4 Digital filters

4.1 Introduction

In the previous chapters we studied different ways of describing discrete-time systems that are linear and time invariant. It was verified that the z transform greatly simplifies the analysis of discrete-time systems, especially those initially described by a difference equation.

In this chapter we study several structures used to realize a given transfer function associated with a specific difference equation through the use of the z transform. The transfer functions considered here will be of the polynomial form (nonrecursive filters) and of the rational-polynomial form (recursive filters). In the nonrecursive case we emphasize the existence of the important subclass of linear-phase filters. Then we introduce some tools to calculate the digital network transfer function, as well as to analyze its internal behavior. We also discuss some properties of generic digital filter structures associated with practical discrete-time systems. The chapter also introduces a number of useful building blocks often utilized in practical applications. A Do-it-yourself section is included in order to enlighten the reader on how to start from the concepts and generate some possible realizations for a given transfer function.

4.2 Basic structures of nonrecursive digital filters

Nonrecursive filters are characterized by a difference equation in the form

$$y(n) = \sum_{l=0}^{M} b_l x(n-l), \tag{4.1}$$

where the b_l coefficients are directly related to the system impulse response; that is, $b_l = h(l)$. Owing to the finite length of their impulse responses, nonrecursive filters are also referred to as finite-duration impulse response (FIR) filters. We can rewrite Equation (4.1) as

$$y(n) = \sum_{l=0}^{M} h(l)x(n-l). \tag{4.2}$$

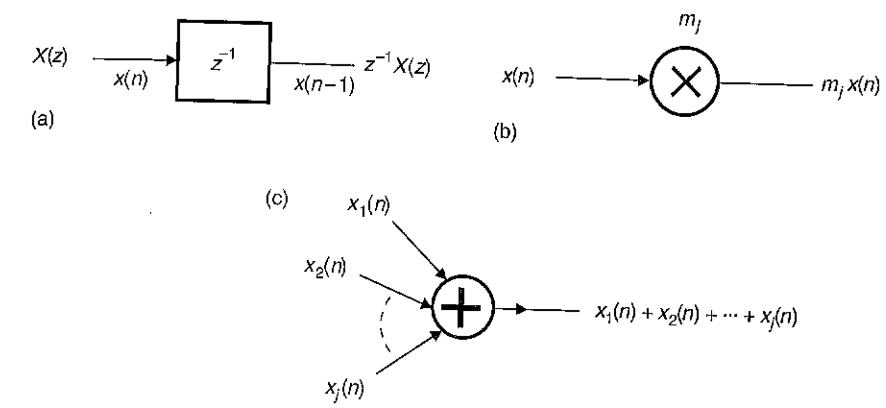


Fig. 4.1 Classic representation of basic elements of digital filters: (a) delay; (b) multiplier; (c) adder.

Applying the z transform to Equation (4.2), we end up with the following input–output relationship:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{l=0}^{M} b_l z^{-l} = \sum_{l=0}^{M} h(l)z^{-l}. \tag{4.3}$$

In practical terms, Equation (4.3) can be implemented in several distinct forms, using as basic elements the delay, the multiplier, and the adder blocks. These basic elements of digital filters and their corresponding standard symbols are depicted in Figure 4.1. An alternative way of representing such elements is the so-called signal flowgraph shown in Figure 4.2. These two sets of symbolisms representing the delay, multiplier, and adder elements, are used throughout this book interchangeably.

4.2.1 Direct form

The simplest realization of an FIR digital filter is derived from Equation (4.3). The resulting structure, seen in Figure 4.3, is called the direct-form realization, as the multiplier coefficients are obtained directly from the filter transfer function. Such a structure is also referred to as the canonic direct form, where we understand canonic form to mean any structure that realizes a given transfer function with the minimum number of delays, multipliers, and adders. More specifically, a structure that utilizes the minimum number of delays is said to be canonic with respect to the delay element, and so on.

An alternative canonic direct form for Equation (4.3) can be derived by expressing $H(z)$ as

$$H(z) = \sum_{l=0}^{M} h(l)z^{-l} = h(0)+z^{-1}(h(1)+z^{-1}(h(2)+\cdots+z^{-1}(h(M-1)+z^{-1}h(M))\cdots)). \tag{4.4}$$

The implementation of this form is shown in Figure 4.4.

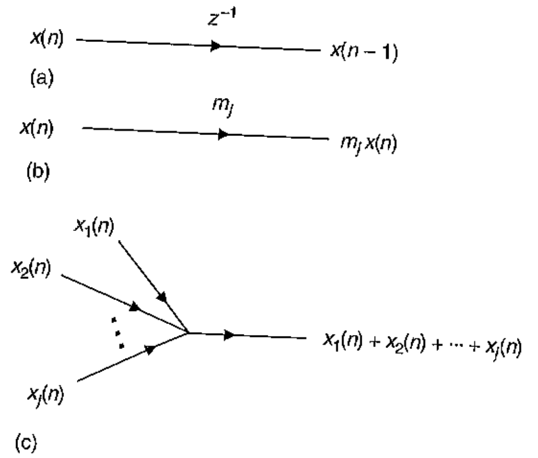


Fig. 4.2 Signal-flowgraph representation of basic elements of digital filters: (a) delay; (b) multiplier; (c) adder.

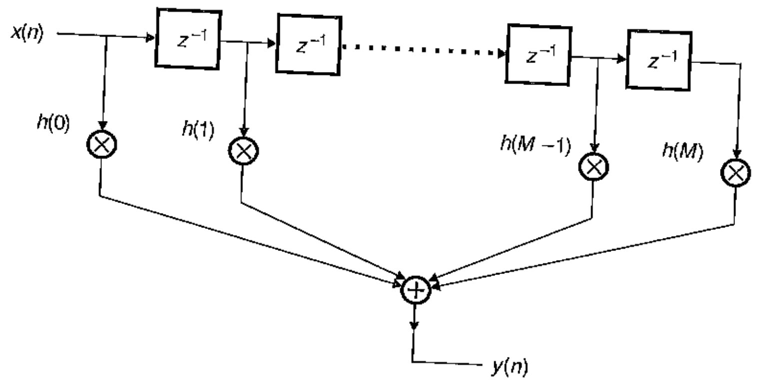


Fig. 4.3 Direct form for FIR digital filters.

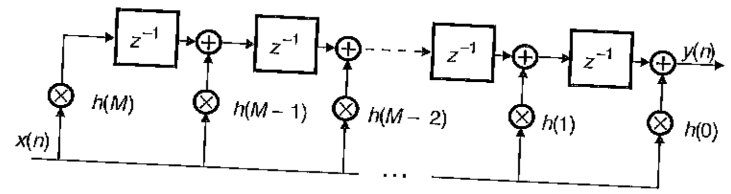


Fig. 4.4 Alternative direct form for FIR digital filters.

