## → Eigenfunction

First of all, refresh what are eigenvalues and eigenvector

$$\underset{\sim}{A}\ \underline{x} = \lambda\ \underline{x}$$

$\lambda \rightarrow$ scalar (can be real or imaginary)

$\underset{\sim}{A}$ belongs to $\in \mathbb{R}^{M\times M}$ or $\mathbb{C}^{M\times M}$

$\underline{x}$ → Eigenvector $\in \mathbb{R}^{M\times 1}$ or $\mathbb{C}^{M\times 1}$

$\underline{x}$ → $\in \mathbb{R}^{M\times 1}$ or $\mathbb{C}^{M\times 1}$

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**Question** :- Can a real valued matrix $\underset{\sim}{A}$ has complex eigenvalues / eigenvector?

Try finding the eigenvalues & eigenvetors for

$$\underset{\sim}{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

**Example:-**

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

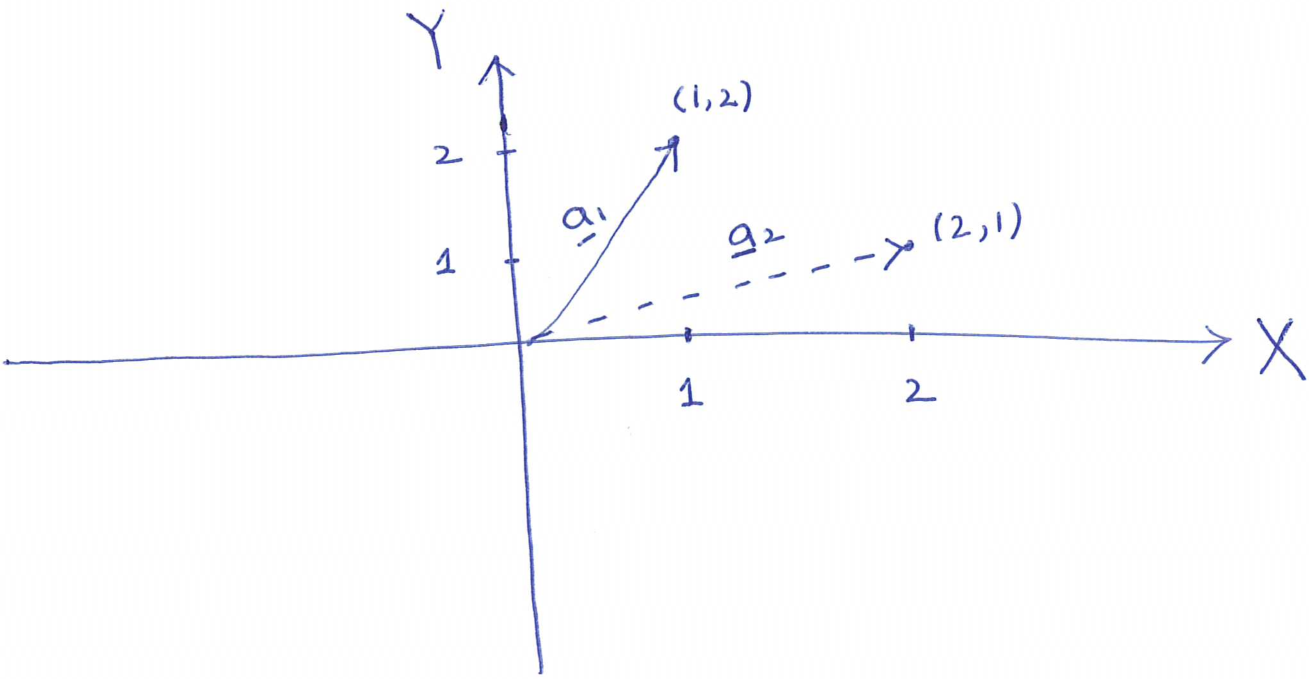
**Visualize:-**

Think columns of matrix $\underset{\sim}{A}$ as two vectors

$$\underset{\sim}{A} = \begin{array}{c} \begin{array}{cc} \underline{a_1} & \underline{a_2} \end{array} \\ \left[ \begin{array}{cc} \boxed{\begin{matrix}1\\2\end{matrix}} & \boxed{\begin{matrix}2\\1\end{matrix}} \end{array} \right] \end{array}$$

