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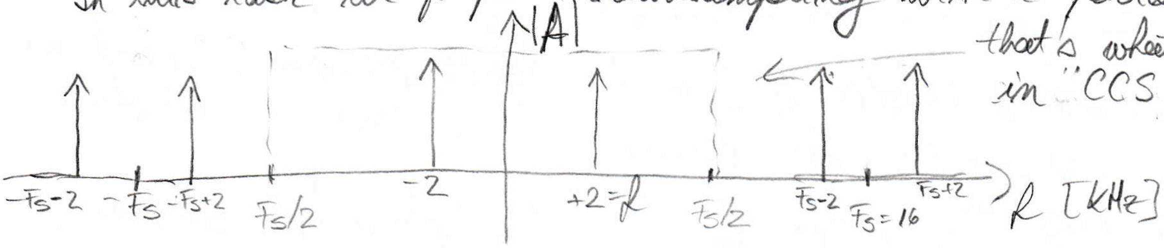
Lab 8: Two-channel filter bank

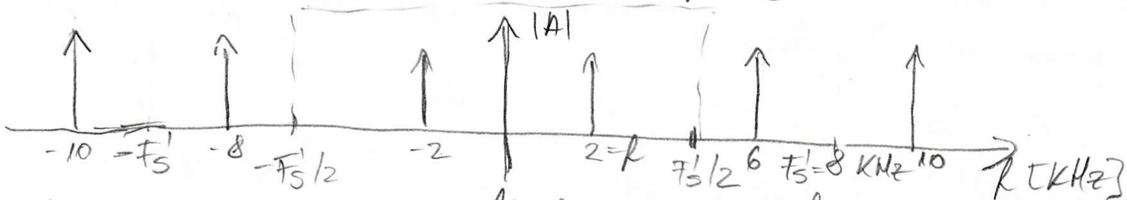
1) Upsampling and downsampling

a) $f = 2\,kHz$; $F_s = 16\,kHz$

In this task we perform downsampling with a factor of 2.

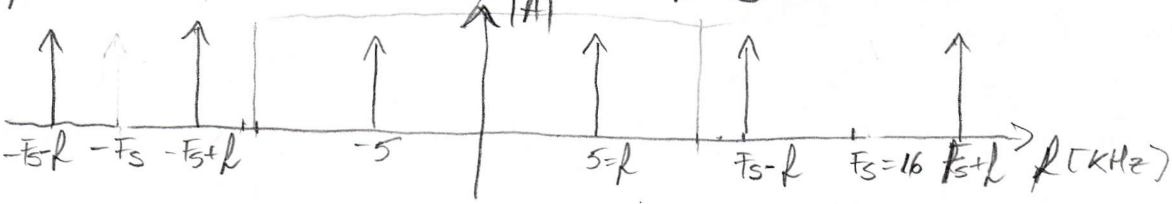


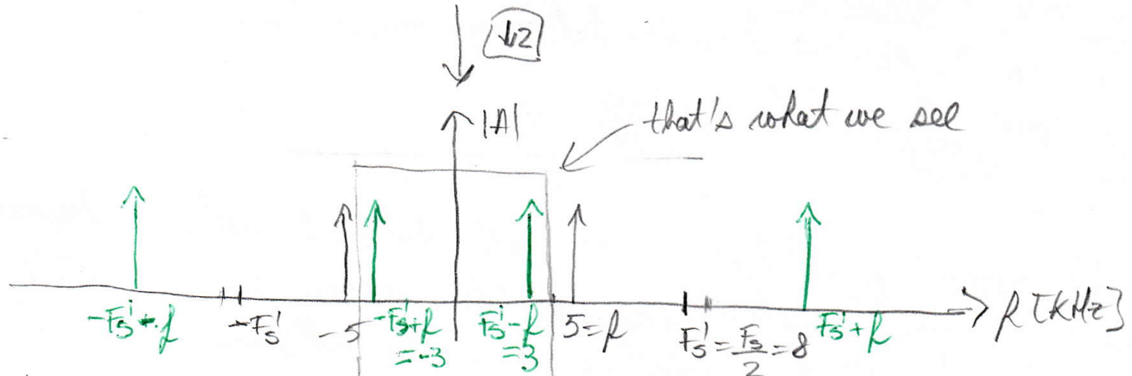
↓ downsampling with factor of 2 => $F_s' = \frac{F_s}{2} = 8\,kHz$



As we can see, no aliasing occurs, because $f < \frac{F_s'}{2}$, so the Nyquist criterion is obeyed also after downsampling.