4.2.2 Cascade form

Equation (4.3) can be realized through a series of equivalent structures. However, the coefficients of such distinct realizations may not be explicitly the filter impulse response or the corresponding transfer function. An important example of such a realization is the

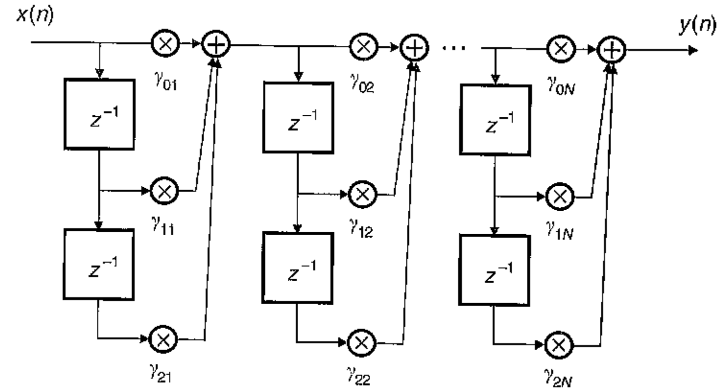


Fig. 4.5 Cascade form for FIR digital filters.

so-called cascade form, which consists of a series of second-order FIR filters connected in cascade, thus the name of the resulting structure, as seen in Figure 4.5.

The transfer function associated with such a realization is of the form

$$H(z) = \prod_{k=1}^{N} (\gamma_{0k} + \gamma_{1k} z^{-1} + \gamma_{2k} z^{-2}), \tag{4.5}$$

where if M is the filter order, then $N = M/2$ when M is even and $N = (M+1)/2$ when M is odd. In the latter case, one of the γ_{2k} becomes zero.

4.2.3 Linear-phase forms

An important subclass of FIR digital filters is the one that includes linear-phase filters. Such filters are characterized by a constant group delay τ; therefore, they must present a frequency response of the following form:

$$H(e^{j\omega}) = B(\omega) e^{-j\omega\tau + j\phi}, \tag{4.6}$$

where $B(\omega)$ is real and τ and ϕ are constant. Hence, the impulse response $h(n)$ of linear-phase filters satisfies

$$\begin{aligned} h(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} B(\omega) e^{-j\omega\tau + j\phi} e^{j\omega n}\, d\omega \\ &= \frac{e^{j\phi}}{2\pi} \int_{-\pi}^{\pi} B(\omega) e^{j\omega(n-\tau)}\, d\omega. \end{aligned} \tag{4.7}$$

We are considering filters here where the group delay is a multiple of half a sample; that is,

$$\tau = \frac{k}{2}, \quad k \in \mathbb{Z}. \tag{4.8}$$

Thus, for such cases when 2τ is an integer, Equation (4.8) implies that

$$h(2\tau - n) = \frac{e^{j\phi}}{2\pi}\int_{-\pi}^{\pi} B(\omega)e^{j\omega(2\tau-n-\tau)}\,d\omega = \frac{e^{j\phi}}{2\pi}\int_{-\pi}^{\pi} B(\omega)e^{j\omega(\tau-n)}\,d\omega. \tag{4.9}$$

Since $B(\omega)$ is real, we have

$$h^*(2\tau - n) = \frac{e^{-j\phi}}{2\pi}\int_{-\pi}^{\pi} B^*(\omega)e^{-j\omega(\tau-n)}\,d\omega = \frac{e^{-j\phi}}{2\pi}\int_{-\pi}^{\pi} B(\omega)e^{j\omega(n-\tau)}\,d\omega. \tag{4.10}$$

Then, from Equations (4.7) and (4.10), in order for a filter to have linear phase with a constant group delay τ, its impulse response must satisfy

$$h(n) = e^{2j\phi}h^*(2\tau - n). \tag{4.11}$$

We now proceed to show that linear-phase FIR filters present impulse responses of very particular forms. In fact, Equation (4.11) implies that $h(0) = e^{2j\phi}h^*(2\tau)$. Hence, if $h(n)$ is causal and of finite duration, for $0 \le n \le M$, we must necessarily have that

$$\tau = \frac{M}{2} \tag{4.12}$$

and then Equation (4.11) becomes

$$h(n) = e^{2j\phi}h^*(M - n). \tag{4.13}$$

This is the general equation that the coefficients of a linear-phase FIR filter must satisfy.

In the common case where all the filter coefficients are real, then $h(n) = h^*(n)$ and Equation (4.13) implies that $e^{2j\phi}$ must be real. Thus:

$$\phi = \frac{k\pi}{2}, \; k \in \mathbb{Z} \tag{4.14}$$

and Equation (4.13) becomes

$$h(n) = (-1)^k h(M - n), \; k \in \mathbb{Z}. \tag{4.15}$$

That is, the filter impulse response must be either symmetric or antisymmetric.

From Equation (4.6), the frequency response of linear-phase FIR filters with real coefficients becomes

$$H(e^{j\omega}) = B(\omega)e^{-j\omega(M/2)+j(k\pi/2)} \tag{4.16}$$

For all practical purposes, we only need to consider the cases when $k = 0, 1, 2, 3$, as all other values of k will be equivalent to one of these four cases. Furthermore, as $B(\omega)$ can be either positive or negative, the cases $k = 2$ and $k = 3$ are obtained from cases $k = 0$ and $k = 1$ respectively by making $B(\omega) \leftarrow -B(\omega)$.

Therefore, we consider solely the four distinct cases described by Equations (4.13) and (4.16). They are referred to as follows:

- Type I: $k = 0$ and M even.
- Type II: $k = 0$ and M odd.
- Type III: $k = 1$ and M even.
- Type IV: $k = 1$ and M odd.

We now proceed to demonstrate that $h(n) = (-1)^k h(M - n)$ is a sufficient condition for an FIR filter with real coefficients to have a linear phase. The four types above are considered separately.

- Type I: $k = 0$ implies that the filter has symmetric impulse response; that is, $h(M - n) = h(n)$. Since the filter order M is even, Equation (4.3) may be rewritten as

$$\begin{aligned} H(z) &= \sum_{n=0}^{(M/2)-1} h(n)z^{-n} + h\left(\frac{M}{2}\right)z^{-M/2} + \sum_{n=(M/2)+1}^{M} h(n)z^{-n} \\ &= \sum_{n=0}^{(M/2)-1} h(n)\left[z^{-n} + z^{-(M-n)}\right] + h\left(\frac{M}{2}\right)z^{-M/2}. \end{aligned} \tag{4.17}$$

Evaluating this equation over the unit circle (that is, using the variable transformation $z \to e^{j\omega}$), one obtains

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{(M/2)-1} h(n)\left(e^{-j\omega n} + e^{-j\omega M + j\omega n}\right) + h\left(\frac{M}{2}\right)e^{-j\omega M/2} \\ &= e^{-j\omega M/2}\left\{h\left(\frac{M}{2}\right) + \sum_{n=0}^{(M/2)-1} 2h(n)\cos\left[\omega\left(n - \frac{M}{2}\right)\right]\right\}. \end{aligned} \tag{4.18}$$

Substituting n by $(M/2) - m$, we get

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega M/2}\left[h\left(\frac{M}{2}\right) + \sum_{m=1}^{M/2} 2h\left(\frac{M}{2} - m\right)\cos(\omega m)\right] \\ &= e^{-j\omega M/2}\sum_{m=0}^{M/2} a(m)\cos(\omega m), \end{aligned} \tag{4.19}$$

with $a(0) = h(M/2)$ and $a(m) = 2h[(M/2) - m]$, for $m = 1, 2, \ldots, M/2$.

