Table 6.7. Characteristics of the digital elliptic bandpass filter. Gain constant: $H_0 = 1.3461 \times 10^{-4}$.

Numerator coefficients	Denominator coefficients
$b_0 = -1.0$	$a_0 = 0.9691$
$b_1 = 3.2025$	$a_1 = -4.7285$
$b_2 = -3.5492$	$a_2 = 10.6285$
$b_3 = 0.0$	$a_3 = -13.7261$
$b_4 = 3.5492$	$a_4 = 10.7405$
$b_5 = -3.2025$	$a_5 = -4.8287$
$b_6 = 1.0$	$a_6 = 1.0$
Filter zeros	**Filter poles**
$z_1 = 0.7399 + \mathrm{j}0.6727$	$p_1 = 0.7982 + \mathrm{j}0.5958$
$z_2 = 0.7399 - \mathrm{j}0.6727$	$p_2 = 0.7982 - \mathrm{j}0.5958$
$z_3 = 0.8613 + \mathrm{j}0.5081$	$p_3 = 0.8134 + \mathrm{j}0.5751$
$z_4 = 0.8613 - \mathrm{j}0.5081$	$p_4 = 0.8134 - \mathrm{j}0.5751$
$z_5 = 1.0 + \mathrm{j}0.0$	$p_5 = 0.8027 + \mathrm{j}0.5830$
$z_6 = -1.0 + \mathrm{j}0.0$	$p_6 = 0.8027 - \mathrm{j}0.5830$

the expression

$$\omega = \Omega \frac{2\pi}{\Omega_s}. \tag{6.138}$$

In the discussion that follows, we assume that $\Omega_s = 2\pi$ rad/s, unless specified otherwise.

6.4 Frequency transformation in the discrete-time domain

Usually, in the approximation of a continuous-time filter, we begin by designing a normalized lowpass filter and then, through a frequency transformation, the filter with the specified magnitude response is obtained. In the design of digital filters, we can also start by designing a digital lowpass filter and then apply a frequency transformation in the discrete-time domain.

The procedure consists of replacing the variable z by an appropriate function $g(z)$ to generate the desired magnitude response. The function $g(z)$ needs to meet some constraints to be a valid transformation (Constantinides, 1970), namely:

- The function $g(z)$ must be a ratio of polynomials, since the digital filter transfer function must remain a ratio of polynomials after the transformation.
- The mapping $z \to g(z)$ must be such that the filter stability is maintained: that is, stable filters generate stable transformed filters and unstable filters generate unstable transformed filters. This is equivalent to saying that the transformation maps the interior of the unit circle onto the interior of the unit circle and the exterior of the unit circle onto the exterior of the unit circle.

It can be shown that a function $g(z)$ satisfying the above conditions is of the form

$$g(z) = \pm \left[\prod_{i=1}^{n} \frac{(z-\alpha_i)}{(1-z\alpha_i^*)} \frac{(z-\alpha_i^*)}{(1-z\alpha_i)} \right] \left(\prod_{i=n+1}^{m} \frac{z-\alpha_i}{1-z\alpha_i} \right), \tag{6.139}$$

where α_i^* is complex conjugate of α_i and α_i is real for $n+1 \leq i \leq m$.

In the following subsections, we analyze special cases of $g(z)$ that generate lowpass-to-lowpass, lowpass-to-highpass, lowpass-to-bandpass, and lowpass-to-bandstop transformations.

6.4.1 Lowpass-to-lowpass transformation

One necessary condition for a lowpass-to-lowpass transformation is that a magnitude response must keep its original values at $\omega = 0$ and $\omega = \pi$ after the transformation. Therefore, we must have

$$g(1) = 1 \tag{6.140}$$

$$g(-1) = -1. \tag{6.141}$$

Another necessary condition is that the frequency response should only be warped between $\omega = 0$ and $\omega = \pi$; that is, a full turn around the unit circle in z must correspond to a full turn around the unit circle in $g(z)$.

