Nonabelian Fourier transform for unipotent representations

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December 8, 2025

1 Unipotent representations for finite groups of Lie type

1.1 Frobenius maps

Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a connected reductive group over k. The structure of G can be understood to a large extent by looking at its maximal connected solvable subgroups of G, denoted as Borel subgroups. If we fix some Borel subgroup B, any maximal torus T in B is also a maximal torus in G, and it determines a set of roots $\Phi = \Phi(G,T) \subset X(T)$. The choice of the Borel B containing T corresponds to a choice of positive roots Φ^+ and therefore of an integral basis $\Delta \subseteq \Phi^+$. Moreover, the subgroups $B, N := N_G(T)$ satisfy the axioms of a BN-pair, as described by Tits whose corresponding Weyl group is $W = N/T = \langle w_{\alpha_i} \mid \alpha_i \in \Delta \rangle$.

Let $F : G \to G$ be a Frobenius map and let G^F be the fixed points under the Frobenius map. One can show that G contains F-stable Borel subgroups, and that inside any F-stable Borel there are F-stable maximal tori. Thus, we may assume that the Borel subgroup B and maximal torus T fixed in the previous paragraph are F-stable. Under these assumptions, the Frobenius map acts on the simple roots by permuting the corresponding the root spaces. Thus, F corresponds to some permutation ρ of Δ satisfying

$$F(\mathcal{X}_\alpha) = \mathcal{X}_{\rho(\alpha)} \quad \text{for all } \alpha \in \Delta.$$

Moreover, one can easily check that ρ is in fact a symmetry of the Dynkin diagram, and these can be completely classified. For each orbit $J \subseteq \Delta$ of ρ, let $w_J \in W_J = \{w_{\alpha_i} \mid \alpha_i \in J\}$ be the unique element such that $w_J(J) = -J$. Moreover, it satisfies that $w_J^2 = 1$. In then follows that the group G^F has a natural BN-pair given by the groups B^F and N^F, whose Weyl group is

$$N^F/T^F = (N/T)^F = W^F = \langle w_J \mid J \subseteq \Delta \text{ is an orbit of } \rho \rangle.$$

Any F-stable Borel subgroup contains an F-stable maximal torus, but the converse might not be true. Any F-stable maximal torus that is contained in an F-stable Borel subgroup is called *maximally split*, and since any two F-stable Borel are conjugate under G^F, any two F-stable maximally split tori are also conjugate under G^F. In fact, one can easily determine the G^F-conjugacy classes of F-stable maximal tori by looking at the Weyl group. To state this result, we first introduce the notion of F-conjugacy classes in W. Given two $w_1, w_2 \in W$, we say that they are F-conjugate if there is some $x \in W$ such that $F(x)w_1x^{-1} = w_2$. Note that if F acts on W trivially, then the F-conjugacy classes are the standard conjugacy classes.

Lemma 1.1. *There is a bijection between*

$$\{G^F\text{-conjugacy classes of } F\text{-stable maximal tori}\} \longrightarrow \{F\text{-conjugacy classes of } W\}$$
$$T' = {}^gT \longmapsto \pi(g^{-1}F(g))$$

From now, we will write T_1 for a maximally split F-stable maximal torus and T_w for any F-stable torus obtained from T_1 by conjugating by some element $g \in G$ such that $\pi(g^{-1}F(g)) = w$. By the previous result, these objects are uniquely defined up to G^F-conjugation.

1.2 Deligne–Lusztig characters and unipotent representations

In their groundbreaking paper from 1976, Deligne and Lusztig attached to each pair (T, θ) of F-stable maximal torus T and character θ of T^F, a virtual character $R_{T,\theta}$ of the group G^F. These virtual characters were constructed using the action of G^F on certain ℓ-adic cohomology groups associated to certain Deligne–Lusztig varieties. We shall not consider the explicit definition of the characters, but we will rather recall without proof some important properties.

1. If the pair (T', θ') is obtained from (T, θ) by conjugation on some element of G^F, then $R_{T,\theta} = R_{T',\theta'}$.

2. If T_1 is a maximally split torus inside some F-stable Borel B, then $R_{T_1,\theta} = \theta_{B^F}^{G^F}$, where $\theta_{B^F}^{G^F}$ is the character of the parabolically induced representation $\mathrm{Ind}_{B^F}^{G^F}\theta$.

3. $R_{T,\theta}(u)$ is independent of θ if u is unipotent. We write $Q_T(u)$ for this common value.

4. The orthogonality relations $(R_{T,\theta}, R_{T',\theta'}) = |\{w \in W(T,T')^F \mid {}^w\theta' = \theta\}|$ hold. In particular, if T, T' are not G^F-conjugate, then $(R_{T,\theta}, R_{T',\theta'}) = 0$.

5. If (T, θ) is in general position, then one of $\pm R_{T,\theta}$ is an irreducible character.

6. If (T, θ) and (T', θ') are not geometrically conjugate, then $R_{T,\theta}$ and $R_{T',\theta'}$ do not share any irreducible component. This is a stronger assumption than not being G^F-conjugate.