Since this equation is in the form of Equation (4.16), this completes the sufficiency proof for Type I filters.

- Type II: $k = 0$ implies that the filter has a symmetric impulse response; that is, $h(M-n) = h(n)$. Since the filter order M is odd, Equation (4.3) may be rewritten as

$$H(z) = \sum_{n=0}^{(M-1)/2} h(n)z^{-n} + \sum_{n=(M+1)/2}^{M} h(n)z^{-n}$$
$$= \sum_{n=0}^{(M-1)/2} h(n)[z^{-n} + z^{-(M-n)}]. \tag{4.20}$$

Evaluating this equation over the unit circle, one obtains

$$H(e^{j\omega}) = \sum_{n=0}^{(M-1)/2} h(n)\left(e^{-j\omega n} + e^{-j\omega M + j\omega n}\right)$$
$$= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} h(n)\left\{e^{-j\omega[n-(M/2)]} + e^{j\omega[n-(M/2)]}\right\}$$
$$= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} 2h(n)\cos\left[\omega\left(n - \frac{M}{2}\right)\right]. \tag{4.21}$$

Substituting n with $[(M+1)/2] - m$:

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{m=1}^{(M+1)/2} 2h\left(\frac{M+1}{2} - m\right)\cos\left[\omega\left(m - \frac{1}{2}\right)\right]$$
$$= e^{-j\omega M/2} \sum_{m=1}^{(M+1)/2} b(m)\cos\left[\omega\left(m - \frac{1}{2}\right)\right], \tag{4.22}$$

with $b(m) = 2h[(M+1/2) - m]$, for $m = 1, 2, \ldots, (M+1)/2$.
Since this equation is in the form of Equation (4.16), this completes the sufficiency proof for Type II filters.

Notice that, at $\omega = \pi$, $H(e^{j\omega}) = 0$, as it consists of a summation of cosine functions evaluated at $\pm\pi/2$, which are obviously null. Therefore, highpass and bandstop filters cannot be approximated as Type II filters.

- Type III: $k = 1$ implies that the filter has an antisymmetric impulse response; that is, $h(M-n) = -h(n)$. In this case, $h(M/2)$ is necessarily null. Since the filter order M is even, Equation (4.3) may be rewritten as

$$H(z) = \sum_{n=0}^{(M/2)-1} h(n)z^{-n} + \sum_{n=(M/2)+1}^{M} h(n)z^{-n}$$
$$= \sum_{n=0}^{(M/2)-1} h(n)\left[z^{-n} - z^{-(M-n)}\right], \tag{4.23}$$

which, when evaluated over the unit circle, yields

$$H(e^{j\omega}) = \sum_{n=0}^{(M/2)-1} h(n)\left(e^{-j\omega n} - e^{-j\omega M + j\omega n}\right)$$
$$= e^{-j\omega M/2} \sum_{n=0}^{(M/2)-1} h(n)\left\{e^{-j\omega[n-(M/2)]} - e^{j\omega[n-(M/2)]}\right\}$$
$$= e^{-j\omega M/2} \sum_{n=0}^{(M/2)-1} -2jh(n)\sin\left[\omega\left(n - \frac{M}{2}\right)\right].$$
$$= e^{-j[\omega(M/2)-(\pi/2)]} \sum_{n=0}^{(M/2)-1} -2h(n)\sin\left[\omega\left(n - \frac{M}{2}\right)\right]. \tag{4.24}$$

Substituting n by $(M/2) - m$:

$$H(e^{j\omega}) = e^{-j[\omega(M/2)-(\pi/2)]} \sum_{m=1}^{M/2} -2h\left(\frac{M}{2} - m\right)\sin[\omega(-m)]$$
$$= e^{-j[\omega(M/2)-(\pi/2)]} \sum_{m=1}^{M/2} c(m)\sin(\omega m), \tag{4.25}$$

with $c(m) = 2h[(M/2) - m]$, for $m = 1, 2, \ldots, M/2$.
Since this equation is in the form of Equation (4.16), this completes the sufficiency proof for Type III filters.

Notice, in this case, that the frequency response becomes null at $\omega = 0$ and at $\omega = \pi$. That makes this type of realization suitable for bandpass filters, differentiators, and Hilbert transformers, these last two due to the phase shift of $\pi/2$, as will be seen in Chapter 5.

- Type IV: $k = 1$ implies that the filter has an antisymmetric impulse response; that is, $h(M-n) = -h(n)$. Since the filter order M is odd, Equation (4.3) may be rewritten as

$$H(z) = \sum_{n=0}^{(M-1)/2} h(n)z^{-n} + \sum_{n=(M+1)/2}^{M} h(n)z^{-n}$$
$$= \sum_{n=0}^{(M-1)/2} h(n)\left[z^{-n} - z^{-(M-n)}\right]. \tag{4.26}$$

Evaluating this equation over the unit circle:

$$H(e^{j\omega}) = \sum_{n=0}^{(M-1)/2} h(n)\left(e^{-j\omega n} - e^{-j\omega M + j\omega n}\right)$$
$$= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} h(n)\left\{e^{-j\omega[n-(M/2)]} - e^{j\omega[n-(M/2)]}\right\}$$

$$= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} -2jh(n) \sin\left[\omega\left(n - \frac{M}{2}\right)\right]$$

$$= e^{-j[\omega(M/2)-(\pi/2)]} \sum_{n=0}^{(M-1)/2} -2h(n) \sin\left[\omega\left(n - \frac{M}{2}\right)\right]. \quad (4.27)$$

Substituting n by $[(M+1)/2] - m$:

$$H(e^{j\omega}) = e^{-j[\omega(M/2)-(\pi/2)]} \sum_{m=1}^{(M+1)/2} -2h\left(\frac{M+1}{2} - m\right) \sin\left[\omega\left(\frac{1}{2} - m\right)\right]$$

$$= e^{-j[\omega(M/2)-(\pi/2)]} \sum_{m=1}^{(M+1)/2} d(m) \sin\left[\omega\left(m - \frac{1}{2}\right)\right], \quad (4.28)$$

with $d(m) = 2h\{[(M+1)/2] - m\}$, for $m = 1, 2, \ldots, (M+1)/2$. Since this equation is in the form of Equation (4.16), this completes the sufficiency proof for Type IV filters, thus finishing the whole proof.