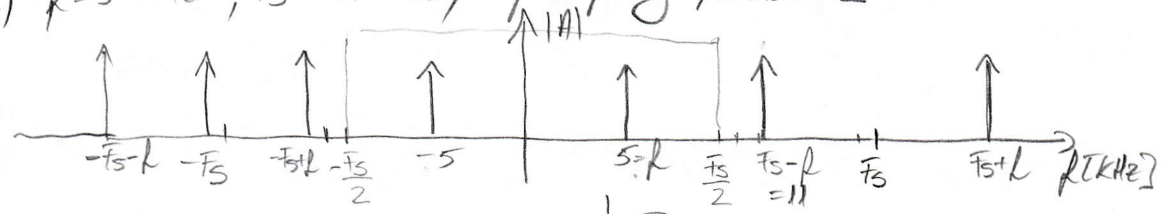
b) $f = 5\,kHz$, $F_s = 16\,kHz$; downsampling factor: 2

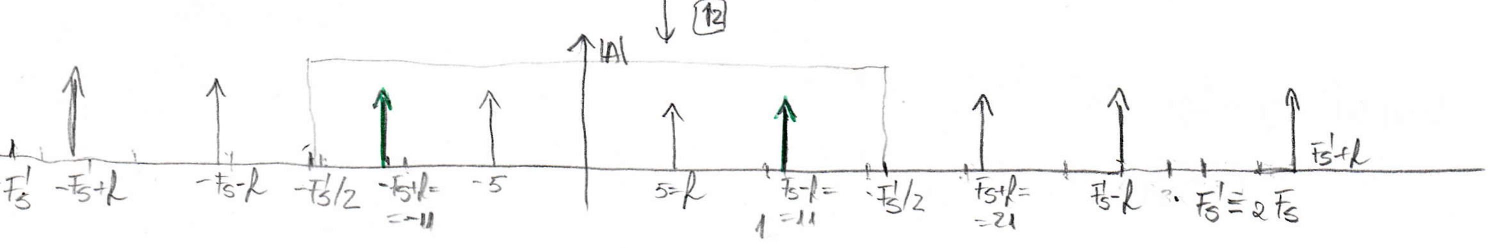




Here, aliasing occurs because $f > \frac{F_s'}{2}$. The imaged spectra are shown in the above image with green.

c) $f = 5\,kHz$; $F_s = 16\,kHz$; upsampling factor: 2





In this case, an additional component appears at 11 KHz which is cause by the fact that at the initial $F_S = 16$ KHz we had a component at $f' = F_S - f = 11$ KHz, which appears now in the observable spectrum due to the doubling of the sampling frequency.

d) In this exercise I designed a LPF with:

$$\omega_c = \frac{\pi}{L} = \frac{\pi}{2} \; ; \; \omega_c = 2\cancel{\pi}\frac{f_c}{F_S} = \frac{\cancel{\pi}}{2} \Rightarrow f_c = \frac{F_S}{4} = \frac{32000 \text{ Hz}}{4} = 8 \text{ KHz}$$

I chose $f_p = 5$ KHz because this is the highest frequency which we want to pass unattenuated and $f_{stopband} = 11$ KHz because ~~we~~ want to elliminate this frequency.

I chose the ripple to be the same in the stopband and passband, namely $\delta_p = \delta_s = 0.001$.

Then I proceede with the design of the LPF with the Kaiser window

$$\omega_c = \frac{\omega_p + \omega_s}{2} \Rightarrow F_c = \frac{F_p + F_s}{2} = \frac{5 + 11}{2} \text{ KHz} = 8 \text{ KHz}$$

$$\Delta\omega = \omega_s - \omega_p = \frac{\cancel{2}\pi}{\cancel{32}_{16} \text{ KHz}}(11 \text{ KHz} - 5 \text{ KHz}) = \frac{6\pi}{16} = \frac{3\pi}{8} = 0.375\pi$$

$$A = -20\log_{10}\delta = 60$$

$$\beta = 0.1102(A - 8.7) = 5.65326$$

$$M = \frac{A - 8}{2.285\Delta\omega} = 19.31 \Rightarrow M = 20$$

2) <u>Sampling rate conversion</u>

In this exercise I started from the following schematic:

$$\underset{F_s}{x[n]} \rightarrow \boxed{\uparrow L} \xrightarrow{y[n]} \underset{F_s'}{\boxed{\text{LPF}}}^{\pi/L} \rightarrow \boxed{\text{LPF}}^{\pi/M} \xrightarrow{x[n']} \boxed{\downarrow M} \xrightarrow{y[n'']} F_s''$$