7. We have
$$(R_{T,\theta}, 1) = \begin{cases} 1 \text{ if } \theta = 1 \\ 0 \text{ if } \theta \neq 1 \end{cases}$$

8. The dimension of $R_{T,\theta}$ equals
$$R_{T,\theta} = \varepsilon_G \varepsilon_T |G^F : T^F|$$

Let's give a couple of examples for the decomposition of the Deligne–Lusztig characters.

Example 1.2. Suppose first that $G = \mathrm{GL}_2(k)$ and $F = F_q : G \to G$ is the standard Frobenius. Then $G^F = \mathrm{GL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 \neq 0 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ while $T_2^F \cong \mathbb{F}_{q^2}^\times$. Now, if $\theta = \theta_1 \otimes \theta_2$ is a character of T_1^F, then

$$R_{T_1,\theta} = \theta_{B^F}{}^{G^F} = \begin{cases} \overline{\theta} \otimes (1 \oplus \mathrm{St}) & \text{if } \theta_1 = \theta_2 \\ \text{irreducible principal series} & \text{if } \theta_1 \neq \theta_2, \end{cases}$$

where $\overline{\theta}$ is the unique extension of θ to all of $\mathrm{GL}_2(\mathbb{F}_q)$ (this is only possible if $\theta_1 = \theta_2$). On the other hand, suppose that θ' is a character of T_w^F. Then

$$R_{T_w,\theta'} = \begin{cases} \overline{\theta} \otimes (1 \ominus \mathrm{St}) & \text{if } \theta'^q = \theta' \\ \text{irreducible cuspidal} & \text{if } \theta'^q \neq \theta', \end{cases}$$

where $\overline{\theta}$ is the extension of the unique character θ of T_1^F for which (θ, T_1) is geometrically conjugate to (θ', T_w).

Example 1.3. Now suppose that $G = \mathrm{SL}_2(k)$ and $F = F_q : G \to G$ to be the standard Frobenius again. Similarly, $G^F = \mathrm{SL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 = 1 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times$ while $T_2^F \cong C_{q+1}$. If θ is a character of T_1^F then

$$R_{T_1,\theta} = \theta_{B^F}{}^{G^F} = \begin{cases} 1 \oplus \mathrm{St} & \text{if } \theta = 1, \\ R_+(\xi) \oplus R_-(\xi) & \text{if } \theta = \xi = \mathrm{sgn}, \\ \text{irreducible principal series} & \text{if } \theta \neq \theta^{-1}, \end{cases}$$

where $R_+(\xi) \neq R_-(\xi)$ are conjugate under $\mathrm{GL}_2(\mathbb{F}_q)$ and so they have the same dimension $(q+1)/2$. On the other hand, if θ' is a character of T_w^F, then

$$R_{T_w,\theta'} = \begin{cases} 1 \ominus \mathrm{St} & \text{if } \theta' = 1, \\ \ominus R'_+(\xi) \ominus R'_-(\xi) & \text{if } \theta' = \xi = \mathrm{sgn}, \\ \ominus \text{ irreducible cuspidal} & \text{if } \theta' \neq \theta'^{-1}, \end{cases}$$

One can show that for a reductive group G over k with centre Z and semisimple rank l, there are exactly $|Z^F|q^l$ geometric conjugacy classes of pairs (T, θ). Moreover, one can define a geometric conjugacy on the set of irreducible characters of G^F as follows. We say that two characters χ_1, χ_2 are related if there are geometrically conjugate pairs (T, θ) and (T', θ') such that

$$(\chi_1, R_{T,\theta}) \neq 0 \quad \text{and} \quad (\chi_2, R_{T',\theta'}) \neq 0.$$

Clearly, there are then $|Z^F|q^l$ geometric conjugacy classes of characters. We are now ready to give the characterization of a semisimple character and a unipotent one.

Definition 1.4. An irreducible character χ of G^F is called *unipotent* if there is some maximal F-stable torus T of G such that $(R_{T,1}, \chi) \neq 0$. An irreducible character χ of the group G^F is called *semisimple* if

$$\sum_{\substack{u \in G^F \\ u \text{ reg unipotent}}} \chi(u) \neq 0.$$

It is clear from the definitions that unipotent characters form one geometric conjugacy class of irreducible characters, and that if χ is a unipotent character, then $(\chi, R_{T,\theta}) = 0$ for any $\theta \neq 1$. Semisimple characters, on the other hand, have the opposite property. To explain this, we define the class function Ξ to be supported on regular unipotent elements with constant value of $|Z^F|q^l$. By using properties of character duality, one can show that $(\Xi, \Xi) = |Z^F|q^l$ and that $(\Xi, \chi) \in \{-1, 0, 1\}$ for all irreducible characters χ of G^F. Note that this implies that there are exactly $|Z^F|q^l$ semisimple characters. In fact, one can furthermore show that

$$\Xi = \sum_{\kappa} \varepsilon_\kappa \chi_\kappa^{ss} \quad \text{where } \chi_\kappa^{ss} \text{ is irreducible and} \quad \varepsilon_\kappa \chi_\kappa^{ss} = \sum_{\substack{(T,\theta) \in \kappa \\ \text{mod } G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})},$$

where κ runs over the conjugacy classes of pairs (T, θ). These results show that each geometric conjugacy class contains one unique semisimple irreducible character.