One possible $g(z)$ in the form of Equation (6.139) that satisfies these conditions is

$$g(z) = \frac{z-\alpha}{1-\alpha z}, \tag{6.142}$$

where α is real such that $|\alpha| < 1$.

Assuming that the passband edge frequency of the original lowpass filter is given by ω_p, and that we wish to transform the original filter into a lowpass filter with cutoff frequency at ω_{p_1}, that is, $g(e^{j\omega_{p_1}}) = e^{j\omega_p}$, the following relation must be valid:

$$e^{j\omega_p} = \frac{e^{j\omega_{p_1}} - \alpha}{1 - \alpha e^{j\omega_{p_1}}} \tag{6.143}$$

and then

$$\alpha = \frac{e^{-j[(\omega_p - \omega_{p_1})/2]} - e^{j[(\omega_p - \omega_{p_1})/2]}}{e^{-j[(\omega_p + \omega_{p_1})/2]} - e^{j[(\omega_p + \omega_{p_1})/2]}} = \frac{\sin[(\omega_p - \omega_{p_1})/2]}{\sin[(\omega_p + \omega_{p_1})/2]}. \tag{6.144}$$

The desired transformation is then implemented by replacing z by $g(z)$ given in Equation (6.142) with α calculated as indicated in Equation (6.144).

6.4.2 Lowpass-to-highpass transformation

If ω_{p_1} is the highpass filter band edge and ω_p is the lowpass filter cutoff frequency, then the lowpass-to-highpass transformation function is given by

$$g(z) = -\frac{z+\alpha}{\alpha z + 1}, \tag{6.145}$$

where

$$\alpha = -\frac{\cos\left[(\omega_p + \omega_{p_1})/2\right]}{\cos\left[(\omega_p - \omega_{p_1})/2\right]}. \tag{6.146}$$

6.4.3 Lowpass-to-bandpass transformation

The lowpass-to-bandpass transformation is accomplished if the following mappings occur:

$$g(1) = -1 \tag{6.147}$$

$$g(e^{-j\omega_{p_1}}) = e^{j\omega_p} \tag{6.148}$$

$$g(e^{j\omega_{p_2}}) = e^{j\omega_p} \tag{6.149}$$

$$g(-1) = -1, \tag{6.150}$$

where ω_{p_1} and ω_{p_2} are the band edges of the bandpass filter and ω_p is the band edge of the lowpass filter. Since the bandpass filter has two passband edges, we need a second-order function $g(z)$ to accomplish the lowpass-to-bandpass transformation. After some manipulation, it can be inferred that the required transformation and its parameters are given by (Constantinides, 1970)

$$g(z) = -\frac{z^2 + \alpha_1 z + \alpha_2}{\alpha_2 z^2 + \alpha_1 z + 1}, \tag{6.151}$$

with

$$\alpha_1 = -\frac{2\alpha k}{k+1} \tag{6.152}$$

$$\alpha_2 = \frac{k-1}{k+1}, \tag{6.153}$$

where

$$\alpha = \frac{\cos\left[(\omega_{p_2} + \omega_{p_1})/2\right]}{\cos\left[(\omega_{p_2} - \omega_{p_1})/2\right]} \tag{6.154}$$

$$k = \cot\left[(\omega_{p_2} - \omega_{p_1})/2\right]\tan\left(\omega_p/2\right). \tag{6.155}$$

6.4.4 Lowpass-to-bandstop transformation

The lowpass-to-bandstop transformation function $g(z)$ is given by

$$g(z) = \frac{z^2 + \alpha_1 z + \alpha_2}{\alpha_2 z^2 + \alpha_1 z + 1}, \tag{6.156}$$

with

$$\alpha_1 = -\frac{2\alpha}{k+1} \tag{6.157}$$

$$\alpha_2 = \frac{1-k}{1+k}, \tag{6.158}$$

where

$$\alpha = \frac{\cos\left[(\omega_{p_2} + \omega_{p_1})/2\right]}{\cos\left[(\omega_{p_2} - \omega_{p_1})/2\right]} \tag{6.159}$$

$$k = \tan\left[(\omega_{p_2} - \omega_{p_1})/2\right]\tan\left(\omega_p/2\right). \tag{6.160}$$

