Bruhat-Tits Study Group

December 22, 2024

1 Introductory Talk

The aim of this introductory talk is to provide three simple examples that fit within the general framework that will be covered during the study group. These examples have the advantage of being very explicit while introducing the main objects that appear within the general theory. The logical developments of all three examples are identical: we start with a complex simple Lie algebra $\mathfrak{g}$ and a fixed field K. From these objects, we construct a linear group G_K over K associated to the Lie algebra $\mathfrak{g}$. We then study the structure of these groups through their generators.

The first two examples are considered classical and hold for general fields K, while the last one crucially uses properties of fields endowed with non-archimedean valuations. First of all, we begin with a short introduction of Lie algebras and their representations.

1.1 Lie algebras

Recall that a Lie algebra over a field K is a K-vector space together with a bilinear map (called the *Lie bracket*)

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

such that

1. Antisymmetry: $[x,x] = 0$ for all $x \in \mathfrak{g}$.
2. Jacobi identity: $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Example 1.1. Here are some important examples of Lie algebras.

1. Let V be a finite dimensional K-vector space. Then $\mathfrak{g} = \mathrm{End}(V)$, together with $[xy] = xy - yx$ is a Lie algebra. This Lie algebra is written as $\mathfrak{gl}(V)$ and denoted by *general linear algebra*, and is clearly isomorphic to the Lie algebra $\mathfrak{gl}_n(K)$ of $n \times n$ matrices with coefficients in K with the same bracket.
2. Lie subalgebras (i.e vector subspaces closed under the bracket of $\mathfrak{gl}(V)$) are called *linear Lie algebras.* Of particular interest, we have
   - The *special linear algebra* $\mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) : \mathrm{tr}(x) = 0\}$ of traceless matrices.

- Suppose that V is a finite dimensional $\mathbb{C}$-vector space and let $\langle\cdot,\cdot\rangle$ be a bilinear form in V. Then the subspace
$$\{x \in \mathfrak{gl}(V) : \langle xv, w\rangle = -\langle v, xw\rangle \text{ for all } v, w \in V\}$$
is a linear Lie algebra. If $\langle\cdot,\cdot\rangle$ is symmetric and non-degenerate we obtain the special orthogonal linear algebra $\mathfrak{so}(V)$, while if $\langle\cdot,\cdot\rangle$ is antisymmetric and non-degenerate we obtain the simplectic linear algebra $\mathfrak{sp}(V)$.

Naturally, we say that a map $\psi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras over K is a homomorphism of Lie algebras if ψ is a K-linear map preserving the Lie bracket; that is, $[\psi(x)\psi(y)] = \psi([xy])$ for all $x, y \in \mathfrak{g}$. We say that ψ is an isomorphism if it is an isomorphism of K-vector spaces. Importantly, this leads to the notion of a *representation of a Lie algebra over K*. This is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$ for some K-vector space V.

We often write $x \cdot v$ for $x \in \mathfrak{g}$ and $v \in V$ instead of $\psi(x)(v)$, and we note that
$$[xy] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v).$$

Example 1.2.

1. The *trivial representation* is the homomorphism $\mathfrak{g} \to \mathfrak{gl}_1(K)$ given by $x \cdot v = 0$ for all $x \in \mathfrak{g}$ and $v \in V$.
2. If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, then the *defining representation* is the natural inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$.
3. The *adjoint representation* is the homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $\mathrm{ad}(x)(y) = [xy]$ for all $x, y \in \mathfrak{g}$. The fact that this is a representation is comes from the Jacobi identity.

With these ingredients, we are now ready to give an explicit construction of the linear Lie groups G_K associated to the Lie algebras $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$.

1.2 Adjoint Chevalley groups associated to $\mathfrak{sl}_2(\mathbb{C})$

The complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is a three-dimensional Lie algebra with basis given by
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
and Lie bracket $[he] = 2e$, $[hf] = -2f$ and $[ef] = h$. Thus, by choosing $\{e, h, f\}$ as basis of $\mathfrak{sl}_2(\mathbb{C})$, the adjoint representation $\mathrm{ad} : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{gl}(\mathfrak{sl}_2(\mathbb{C}))$ is given by
$$e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Importantly, the matrix h acts on $\mathfrak{sl}_2(\mathbb{C})$ by a semisimple endomorphism and decomposes into the eigenspaces
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C}) = \langle h\rangle \oplus \langle e\rangle \oplus \langle f\rangle.$$

This decomposition is called the *root space decomposition* of $\mathfrak{sl}_2$, and can be generalized to higher dimensional semisimple Lie algebras. The subspace spanned by h is a maximal abelian semisimple subalgebra of $\mathfrak{sl}_2(\mathbb{C})$, called a *Cartan subalgebra* and denoted by $\mathfrak{t} = \langle h \rangle$. In order to visualize this decomposition, we note that

$$\langle e \rangle = \{x \in \mathfrak{sl}_2(\mathbb{C}) : [hx] = 2x\} = \{x \in \mathfrak{sl}_2(\mathbb{C}) : [yx] = \alpha(y)x \text{ for all } y \in \langle h \rangle\}$$

while

$$\langle f \rangle = \{x \in \mathfrak{sl}_2(\mathbb{C}) : [hx] = -2x\} = \{x \in \mathfrak{sl}_2(\mathbb{C}) : [yx] = -\alpha(y)x \text{ for all } y \in \langle h \rangle\},$$

where $\alpha : \mathfrak{t} \to \mathbb{C}$ is a linear form satisfying $\alpha(h) = 2$. The set $\Phi = \{\alpha, -\alpha\} \subset \mathfrak{t}^*$ is called the *root system* of $\mathfrak{sl}_2(\mathbb{C})$. Moreover, we say that $Q = \mathbb{Z}\alpha$ is the *root lattice* while $P = \{\lambda \in \mathfrak{t}^* : \lambda(h) \in \mathbb{Z}\} = \mathbb{Z}\frac{\alpha}{2}$ is called the *weight lattice*.

Since $[he] = 2e$, $[hf] = 2f$ and $[ef] = h$, the $\mathbb{Z}$-module $\mathfrak{g}_{\mathbb{Z}} = h\mathbb{Z} \oplus e\mathbb{Z} \oplus f\mathbb{Z}$ is a Lie algebra over $\mathbb{Z}$. Hence, by fixing a field K, one obtains $\mathfrak{g}_K := K \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$, naturally a Lie algebra over K with Lie bracket

$$[\mu_1 \otimes x_1, \mu_2 \otimes x_2] := \mu_1\mu_2 \otimes [x_1, x_2],$$

making it isomorphic to $\mathfrak{sl}_2(K)$. We are now ready to give the main definition of this subsection.

Definition 1.3. The adjoint Chevalley group of type A_1 over K is defined as the subgroup of G_K of $\mathrm{GL}(\mathfrak{sl}_2(K))$ generated by $\{x_\alpha(t), x_{-\alpha}(t) : t \in K\}$ and $\{h(\chi) : \chi \in \mathrm{Hom}(\alpha\mathbb{Z}, K^*)\}$ where

$$x_\alpha(t) = \exp(\mathrm{ad}(te)) = \begin{pmatrix} 1 & -2t & -t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) = \exp(\mathrm{ad}(tf)) = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^2 & 2t & 1 \end{pmatrix}, \quad h(\chi) = \begin{pmatrix} \chi(\alpha) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \chi(-\alpha) \end{pmatrix}.$$

Here, the matrices are with respect to the basis $\{1 \otimes e, 1 \otimes h, 1 \otimes f\}$.

One can check by direct computation that all generators above preserve the Lie bracket and have determinant 1. Thus, in fact, G_K is a subgroup of $\mathrm{Aut}(\mathfrak{sl}_2(K)) \cap \mathrm{SL}_3(K)$. The abstract definition of G_K gives little insight of its group structure, which we investigate now.