1.3 Jordan decomposition for irreducible characters

Finally, we are ready to describe the *Jordan decomposition for characters.* To simplify the discussion, we shall assume that the centre $(Z(G))^F$ of G^F is connected. The idea is that one can completely understand all characters of a finite group of Lie type by understanding its semisimple representations and unipotent representations of its Levi subgroups. To state it, we first recall that there is a natural bijection between geometric conjugacy classes of (T, θ) and conjugacy classes of semisimple elements in the dual group $(G^*)^F$, both sets having size $|Z(G)^F|q^l$.

Definition 1.5. Let (s) be a semisimple conjugacy class of the dual group $(G^*)^{F^*}$. Then the *Lusztig series* $\mathcal{E}(G^F, (s))$ associated to (s) is the set of irreducible characters of G^F appearing in $R_{T,\theta}$ for some pair (T, θ) corresponding to (s).

The Lusztig series $\mathcal{E}(G^F, (s))$ are the geometric conjugacy classes of characters defined in the previous section. If (s) is regular semisimple, then (T, θ) is in general position and $\mathcal{E}(G^F, (s))$ is a singleton. On the other end, the series $\mathcal{E}(G^F, (1))$ contains the unipotent characters.

Theorem 1.6. *Let (s) be a semisimple conjugacy class of $(G^*)^F$ and let H be the dual group of the centralizer $Z_{G^*}(s)$. Then there is a bijection*

$$\mathcal{E}(G^F, (s)) \to \mathcal{E}(H^F, (1)), \quad \chi \mapsto \chi_u,$$

such that for any pair (T,θ) corresponding to $s \in (G^*)^{F^*}$ and any pair (S,ψ) corresponding to $s \in (H^*)^{F^*}$,

$$(\chi, \varepsilon_G \varepsilon_T \cdot R^G_{T,\theta}) = (\chi_u, \varepsilon_H \varepsilon_S \cdot R^H_S(\psi)).$$

In addition, the unique semisimple character $\chi_s \in \mathcal{E}(G^F,(s))$ corresponds to the trivial character of H^F and for any $\chi \in \mathcal{E}(G^F,(s))$, we have that

$$\chi(1) = \chi_s(1)\chi_u(1).$$

To summarize, for any irreducible character χ, there is one unique semisimple character χ_s geometrically conjugate to χ, corresponding to some semisimple conjugacy class of $(G^*)^{F^*}$. One can in fact show that

$$\chi_s(1) = |(G^*)^{F^*} : C^{F^*}|_{p'},$$

where C is the centralizer of s^* in G^*. Finally, there is a natural bijection $\chi \mapsto \chi_u$ between characters in the class containing χ_s and unipotent characters of the dual group of C^{F^*} satisfying

$$\chi(1) = \chi_s(1)\chi_u(1).$$

As it turns out, studying semisimple characters is easy since we have explicit formulas to understand them. So Lusztig turned his attention into understanding unipotent representations of finite groups of Lie type. Lusztig first observed that the study of unipotent characters of G^F can be reduced to the case when G is simple of adjoint type. That's because every unipotent character appears as a component of some $R_{T,1}$, where $R_{T,1}(g) = \mathcal{L}(g, \mathfrak{B}_w)$. But Z^F acts trivially on $\mathfrak{B}_w$, so it lies in the kernel of every unipotent representation. So G can be assumed to be semisimple, and a simple argument shows that G can be further assumed to be simple.

Secondly, following the same approach as Harish–Chandra to classify irreducible characters of G^F, Lusztig showed the following result.

Proposition 1.7. *Let χ be an irreducible character of G^F.*

1. *There is an F-stable parabolic subgroup P of G with F-stable Levi decomposition $P = LN$ and a cuspidal character ϕ of L^F such that $(\chi, \phi_{P^F}{}^{G_F}) \neq 0$.*

2. *Moreover, the pair (P,ϕ) is unique up to G^F-conjugacy.*

3. *The character χ of G^F is unipotent if and only if ϕ is a unipotent character of L^J.*

Proof. The proof of parts 1. and 2. are classical, so we only give a sketch. We fix some F-stable maximal torus T and some integral basis $\Delta \subset \Phi(G,T)$. For each $J \subseteq \Delta$, let $P_J = L_J U_J$ be the standard F-stable parabolic with standard F-stable Levi L_J. Since any parabolic subgroup of G^F is conjugate to some P_J^F, it is enough to prove the assertions for standard parabolics of G^F.

Let V be the G^F representation affording χ and let $\mathcal{J} = \{J \subseteq \Delta \mid (1_{U_J}, \chi|_{U_J}) \neq 0\} = \{J \subseteq \Delta \mid V^{U_J} \neq 0\}$, which is non-empty since $\Delta \in \mathcal{J}$. If $J \in \mathcal{J}$ is minimal with respect to inclusion, we may write $V^{U_J} = \oplus_{i=1}^k U_i$ as a direct sum of irreducible L_J-representations, all of which are cuspidal. The character ϕ afforded by U_1 satisfies the conditions of 1., and part 2. is contained in Carter 9.1.5.

