor up-scaled image? What is your interpretation of this result, pertaining to the preservation of information after up-scaling? Show your calculations/reasoning. **HINT:** The process of up-scaling can be represented as a two-step process. Firstly, the image is up-sampled. Secondly, missing samples are estimated using convolutions with some filters; one separate filter for each of the two methods.

Submission

You should submit one ZIP-file via iCorsi containing:

- All your code in MATLAB appropriately commented, and the processed pictures that you obtained.
- A complete PDF report detailing your solution, partial results for each exercise, and answers to the theoretical questions poised.

Grading will be mostly based on the provided PDF report so we encourage clarity and detailed answers. We recommend using LaTeX or Overleaf to write the report. Usage of ChatGPT or any other natural language model is strictly prohibited and will be severely punished.

Solutions must be returned on May 12, 2023 via iCorsi3