To begin with, we consider the *root subgroups*

$$X_\alpha = \{x_\alpha(t) : t \in K\} \quad \text{and} \quad X_{-\alpha} = \{x_{-\alpha}(t) : t \in K\},$$

both isomorphic to K via the isomorphism $t \mapsto x_\alpha(t)$ together with the diagonal subgroup

$$H = \{h(\chi) : \chi \in \mathrm{Hom}(Q, K^*)\},$$

isomorphic to K^* via the isomorphism $s \mapsto h(\chi_s)$ where $\chi_s(\alpha) = s$. In fact, H are all Lie group automorphisms of $\mathfrak{g}_K$ fixing $\mathfrak{t}_K$ element-wise and preserving the eigenspaces. A simple calculation shows that H normalizes X_α and $X_{-\alpha}$ and, using that H is abelian, it also follows that $G'_K := \langle X_\alpha, X_{-\alpha} \rangle$ is the commutator subgroup of G_K. The following theorem is crucial to understand the structure of G_K.

Proposition 1.4. *There exists a homomorphism of groups* $\Psi' : \mathrm{SL}_2(K) \to \langle X_\alpha, X_{-\alpha}\rangle = G'_K$ *such that*

$$\Psi'\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = x_\alpha(t) \quad \textit{and} \quad \Psi'\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = x_{-\alpha}(t) \quad \textit{for all } t \in K.$$

Proof. The map Ψ' can be realized by an explicit representation of $\mathrm{SL}_2(K)$. Consider the action of $\mathrm{SL}_2(K)$ on the space of polynomials $K[x, y]$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x, y) = f(ax + cy, bx + dy),$$

and restrict this action to the three dimensional subspace $K[x, y]_2$ of degree 2 polynomials. By choosing the basis $\{-x^2, 2xy, y^2\}$, one can easily check that the action is given by the homomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad + bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix},$$

and this is precisely the desired homomorphism Ψ'. □

Moreover, one can easily check that $\ker \Psi' = \pm\{I\}$, so

$$G'_K = \langle X_\alpha, X_{-\alpha}\rangle \cong \mathrm{SL}_2(K)/\{\pm I\} = \mathrm{PSL}_2(K),$$

and the bijection is explicitly given by Ψ'. In addition, this gives an explicit description of the diagonal subgroup $H' := H \cap G'$ of G' since $\Psi'\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)$ is diagonal if and only if $b = c = 0$, in which case $a = d^{-1}$ and

$$h_\alpha(\lambda) := \Psi'\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right) = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

Hence, $H' = H \cap G'$ contains the elements $h(\chi)$ such that $\chi \in \mathrm{Hom}(\alpha\mathbb{Z}, (K^*)^2)$. Equivalently, $h(\chi) \in H'$ if and only if there is some $\bar{\chi} \in \mathrm{Hom}(\frac{\alpha}{2}\mathbb{Z}, K^*)$ such that $\bar{\chi}(\alpha) = \chi(\alpha)$. This happens to be a general phenomenon.

Finally, this result allows us to give a global description for G_K.

Theorem 1.5. *There exists a unique homomorphism of groups* $\Psi : \mathrm{GL}_2(K) \to \langle X_\alpha, X_{-\alpha}, H\rangle = G_K$ *extending* Ψ' *such that*

$$\Psi\left(\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^{-1} \end{pmatrix} \in H$$

for all $s \in K^*$. *Moreover,* Ψ *is surjective and* $\ker \Psi = \{\lambda I : \lambda \in K^*\}$. *In particular,* $G_K \cong \mathrm{PSL}_2(K)$.

Proof. The desired homomorphism Ψ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K) \longmapsto \frac{1}{ad - bc}\begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad + bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix}.$$

This map is clearly surjective, extends Ψ' and maps $\left(\begin{smallmatrix} s & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\mathrm{Diag}(s, 1, s^{-1})$. Finally, a tedious calculation shows it is a homomorphism. From the description above, it is easy to check that $\ker \Psi = \{\lambda I : \lambda \in K^*\}$, so $G \cong \mathrm{GL}_2(K)/\{\lambda I, \lambda \in K^*\} = \mathrm{PGL}_2(K)$, where the isomorphism is explicitly given by Ψ. $\square$

Under this isomorphism, one can identify important subgroups of G with subgroups of $\mathrm{PGL}_2(K)$. Indeed,

$$X_\alpha \longleftrightarrow \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad X_{-\alpha} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \quad H \longleftrightarrow \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Moreover, one also defines the *monoidal* subgroup $N = \langle H, n_\alpha \rangle$, where $n_\alpha = \Psi_\alpha\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the *Borel* subgroup $B = X_\alpha H$. Under Ψ, these correspond to

$$N \longleftrightarrow \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \sqcup \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad B \longleftrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Of course, by intersecting these subgroups with G'_K, we get the same identifications inside $\mathrm{PSL}_2(K) \subseteq \mathrm{PGL}_2(K)$. Therefore, the following results also hold if we intersect the subgroups with G'_K.

We also note that N is the normalizer of H inside G. Therefore $H \triangleleft N$ and $N/H \cong C_2$, generated by $n_\alpha H$. Finally, a simple calculation shows the following.

Theorem 1.6 (Bruhat Decomposition for $\mathfrak{sl}_2$). *Let G, B, N be as above. Then*

$$G = BNB = B \sqcup Bn_\alpha B.$$

The same is true if we replace G, B, N for G', B', N'.

Proof. By using the identification given by Ψ, a simple calculation shows that

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} -b_1 a_2 & -b_1 b_2 + a_1 d_2 \\ -d_1 a_2 & -d_1 b_2 \end{pmatrix}.$$

Note that $d_1 a_2 \neq 0$ and, in fact, $Bn_\alpha B = G \setminus B$, as desired. $\square$

1.3 Adjoint Chevalley groups associated to $\mathfrak{sl}_3(\mathbb{C})$

We now investigate how the above construction may be generalized to $\mathfrak{sl}_3(\mathbb{C})$, an 8-dimensional Lie algebra with basis given by

$$\mathcal{B} = \{E_{11} - E_{22}, E_{22} - E_{33}, E_{12}, E_{23}, E_{13}, E_{21}, E_{32}, E_{31}\}.$$

The diagonal matrices $\mathfrak{t}$ inside $\mathfrak{sl}_3(\mathbb{C})$ are a Cartan subalgebra and are spanned by $h_1 := E_{11} - E_{22}$ and $h_2 := E_{22} - E_{33}$. In some sense that will be made precise in later talks, h_1 and h_2 are a natural basis for $\mathfrak{t}$. Importantly, their action on $\mathfrak{sl}_3(\mathbb{C})$ under the adjoint representation is simultaneously diagonalizable. The root space decomposition is

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{t} \oplus \langle E_{12} \rangle \oplus \langle E_{23} \rangle \oplus \langle E_{13} \rangle \oplus \langle E_{21} \rangle \oplus \langle E_{32} \rangle \oplus \langle E_{31} \rangle,$$

where

$$\langle E_{12}\rangle = \mathfrak{g}_{\alpha_1},\ \langle E_{23}\rangle = \mathfrak{g}_{\alpha_2},\ \langle E_{31}\rangle = \mathfrak{g}_{\alpha_1+\alpha_2},\ \langle E_{21}\rangle = \mathfrak{g}_{-\alpha_1},\ \langle E_{32}\rangle = \mathfrak{g}_{-\alpha_2},\ \langle E_{31}\rangle = \mathfrak{g}_{-\alpha_1-\alpha_2}$$

and the linear functionals satisfy $\alpha_1(h_{\alpha_1}) = \alpha_2(h_{\alpha_2}) = 2$ and $\alpha_1(h_{\alpha_2}) = \alpha_2(h_{\alpha_1}) = -1$. Therefore, the root system of $\mathfrak{sl}_3(\mathbb{C})$ is $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$ of type A_2 and has root lattice $Q = \alpha_1\mathbb{Z} \oplus \alpha_2\mathbb{Z}$ and weight lattice

$$P = \{\lambda \in \mathfrak{t}^* : \lambda(h_1), \lambda(h_2) \in \mathbb{Z}\} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

where $w_1 = (2\alpha_1+\alpha_2)/3$ and $w_2 = (\alpha_1+2\alpha_2)/3$. To ease notation, one normally writes e_α for E_{ij} if $\mathfrak{g}_\alpha = \langle E_{ij}\rangle$.