To prove the last assertion, fix some $J \subseteq \Delta$ and some irreducible character ϕ of L_J. Let (T,θ) be such that $(\phi, R_{T,\theta}^{L_J^F})_{L_J^F} \neq 0$, and let χ be an irreducible component of ${\phi_{P_J^F}}^{G^F}$. Then by Frobenius reciprocity, we have that

$$(\chi|_{P_J^F}, \phi_{P_J^F})_{P_J^F} = (\chi, {\phi_{P_J^F}}^{G^F})_{G^F} \neq 0,$$

and since $\phi_{P_J^F}$ is an irreducible P_J^F representation, we have that

$$(\chi, R_{T,\theta}^{G^F}) = (\chi, {(R_{T,\theta}^{L_J^F})_{P_J^F}}^{G^F})_{G^F} = (\chi|_{P_J^F}, (R_{T,\theta}^{L_J^F})_{P_J^F})_{P_J^F} \neq 0.$$

This calculation, together with the fact that unipotent representations form a geometric conjugacy class yield the last part. □

Therefore, to classify the unipotent characters of G^F, it is enough to determine the cuspidal unipotent representations ϕ of the standard Levi subgroups L_J^F of G^F and then calculate the decomposition of ${\phi_{P_J^F}}^{G^F}$ into irreducible characters. The later task can be achieved by Howlett–Lehrer theory (Carter, §10), while the former was achieved by Lusztig by a case by case analysis. For example, Lusztig showed that if G^F is of classical type, then the number of cuspidal unipotent characters is either 0 or 1.

1.4 Families of unipotent characters

Lusztig further observed that the unipotent characters of G^F naturally form families in a remarkable way. Firstly, he parametrized the principal series unipotent characters with irreducible characters of W by showing that there is a natural bijection

$$\{\text{Irreducible characters of } W\} \longrightarrow \{\text{Irreducible components of } \mathrm{Ind}_{B^F}^{G^F} 1\}$$
$$\phi \longmapsto \chi_\phi.$$

To prove this, we first note that

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1) \cong \mathrm{Hom}_{B^F}(1, \mathrm{Ind}_{B^F}^{G^F} 1|_{B^F}) \cong \bigoplus_{w \in B\backslash G/B} \mathrm{Hom}_{B^F \cap {}^w B^F}(1,1),$$

where in the first step we have applied Frobenius reciprocity and the Mackey decomposition formula for the second one. By the Bruhat decomposition, W is canonically isomorphic to $B\backslash G/B$ and the borel subgroup B gives a natural choice of simple roots, and therefore of simple reflections $S \subset W$.

If we let $T_w \in \mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ be the image of the identity map in $\mathrm{Hom}_{B^F \cap {}^w B^F}(1,1)$, then one can prove that $\{T_w : w \in W\}$ is a basis for $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ satisfying

$$\begin{aligned} T_s^2 &= (q-1)T_s + qT_1 \quad && \text{if } s \in S, \\ T_{w_1}T_{w_2} &= T_{w_1 w_2} \quad && \text{if } l(w_1 w_2) = l(w_1) + l(w_2). \end{aligned}$$

And therefore, $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ is isomorphic to the coxeter algebra $\mathcal{H}(W, S, q)$ of the pair (W, S) with constant parameter q. It is possible to do a change of variables that give the important isomorphism

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1) \cong \mathbb{C}[W].$$

The algebra $\mathbb{C}[W]$ acts on itself by left multiplication, and the irreducible submodules are precisely the irreducible representations of W. By the isomorphism above, $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F}1)$ also decomposes into a direct sum of irreducible submodules under post composition. If $\mathrm{Ind}_{B^F}^{G^F}1 = \oplus_{i=1}^k V_i^{a_i}$ is a direct sum into G^F irreducible components, then

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F}1) = \bigoplus_{i=1}^{k} \mathrm{Hom}_{G^F}(V_i, \oplus_{j=1}^k V_j^{a_j})^{a_i},$$

and the modules on the right hand side are precisely the irreducible submodules, each one corresponding to one unique irreducible component of $\mathrm{Ind}_{B^F}^{G^F}1$. Thus, irreducible characters of W parametrize principal series cuspidal characters of G^F.

Example 1.8. Let G be a reductive group of type A_l. Then G^F has no cuspidal unipotent representations. Consequently, all unipotent representations of G^F are in the principal series. By the above discussion, this means that the irreducible characters of W completely parametrize all unipotent representations of G^F. Explicitly, given some irreducible character ϕ of W,

$$\chi_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1}$$

For $G = \mathrm{GL}_2(k)$ or $\mathrm{SL}_2(k)$, $\chi_1 = 1$ and $\chi_{\mathrm{sgn}} = \mathrm{St}$.

In general, however, finite groups of Lie type do have cuspidal unipotent characters, and the virtual characters

$$R_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1}$$

as defined above are not irreducible. Lusztig then divided unipotent representations of G^F into families by the rule that two unipotent characters appearing in the same R_ϕ are in the same family and then extending by transitivity. Similarly, we can define an equivalence relation on the irreducible characters of W by the rule that two characters ϕ_1, ϕ_2 are related if R_{ϕ_1} and R_{ϕ_2} share an irreducible component. It is clear that there is a bijection between families of unipotent characters of G^F and families of characters of W. Remarkably, Lusztig proved that these families can be parametrized in the following manner.