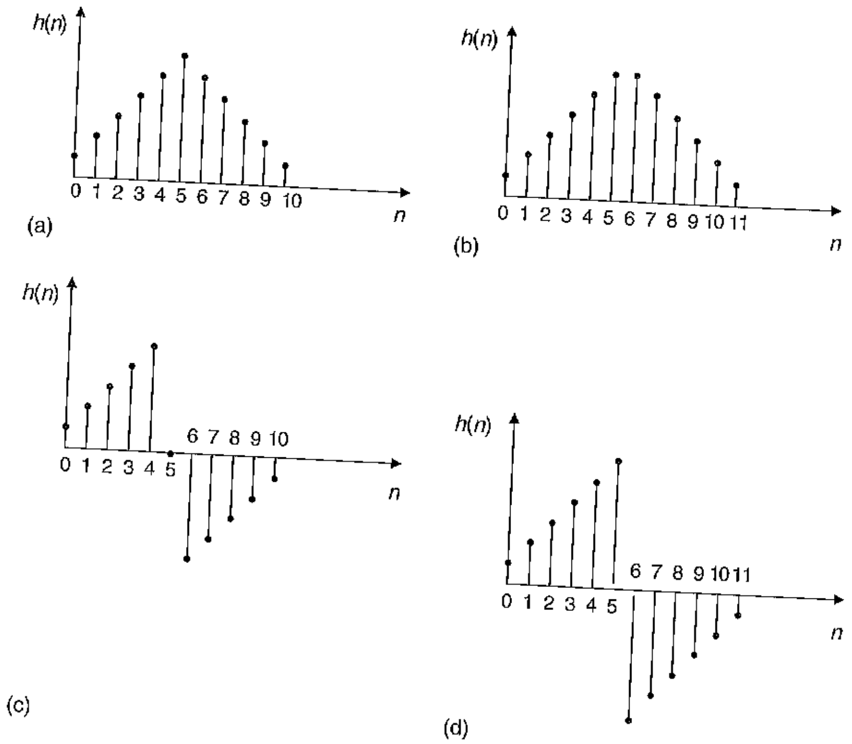


Fig. 4.6 Example of impulse responses of linear-phase FIR digital filters: (a) Type I; (b) Type II; (c) Type III; (d) Type IV.

Table 4.1 Main characteristics of linear-phase FIR filters: order, impulse response, frequency response, phase response, and group delay.

Type	M	$h(n)$	$H(e^{j\omega})$	$\Theta(\omega)$	τ
I	Even	Symmetric	$e^{-j\omega M/2} \sum_{m=0}^{M/2} a(m) \cos(\omega m)$	$-\omega \frac{M}{2}$	$\frac{M}{2}$
			$a(0) = h(M/2)$; $a(m) = 2h[(M/2) - m]$		
II	Odd	Symmetric	$e^{-j\omega M/2} \sum_{m=1}^{(M+1)/2} b(m) \cos[\omega(m - \frac{1}{2})]$	$-\omega \frac{M}{2}$	$\frac{M}{2}$
			$b(m) = 2h\{[(M+1)/2] - m\}$		
III	Even	Antisymmetric	$e^{-j[\omega(M/2)-(\pi/2)]} \sum_{m=1}^{M/2} c(m) \sin(\omega m)$	$-\omega \frac{M}{2} + \frac{\pi}{2}$	$\frac{M}{2}$
			$c(m) = 2h[(M/2) - m]$		
IV	Odd	Antisymmetric	$e^{-j[\omega(M/2)-(\pi/2)]} \sum_{m=1}^{(M+1)/2} d(m) \sin[\omega(m - \frac{1}{2})]$	$-\omega \frac{M}{2} + \frac{\pi}{2}$	$\frac{M}{2}$
			$d(m) = 2h\{[(M+1)/2] - m\}$		

Notice that $H(e^{j\omega}) = 0$, at $\omega = 0$. Hence, lowpass filters cannot be approximated as Type IV filters, although this filter type is still suitable for differentiators and Hilbert transformers, like the Type III form.

Typical impulse responses of the four cases of linear-phase FIR digital filters are depicted in Figure 4.6. The properties of all four cases are summarized in Table 4.1.

One can derive important properties of linear-phase FIR filters by representing Equations (4.17), (4.20), (4.23), and (4.26) in a single framework as

$$H(z) = z^{-M/2} \sum_{n=0}^{K} h(n) \left\{ z^{(M/2)-n} \pm z^{-[(M/2)-n]} \right\} \quad (4.29)$$

where $K = M/2$ if M is even, or $K = (M-1)/2$ if M is odd. From Equation (4.29), it is easy to observe that if z_γ is a zero of $H(z)$, then so is z_γ^{-1}. This implies that all zeros of $H(z)$ occur in reciprocal pairs. Considering that if the coefficients $h(n)$ are real, all complex zeros occur in conjugate pairs, and then one can infer that the zeros of $H(z)$ must satisfy the following relationships:

- All complex zeros which are not on the unit circle occur in conjugate and reciprocal quadruples. In other words, if z_γ is complex, then z_γ^{-1}, z_γ^*, and $(z_\gamma^{-1})^*$ are also zeros of $H(z)$.

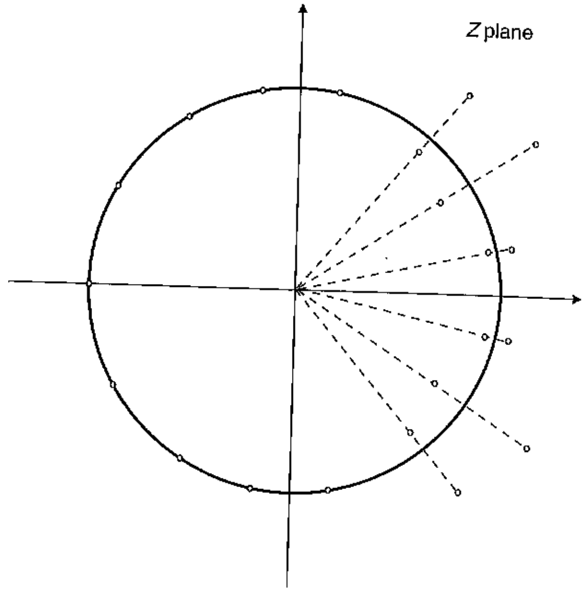


Fig. 4.7 Typical zero plot of a linear-phase FIR digital filter.

- There can be any given number of zeros over the unit circle, in conjugate pairs, since in this case we automatically have that $z_\gamma^{-1} = z_\gamma^*$.
- All real zeros outside the unit circle occur in reciprocal pairs.
- There can be any given number of zeros at $z = z_\gamma = \pm 1$, since in this case we necessarily have that $z_\gamma^{-1} = \pm 1$.

A typical zero plot for a linear-phase lowpass FIR filter is shown in Figure 4.7.