Because the cascading of two LPF with different cutoff-frequencies is impractical, I implemented only one LPF whose $\omega_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$

Thus, the schematic that I implemented is the following:

$$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \underset{\omega_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)}{\boxed{\text{LPF}}} \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

conversion ratio $1.5 \Rightarrow L = 3, M = 2$; $\omega_c = \frac{\cancel{\pi}}{3} = \frac{2\cancel{\pi} F_c}{F_S} \Rightarrow F_c = \frac{F_S}{6} = \frac{8 \text{KHz} \cdot \cancel{3}}{\cancel{6}\,2}$

$F_c = 4$ KHz

conversion ratio $0.75 \Rightarrow L = 3, M = 4$; $\omega_c = \frac{\cancel{\pi}}{4} = \frac{2\cancel{\pi} F_c}{F_S} \Rightarrow F_c = \frac{F_S}{8} = \frac{8 \text{KHz} \cdot 3}{8}$

$F_c = 3$ KHz

Initial sampling rate is $F_S = 8$ KHz

3) Analysis filter bank and synthesis filter bank coefficients

Prototype filter design 1

In this exercise we are given h_0, which is the impulse response of the analysis FIR LPF and f_0, which is the impulse response of the synthesis FIR LPF and we are required to find h_1, the impulse response of the analysis HPF and f_1, the impulse response of the synthesis HPF.

I did this by using the following condition:

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases}$$

Knowing $\boxed{h_0 = [-1\ 2\ 6\ 2\ -1]/8} \Rightarrow H_0(z) = (-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4})/8$

$-H_0(-z) = -(-1 - 2z^{-1} + 6z^{-2} - 2z^{-3} - z^{-4})/8 = (1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4})/8$

$\Rightarrow F_1(z) = -H_0(-z) \Rightarrow \boxed{f_1 = [1\ 2\ -6\ 2\ 1]/8}$

Knowing $\boxed{f_0 = [1\ 2\ 1]/2} \Rightarrow F_0(z) = (1 + 2z^{-1} + z^{-2})/2$

$H_1(-z) = F_0(z) \Rightarrow H_1(z) = F_0(-z) = (1 - 2z^{-1} + z^{-2})/2$

$\Rightarrow h_1 = [1\ -2\ 1]/2$

Prototype filter design 2

In this exercise we are given only h_0 and we are required to find h_1, f_0 and f_1. To do so, we need to use both of the following conditions:

(I) $\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases}$

(II) $P_0(z) - P_0(-z) = 2z^{-l}$, where $P_0(z) = H_0(z)F_0(z)$

$$H_0 = \frac{1}{4\sqrt{2}}\left[(1+\sqrt{3}) + (3+\sqrt{3})z^{-1} + (3-\sqrt{3})z^{-2} + (1-\sqrt{3})z^{-3}\right]$$

Assume $F_0 = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$

$$\Rightarrow [a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}]\frac{1}{4\sqrt{2}}\left[(1+\sqrt{3}) + (3+\sqrt{3})z^{-1} + (3-\sqrt{3})z^{-2} + (1-\sqrt{3})z^{-3}\right]$$

$$-[a_0 - a_1 z^{-1} + a_2 z^{-2} - a_3 z^{-3}]\frac{1}{4\sqrt{2}}\left[(1+\sqrt{3}) - (3+\sqrt{3})z^{-1} + (3-\sqrt{3})z^{-2} - (1-\sqrt{3})z^{-3}\right]$$

$= 2z^{-3}$. We choose this value for the delay l because all filters that we considered have the maximum delay of 3, so the cascading of such filters cannot yield a lower delay.