Theorem 1.9. *For each family of unipotent representations $\mathcal{F}$ there is a group $\Gamma = \Gamma_{\mathcal{F}} \in \{1, C_2 \times \cdots \times C_2, S_3, S_4, S_5\}$ and a bijection*

$$\begin{aligned} M(\Gamma) &\longrightarrow \mathcal{F} \\ (x,\sigma) &\longmapsto \chi^{\mathcal{F}}_{(x,\sigma)} \end{aligned}$$

satisfying that

$$(\chi^{\mathcal{F}}_{(x,\sigma)}, R_\phi) = \begin{cases} \{(x,\sigma),(y,\tau)\} & \text{if } \chi_\phi = \chi^{\mathcal{F}}_{(y,\tau)} \in \mathcal{F}, \\ 0 & \text{if } \chi_\phi \notin \mathcal{F}. \end{cases}$$

Since $R_{T_1,1} = \sum_{\phi\in\hat{W}} R_\phi$, it follows that for any family $\mathcal{F}$, $(\chi^{\mathcal{F}}_{(1,1)}, R_{T_1,1}) > 0$, so $\chi^{\mathcal{F}}_{(1,1)} = \chi_\phi$ for some character ϕ of W. Characters arising this way are called *special characters* of W and they have distinct characterizations. They are the distinguished elements of the families of characters of W as described above. The upshot of this discussion is that families of unipotent characters can be parametrized by special characters of the Weyl group.

Example 1.10. Let $G = G_2(k)$ and let $F = F_q : G \to G$ be the standard Frobenius. Then $G^F = G_2(\mathbb{F}_q)$, whose Weyl group W is isomorphic to D_{12}. Following Carter, we label the six irreducible representations by $\phi_{1,0}, \phi'_{1,3}, \phi''_{1,3}, \phi_{1,6}, \phi_{2,1}, \phi_{2,2}$, where the first subindex gives the dimension and $\phi_{1,0} = 1$ and $\phi_{1,6} = \det$. The special characters are $\phi_{1,0}, \phi_{1,6}, \phi_{2,1}$ and the families are

$$(\phi_{1,0}), (\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}), (\phi_{1,6}).$$

On the other hand, G_2 has 10 unipotent characters, 6 of which are principal series and 4 are cuspidal. The principal series way can have the same labels as the irreducible characters of W, while the unipotent cuspidal are labelled by $G_2[-1], G_2[\theta], G_2[\theta^2], G_2[1]$. They fall into three families, parametrized as follows.

Description in terms of cuspidal characters	Degree	Pair (x, σ)
$\phi_{1,0}$	1	
$\phi_{2,1}$	$\frac{1}{6}q\Phi_2{}^2\Phi_3$	$(1, 1)$
$G_2[1]$	$\frac{1}{6}q\Phi_1{}^2\Phi_6$	$(1, \varepsilon)$
$\phi_{2,2}$	$\frac{1}{2}q\Phi_2{}^2\Phi_6$	$(g_2, 1)$
$\phi'_{1,3}$	$\frac{1}{3}q\Phi_3\Phi_6$	$(1, r)$
$\phi''_{1,3}$	$\frac{1}{3}q\Phi_3\Phi_6$	$(g_3, 1)$
$G_2[-1]$	$\frac{1}{2}q\Phi_1{}^2\Phi_3$	(g_2, ε)
$G_2[\theta]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ)
$G_2[\theta^2]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ^2)
$\phi_{1,6}$	q^6	

Finally, Lusztig defined a nonabelian Fourier transform on the set of irreducible characters of G^F. For each family $\mathcal{F}$ of unipotent characters parametrized by the group Γ, he considered the $|M(\Gamma)| \times |M(\Gamma)|$ matrix, whose $((x, \sigma), (y, \tau))$ entry is the value

$$\{(x, \sigma), (y, \tau)\} = \frac{1}{|C_\Gamma(x)||C_\Gamma(y)|} \sum_{\substack{g \in \Gamma \\ xgyg^{-1} = gyg^{-1}x}} \sigma(gyg^{-1})\overline{\tau(g^{-1}xg)}.$$

Then he proved that this matrix is Hermitian and that it squares to the identity. It therefore induces an involution on the space $\mathbb{C}[G^F]^{G^F}_{\mathcal{F}}$ of class functions spanned by the characters in $\mathcal{F}$, where we take the natural basis $\{\chi^{\mathcal{F}}_{(x,\sigma)} \mid (x, \sigma) \in M(\Gamma)\}$. Combining for each family, this gives an involution

$$R : \mathbb{C}_{un}[G^F]^{G^F} \longrightarrow \mathbb{C}_{un}[G^F]^{G^F}$$

on the space $\mathbb{C}_{un}[G^F]^{G^F}$ of class functions spanned by unipotent characters. This forces, for example, that $R(\chi_\phi) = R_\phi$ for all characters of W. The involution R transforms unipotent characters into *unipotent almost characters* that also satisfy the orthogonality relations and have a geometrical significance. By this we mean that every almost character agrees up to a scalar with a characteristic function associated to an F-stable character sheaf on G^F (see Shoji's article).