$$\underline{a_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} \rightarrow x_1 \\ \rightarrow y_1 \end{matrix} \quad , \quad \underline{a_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{matrix} \rightarrow x_2 \\ \rightarrow y_2 \end{matrix}$$

We can plot in X-Y Plane



The two eigenvectors are

$$\underline{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

We can check whether this is correct

$$\underset{\sim}{A}\, \underline{x}_1 = \lambda_1 \underline{x}_1$$

L.H.S =

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$$

$$= 3 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= 3\, x_1$$

$\boxed{\text{Questio}}$ Check the other eigenvalue, $\lambda_2$

$$\boxed{\therefore \lambda_1 = 3}$$

In vector space, eigenvectors and eigenvalues are written as

$$\boxed{\underset{\sim}{A}\,\underline{x} = \lambda\,\underline{x}} \rightarrow \text{Eq } 1$$

Similarly, in function space

eigenvalue corresponding to function

$$\boxed{A f = \lambda f} \rightarrow \text{Eq } 2$$

Any Mathematical Operator

function of variable $x, t, \ldots$

<u>Example</u>

↳ $\frac{d}{dt}(f)$

↳ $\int f \, .dt$

↳ $\sqrt{f}$

⋮

and so on

Note i) Not all the operators satisfy Eq 2.

ii) You need to find a function, e.g, $f(t)$ such that when operator $A$ applies, you get the same function back multiplied by some scaling factor $\lambda$ (which is an eigenvalue) in this case.

Example :-

$$\mathcal{A}\, f(t) = \frac{d}{dt} f(t)$$

derivative operator

$f(t) \rightarrow$ function of time

Only exponential functions satisfy Eq (2).

$$f(t) = e^{st}$$

$$\frac{d}{dt} f(t) = s\, e^{st}$$

$$\downarrow$$

eigenvalue

$$\frac{d}{dt} e^{st} = s\, e^{st}$$

$s = \sigma + j\omega$

Question :- What is an eigenfunction for operator $\mathcal{A}\, f(t) = \int f(t) \cdot dt$

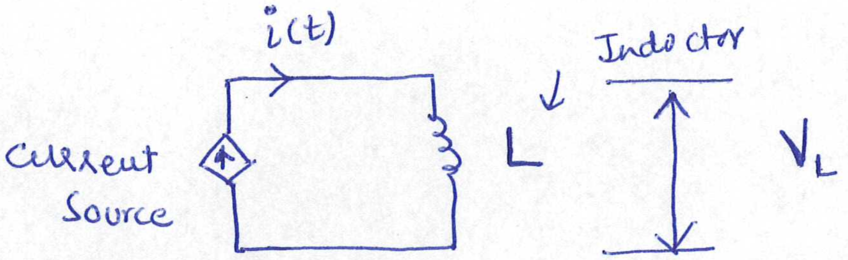
→ For LTI ( Linear Time Invariant ) Systems, exponential function $\boxed{e^{st}}$ are the eigenfunctions.

→ It is easy to characterize such Systems with exponential functions. Particularly characterizing the input and Output relationship.

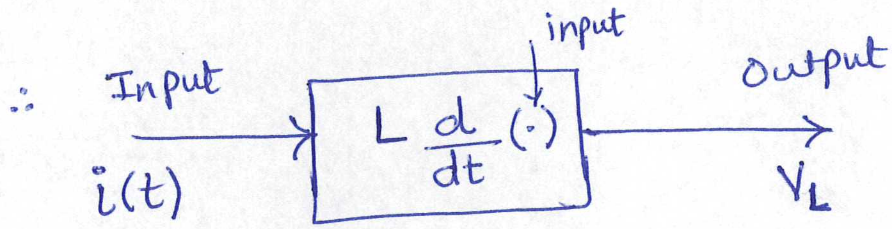
→ In continuous time, differential equations models the dynamic behavior of the System. For Differential operator, exponential functions are eigenfunctions.

