

Arithmetic Applications of Artin Twist and BSD

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Introduction

Notation

We use the following notation for characters:

$R_{\mathbb{C}}(G)$	the ring of characters of representations of G over \mathbb{C} ,
$R_{\mathbb{Q}}(G)$	the ring of characters of representations of G over \mathbb{Q} ,
$\text{Irr}_{\mathbb{C}}(G)$	the set of characters of complex irreducible representations of G ,
$\text{Irr}_{\mathbb{Q}}(G)$	the set of characters of \mathbb{Q} -irreducible representations of G ,
$\mathbb{Q}(\rho)$	the field of character values of a complex character ρ ,
$m(\rho)$	the Schur Index of an irreducible complex character ρ over $\mathbb{Q}(\rho)$,

1 Birch and Swinnerton-Dyer Conjecture

2 Algebraic number theory and representation theory background

2.1 Representation theory of finite groups

Let G be a finite group. Recall that a **representation** of G is a group homomorphism $\rho: G \rightarrow \mathrm{GL}(V)$ where V is a complex vector space. Associated to a representation ρ is a **character** $\chi: G \rightarrow \mathbb{C}^\times$, defined by letting $\chi(g) = \mathrm{Tr} \rho(g)$ for $g \in G$. For complex representations, ρ is determined by its character; if ρ, ρ' are representations with identical characters, then ρ and ρ' are isomorphic as representations.

Given an irreducible $\mathbb{Q}G$ -representation with character ψ , we have that

$$\psi = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q})} m(\rho) \cdot \rho^\sigma$$

for ρ the character of an irreducible $\mathbb{C}G$ -representation, and $m(\rho)$ the Schur index.

In particular, the map $R_{\mathbb{C}}(G) \rightarrow R_{\mathbb{Q}}(G)$ given by sending an irreducible complex character ρ to $\tilde{\rho} = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q})} m(\rho) \cdot \rho^\sigma$ is surjective.

Induction, Restriction...

Theorem 2.1 (Mackey Decomposition).

2.1.1 Permutation representations and the Burnside ring

Let G be a finite group. The **Burnside ring** $B(G)$ is the ring of formal sums of isomorphism classes of finite G -sets. We have addition by disjoint union: $[S] + [T] = [S \sqcup T]$, and multiplication by Cartesian product: $[S] \times [T] = [S \times T]$ for S, T finite G -sets.

There exists a bijection between the isomorphism classes of transitive G -sets and the conjugacy classes of subgroups $H \leq G$, where H is the stabilizer of a point on which G acts. Then any transitive G -set X is isomorphic to the action of G on G/H for $H \leq G$, so that we can consider $B(G)$ to be a \mathbb{Z} -module generated by the orbits of the action of G on the elements $\{G/H: H \leq G\}$, where we consider H up to conjugacy. For notational purposes, we then write elements $\Theta \in B(G)$ as $\Theta = \sum_i n_i H_i$ with $n_i \in \mathbb{Z}$, $H_i \leq G$.

Given a transitive G -set G/H for $H \leq G$, we can look at the permutation representation $\mathbb{C}[G/H]$. This defines a homomorphism from the Burnside ring to the rational representation ring $R_{\mathbb{Q}}(G)$ of G :

$$a: B(G) \rightarrow R_{\mathbb{Q}}(G), \quad \sum_i n_i H_i \mapsto \sum_i n_i \mathrm{Ind}_{H_i}^G \mathbb{1}_{H_i}.$$

Elements in the kernel of this map are known as **Brauer relations**

2.2 Decompositions of primes in field extensions

2.3 Class field theory

3 Proving things...

3.1 Norm relations

Recall that in Section 2.1, we associated to $\rho \in R_{\mathbb{C}}(G)$ the character

$$\tilde{\rho} = \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\rho)/\mathbb{Q})} m(\rho) \rho^{\sigma}.$$

We call $\sum_i n_i H_i \in B(G)$ a ρ -**relation** if

$$\sum_i n_i \text{Ind}_{H_i}^G \mathbb{1} \simeq \tilde{\rho}.$$

Given such a ρ , consider functions $\psi: B(G) \rightarrow \mathbb{Q}^{\times}/N_{\mathbb{Q}(\rho)/\mathbb{Q}}(\mathbb{Q}(\rho)^{\times})$ (written multiplicatively). We say two functions ψ, ψ' are equivalent, written $\psi \sim_{\rho} \psi'$, if ψ/ψ' is trivial on all ρ -relations.

Remark 3.1. If $\rho = 0$ then we call functions $\psi \sim_{\rho} 1$ **representation theoretic**. These have been studied in [cite](#).

Example 3.2. Consider $G = C_2 \times C_2$.

3.1.1 D-local functions

3.2 Compatibility in odd order extensions

In this section we work towards proving the following:

Theorem 3.3. *Let F/\mathbb{Q} be a Galois extension of odd degree, with $G = \text{Gal}(F/\mathbb{Q})$. Consider a semistable elliptic curve E/\mathbb{Q} with good reduction at primes that are wildly ramified in F/\mathbb{Q} .*

Then, for any $\rho \in R_{\mathbb{C}}(G)$ with $[\mathbb{Q}(\rho): \mathbb{Q}] > 1$, the function $f: B(G) \rightarrow \mathbb{Q}^{\times}/N_{\mathbb{Q}(\rho)/\mathbb{Q}}(\mathbb{Q}(\rho)^{\times})$ sending $H \mapsto C(E/F^H)$ satisfies $f \sim_{\rho} 1$.

References

- [BH06] C. J. Bushnell and G. Henniart, *The Local Langlands Conjecture for $GL(2)$* , Grundlehren der mathematischen Wissenschaften, Springer Berlin, 2006.