Example 1.11. If G^F is of type A_l, then the unipotent characters coincide with the almost characters.

Example 1.12. If G^F is of type G_2, then R fixes the characters $\phi_{1,0}$ and $\phi_{1,6}$ but transforms the third family according to the Fourier transform matrix

If $\Gamma \cong S_3$ the 8×8 Fourier transform matrix is:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} (1,1) & (1,r) & (1,\varepsilon) & (g_2,1) & (g_2,\varepsilon) & (g_3,1) & (g_3,\theta) & (g_3,\theta^2) \end{array} \\
\begin{array}{c} (1,1) \\ (1,r) \\ (1,\varepsilon) \\ (g_2,1) \\ (g_2,\varepsilon) \\ (g_3,1) \\ (g_3,\theta) \\ (g_3,\theta^2) \end{array} &
\begin{bmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\end{array}
$$

The almost characters satisfy certain *stability properties*. Search what do they exactly mean by this. The aim of the next chapter is to discuss a lift of this map for p-adic groups.

2 Structure theory and representations of p-adic groups

2.1 The Bernstein decomposition

Let F be a nonarchimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k of cardinality q, a power of a prime p. Let $\mathbf{G}$ be a connected, almost simple, split algebraic group over F and let $G = \mathbf{G}(F)$. We denote by $\mathrm{Rep}(G)$ the set of smooth admissible complex representations of G. We begin this chapter by discussing a fundamental result that instrumental in the study of the category $\mathrm{Rep}(G)$. The starting point is the following well-known fact.

Proposition 2.1. *Let (π, V) be an irreducible smooth representation of G. Then there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal representation (σ, M) of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$. Moreover, if P' is another parabolic subgroup with Levi subgroup M and supercuspidal representation (σ', W') such that $\pi \hookrightarrow \mathrm{Ind}_{P'}^{G'} \sigma'$, then there exists $g \in G$ such that $M' = gMg^{-1}$ and $\sigma' \cong {}^g\sigma$.*

Given an irreducible smooth representation (π, V), we denote the G-conjugacy class of $(M, (\sigma, W))$ as above the *supercuspidal support* of (π, V).

The Bernstein decomposition naturally arises when we study whether two irreducible representations with distinct supercuspidal support can have non-trivial extensions between them. This is indeed possible, but only to a very limited extent.

Lemma 2.2. *Let $(\pi, V), (\pi', V')$ be two irreducible representations with supercuspidal support (M, σ), (M', σ'), respectively. If there is a non-trivial extension between V and V', then there exists $g \in G$ and an unramified character χ of $M'(F)$ such that $M' = gMg^{-1}$ and $\sigma' \cong {}^g\sigma \otimes \chi$.*

If the conclusion of the lemma is satisfied, then we say that the pairs (M, σ) and (M', σ') are *inertially equivalent*, and we denote the equivalence by $\sim$ and the inertial equivalence class by $[M, \sigma]_G$. Finally, we let $\mathfrak{J}(G)$ be the set of inertial equivalence classes.

If $[M, \sigma] \in \mathfrak{J}(G)$, then we denote $\mathrm{Rep}(G)_{[M,\sigma]}$ the full subcategory of $\mathrm{Rep}(G)$ whose objects are representations (π, V) satisfying that all for any irreducible subquotient π' of π, there is a parabolic subgroup P' with Levi subgroup M' and supercuspidal representation σ' of M' such that $\pi' \hookrightarrow \mathrm{Ind}_{P'}^{G'}$ and $(M', \sigma') \in [M, \sigma]_G$.

These results are summarized in the following theorem.

Theorem 2.3 (Bernstein decomposition). *We have an equivalence of categories*

$$\mathrm{Rep}(G) \cong \prod_{[M,\sigma] \in \mathfrak{J}(G)} \mathrm{Rep}(G)_{[M,\sigma]} \tag{1}$$

and each full subcategory $\mathrm{Rep}(G)_{[M,\sigma]}$ is indecomposable.

2.2 The apartment of a split maximal torus

Before continuing with representation theoretic aspects of p-adic groups, we first shift towards a more structural focus. As in the previous section, let $\mathbf{G}$ be a connected, almost simple, split algebraic group over F and let

$G = \mathbf{G}(F)$. Let T be a split maximal torus of G over F, and let $X^*(T)$ (resp. $X_*(T)$) be its character (resp. cocharacter) lattice and let

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

the natural perfect pairing between characters and cocharacters of T. Let $\Phi(G,T) \subset X^*(T)$ be the set of roots associated to T, with the corresponding set of coroots $\Phi^\vee(G,T) \subset X_*(T)$. We recall from the previous chapter that a choice of a Borel subgroup B of G containing T is equivalent to the choice of simple roots $\Delta(G,T) = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(G,T)$, which we fix throughout. In addition, the group B together with the normalizer $N := N_G(T)$ form a BN-pair with corresponding Weyl group $W = N(F)/T(F)$.