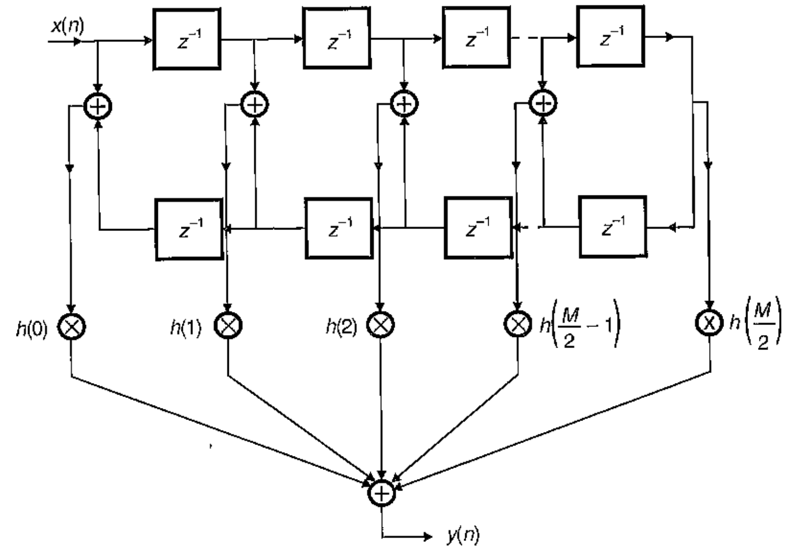
An interesting property of linear-phase FIR digital filters is that they can be realized with efficient structures that exploit their symmetric or antisymmetric impulse-response characteristics. In fact, when M is even, these efficient structures require $(M/2)+1$ multiplications, while when M is odd, only $(M+1)/2$ multiplications are necessary. Figure 4.8 depicts two of these efficient structures for linear-phase FIR filters when the impulse response is symmetric.

4.3 Basic structures of recursive digital filters

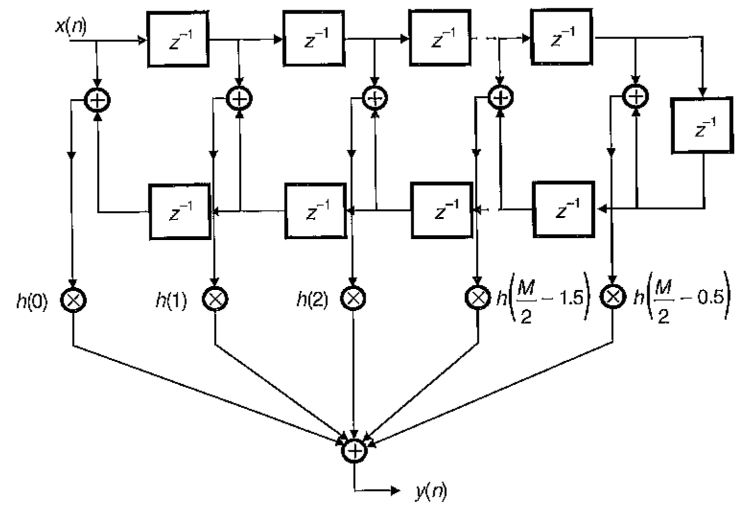
4.3.1 Direct forms

The transfer function of a recursive filter is given by

$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 + \sum_{i=1}^{N} a_i z^{-i}}. \tag{4.30}$$



(a)



(b)

Fig. 4.8 Realizations of linear-phase filters with symmetric impulse response: (a) even order; (b) odd order.

Since, in most cases, such transfer functions give rise to filters with impulse responses having infinite durations, recursive filters are also referred to as infinite-duration impulse response (IIR) filters.[1]

We can consider that $H(z)$ as above results from the cascading of two separate filters of transfer functions $N(z)$ and $1/D(z)$. The $N(z)$ polynomial can be realized with the FIR direct form, as shown in the previous section. The realization of $1/D(z)$ can be performed

[1] It is important to note that in the cases where $D(z)$ divides $N(z)$, the filter $H(z)$ turns out to have a finite-duration impulse response and is actually an FIR filter.

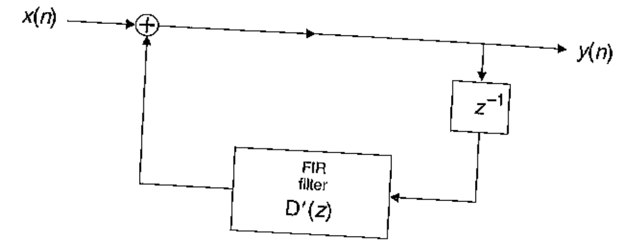


Fig. 4.9 Block diagram realization of $1/D(z)$.

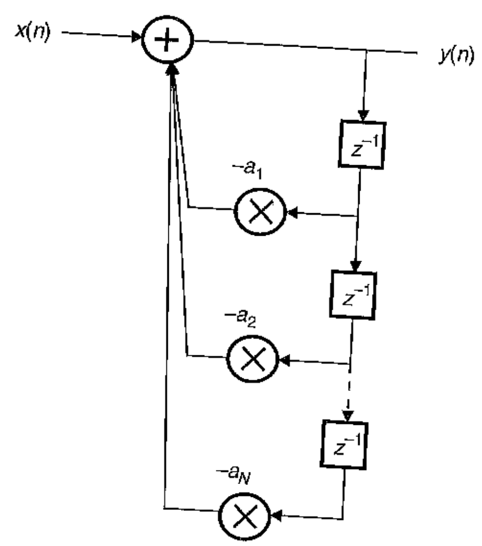


Fig. 4.10 Detailed realization of $1/D(z)$.

as depicted in Figure 4.9, where the FIR filter shown will be an $(N-1)$th-order filter with transfer function

$$D'(z) = z(1 - D(z)) = -z \sum_{i=1}^{N} a_i z^{-i}, \tag{4.31}$$

which can be realized as in Figure 4.3. The direct form for realizing $1/D(z)$ is then shown in Figure 4.10.

The complete realization of $H(z)$, as a cascade of $N(z)$ and $1/D(z)$, is shown in Figure 4.11. Such a structure is not canonic with respect to the delays, since for an (M, N)th-order filter this realization requires $(N + M)$ delays.

Clearly, in the general case we can change the order in which we cascade the two separate filters; that is, $H(z)$ can be realized as $N(z) \times 1/D(z)$ or $(1/D(z)) \times N(z)$. In the second option, all delays employed start from the same node, which allows us to eliminate the consequent redundant delays. In that manner, the resulting structure, usually referred to as the Type 1 canonic direct form, is the one depicted in Figure 4.12, for the special case when $N = M$.

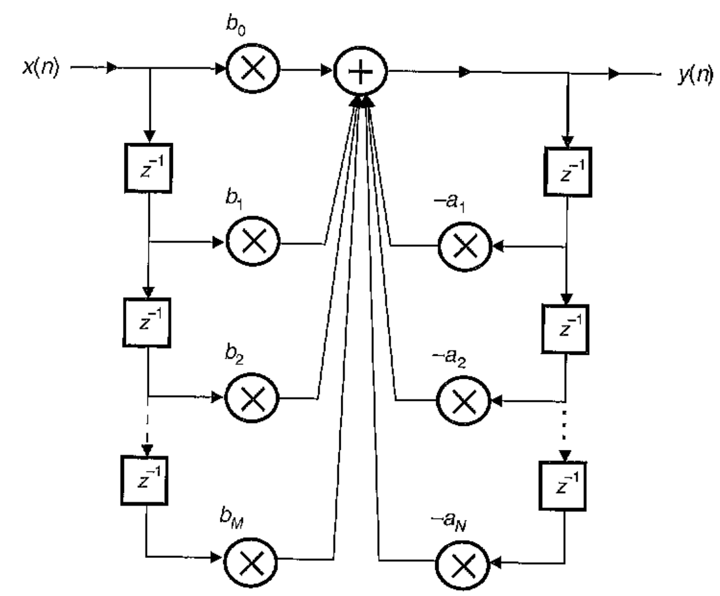


Fig. 4.11 Noncanonic IIR direct-form realization.