→ Let look at the following example

Examp 1




Input $i(t)$ , output Voltage across inductor




$$i(t) = e^{st}$$

$$\boxed{V_L = sLe^{st}} \quad \boxed{= j\omega L e^{j\omega t}}$$

→ Getting same ~~outp~~ output frequency $(\omega)$ as input but scaled by $\boxed{j\omega L}$.

∴ $j\omega L$ is an eigenfunction and it is also called ~~known~~ as Transfer function (T.F)

→ T.F tell us about the output of a system (magnitude & phase) when particular frequency is applied at input

<u>Question :-</u>

Pass a periodic square wave through differentiator.

Square wave ——— $\boxed{L\frac{d}{dt}(\cdot)}$ ——— output ?

Plot/ Tell which frequencies are present at the output. Also Plot/ tell the magnitude and phase of those frequencies.

Take $L = 1\,mH$.

$\boxed{\text{Note}}$ Square wave is shown in the next slide.

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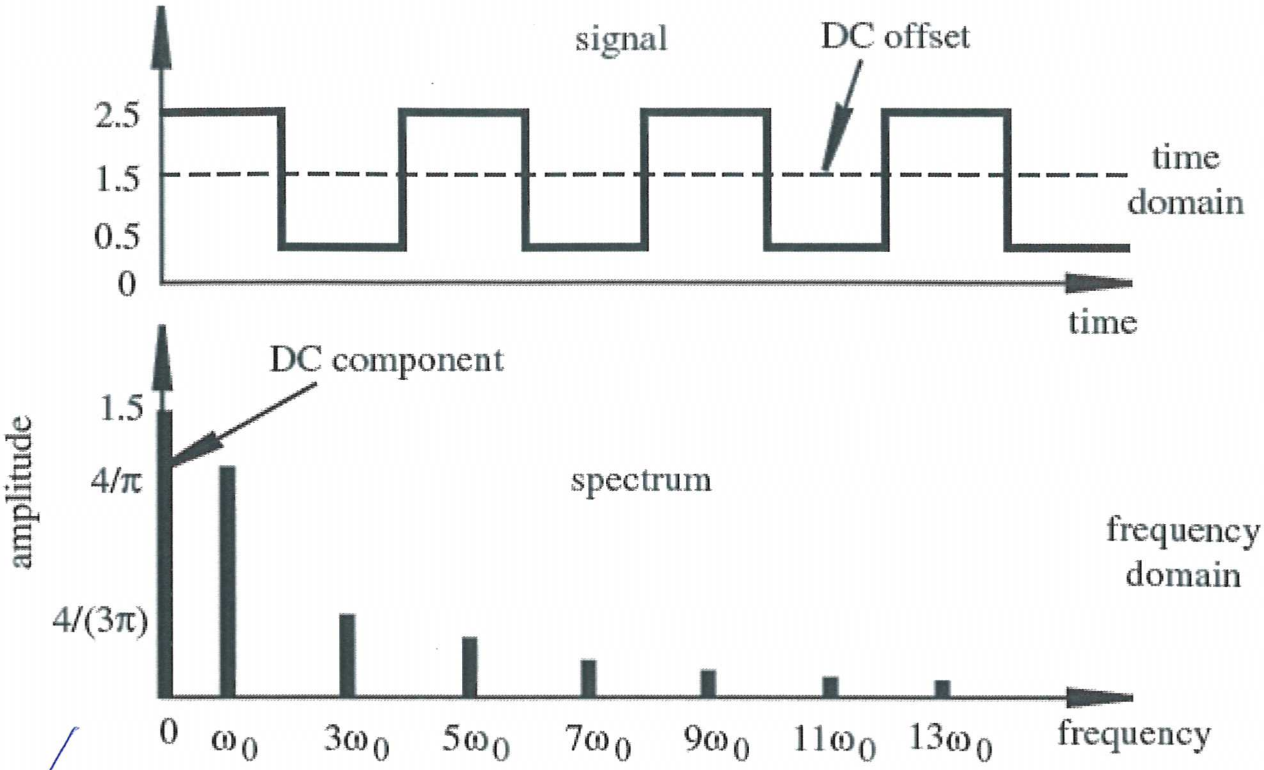