6.4.5 Variable-cutoff filter design

An interesting application for the frequency transformations, first proposed in Constantinides (1970), is to design highpass and lowpass filters with variable cutoff frequency with the cutoff frequency being directly controlled by a single parameter α. This method can be best understood through Example 6.4.

Example 6.4. Consider the lowpass notch filter

$$H(z) = 0.004\frac{z^2 - \sqrt{2}z + 1}{z^2 - 1.8z + 0.96} \tag{6.161}$$

whose zeros are located at $z = (\sqrt{2}/2)(1 \pm j)$. Transform this filter into a highpass notch with a zero at frequency $\omega_{p_1} = \pi/6$ rad/sample. Plot the magnitude responses before and after the frequency transformation.

Solution

Using the lowpass-to-highpass transformation given in Equation (6.145), the highpass transfer function is of the form

$$H(z) = H_0\frac{(\alpha^2+\sqrt{2}\alpha+1)(z^2+1)+(\sqrt{2}\alpha^2+4\alpha+\sqrt{2})z}{(0.96\alpha^2+1.8\alpha+1)z^2+(1.8\alpha^2+3.92\alpha+1.8)z+(\alpha^2+1.8\alpha+0.96)} \tag{6.162}$$

with $H_0 = 0.004$. The parameter α can control the position of the zeros of the highpass notch filter. For instance, in this example, as the original zero is at $\omega_p = \pi/4$ rad/sample

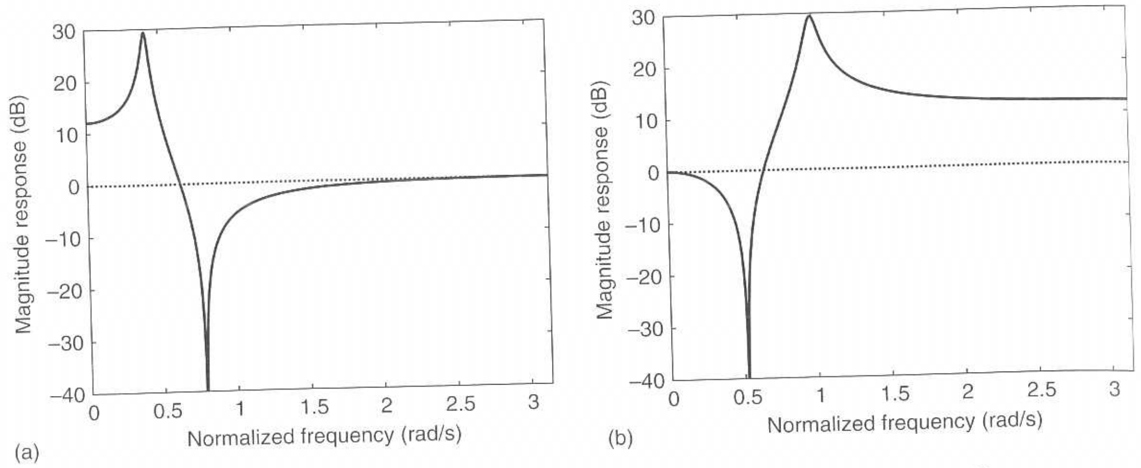

Fig. 6.11. Magnitude responses of notch filters: (a) lowpass notch filter; (b) highpass notch filter.

and the desired zero is at $\omega_{p_1} = \pi/6$ rad/sample, the parameter α should be, as given in Equation (6.146), equal to

$$\alpha = -\frac{\cos\left[\left(\frac{\pi}{4} + \frac{\pi}{6}\right)/2\right]}{\cos\left[\left(\frac{\pi}{4} - \frac{\pi}{6}\right)/2\right]} = -0.8002. \tag{6.163}$$