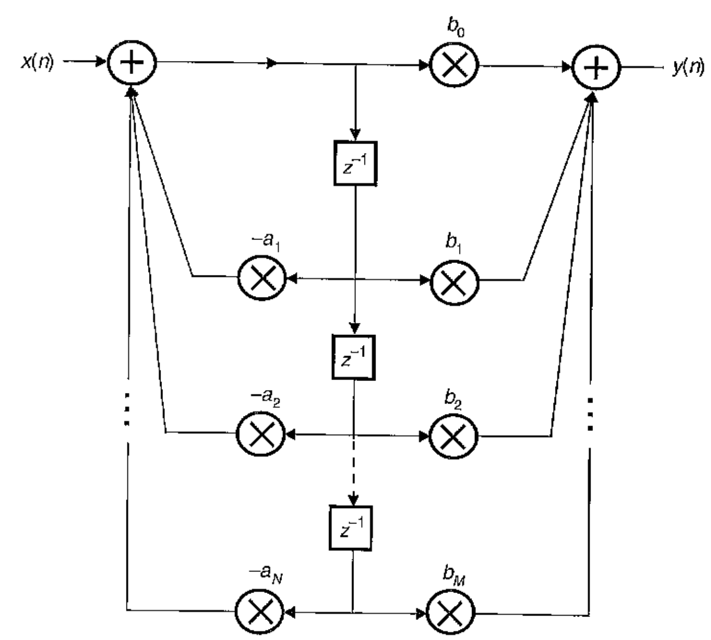
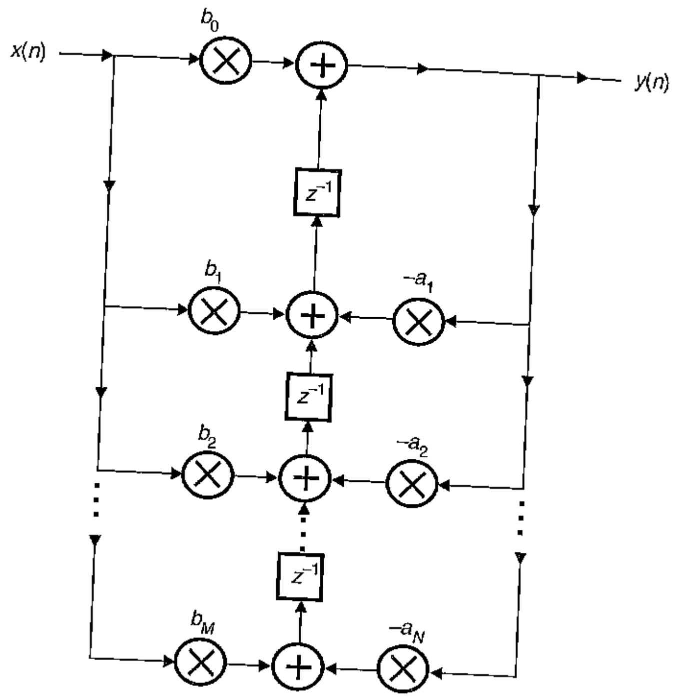


Fig. 4.12 Type 1 canonic direct form for IIR filters.

An alternative structure, the so-called Type 2 canonic direct form, is shown in Figure 4.13. Such a realization is generated from the nonrecursive form in Figure 4.4.

The majority of IIR filter transfer functions used in practice present a numerator degree M smaller than or equal to the denominator degree N. In general, one can consider, without



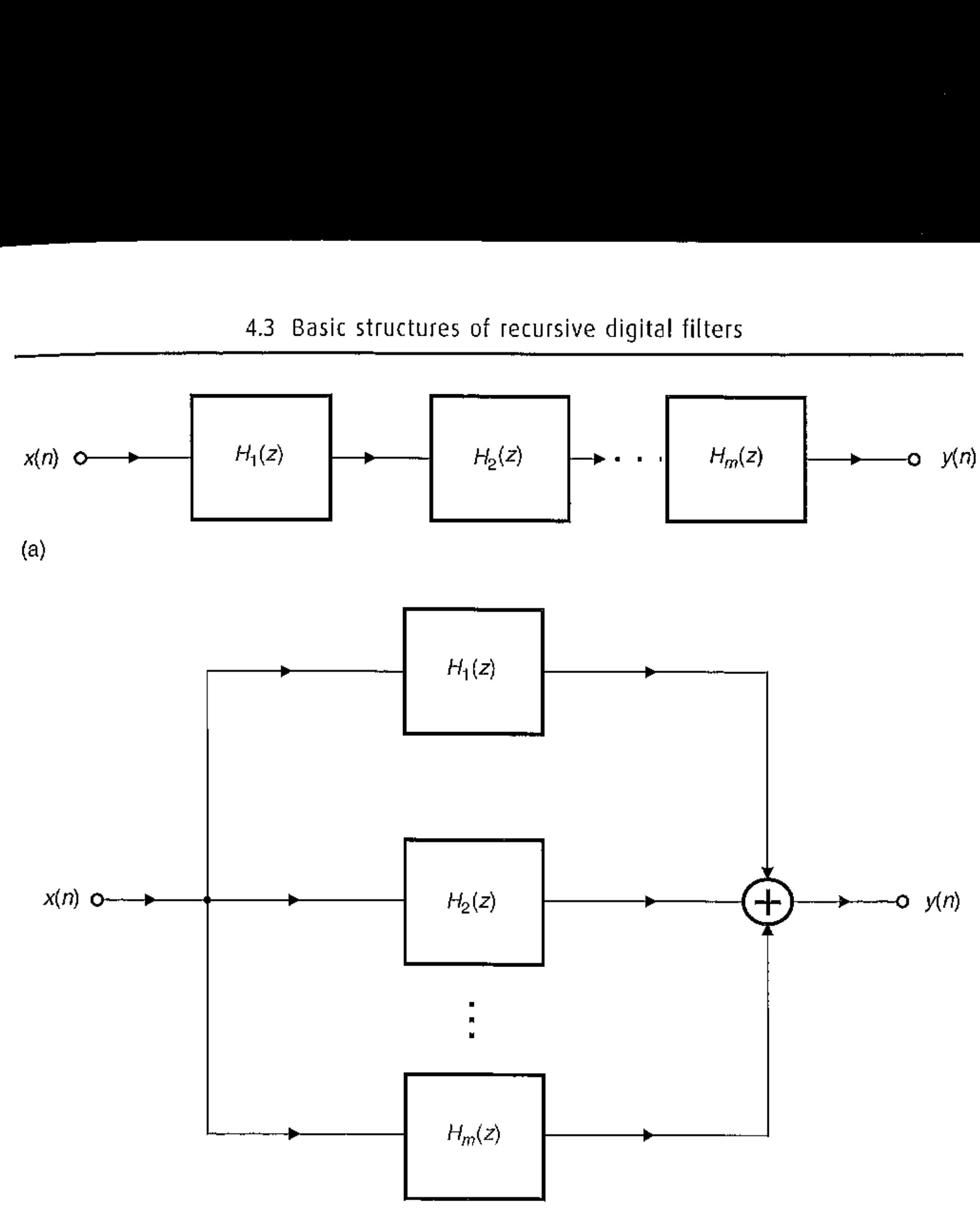
Type 2 canonic direct form for IIR filters.

much loss of generality, that $M = N$. In the case where $M < N$, we just make the coefficients $b_{M+1}, b_{M+2}, \ldots, b_N$ in Figures 4.12 and 4.13 equal to zero.