Again, one can easily check that the free $\mathbb{Z}$-module

$$\mathfrak{g}_{\mathbb{Z}} = h_1\mathbb{Z} \oplus h_2\mathbb{Z} \oplus \bigoplus_{1\le i\ne j\le 3} E_{ij}\mathbb{Z}$$

is closed under the bracket and therefore is a Lie algebra over $\mathbb{Z}$. For any fixed field K, the vector space $\mathfrak{g}_K := K \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ is a Lie algebra over K isomorphic to $\mathfrak{sl}_3(K)$. Similarly to the previous example, we define the *adjoin Chevalley group of type A_2 over K* to be the subgroup G_K of $\mathrm{Aut}(\mathfrak{sl}_3(\mathbb{C})) \cap \mathrm{SL}_8(K)$ generated by

$$\{x_\alpha(t) : \alpha \in \Phi, t \in K\} \quad \text{and} \quad \{h(\chi) : \chi \in \mathrm{Hom}(Q, K^*)\},$$

where $x_\alpha(t) = \exp(\mathrm{ad}(te_\alpha))$ and $h(\chi)$ satisfies $h(\chi)(t) = t$ for all $t \in \mathfrak{t}$ and $h(\chi)(e_\alpha) = \chi(\alpha)e_\alpha$ for all $\alpha \in \Phi$.

We now study the structure of G_K in an analogous way to the previous section. We define the *root subgroups*

$$X_\alpha = \{x_\alpha(t) : t \in K\} \cong K$$

for each $\alpha \in \Phi$ and the *diagonal subgroup* (or *torus*)

$$H = \{h(\chi) : \chi \in \mathrm{Hom}(Q, K^*)\}.$$

Moreover, we define the *unipotent subgroups*

$$U = \langle X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_1+\alpha_2}\rangle \quad \text{and} \quad V = \langle X_{-\alpha_1}, X_{-\alpha_2}, X_{-\alpha_1-\alpha_2}\rangle,$$

which are normalized by H since each $X_\alpha, \alpha \in \Phi$ is. Hence, $B = UH = HU$ is called the *Borel subgroup* and since H is abelian, it follows that $\langle X_\alpha : \alpha \in \Phi\rangle = G'_K \lhd G_K$ is the commutator subgroup of G_K.

The following Proposition, analogous to Proposition 1.4 gives a 'local' description of G.

Proposition 1.7. *For each $\alpha \in \Phi$, there exists an isomorphism of groups $\Psi_\alpha : \mathrm{SL}_2(K) \to \langle X_\alpha, X_{-\alpha}\rangle \le G'$ such that*

$$\Psi_\alpha\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right) = x_\alpha(t) \quad \textit{and} \quad \Psi_\alpha\left(\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\right) = x_{-\alpha}(t) \quad \textit{for all} \quad t \in K.$$

This result is very useful in practice, but to give a global description, we need the following result that also holds more generally for all semisimple Lie algebras with small modifications.

Lemma 1.8. *Let $\alpha \in \Phi$ and $t \in K$. Then, for all $y \in \mathfrak{sl}_3(\mathbb{C})$, we have that*

$$x_\alpha(t)(y) = \exp(\mathrm{ad}(te_\alpha))(y) = \exp(te_\alpha) \circ y \circ \exp(te_\alpha)^{-1}.$$

We note that for all $t \in K$,

$$\exp(te_{\alpha_1}) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(te_{\alpha_2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(te_{\alpha_1}) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(te_{-\alpha}) = \exp(te_\alpha)^T, \tag{1}$$

and these matrices generate $\mathrm{SL}_3(K)$. This gives a surjective group homomorphism

$$\Psi' : \mathrm{SL}_3(K) \longrightarrow G'$$

such that $\Psi'(A)(y) = AyA^{-1}$ for all $A \in \mathrm{SL}_3(K)$ and $y \in \mathfrak{sl}_3(K)$. Moreover, $\ker \Psi' = \{\lambda I : \lambda^3 = 1\}$ and therefore $G'_K \cong \mathrm{PSL}_3(K)$. Under this isomorphism, we can identify subgroups of G'_K with subgroups of $\mathrm{PSL}_3(K)$. Indeed,

$$X_{\alpha_1} \longleftrightarrow \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{\alpha_2} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{\alpha_1+\alpha_2} \longleftrightarrow \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U \longleftrightarrow \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

and analogously for negative roots. It is also possible to extend the domain of Ψ' to obtain a global description of G_K. The proof of the following result will be postponed to a later talk.

Theorem 1.9. *There exists a unique surjective group homomorphism*

$$\Psi : \mathrm{GL}_3(K) \longrightarrow G_K$$

such that it agrees with Ψ' on $\mathrm{SL}_3(K)$ and

$$\Psi\left(\begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = h(\chi_s), \quad \text{where } \chi_s(\alpha_1) = s \text{ and } \chi_s(\alpha_2) = 1.$$

Moreover, $\ker \Psi = \{\lambda I : \lambda \in K^*\}$ *and, in particular,* $G_K \cong \mathrm{PGL}_3(K)$.

One can easily deduce from the defining properties of Ψ that for $A \in \mathrm{GL}_3(K)$, $\Psi(A)$ preserves all root spaces if and only if A is diagonal. Hence,

$$H \longleftrightarrow \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad B = UH \longleftrightarrow \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

These observations motivate one last important question. Can we explicitly describe the diagonal subgroup of $H' = H \cap G'_K$ inside G'_K? This can be answered by noting that, with respect to the basis $\mathcal{B}$,

$$\Psi(\mathrm{Diag}(s, u, v)) = h(\chi) \quad \text{where} \quad \chi(\alpha_1) = su^{-1} \text{ and } \chi(\alpha_2) = uv^{-1}.$$

We note that $\Psi(\mathrm{Diag}(s,u,v)) \in G'_K$ if and only if $suv \in (K^*)^3$. By multiplying by an appropriate constant, we may assume that $suv = 1$. Thus,

$$\Psi(\mathrm{Diag}(s,(sv)^{-1},v)) = h(\chi) \quad \text{where} \quad \chi(\alpha_1) = s^2 v \text{ and } \chi(\alpha_2) = s^{-1}v^{-2}.$$

Importantly, such a character can be extended to a character $\bar{\chi}$ of the weight lattice by setting $\bar{\chi}(\omega_1) = s$ and $\bar{\chi}(\omega_2) = v^{-1}$. By the construction above, this is also a sufficient condition, so have proved the following.