$$\Rightarrow \frac{1}{4\sqrt{2}}\Big[\cancel{a_0(1+\sqrt{3})} + a_0(3+\sqrt{3})z^{-1} + \cancel{a_0(3-\sqrt{3})z^{-2}} + a_0(1-\sqrt{3})z^{-3} +$$

$$+ a_1(1+\sqrt{3})z^{-1} + \cancel{a_1(3+\sqrt{3})z^{-2}} + a_1(3-\sqrt{3})z^{-3} + \cancel{a_1(1-\sqrt{3})z^{-4}} +$$

$$+ \cancel{a_2(1+\sqrt{3})z^{-2}} + a_2(3+\sqrt{3})z^{-3} + \cancel{a_2(3-\sqrt{3})z^{-4}} + a_2(1-\sqrt{3})z^{-5} +$$

$$+ a_3(1+\sqrt{3})z^{-3} + \cancel{a_3(3+\sqrt{3})z^{-4}} + a_3(3-\sqrt{3})z^{-5} + \cancel{a_3(1-\sqrt{3})z^{-6}} -$$

$$- \cancel{a_0(1+\sqrt{3})} + a_0(3+\sqrt{3})z^{-1} - \cancel{a_0(3-\sqrt{3})z^{-2}} + a_0(1-\sqrt{3})z^{-3} +$$

$$+ a_1(1+\sqrt{3})z^{-1} - \cancel{a_1(3+\sqrt{3})z^{-2}} + a_1(3-\sqrt{3})z^{-3} - \cancel{a_1(1-\sqrt{3})z^{-4}} -$$

$$- \cancel{a_2(1+\sqrt{3})z^{-2}} + a_2(3+\sqrt{3})z^{-3} - \cancel{a_2(3-\sqrt{3})z^{-4}} + a_2(1-\sqrt{3})z^{-5} +$$

$$+ a_3(1+\sqrt{3})z^{-3} - \cancel{a_3(3+\sqrt{3})z^{-4}} + a_3(3-\sqrt{3})z^{-5} - \cancel{a_3(1-\sqrt{3})z^{-6}}\Big] = 2z^{-3}$$

$$\begin{cases} \frac{1}{4\sqrt{2}} \cdot 2 2^{-1} \left[a_0 (3+\sqrt{3}) + a_1 (1+\sqrt{3}) \right] = 0 \\ \frac{1}{4\sqrt{2}} \cdot \cancel{2}\cancel{2}^{-3} \left[a_0 (1-\sqrt{3}) + a_1 (3-\sqrt{3}) + a_2 (3+\sqrt{3}) + a_3 (1+\sqrt{3}) \right] = \cancel{2}\cancel{2}^{-3}\, 1 \\ \frac{1}{4\sqrt{2}}\, 2 2^{-5} \left[a_2 (1-\sqrt{3}) + a_3 (3-\sqrt{3}) \right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 1)\ a_0 (3+\sqrt{3}) + a_1 (1+\sqrt{3}) = 0 \Rightarrow a_1 = - \frac{a_0 \sqrt{3}\, \cancel{(1+\sqrt{3})}}{\cancel{1+\sqrt{3}}} = -a_0 \sqrt{3} \\ 2)\ a_0 (1-\sqrt{3}) + a_1 (3-\sqrt{3}) + a_2 (3+\sqrt{3}) + a_3 (1+\sqrt{3}) = 4\sqrt{2} \\ 3)\ a_2 (1-\sqrt{3}) + a_3 (3-\sqrt{3}) = 0 \Rightarrow a_3 = a_2 \frac{\sqrt{3}-1}{\sqrt{3}(\sqrt{3}-1)} = \frac{a_2}{\sqrt{3}} \end{cases}$$

Choose $a_0 = \frac{1-\sqrt{3}}{4\sqrt{2}} \overset{1)}{\Rightarrow} a_1 = \frac{-\sqrt{3}+3}{4\sqrt{2}}$

$$2)\ \frac{(1-\sqrt{3})^2}{4\sqrt{2}} + \frac{(3-\sqrt{3})^2}{4\sqrt{2}} + a_2 (3+\sqrt{3}) + a_3 (1+\sqrt{3}) = 4\sqrt{2}$$

$$\Rightarrow \frac{1-2\sqrt{3}+3+9-6\sqrt{3}+3}{4\sqrt{2}} + a_2 (3+\sqrt{3}) + a_3 (1+\sqrt{3}) = 4\sqrt{2}$$

$$\Rightarrow \frac{16-8\sqrt{3}}{4\sqrt{2}} + a_2 (3+\sqrt{3}) + a_3 (1+\sqrt{3}) = 4\sqrt{2} \Rightarrow$$

$$\Rightarrow a_2 (3+\sqrt{3}) + a_3 (1+\sqrt{3}) = \frac{32-16+8\sqrt{3}}{4\sqrt{2}} = \frac{16+8\sqrt{3}}{4\sqrt{2}}$$