Figure is taken from

→ Example 2
Let look at more complicated
example :) !!

Source
mechatronics.colostate.edu/figures/4-5.jpg

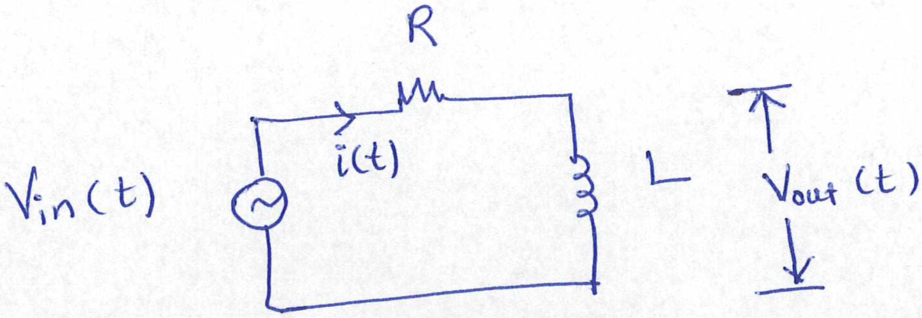



Fig 1

$$\boxed{V_{out}(t) = L\frac{di(t)}{dt}} \rightarrow Eq\ 3$$

But if $V_{in}(t) = e^{j\omega t}$ is applied,

we known that

$$V_{out}(t) = \underset{\substack{\uparrow \\ T.F}}{\lambda}\ e^{j\omega t}$$

But how to $\lambda$ ?

Method 1

Well, first we need to solve the following differential equation to find $i(t)$,

$$R\,i(t) + L\frac{di(t)}{dt} = V_{in}(t)$$

$$R\,i(t) + L\frac{di}{dt}(t) = e^{j\omega t}$$

and then substitute it in Eq 3 to find

Vout (t). Since system is LTI,

you will get ~~the~~ ~~same~~ ~~out~~ $e^{j\omega t}$

with scale factor. The scaling will

be eigenfunction $\lambda$ or T.F.

$\boxed{\text{METHOD 2}}$

~~The~~ Convert all the stuff into

Fourier domain

Impedance of Resistor $= R$

" " ~~Tran~~ Inductor $= j\omega L$

$$\mathcal{F}\{V_{in}(t)\} = V_{in}(\omega)$$

$$\mathcal{F}\{V_{out}(t)\} = V_{out}(\omega)$$

Circuit shown in Fig 1 is in time domain and has a differential equation representation.

Following is the representation of a circuit in frequence domain. Nice thing is one can use KVL or KCL since frequency domain representation is algebraic.