Lemma 1.10. *Let $\chi \in \mathrm{Hom}(Q, K^*)$ be a character of the root lattice. Then $h(\chi) \in H' = H \cap G'$ if and only if χ can be extended to a character of the weight lattice P.*

We are finally ready to give the Bruhat decomposition of G (and G'). For each $\alpha \in \Phi$, let $n_\alpha = \Psi_\alpha\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$, and define the monoidal subgroups

$$N = \langle H, n_\alpha : \alpha \in \Phi \rangle, \quad N' = N \cap G' = \langle H', n_\alpha : \alpha \in \Phi \rangle.$$

Under the isomorphism given by Ψ, we have that

$$n_{\alpha_1} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_{\alpha_2} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad n_{\alpha_1+\alpha_2} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and therefore

$$N \longleftrightarrow \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \sqcup \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \sqcup \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \sqcup \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \sqcup \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \sqcup \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$$

From this perspective, it is clear that N is in fact the normalizer of $H = B \cap N$ inside G_K, and that the quotient N/H is isomorphic to S_3, generated by the elements $n_{\alpha_1}H$ and $n_{\alpha_2}H$. Now the proof of the Bruhat decomposition for G_K is a tedious analogous to Theorem 1.6.

Theorem 1.11 (Bruhat decomposition for $\mathfrak{sl}_3$)**.** *Let G_K, B, N as above. Then*

$$G_K = BNB = B \sqcup Bn_{\alpha_1}B \sqcup Bn_{\alpha_2}B \sqcup Bn_{\alpha_1}n_{\alpha_2}B \sqcup Bn_{\alpha_2}n_{\alpha_1}B \sqcup Bn_{\alpha_1}n_{\alpha_2}n_{\alpha_1}B.$$

The same is true if we replace G_K, B, N for G'_K, B', N'.

1.4 Affine bruhat decomposition of $\mathfrak{sl}_2$

We are now ready to discuss the last example of the talk. We start again with the simple complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and we fix a field K endowed with a discrete non-archimedean absolute value $|\cdot|$. Let $\mathfrak{O} = \{x \in K : |x| \leq 1\}$ be its ring of integers, $\mathfrak{p} = \varpi\mathfrak{O} = \{x \in K : |x| < 1\}$ its unique maximal ideal with uniformizer ϖ and $k = \mathfrak{O}/\mathfrak{p}$ its residue field.

The aim of this final example is to exhibit subgroups of $G'_K \cong \mathrm{PSL}_2(K)$ satisfying analogous properties to the Borel subgroup B' and the monoidal subgroup N', in a way that we will make precise. In particular, this will lead us to *affine Bruhat decompositions* of G'_K. We also remark that such decompositions also exist for G_K, but its treatment is more complicated. Therefore, we will content ourselves with studying G'_K, while the rest will be discussed in later talks. The idea is quite simple: from the natural reduction map $\mathcal{O} \twoheadrightarrow k$, we have natural surjective homomorphism $G'_{\mathfrak{O}} \twoheadrightarrow G'_k$ by reducing each matrix entry modulo $\mathfrak{p}$. This homomorphism is in fact the reduction modulo $\mathfrak{p}$ map

$$\phi : \mathrm{PSL}_2(\mathfrak{O}) \twoheadrightarrow \mathrm{PSL}_2(k).$$

Let B'_k be the Borel subgroup of G'_k and let

$$I' = \phi^{-1}(B'_k) = \begin{pmatrix} \mathfrak{O}^* & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O}^* \end{pmatrix} \leq G'_{\mathfrak{O}} \leq G'_K$$

be the *Iwahori subgroup* of G'_K. Of course, from the Bruhat decomposition of B'_k, we have a decomposition

$$G'_{\mathfrak{O}} = I' \sqcup I' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I'.$$

The question is whether there is an explicit double cosset decomposition of I' for G'_K. Firstly, we may note that under the natural topology induced by $|\cdot|$, the Iwahori subgroup is compact, while G'_K is certainly not. Thus, the decomposition must have infinitely many double cossets.

Proposition 1.12 (Iwahori decomposition)**.** *We have that*

$$\mathrm{PSL}_2(K) = \bigcup_{a \in \mathbb{Z}^{\geq 0}} \mathrm{PSL}_2(\mathfrak{O}) \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^a \end{pmatrix} \mathrm{PSL}_2(\mathfrak{O}),$$

and the union is disjoint.

Sketch. One can diagonalize any matrix of $\mathrm{PSL}_2(K)$ by pre and post multiplication of matrices in $\mathrm{PSL}_2(\mathfrak{O})$, and we can also absorb units. This proves the union, and it is disjoint since the index

$$\left[\mathrm{PSL}_2(\mathfrak{O}) \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^a \end{pmatrix} \mathrm{PSL}_2(\mathfrak{O}) : \mathrm{PSL}_2(\mathfrak{O}) \right]$$

is distinct for each $a \in \mathbb{Z}^{\geq 0}$. $\square$

Therefore, we have that

$$\mathrm{PSL}_2(K) = \bigcup_{a \in \mathbb{Z}^{\geq 0}} \left(I' \sqcup I' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I' \right) \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^a \end{pmatrix} \left(I' \sqcup I' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I' \right).$$

To simplify this expression, we need the following result, which can be proven by direct computation.

Lemma 1.13. *For each $a \in \mathbb{Z}^{\geq 0}$, we have that*

$$\left(I' \sqcup I' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I' \right) \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^a \end{pmatrix} \left(I' \sqcup I' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I' \right) =$$

$$= I' \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^a \end{pmatrix} I' \sqcup I' \begin{pmatrix} 0 & \varpi^a \\ -\varpi^{-a} & 0 \end{pmatrix} I' \sqcup I' \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^{-a} \end{pmatrix} I' \sqcup I' \begin{pmatrix} 0 & \varpi^{-a} \\ -\varpi^a & 0 \end{pmatrix} I'.$$

The key observation is the matrices

$$\left\{ \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^{a} \end{pmatrix}, \begin{pmatrix} 0 & \varpi^{a} \\ -\varpi^{-a} & 0 \end{pmatrix} : a \in \mathbb{Z} \right\}$$

form a multiplicative group isomorphic to the infinite dihedral group $D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. Hence, with a slight abuse of notation, we have shown:

Theorem 1.14 (Affine Bruhat decompsition for $\mathfrak{sl}_2(K)$)**.** *We have that*

$$G'_K = \bigcup_{w \in D_\infty} I' w I',$$

and the union is disjoint.

As a final remark, it is worth mentioning that all of the examples above can be explained by a more general framework. Both pairs of subgroups (B', N') and (I', N') of G'_K are what is generally called a (B, N) pair of G'_K, which will be abstractly described later during the study group. This general theory introduced by Tits is extremely powerful; in particular, any such pair (B, N) of a group G automatically gives a Bruhat-type decomposition of G in terms of double cossets of B, and the representatives can be explicitly chosen inside N. The above theorems are just instances of this phenomenon.

2 Week 4: Chevalley basis and Chevalley groups

2.1 Introduction and Recap on Lie Algebras

The study of the structure of Lie algebras and their representation theory is a central tool in the representation theory of groups of Lie type. This is a consequence of the fact that the algebraic structure of the Lie algebra encapsulates to a large extent the interplay between the algebraic and topological properties of the group. To understand how one side helps understand the other, it is essential to give explicit methods that allows us to construct groups of Lie type from a Lie algebra and vice-versa. The construction of complex (or real) Lie groups from a complex (or real) Lie algebra has been known for a long time, and it has sparked remarkable progress in the understanding of the representations of complex (or real) Lie groups.

In the 1950s, Chevalley became interested in the connection between complex Lie algebras and finite groups, an unexplored link at the time. In a fundamental paper, Chevalley constructed, for each complex simple Lie algebra $\mathfrak{g}$, a corresponding linear group over any field K. In this chapter, we investigate the main ingredient of his construction; the existence of a Chevalley basis. By using the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ together with the exponential map, we will be able to construct adjoint groups of Lie type over an arbitrary field K.