A natural object arising in the representation theory of G is the apartment $\mathcal{A}(G,T) := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, a real vector space containing all coroots. Moreover, $\mathcal{A}(G,T)$ has the structure of a simplicial complex given by the hyperplanes

$$H_{\alpha,n} = \{x \in \mathcal{A}(G,T) \mid \langle \alpha, x \rangle = n\}, \quad \text{for each } \alpha \in \Phi(G,T)^+ \text{ and } n \in \mathbb{Z}.$$

Whenever the torus T is clear from context, we will omit it from the notation. The complexes on the apartment are called *facets*, and the facets of largest dimension (equivalently, they are open in the apartment) are called *alcoves*. Our choice of simple roots Δ determines a canonical alcove

$$\mathcal{C}_0 = \{x \in \mathcal{A} \mid \langle \alpha, x \rangle > 0, \alpha \in \Delta \ \text{ and } \langle \alpha_0, x \rangle < 1\},$$

commonly referred to as the *fundamental alcove*.

Another important property of the apartment is that it carries a natural action of the group N satisfying

- For any $\alpha \in \Phi$ and $\lambda \in F$, the element $\check{\alpha}(\lambda) \in T \subset N$ acts on $\mathcal{A}$ by a translation $-\nu_p(\lambda)\check{\alpha}$.
- The centre of G acts faithfully and fixes every alcove. maybe important to explain this better?
- For any $\alpha \in \Phi$, the element $w_\alpha(1) \in N$ acts on $\mathcal{A}$ by a reflection along $H_{\alpha,0}$. This coincides with the natural action of W on $\mathcal{A}$.

This action preserves the simplicial structure of the apartment and is transitive on the set of alcoves of $\mathcal{A}$. Moreover, the kernel of this action is $T(\mathcal{O})$ and therefore the *extended Weyl group*

$$\widetilde{W} := N(F)/T(\mathcal{O}) \cong W \ltimes X_*(T)$$

acts faithfully on the apartment $\mathcal{A}$ and transitively on the set of alcoves. We denote by $w_{\alpha,n}$ the unique element in $\widetilde{W}$ acting on $\mathcal{A}$ by a reflection on the hyperplane $H_{\alpha,n}$.

In general, however, this action is not simple on the set of alcoves and the group $\Omega = \{w \in \widetilde{W} \mid w(\mathcal{C}_0) = \mathcal{C}_0\}$ is non-empty. These groups fit in a **splitting** short exact sequence

$$1 \longrightarrow W_{\text{aff}} \longrightarrow \widetilde{W} \longrightarrow \Omega \longrightarrow 1,$$

where $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter group generated by the simple reflections $s_0 := w_{\alpha_0,1}$, $s_i = w_{\alpha_i,0}$, $i = 1, \dots, r$ along the walls of the fundamental alcove $\mathcal{C}_0$ and acting simply transitively on the set of alcoves of $\mathcal{A}$. The group

W_{aff} is the *affine Weyl group* associated to the group G. The Weyl groups W, $\widetilde{W}$ and W_{aff} are independent of T, up to isomorphism.

Example 2.4. 1. Let $G = \mathrm{SL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = t^2 \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \frac{\alpha}{2}\mathbb{Z}$ and $X_*(T) = \check{\alpha}\mathbb{Z}$. Moreover, we have that

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T \quad \text{and} \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right),\ t \in F^\times.$$

The apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. It is easy to check that $\Omega = \{1\}$ so that $\widetilde{W} = W_{\text{aff}}$ is generated by $s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ -\varpi & 0 \end{smallmatrix}\right)$ and $s_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

2. Let $G = \mathrm{PGL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) = t \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \alpha\mathbb{Z}$ and $X_*(T) = \frac{\check{\alpha}}{2}\mathbb{Z}$. Similarly,

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)T, \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right),\ t \in F^\times$$

and the apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. This time, however, $\Omega = \{1, \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)\}$ is non-trivial, and

$$W_{\text{aff}} = \left\langle s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{smallmatrix}\right), s_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle = \{w \in \widetilde{W} \mid \nu(\det(w)) \text{ is even}\}$$

is an index 2 normal subgroup of $\widetilde{W}$.

3. Let $G = \mathrm{GL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & s \end{smallmatrix}\right) = ts^{-1} \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \quad \text{for any } t \in F^\times.$$

In this case, $\Omega \cong \mathbb{Z}$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ and therefore $W_{\text{aff}} = \langle s_0, s_1 \rangle = \{w \in \widetilde{W} \mid \det(w) \in \mathcal{O}^\times\}$ is a normal subgroup of $\widetilde{W}$ of infinite index.

Some of the behaviour observed in the previous example holds in much greater generality. For example, Ω is an abelian group, and it has finite order if and only if G is a simple group. In that case, Ω is in bijection with the centre of complex dual group $G^\vee(\mathbb{C})$ of G. In particular, if G is simply connected, then Ω is trivial, while Ω has the largest size within the isogeny class when G is adjoint. On the other hand, W_{aff} only depends on the isogeny class, and therefore only on the root system of G.