$$\left.\begin{cases} \sqrt{3}\, a_2 (1+\sqrt{3}) + a_3 (1+\sqrt{3}) = \frac{16+8\sqrt{3}}{4\sqrt{2}} \\ a_3 = \frac{a_2}{\sqrt{3}} \end{cases}\right| \Rightarrow$$

$$\Rightarrow \sqrt{3}\, a_2 (1+\sqrt{3}) + a_2 \frac{\sqrt{3}+1}{\sqrt{3}} = \frac{16+8\sqrt{3}}{4\sqrt{2}} \Rightarrow \frac{3 a_2 (1+\sqrt{3}) + a_2 (1+\sqrt{3})}{\sqrt{3}} = \frac{16+8\sqrt{3}}{4\sqrt{2}}$$

$$\Rightarrow \frac{\cancel{4}(1+\sqrt{3})\, a_2}{\sqrt{3}} = \frac{\overset{2}{\cancel{8}}(\sqrt{3}+2)}{4\sqrt{2}} \Rightarrow (1+\sqrt{3})\, a_2 = \frac{6+4\sqrt{3}}{4\sqrt{2}}$$

$$\Rightarrow a_2 = \frac{3+\sqrt{3}}{4\sqrt{2}} \Rightarrow a_3 = \frac{\sqrt{3}+1}{4\sqrt{2}}$$

So $h_0 = \frac{1}{4\sqrt{2}} \left[(1+\sqrt{3}), (3+\sqrt{3}), (3-\sqrt{3}), (1-\sqrt{3}) \right]$

$h_1 = \frac{1}{4\sqrt{2}} \left[(1-\sqrt{3}), (\sqrt{3}-3), (3+\sqrt{3}), -(1+\sqrt{3}) \right]$

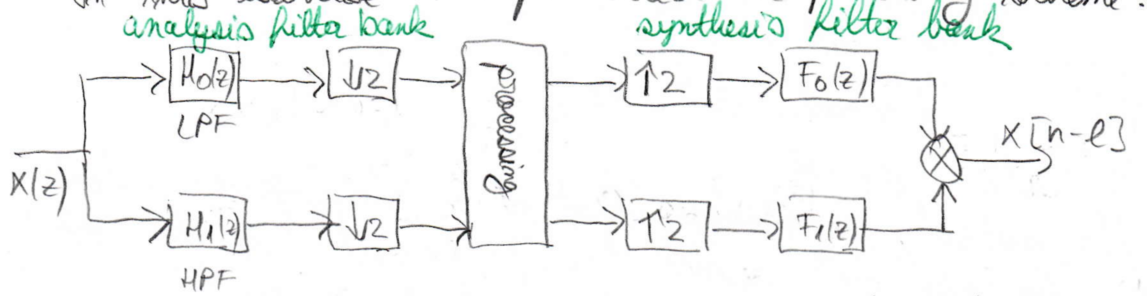
$f_0 = \frac{1}{4\sqrt{2}} \left[(1-\sqrt{3}), (3-\sqrt{3}), (3+\sqrt{3}), (1+\sqrt{3}) \right]$

$f_1 = \frac{1}{4\sqrt{2}} \left[-(1+\sqrt{3}), 3+\sqrt{3}, -(3-\sqrt{3}), 1-\sqrt{3} \right]$

Having a look on the output yielded by fvtool, we can see that the perfect reconstruction condition is verified (Th 4.1 / pag 105): the magnitud response is a straight line at around 6 dB, and the impulse response is a Dirac delayed with 3 samples, as desired.

4) Non-efficient implementation of two-channel filter bank

In this exercise I implemented the following scheme:



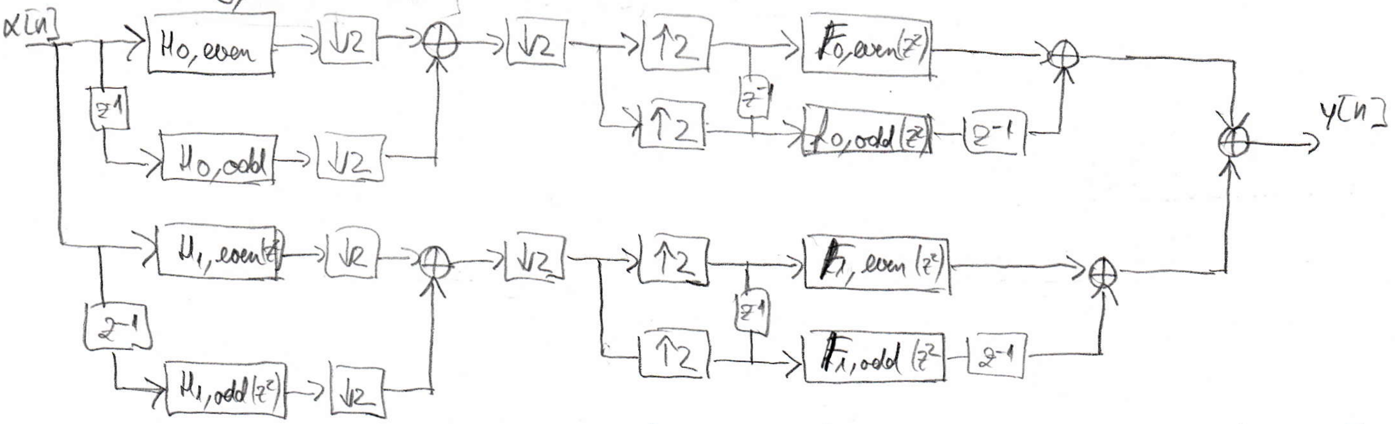
I chose for h_0, h_1, f_0, f_1 the values obtained a 3.1.

At the output we observe an impulse which has a delay of 3 with respect to the input.

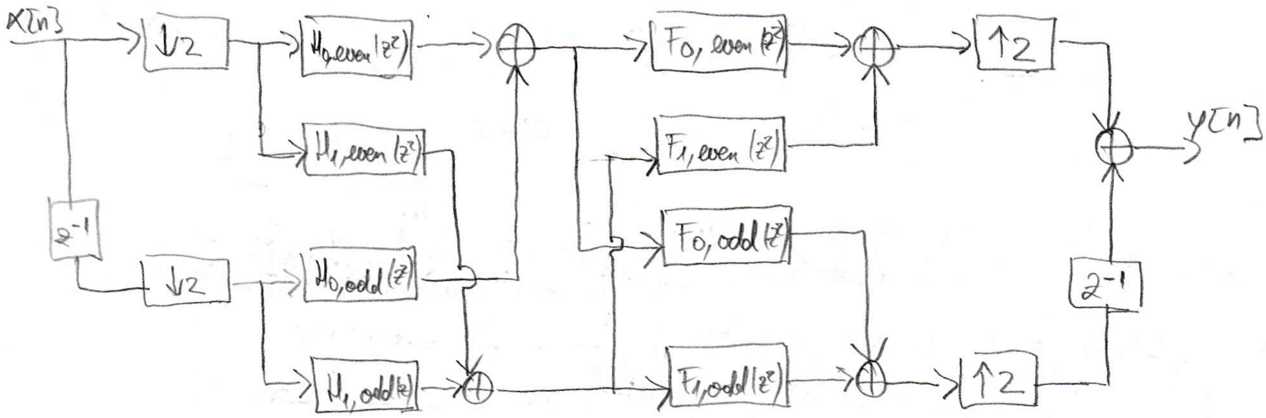
5) Efficient-implementation of two-channel filter bank

The main problem in the previous implementation is that we downsample after the LPF/HPF and upsample before it. This is of course not desirable: it is better to first downsample and then filter the signal. Thus, moving the downsampling before the filter and the upsampling after the filter, we obtain the following schematics:

1° firstly, we split each filter in an odd and even part:



2° moving the downsampling before the filter and the upsampling after the filter we get:



$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} = (h_0 + h_2 z^{-2}) + (h_1 + h_3 z^{-2}) z^{-1}$$

$$H_{even}(z^2) = h_0 + h_2 z^{-2}$$

$$H_{odd}(z^2) = h_1 + h_3 z^{-2}$$

thus,

$h_{0,even} = [-1\ 6\ -1]/8$; $h_{0,odd} = [2\ 2]/8$

$h_{1,even} = [1\ 1]/2$, $h_{1,odd} = [-2]/2$;

$f_{0,even} = [1\ 1]/2$, $f_{0,odd} = [2]/2$;

$f_{1,even} = [1\ -6\ 1]/8$, $f_{1,odd} = [2\ 2]/8$

Looking at the output, we observe that it is an impulse which is delayed with 4 samples with respect to the input impulse we provide to the filter bank.