Let's begin by considering a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with Cartan subalgebra $\mathfrak{t}$ and root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We recall that for each $\alpha \in \Phi$, the subalgebra

$$m_\alpha := \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$$

is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In particular, each $\mathfrak{g}_\alpha$ is one-dimensional, and we can pick elements $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, e_{-\alpha}] = -2e_{-\alpha}, \quad [e_\alpha, e_{-\alpha}] = h_\alpha.$$

We remark that h_α is uniquely determined, while e_α and $e_{-\alpha}$ are determined up to a constant. Next, fix an integral basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of Φ.

Lemma 2.1. *With the notation as above, the elements $h_{\alpha_1}, \ldots, h_{\alpha_l}$ are a basis of $\mathfrak{t}$ and for any $\alpha \in \Phi^+$ (resp Φ^-), h_α is a linear combination of $h_{\alpha_1}, \ldots, h_{\alpha_l}$ with non-negative (resp. non-positive) integer coefficients.*

Proof. The Killing form gives $\mathfrak{t}$ the structure of an euclidean space and induces isometries $\phi : \mathfrak{t} \to \mathfrak{t}^*$ and $\phi^* : \mathfrak{t}^* \to \mathfrak{t}^{**}$. Note that for any $\alpha \in \Phi$, $\phi(t_\alpha) = \alpha$ where $K(t_\alpha, x) = \alpha(x)$ for all $x \in \mathfrak{t}$. Since $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$, then $\phi(h_\alpha) = 2\alpha/(\alpha, \alpha)$, where $(\cdot, \cdot)$ is the inner product on $\mathfrak{t}^*$ induced by the Killing form. This implies that $\phi^*\phi(h_\alpha) = \check{\alpha}$. Hence, the statement of the lemma is equivalent to the fact that $\check{\Phi} = \{\check{\beta} : \beta \in \Phi\}$ is a root system in $\mathfrak{t}^{**}$ with integral basis $\check{\Delta} = \{\check{\alpha}_i : 1 \le i \le l\}$, which is true. Hence, the result follows. $\square$

To simplify notation, we write h_i instead of h_{α_i} for each $1 \leq i \leq l$. The lemma implies that the set $\mathcal{B} = \{e_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ is a basis for $\mathfrak{g}_\mathbb{C}$. In addition, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \Phi$, we have that

$$[h_i, h_j] = 0, \quad [h_i, e_\alpha] = \alpha(h_i)e_\alpha = \langle \alpha, \check{\alpha}_i \rangle e_\alpha, \quad [e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{if } \alpha + \beta \notin \Phi. \end{cases}$$

The constants $N_{\alpha,\beta}$ are called the *structure constants* associated to $\mathcal{B}$.

Definition 2.2. A basis $\mathcal{B} = \{e_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ with $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Phi$ is said to be a *Chevalley basis* if the structure constants satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ for all $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$.

The central aim of this chapter is to prove that any simple Lie algebra has a Chevalley basis, and that its structure constants are integral. First, however, we give an explicit example of Chevalley basis for the Lie algebras $\mathrm{sl}_{n+1}(\mathbb{C})$ (of type A_n).

2.2 Chevalley Basis on $\mathrm{sl}_{n+1}(\mathbb{C})$

Let's begin with a motivating example in which Chevalley basis arise naturally. The Lie algebra $\mathfrak{g}_\mathbb{C} = \mathrm{sl}_{n+1}(\mathbb{C})$ has a Cartan subalgebra $\mathfrak{t}$ consisting of the diagonal matrices, which is spanned by $E_{ii} - E_{i+1,i+1}$ for $1 \leq i \leq n$.

The root space decomposition of $\mathrm{sl}_{n+1}(\mathbb{C})$ with this choice of Cartan subalgebra is

$$\mathrm{sl}_{n+1}(\mathbb{C}) = \mathfrak{t} \oplus \sum_{1 \leq i \neq j \leq n+1} \langle E_{ij} \rangle,$$

and has associated root system Φ of type A_n. For each $1 \leq i \leq n$, let $\alpha_i \in \Phi$ be the root associated such that $\mathfrak{g}_{\alpha_i} = E_{i,i+1}$. An easy computation shows that

$$\alpha_i(h_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3. *The roots $\alpha_1, \ldots, \alpha_n$ are an integral basis of the root system Φ.*

Proof. Firstly, we note that if $1 \leq i < j \leq n+1$ and $1 \leq k < l \leq n+1$, then $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$. Therefore, if $\alpha \in \Phi$ is the root such that $\mathfrak{g}_\alpha = \langle E_{ij} \rangle$ for some $i < j$, then $\mathfrak{g}_{-\alpha} = \langle E_{ji} \rangle$ and $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, which is what we wanted to show. $\square$

Proposition 2.4. *The set $\{E_{ij}, 1 \leq i \neq j \leq n+1;\ E_{ii} - E_{i+1,i+1}, 1 \leq i \leq n\}$ is a Chevalley basis of* $\mathrm{sl}_{n+1}(\mathbb{C})$.

Proof. Recall that $\alpha_1, \ldots, \alpha_n$ is an integral basis, where $\mathrm{sl}_{n+1}(\mathbb{C})_{\alpha_i} = \langle E_{i,i+1} \rangle$. Hence, the matrices $E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}]$ are the desired basis for the Cartan subalgebra $\mathfrak{t}$. Consider now two roots $\alpha, \beta \in \Phi$ and let $E_{ij} = e_\alpha \in \mathfrak{g}_\alpha$ and $E_{kl} = e_\beta \in \mathfrak{g}_\beta$. Note that $\alpha + \beta \in \Phi$ if and only if $j = k$ and $l \neq i$ or $l = i$ and $k \neq j$. Without loss of generality, we may assume the latter and therefore

$$[e_\alpha, e_\beta] = [E_{ij}, E_{ki}] = -E_{kj} = -e_{\alpha+\beta} \quad \text{and} \quad [e_{-\alpha}, e_{-\beta}] = [E_{ji}, E_{ik}] = E_{jk} = e_{-\alpha-\beta}.$$

This calculation shows that $N_{\alpha,\beta} = -N_{-\alpha,-\beta} \in \{\pm 1\}$, so $\{E_{ij}, 1 \leq i \neq j \leq n+1;\ E_{ii} - E_{i+1,i+1}, 1 \leq i \leq n\}$ is a Chevalley basis of $\mathrm{sl}_{n+1}(\mathbb{C})$. $\square$

2.3 Existence of Chevalley Basis

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra with with root system Φ and integral basis $\Delta = \{\alpha_1, \dots, \alpha_l\}$. As in the previous discussion, we let $h_i = h_{\alpha_i}$. Chevalley's groundbreaking result (Corollary 2.6 Theorem 2.8) was the fact that any complex Lie algebra has a Chevalley basis and that its structure constants are always integral.

Firstly, we prove that $\mathfrak{g}_{\mathbb{C}}$ has a Chevalley basis. This is a consequence of the following result.

Proposition 2.5. *For each $1 \leq i \leq l$, fix some $e_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and let $e_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ be the unique element such that $[e_{\alpha_i}, e_{-\alpha_i}] = h_i$. Then there exists an automorphism σ of $\mathfrak{g}_{\mathbb{C}}$, of order 2, satisfying $\sigma(e_{\alpha_i}) = -e_{-\alpha_i}$, $\sigma(e_{-\alpha_i}) = -e_{\alpha_i}$ and $\sigma(h) = -h$ for all $h \in \mathfrak{t}$.*

We state the first result without proof, since it depends on a lenghty calculation can be easily found in many references.