4.3.2 Cascade form

In the same way as their FIR counterparts, the IIR digital filters present a large variety of possible alternative realizations. An important one, referred to as the cascade realization, is depicted in Figure 4.14a, where the basic blocks represent simple transfer functions of orders 2 or 1. In fact, the cascade form, based on second-order blocks, is associated with the following transfer function decomposition:

$$\begin{aligned} H(z) &= \prod_{k=1}^{m} \frac{\gamma_{0k} + \gamma_{1k}z^{-1} + \gamma_{2k}z^{-2}}{1 + m_{1k}z^{-1} + m_{2k}z^{-2}} \\ &= \prod_{k=1}^{m} \frac{\gamma_{0k}z^2 + \gamma_{1k}z + \gamma_{2k}}{z^2 + m_{1k}z + m_{2k}} \\ &= H_0 \prod_{k=1}^{m} \frac{z^2 + \gamma'_{1k}z + \gamma'_{2k}}{z^2 + m_{1k}z + m_{2k}}. \end{aligned} \tag{4.32}$$



Block diagrams of: (a) cascade form; (b) parallel form.

4.3.3 Parallel form

Another important realization for recursive digital filters is the parallel form represented in Figure 4.14b. Using second-order blocks, which are the most commonly used in practice, the parallel realization corresponds to the following transfer function decomposition:

$$\begin{aligned} H(z) &= \sum_{k=1}^{m} \frac{\gamma^p_{0k}z^2 + \gamma^p_{1k}z + \gamma^p_{2k}}{z^2 + m_{1k}z + m_{2k}} \\ &= h_0 + \sum_{k=1}^{m} \frac{\gamma^{p'}_{1k}z + \gamma^{p'}_{2k}}{z^2 + m_{1k}z + m_{2k}} \\ &= h'_0 + \sum_{k=1}^{m} \frac{\gamma^{p''}_{0k}z^2 + \gamma^{p''}_{1k}z}{z^2 + m_{1k}z + m_{2k}}, \end{aligned} \tag{4.33}$$

also known as the partial-fraction decomposition. This equation indicates three alternative forms of the parallel realization, where the last two are canonic with respect to the number of multiplier elements.

It should be mentioned that each second-order block in the cascade and parallel forms can be realized by any of the existing distinct structures, as, for instance, one of the direct forms shown in Figure 4.15.

As will be seen in future chapters, all these digital filter realizations present different properties when one considers practical finite-precision implementations; that is, the quantization of the coefficients and the finite precision of the arithmetic operations, such as additions and multiplications (Jackson, 1969, 1996; Oppenheim & Schafer, 1975; Antoniou, 1993). In fact, the analysis of the finite-precision effects in the distinct realizations is

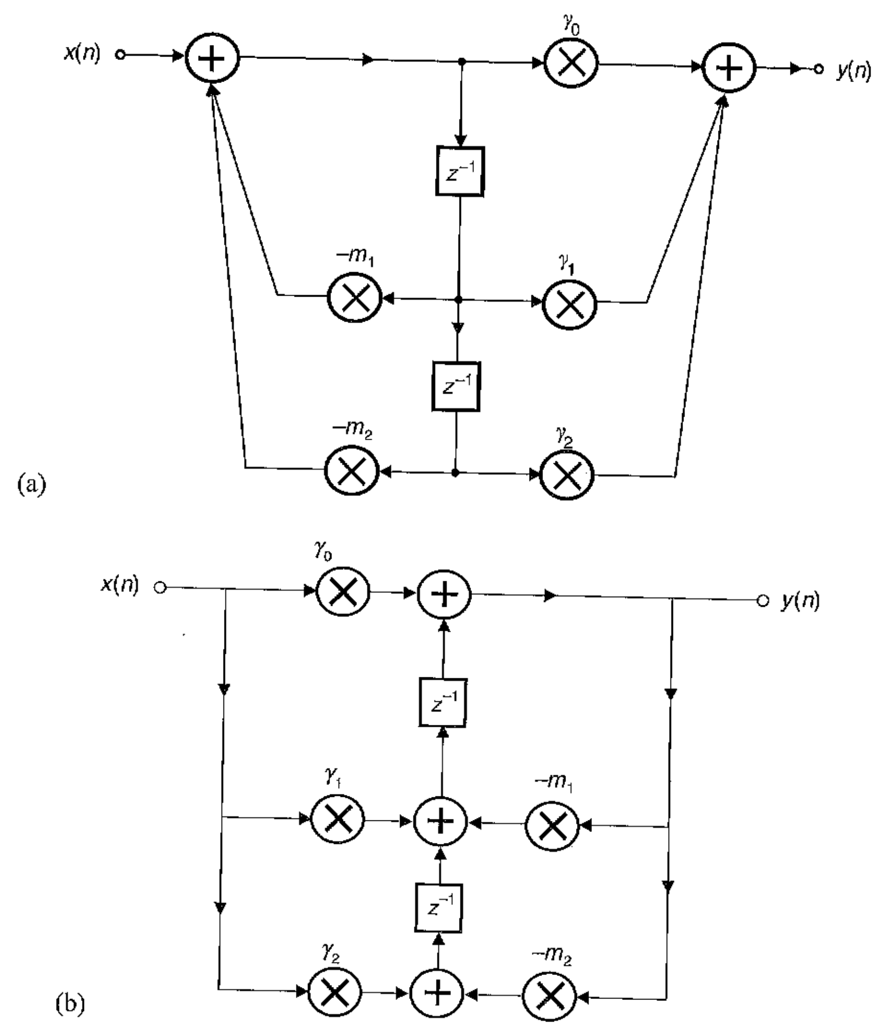


Fig. 4.15 Realizations of second-order blocks: (a) Type 1 direct form; (b) Type 2 direct form.

a fundamental step in the overall process of designing any digital filter, as will be discussed in detail in Chapter 11.

Example 4.1. Describe the digital filter implementation of the transfer function

$$H(z) = \frac{16z^2(z+1)}{(4z^2-2z+1)(4z+3)} \tag{4.34}$$

using:

(a) A cascade realization.
(b) A parallel realization.

Solution

(a) One cascade realization is obtained by describing the original transfer function as a product of second- and first-order building blocks as follows:

$$H(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}\right)\left(\frac{1+z^{-1}}{1+\frac{3}{4}z^{-1}}\right). \tag{4.35}$$

Each section of the decomposed transfer function is implemented using the Type 1 canonic direct-form structure as illustrated in Figure 4.16.