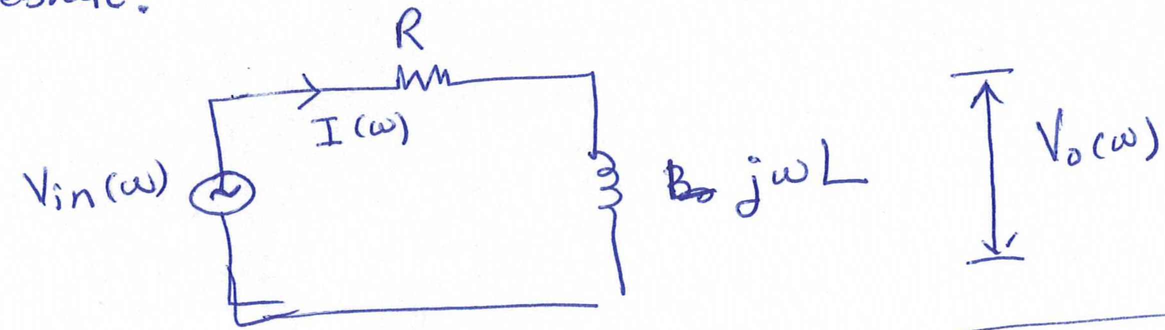
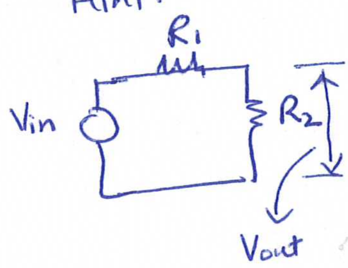



Fig. 2

Use voltage divider rule

Hint:-




$$V_{out} = \frac{R_2}{R_1+R_2} V_{in}$$

$$\boxed{V_{out}(\omega) = \frac{j\omega L}{R+j\omega L} V_{in}(\omega)}$$

Eigenvalue

Eq (4)

$$\Rightarrow \quad T.F = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{j\omega L}{R+j\omega L}$$

To find $V_{out}(t)$, take inverse Fourier Transform

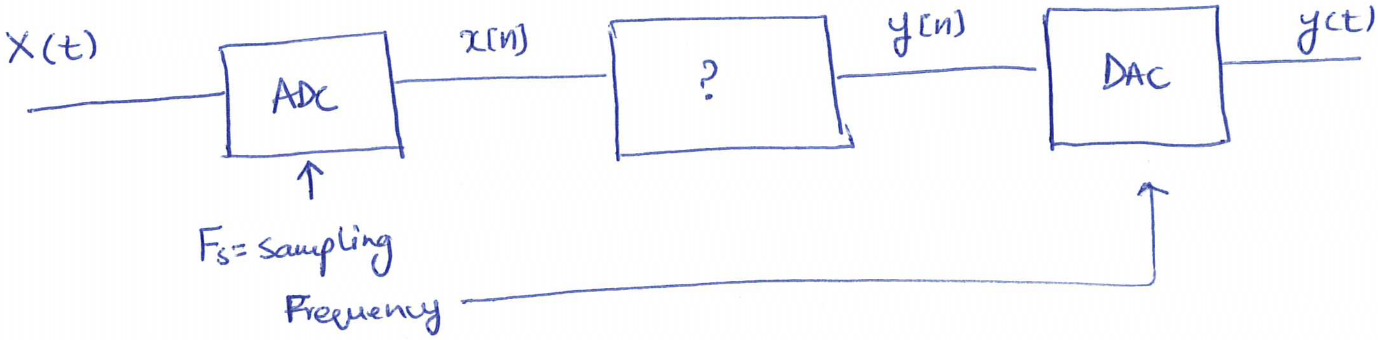
$$V_{out}(t) = \mathcal{F}^{-1}\left\{ \frac{j\omega L}{R+j\omega L} V_{in}(\omega) \right\}$$

Question:- Find $V_{out}(t)$ when $\left.\begin{array}{l} V_{in}(t) = e^{j\omega t} \\ V_{in}(t) = t \end{array}\right\} \; 0 \leq t < \infty$ and

Approximating Derivative IN



Discrete time LTI :-




→ Both ADC and DAC are running at same sampling frequency $F_s$ Hz $= \frac{1}{T_s}$. [$T_s$ = Sampling Time]

→ How to approximate derivative in discrete domain

$$\boxed{y[n] = \frac{1}{T_s}\left[x[n] - x[n-1]\right]} \rightarrow Eq\ 4$$

→ Exponential functions $e^{sn}$ are also the eigenfunctions of discrete-time LTI systems

Question) Verify $x[n] = e^{sn}$ is an eigenfunction of above equation. Find the eigenvalue also.