Corollary 2.6 (Chevalley, 1955)**.** *The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has a Chevalley basis.*

Proof. Consider the automorphism σ of order 2 from the previous proposition and let α be a positive root. Fix some $e_\alpha \in \mathfrak{g}_\alpha$ and let $e_{-\alpha} = -\sigma(e_\alpha)$. By rescaling e_α if necessary, we may assume that $[e_\alpha, e_{-\alpha}] = h_\alpha$ (here we are using that $\mathbb{C}$ contains all of its square roots). With this choice for every positive root, we note that

$$N_{-\alpha,-\beta} e_{-\alpha-\beta} = [e_{-\alpha}, e_{-\beta}] = [-\sigma(e_\alpha), -\sigma(e_\beta)] = \sigma([e_\alpha, e_\beta]) = \sigma(N_{\alpha,\beta} e_{\alpha+\beta}) = N_{\alpha,\beta} e_{-\alpha-\beta}.$$

Hence, $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$, so $\{e_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ is a Chevalley basis. □

Once the existence of Chevalley basis has been settled, we now show that the structure constants are integral. This is a consequence of the following result, for which we omit the proof as it is lenghty but unenlightening.

Theorem 2.7 (Properties of Structure Constants)**.** *Let $\{e_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ be a basis of $\mathfrak{g}_{\mathbb{C}}$ with $h_i = h_{\alpha_i}(1 \leq i \leq l)$ and $e_\alpha \in \mathfrak{g}_\alpha$ satisfying $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Phi$. Then the structure constants $N_{\alpha,\beta}$ satisfy the following properties:*

1. *For all $\alpha, \beta \in \Phi$, $N_{\alpha,\beta} = -N_{\beta,\alpha}$.*
2. *For all $\alpha, \beta, \gamma \in \Phi$ with $\alpha + \beta + \gamma = 0$ we have that*

$$\frac{N_{\alpha,\beta}}{(\gamma,\gamma)} = \frac{N_{\beta,\gamma}}{(\alpha,\alpha)} = \frac{N_{\gamma,\alpha}}{(\beta,\beta)}.$$

3. *Suppose that $\alpha, \beta \in \Phi$ are independent roots and that $\beta - p\alpha, \dots, \beta, \dots, \beta + q\alpha$ is the α string through β. If $q \geq 1$, then*

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = -(p+1)^2.$$

We are now ready to state the main theorem Chevalley proved.

Theorem 2.8 (Chevalley, 1995)**.** *Let $\{e_\alpha, \alpha \in \Phi;\ h_i, 1, \leq i \leq l\}$ be a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$. Then the resulting constants lie in $\mathbb{Z}$. More precisely,*

- $[h_i h_j] = 0$ *for* $1 \leq i, j \leq l$.
- $[h_i x_\alpha] = \langle \alpha, \check{\alpha}_i \rangle e_\alpha$ *for* $1 \leq i \leq l$, $\alpha \in \Phi$.
- $[e_\alpha, e_{-\alpha}] = h_\alpha$ *is a* $\mathbb{Z}$ *linear combination of* $h_1, \ldots, h_l$.
- *If* α, β *are independent roots and* $\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ *the* α*-string through* β, *then* $[e_\alpha, e_\beta] = 0$ *if* $q = 0$ *while* $[e_\alpha, e_\beta] = \pm(p+1)e_{\alpha+\beta}$ *if* $q \geq 1$.

2.4 The Exponential Map

Having established the existence of Chevalley basis, we can now define the exponential maps arising form the adjoint representation of the Lie algebra and study its effect on the basis.

For the remainder of this chapter, fix some complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$ with Cartan subalgebra $\mathfrak{t}$, root system Φ with integral basis Δ and Chevalley basis $\mathcal{B}$. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ and, for every $\alpha \in \Phi$ and $\zeta \in \mathbb{C}$, we define the endomorphism of $\mathfrak{g}_\mathbb{C}$ given by

$$\exp(\mathrm{ad}(\zeta e_\alpha)) := I + \mathrm{ad}(\zeta e_\alpha) + \frac{\mathrm{ad}(\zeta e_\alpha)^2}{2} + \frac{\mathrm{ad}(\zeta e_\alpha)^3}{6} + \cdots + \frac{\mathrm{ad}(\zeta e_\alpha)^k}{k!} + \cdots .$$

This is a well-defined map since $\mathrm{ad}(\zeta e_\alpha)$ is a nilpotent transformation for all $\alpha \in \Phi$ and $\zeta \in \mathbb{C}$. For $\exp(\mathrm{ad}(\zeta e_\alpha))$ to be of interest, we need to show that it is in fact a Lie algebra automorphism of $\mathfrak{g}_\mathbb{C}$. To do this, we first note that for any $x \in \mathfrak{g}_\mathbb{C}$, the Jacobi identity implies that

$$\mathrm{ad}(x)([yz]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]$$

for any $y, z \in \mathfrak{g}_\mathbb{C}$. A map $\delta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ satisfying the condition above is called a *derivation* of the Lie algebra. Importantly, derivations satisfy the following property.

Proposition 2.9. *Let* $\delta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ *be a nilpotent derivation with* $\delta^n = 0$ *for* $n \geq 0$. *Then the map*

$$\exp(\delta) = I + \delta + \frac{\delta^2}{2} + \frac{\delta^3}{3!} + \cdots + \frac{\delta^{n-1}}{(n-1)!}$$

is an automorphism of $\mathfrak{g}$ *as a Lie algebra.*

Proof. The map $\exp(\delta)$ is certainly a $\mathbb{C}$-linear map, with inverse given by $\exp(-\delta)$. It remains to show that it preserves the bracket. Since δ is a derivation, it is easy to check by induction that

$$\frac{\delta^r}{r!}([xy]) = \sum_{\substack{i+j=r \\ i,j \geq 0}} \left[\frac{\delta^i}{i!}(x), \frac{\delta^j}{j!}(y) \right]$$

for all $r \geq 1$ and $x, y \in \mathfrak{g}$. This immediately implies that

$$\exp(\delta)([xy]) = \sum_{r=0}^{\infty} \frac{\delta^r}{r!}([xy]) = \sum_{r=0}^{\infty} \sum_{\substack{i+j=r \\ i,j \geq 0}} \left[\frac{\delta^i}{i!}(x), \frac{\delta^j}{j!}(y) \right] = \left[\sum_{i=0}^{\infty} \frac{\delta^i}{i!}(x), \sum_{j=0}^{\infty} \frac{\delta^j}{j!} \right] = [\exp(\delta)(x), \exp(\delta)(y)],$$

as desired. ∎

Corollary 2.10. *For any root $\alpha \in \Phi$ and $\zeta \in \mathbb{C}$, the map* $\exp(\mathrm{ad}(\zeta e_\alpha))$ *is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ as a Lie algebra.*

To simplify notation, we simply write $x_\alpha(\zeta)$ for $\exp(\mathrm{ad}(\zeta e_\alpha))$.

Definition 2.11. The adjoint Chevalley group over $\mathbb{C}$ associated to $\mathfrak{g}_{\mathbb{C}}$ is the subgroup G of $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ generated by the automorphisms $x_\alpha(\zeta)$ for all $\alpha \in \Phi$ and $\zeta \in \mathbb{C}$.