(b) Let us start by writing the original transfer function in a more convenient form as follows:

$$\begin{aligned} H(z) &= \frac{z^2(z+1)}{\left(z^2-\frac{1}{2}z+\frac{1}{4}\right)\left(z+\frac{3}{4}\right)} \\ &= \frac{z^2(z+1)}{\left(z-\frac{1}{4}-\mathrm{j}\frac{\sqrt{3}}{4}\right)\left(z-\frac{1}{4}+\mathrm{j}\frac{\sqrt{3}}{4}\right)\left(z+\frac{3}{4}\right)}. \end{aligned} \tag{4.36}$$

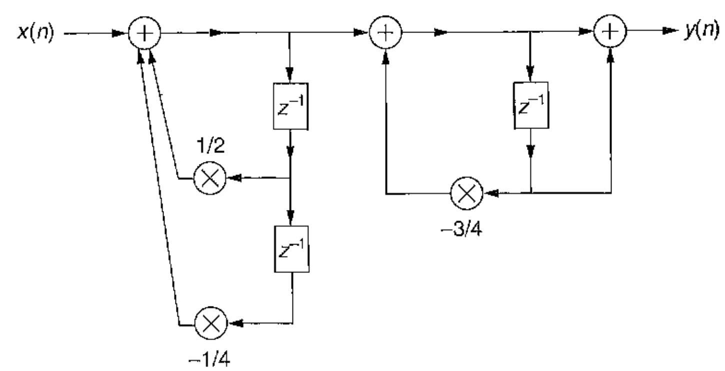


Fig. 4.16 Cascade implementation of $H(z)$ as given in Equation (4.35).

Next, we decompose $H(z)$ as a summation of first-order complex sections as

$$H(z) = r_1 + \frac{r_2}{z - p_2} + \frac{r_2^*}{z - p_2^*} + \frac{r_3}{z + p_3}, \tag{4.37}$$

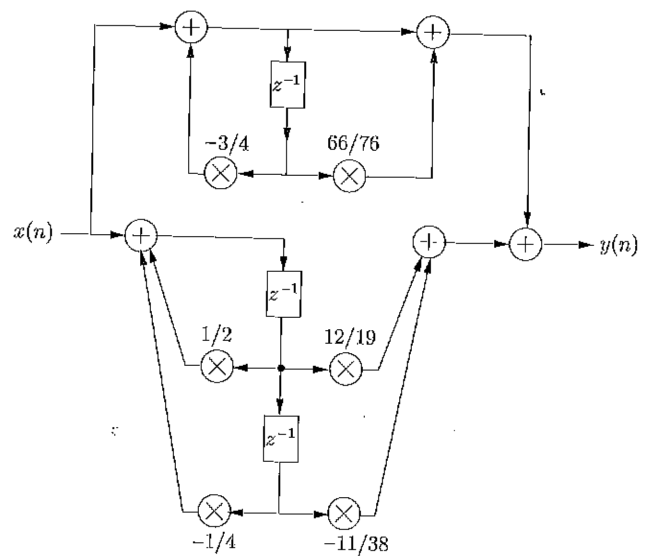
where r_1 is the value of $H(z)$ at $z \to \infty$ and r_i is the residue associated with the pole p_i, for $i = 2, 3$, such that

$$\left.\begin{aligned} p_2 &= \tfrac{1}{4} + \mathrm{j}\sqrt{3}/4 \\ p_3 &= \tfrac{3}{4} \\ r_1 &= 1 \\ r_2 &= \tfrac{6}{19} + \mathrm{j}5/(3\sqrt{19}) \\ r_3 &= \tfrac{9}{76} \end{aligned}\right\}. \tag{4.38}$$

Given these values, the complex first-order sections are properly grouped to form second-order sections with real coefficients, and the constant r_1 is grouped with the real coefficient first-order section, resulting in the following decomposition for $H(z)$:

$$H(z) = \frac{1 + \frac{66}{76}z^{-1}}{1 + \frac{3}{4}z^{-1}} + \frac{\frac{12}{19}z^{-1} - \frac{11}{38}z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}. \tag{4.39}$$

We can then implement each section using the Type 1 canonic direct-form structure leading to the realization shown in Figure 4.17. △



Parallel implementation of $H(z)$ as given in Equation (4.39).

4.4 Digital network analysis

The signal-flowgraph representation greatly simplifies the analysis of digital networks composed of delays, multipliers, and adder elements (Crochiere & Oppenheim, 1975). In practice, the analysis of such devices is implemented by first numbering all nodes of the graph of interest. Then, one determines the relationship between the output signal of each node with respect to the output signals of all other nodes. The connections between two nodes, referred to as branches, consist of combinations of delays and/or multipliers. Branches that inject external signals into the graph are called source branches. Such branches have a transmission coefficient equal to 1. Following this framework, we can describe the output signal of each node as a combination of the signals of all other nodes and possibly an external signal; that is:

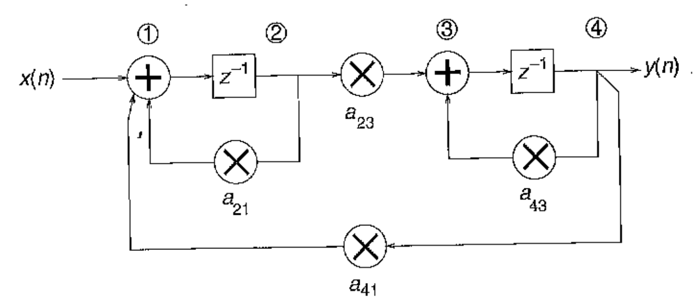
$$Y_j(z) = X_j(z) + \sum_{k=1}^{N}\left(a_{kj}Y_k(z) + z^{-1}b_{kj}Y_k(z)\right), \tag{4.40}$$

for $j = 1, 2, \ldots, N$, where N is the number of nodes, a_{kj} and $z^{-1}b_{kj}$ are the transmission coefficients of the branch connecting node k to node j, $Y_j(z)$ is the z transform of the output signal of node j, and $X_j(z)$ is the z transform of the external signal injected in node j. We can express Equation (4.40) in a more compact form as

$$\mathbf{y}(z) = \mathbf{x}(z) + \mathbf{A}^{\mathrm{T}}\mathbf{y}(z) + \mathbf{B}^{\mathrm{T}}\mathbf{y}(z)z^{-1}, \tag{4.41}$$

where $\mathbf{y}(z)$ is the output signal $N \times 1$ vector and $\mathbf{x}(z)$ is the external input signal $N \times 1$ vector for all nodes in the given graph. Also, $\mathbf{A}^{\mathrm{T}}$ is an $N \times N$ matrix formed by the multiplier coefficients of the delayless branches of the circuit, while $\mathbf{B}^{\mathrm{T}}$ is the $N \times N$ matrix of multiplier coefficients of the branches with a delay.

Example 4.2. Describe the digital filter seen in Figure 4.18 using the compact representation given in Equation (4.41).



Second-order digital filter.