**Note**

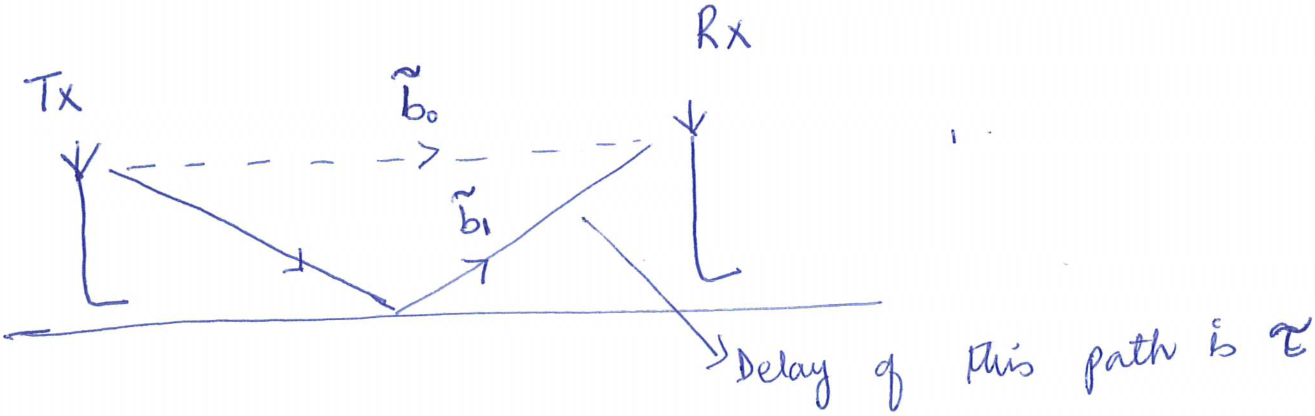
FIR = Finite Impulse Response $\Rightarrow$ Only depends on current and past values of the input

IIR = Infinite Impulse Response

$\Downarrow$

Depends both on current and past values of the input and output

<u>Multipath Effect</u> can also be modeled by the same kind of difference equation (Eq 4)




→ Tx ~~sends~~ transmits $x(t)$

→ Two path are taken by $x(t)$

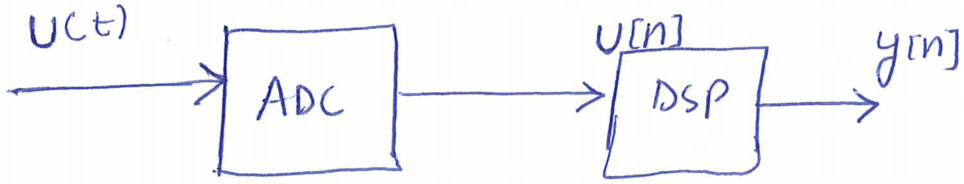
→ Rx signals is given by

$$u(t) = \tilde{b}_0\, x(t) + \tilde{b}_1\, x(t-\tau)$$

→ If we discretize the signal ~~than~~ in time using ADC then above eqution becom

$$\boxed{U[n] = b_0\, x[n] + b_1\, x[n - N_0]}$$

$b_0$ and $b_1$ are the values after the ADC.*




~~when~~
* Note i) $b_0 \neq \tilde{b}_0$ and $b_1 \neq \tilde{b}_1$ is possible because ADC has finite resolution. Therefore, values before and after ADC might be different.

ii) Also, there are filters and amplifiers are present in Receiver chain.

→ If Rx moves, $b_0$ & $b_1$ changes. Therefore, ~~we~~ we need a mechanism to track the changes ⇒ $\boxed{\text{ADAPTIVE SIGNAL PROCESSING}}$