The group G naturally inherits a complex Lie group structure as a subgroup of $\mathrm{Aut}(\mathfrak{g})$, so this construction provides one direction of the bridge between complex Lie algebras and complex Lie groups. However, at the beginning of this chapter we promised a construction of a group of Lie type for an arbitrary field K. The first step, of course, is to consider a Chevalley basis $\mathcal{B} = \{e_\alpha, \alpha \in \Phi;\ h_i, 1 \le i \le l\}$ for $\mathfrak{g}_{\mathbb{C}}$ and to consider the $\mathbb{Z}$-module

$$\mathfrak{g}_{\mathbb{Z}} = \sum_{i=1}^{l} \mathbb{Z}h_i \oplus \sum_{\alpha\in\Phi} \mathbb{Z}e_\alpha,$$

which is closed under the bracket. Then we can construct the K-vector space $\mathfrak{g}_K = K \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$, a Lie algebra over K with basis $\overline{h_i} := 1 \otimes h_i$ for $1 \le i \le n$ and $\overline{e_\alpha} = 1 \otimes e_\alpha$ for $\alpha \in \Phi$. We denote this basis as $\overline{\mathcal{B}}_K$. Ideally, we would like to construct the Chevalley group over K as a subgroup of $\mathrm{Aut}(\mathfrak{g}_K)$ generated by exponential maps (and possibly other automorphisms). This is indeed possible if K has characteristic 0; however, if the characteristic is positive, the exponential maps are not well-defined automorphisms of $\mathfrak{g}$. To circumvent this problem, we first need to analyse how the exponential maps $x_\alpha(\zeta)$ act on the Chevalley basis $\mathcal{B}$. This is simply a matter of tracing through the definitions. Indeed,

$$x_\alpha(\zeta)(e_\alpha) = e_\alpha, \qquad x_\alpha(\zeta)(e_{-\alpha}) = e_{-\alpha} + \zeta h_\alpha - \zeta^2 e_\alpha \qquad \text{and} \qquad x_\alpha(\zeta)(h_i) = h_i - \langle \alpha, \breve{\alpha}_i\rangle e_\alpha$$

for all $1 \le i \le l$. It remains to show the effect of $x_\alpha(\zeta)$ on basis elements e_β for $\beta \ne \pm\alpha$. To do this, suppose that $\beta - p\alpha, \dots, \beta, \dots, \beta + q\alpha$ is the α-string through β, where $p, q \ge 0$. Then

$$x_\alpha(\zeta)(e_\beta) = e_\beta + N_{\alpha,\beta}\zeta e_{\beta+\alpha} + \frac{N_{\alpha,\beta}N_{\alpha,\alpha+\beta}}{2}\zeta^2 e_{\beta+2\alpha} + \cdots + \frac{N_{\alpha,\beta}\cdots N_{\alpha,\beta+(q-1)\alpha}}{q!}\zeta^q e_{\beta+q\alpha}.$$

From Theorem 2.8, we know that $N_{\alpha,\beta} = \pm(p+1)$ and therefore

$$\frac{N_{\alpha,\beta}\cdots N_{\alpha,\beta+(i-1)\alpha}}{i!} = \pm\frac{(p+1)(p+2)\cdots(p+i)}{i!} = \pm\binom{p+i}{i}.$$

Hence, we have shown the following.

Proposition 2.12. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra with Chevalley basis $\mathcal{B} = \{e_\alpha, \alpha \in \Phi;\ h_i, 1 \le i \le l\}$. Then the matrix $A_\alpha(\zeta)$ of the automorphism $x_\alpha(\zeta)$ with respect to the basis $\mathcal{B}$ has entries in $\mathbb{Z}[\zeta]$. In fact, all entries are of the form $a\zeta^i$ for some $a \in \mathbb{Z}$ and $i \ge 0$.*

2.5 Chevalley Groups over Arbitrary Fields

We are finally ready to define the adjoint Chevalley groups associated to a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with Chevalley basis $\mathcal{B}$ over an arbitrary field K. The idea is now simple: for each $\alpha \in \Phi$ and $t \in K$, we consider the matrix $\bar{A}_\alpha(t)$ obtained from $A_\alpha(\zeta)$ by replacing each entry $a\zeta^i$ with $\bar{a}t^i$ where $\bar{a}$ is the image of a under the homomorphism $\mathbb{Z} \to K$. Then, we define $\bar{x}_\alpha(t)$ to be the automorphism of $\mathfrak{g}_K$ with matrix $\bar{A}_\alpha(t)$ with respect to the basis $\overline{\mathcal{B}}_K$. Whenever the field K is clear from context, we simply write h_i, e_α and $x_\alpha(t)$ instead of $\bar{x}_\alpha(t)$, $\overline{h_i}$ and $\overline{e_\alpha}$, respectively. In addition, for each $\chi \in \mathrm{Hom}(\mathbb{Z}\Phi, K^*)$, we define the K-linear map

$$\begin{aligned} h(\chi) : \mathfrak{g}_K &\longrightarrow \mathfrak{g}_K \\ h_i &\longmapsto h_i \qquad\qquad \text{for all } 1 \le i \le l, \\ e_\alpha &\longmapsto \chi(\alpha)e_\alpha \qquad \text{for all } \alpha \in \Phi. \end{aligned}$$

It is easy to see that the maps $h(\chi)$ are all the automorphisms of $\mathfrak{g}_K$ as a Lie algebra that fix every root subspace.

Definition 2.13. The *adjoint Chevalley group* over K associated to $\mathfrak{g}_\mathbb{C}$ is the subgroup G of $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$ generated by the automorphisms $x_\alpha(t)$ for all $\alpha \in \Phi$ and $t \in \mathbb{C}$ **AND** the automorphisms $h(\chi)$ for all $\chi \in \mathrm{Hom}(\mathbb{Z}\Phi, K^*)$.

When the field K is algebraically closed, the automorphisms $h(\chi)$ are redundant; hence the above definition generalizes Definition 2.11.

This definition of G gives very little information about the group. To study its structure, we first consider the *root subgroups* $X_\alpha = \{x_\alpha(t) : t \in K\} \cong K$ for each $\alpha \in \Phi$ and the *diagonal subgroup* (or *torus*) $H = \{h(\chi) : \chi \in \mathrm{Hom}(\mathbb{Z}\Phi, K^*)\} \cong (K^*)^l$. In addition, the following elementary lemma will be very important.

Lemma 2.14. *Let $\mathfrak{g}$ be a Lie algebra over K and let $x \in \mathfrak{g}$ be* ad*-nilpotent. Then, for all automorphisms ϕ of $\mathfrak{g}$, we have that*

$$\phi \circ \exp(\mathrm{ad}(x)) \circ \phi^{-1} = \exp(\mathrm{ad}(\phi(x))).$$

Corollary 2.15. *The diagonal subgroup H is abelian and normalizes each root subgroup X_α. In particular, the subgroup $\langle X_\alpha : \alpha \in \Phi\rangle$ equals the commutator subgroup G' of G. Moreover, if K is algebraically closed, then $G = G'$.*

2.6 Chevalley Groups of Type A_n

In this section, we let $\mathfrak{g}_\mathbb{C} = \mathrm{sl}_{n+1}(\mathbb{C})$ with root system Φ (of type A_n) and Chevalley basis $\mathcal{B} = \{e_\alpha, \alpha \in \Phi;\ h_i, 1 \le i \le n\}$ as in Section 2.2. Recall that $h_i = E_{ii} - E_{i+1,i+1}$ $(1 \le i \le n)$ and the basis elements e_α are the matrices E_{ij} $(1 \le i \ne j \le n+1)$. Importantly, $e_{\alpha_i} = E_{i,i+1}$ where $\alpha_1, \ldots, \alpha_n$ are the simple roots.

Fix a field K and consider $\mathfrak{g}_K \cong \mathrm{sl}_{n+1}(K)$ and consider the Chevalley group G from Definition 2.13, also denoted as the *adjoint Chevalley group of type A_n over K*. In this section, we give a global description of G and its commutator subgroup G'. This result is dependent on the following elementary lemma.

Lemma 2.16. *Let $\mathfrak{g} \subseteq \mathrm{gl}_n(K)$ be a linear Lie algebra over K and let $x \in \mathfrak{g}$ be nilpotent. Then*

$$\exp(\mathrm{ad}(x))(y) = \exp(x) \circ y \circ \exp(x)^{-1}$$

for all $y \in \mathfrak{g}$, where the composition is the standard multiplication of matrices in $\mathrm{gl}_n(K)$.