The magnitude responses corresponding to the lowpass and highpass transfer functions are seen in Figure 6.11. Notice how the new transfer function has indeed a zero at the desired position. △

6.5 Magnitude and phase approximation

In this section we discuss the approximation of IIR digital filters using optimization techniques aimed at the simultaneous approximation of the magnitude and phase responses. The same approach is useful in designing continuous-time filters and FIR filters. However, in the case of FIR filters, more efficient approaches exist, as we have seen in Sections 5.6.2 and 5.6.3.

6.5.1 Basic principles

Assume that $H(z)$ is the transfer function of an IIR digital filter. Then $H(e^{j\omega})$ is a function of the filter coefficients, which are usually grouped into a single vector $\boldsymbol{\gamma}$, and of the independent variable $\theta = \omega$.

The frequency response of a digital filter can be expressed as a function of the filter parameters $\boldsymbol{\gamma}$ and θ, namely $F(\boldsymbol{\gamma}, \theta)$, and a desired frequency response is usually referred to as $f(\theta)$.

The complete specification of an optimization problem involves: definition of an objective function (also known as a cost function), determination of the form of the transfer function $H(z)$ and its coefficients $\boldsymbol{\gamma}$, and the solution methods for the optimization problem. These three items are further discussed below.

- *Choosing the objective function*: A widely used type of objective function in filter design is the weighted L_p norm, defined as (Deczky, 1972)

$$\|L(\boldsymbol{\gamma})\|_p = \left(\int_0^{\pi} W(\theta)|F(\boldsymbol{\gamma}, \theta) - f(\theta)|^p \,d\theta\right)^{1/p} \tag{6.164}$$

where $W(\theta) > 0$ is the so-called weight function.

Problems based on the L_p-norm minimization criteria with different values of p lead, in general, to different solutions. An appropriate choice for the value of p depends on the type of error which is acceptable for the given application. For example, when we wish to minimize the mean-square value of the error between the desired and the designed responses, we should choose $p = 2$. Another problem is the minimization of the maximum deviation between the desired specification and the designed filter by searching the space of parameters. This case, which is known as the Chebyshev or minimax criterion, corresponds to $p \to \infty$. This important result derived from the optimization theory can be stated more formally as Theorem 6.1 (Deczky, 1972).

Theorem 6.1. *For a given coefficient space P and a given angle space X_θ, there is a unique optimal minimax approximation $F(\boldsymbol{\gamma}^*_\infty, \theta)$ for $f(\theta)$. In addition, if the best L_p approximation for the function $f(\theta)$ is denoted by $F(\boldsymbol{\gamma}^*_p, \theta)$, then it can be demonstrated that*

$$\lim_{p\to\infty} \boldsymbol{\gamma}^*_p = \boldsymbol{\gamma}^*_\infty. \tag{6.165}$$

◇

This result shows that we can use any minimization program based on the L_p norm to find a minimax (or approximately minimax) solution, by progressively calculating the L_p optimal solution with, for instance, $p = 2, 4, 6$, and so on, indefinitely.

Specifically, the minimax criterion for a continuous frequency function is best defined as

$$\left\|L(\boldsymbol{\gamma}^*)\right\|_\infty = \min_{\boldsymbol{\gamma}\in P}\{\max_{\theta\in X_\theta}\{W(\theta)|F(\boldsymbol{\gamma}, \theta) - f(\theta)|\}\}. \tag{6.166}$$

In practice, due to several computational aspects, it is more convenient to use a simplified objective function given by

$$L_{2p}(\boldsymbol{\gamma}) = \sum_{k=1}^{K} W(\theta_k)\left(F(\boldsymbol{\gamma}, \theta_k) - f(\theta_k)\right)^{2p}, \tag{6.167}$$