For any $\alpha \in \Phi$, we note that the basis element $e_\alpha \in \Phi$ satisfies $e_\alpha^2 = 0$ so $\exp(te_\alpha) = I + te_\alpha$ for all $t \in K$. In particular, the lemma above applies with $x = te_\alpha$. Finally, since the group $\mathrm{SL}_{n+1}(K)$ is generated by the matrices $\{I + te_\alpha : \alpha \in \Phi, t \in K\}$, we have the following result.

Proposition 2.17. *There exists a surjective group homomorphism*

$$\Psi' : \mathrm{SL}_{n+1}(K) \longrightarrow \langle X_\alpha : \alpha \in \Phi \rangle = G'$$

such that $\Psi'(A)(y) = AyA^{-1}$ *for all* $A \in \mathrm{SL}_{n+1}(K)$ *and* $y \in \mathrm{sl}_{n+1}(K)$. *Moreover,* $\ker \Psi' = Z(\mathrm{SL}_{n+1}(K))$ *consists of the scalar matrices of the identity inside* $\mathrm{SL}_{n+1}(K)$. *In particular,* $G' \cong \mathrm{PSL}_{n+1}(K)$.

Proof. The existence of Ψ' is immediate from our discussion above. The kernel of Ψ' are matrices $A \in \mathrm{SL}_{n+1}(K)$ such that $Ay = yA$ for all $y \in \mathrm{sl}_{n+1}(K)$. These are precisely the scalar multiples of the identity, as desired. ☐

Of course, this motivates the natural question whether there is a similar global description for G extending the one we already have for G'. This is indeed possible, as we now show.

Theorem 2.18. *There exists a unique surjective homomorphism*

$$\Psi : \mathrm{GL}_{n+1}(K) \longrightarrow G = \langle G', H \rangle$$

such that Ψ *agrees with* Ψ' *on* $\mathrm{SL}_{n+1}(K)$ *and*

$$\Psi\left(\begin{pmatrix} s & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\right) = h(\chi_s), \quad \text{where } \chi_s(\alpha_1) = s \text{ and } \chi_s(\alpha_i) = 1 \text{ for all } 2 \leq i \leq n.$$

Proof. To simplify notation, let $D_s = \begin{pmatrix} s & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ and consider the subgroup $H_1 = \{D_s : s \in K^*\}$. Since $\mathrm{GL}_{n+1}(K) = \mathrm{SL}_{n+1}(K) \rtimes H_1$, for any $A \in \mathrm{GL}_{n+1}$, there are unique $D_s \in H_1$ and $A' \in \mathrm{SL}_{n+1}$ such that $A = D_s A'$. Then we can define a map Ψ on $\mathrm{GL}_{n+1}(K)$ as

$$\Psi(A) = h(\chi_s)\Psi(A') \quad \text{where } \chi_s(\alpha_1) = s \text{ and } \chi_s(\alpha_i) = 1 \text{ for all } 2 \leq i \leq n.$$

Of course, it remains to show the non-trivial fact that Ψ is in fact a group homomorphism. Thus, we need to show that Ψ is compatible with the semidirect product between $\mathrm{SL}_{n+1}(K)$ and H_1. More concretely, we need to show that

$$\Psi'(A')h(\chi_s) = h(\chi_s)\Psi'(D_s^{-1}A'D_s) \tag{2.18.1}$$

for all $A' \in \mathrm{SL}_{n+1}(K)$ and $s \in K^*$. Since Ψ' is a group homomorphism, it is enough to show the above condition for a set of generators of $\mathrm{SL}_{n+1}(K)$. Hence, it will be enough to show (2.18.1) for all $I + te_\alpha \in \mathrm{SL}_{n+1}(K)$ where $\alpha \in \Phi$ and $t \in K$ and all $s \in K^*$. Firstly, we note that

$$D_s^{-1}(I + te_\alpha)D_s = \begin{cases} I + s^{-1}te_\alpha & \text{if } \alpha = \alpha_1 + \cdots + \alpha_j \text{ for some } 1 \leq j \leq n, \\ I + ste_\alpha & \text{if } \alpha = -\alpha_1 - \cdots - \alpha_j \text{ for some } 1 \leq j \leq n, \\ I + te_\alpha & \text{otherwise.} \end{cases}$$

Since χ_s is a character of $\mathbb{Z}\Phi$ defined by $\chi_s(\alpha_1) = s$ and $\chi_s(\alpha_i) = 1$ for all $2 \leq i \leq n$, it follows that

$$D_s^{-1}(I + te_\alpha)D_s = I + \chi_s(\alpha)^{-1}te_\alpha$$

for all $\alpha \in \Phi$. Finally, since $\Psi'(I + te_\alpha) = x_\alpha(t)$, the equation (2.18.1) is equivalent to

$$x_\alpha(t) = h(\chi_s)x_\alpha(\chi_s(\alpha)^{-1}t)h(\chi_s)^{-1},$$

which is exactly the content of Lemma 2.14, and so the result follows. □

To get the desired description for G, we need to determine $\ker \Psi$.

Lemma 2.19. *The kernel of Ψ consists of the scalar multiples of the identity. In particular,*

$$G \cong \mathrm{GL}_{n+1}(K)/Z(\mathrm{GL}_{n+1}(K)) = \mathrm{PGL}_{n+1}(K).$$

Proof. Let $A = \lambda\mathrm{Id}$ be a multiple of the identity, where $\lambda \in K^*$. Then

$$\Psi(A) = h(\chi_{\lambda^{n+1}})\Psi'(A'),$$

where A' is a diagonal matrix with entries $\lambda^{-n}, \lambda, \ldots, \lambda$. Both $h(\chi_{\lambda^{n+1}})$ and $\Psi(A')$ fix the set of diagonal matrices in $\mathrm{sl}_{n+1}(K)$. In addition,

$$h(\chi_{\lambda^{n+1}})(e_{\alpha_1}) = \lambda^{n+1}e_{\alpha_1} \quad \text{and} \quad \Psi'(A')(e_{\alpha_1}) = A'E_{12}A'^{-1} = \lambda^{-n}\lambda^{-1}E_{12} = \lambda^{-n-1}e_{\alpha_i},$$

while

$$h(\chi_{\lambda^{n+1}})(e_{\alpha_i}) = e_{\alpha_i} \quad \text{and} \quad \Psi'(A')(e_{\alpha_i}) = A'E_{12}A'^{-1} = \lambda\lambda^{-1}E_{12} = e_{\alpha_i} \quad \text{for all } 2 \leq i \leq n.$$

Hence $\Psi(A) = \mathrm{Id}$, as desired.

On the other hand, if $A \in \ker\Psi$. then $h(\chi_s)\Psi'(A') = \mathrm{Id}$ where $A = D_sA'$ for unique $D_s \in H_1$ and $A' \in \mathrm{SL}_{n+1}(K)$. Therefore, $\Psi'(A') = h(\chi_s)^{-1} = h(\chi_{s^{-1}}) \in H' = H \cap G'$ and thus

$$A'e_\alpha A'^{-1} = \chi_{s^{-1}}(\alpha)e_\alpha \quad \text{for all } \alpha \in \Phi.$$

A simple calculation shows that A' must be diagonal, say with diagonal entries $\lambda_1, \ldots, \lambda_{n+1}$. Then, for all $1 \leq i \neq j \leq n+1$, we have that $A'E_{ij}A'^{-1} = \lambda_i\lambda_j^{-1}E_{ij}$ and therefore

$$\lambda_i\lambda_{i+1}^{-1} = \chi_{s^{-1}}(\alpha_i) \quad \text{for all } 1 \leq i \leq n.$$

Since $\chi_s(\alpha_i) = 1$ for all $2 \leq i \leq n$ and $\chi_s(\alpha_1) = s$ we have that

$$s\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_{n+1} =: \lambda$$

and therefore $A = D_sA' = \lambda\mathrm{Id}$. □