2.3 The Bruhat-Tits building and parahoric subgroups

It is possible to push idea further and construct the Bruhat-Tits building $\mathcal{B}(G)$, a polysimplial space associated to G that contains $\mathcal{A}(G,T)$ for any F-split maximal torus. This is achieved by gluing together the apartments of all F-split maximal tori of G and then gluing them according to some equivalence relation. An important property of the building $\mathcal{B}(G)$ is that it carries a G-action satisfying the following:

1. It extends the action of $N_G(T)$ on $\mathcal{A}(G,T)$ for each F-split maximal torus T.

2. The stabilizer of $\mathcal{A}(G,T)$ is $N_G(T)$ for each F-split maximal torus T.

3. The stabilizer of any facet c of the building is a (maybe disconnected) open compact subgroup of G.

4. The action is strongly transitive on the set $\{(\mathcal{C},\mathcal{A}) \mid \mathcal{C} \text{ is an alcove inside the apartment } \mathcal{A}\}$.

5. For any pair $(\mathcal{C},\mathcal{A})$ as above, its stabilizer acts on $\mathcal{B}(G)$ as the group Ω. In other words

$$\mathrm{Stab}_G(\mathcal{C},\mathcal{A})/T(\mathcal{O}) = (N \cap \mathrm{Stab}_G(\mathcal{C}))/T(\mathcal{O}) = \mathrm{Stab}_N(\mathcal{C})/T(\mathcal{O}) \cong \Omega$$

Example 2.5. The Bruhat-Tits building of $G = \mathrm{SL}_2(\mathbb{Q}_p)$ or $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ is an infinite tree all of whose vertex have degree $p+1$. Each infinite line inside the building is an apartment corresponding to a distinct F-split maximal torus of G. Consider the apartment $\mathcal{A}(G,T)$, where T is the group of diagonal matrices, and let $\Delta = \{\alpha\}$ be the simple root as above. Then $\mathcal{C}_0$ is the segment between the vertices 0 and $\check{\alpha}/2$.

If $G = \mathrm{SL}_2(\mathbb{Q}_p)$, then

$$K_1 := \mathrm{Stab}(0) = \mathrm{SL}_2(\mathbb{Z}_p),\ K_2 := \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},\ \text{and } \mathcal{I} := \mathrm{Stab}(\mathcal{C}_0) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

are all connected, open compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ not conjugate to each other. K_0 and K_1 are the unique maximal compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ – in particular, the stabilizer of any vertex of the building is conjugate to either K_0 or K_1. The subgroup $\mathcal{I}$ is called the **Iwahori subgroup**, it is conjugate to the stabilizer of any facet in the building and is of fundamental importance in the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

On the other hand, if $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, then

$$K_1 := \mathrm{Stab}(0) = \mathrm{PGL}_2(\mathbb{Z}_p),\ K_2 := \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}_{\det \in \mathbb{Z}_p^\times}$$

are both connected open compact subgroups and conjugate in $\mathrm{PGL}_2(F)$ by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, so $\mathrm{PGL}_2(F)$ has one unique *connected* maximal compact subgroup up to conjugacy. Correspondingly,

$$\mathrm{Stab}(\mathcal{C}_0) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det \in \mathbb{Z}_p^\times} \bigsqcup \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det \in \mathbb{Z}_p^\times}$$

is a disconnected open compact subgroup, whose identity component is the Iwahori subgroup.

The example above suggests that the connected components of the stabilizers of facets in the building depend in a subtle way on the group Ω. This is indeed the case, and we discuss this connection now. Since $\Omega = \mathrm{Stab}_G(\mathcal{C}_0)$ and $\mathcal{C}_0$ is bounded by hyperplanes corresponding to S_{aff}, there is a natural homomorphism of groups

$$\Omega \longrightarrow \mathrm{Aut}(S_{\mathrm{aff}}).$$

Moreover, all permutations of S_{aff} induced by Ω can be easily seen to preserve the affine Dynkin diagram associated to S_{aff}, and if G is simple of adjoint type, then all such automorphisms of S_{aff} are induced by Ω. This greatly restricts the size of Ω.

Next, fix some *proper* subset $J \subset S_{\text{aff}}$ and consider the *standard* facet

$$c_J = \{x \in \overline{\mathcal{C}_0} \mid \langle \alpha, x\rangle \in \mathbb{Z} \text{ for } \alpha \in J \text{ and } \langle \alpha, x\rangle \notin \mathbb{Z} \text{ for } \alpha \in S_{\text{aff}} - J\}.$$

Two facets c_{J_1} and c_{J_2} are conjugate under the action of G if and only if J_1 and J_2 lie in the same Ω-orbit. Moreover, any facet c in the building is conjugate to c_J for some proper subset $J \subset S_{\text{aff}}$. In other words, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\text{aff}}\} \longleftrightarrow \{G - \text{orbits of facets } c \text{ in the BT-building}\}$$

For any facet c of the building of G, we let $K_c^+ := \mathrm{Stab}_G(c)$ be the stabilizer of c. There is a short exact sequence

$$1 \longrightarrow U_c \longrightarrow K_c^+ \longrightarrow \overline{K}_c^+ \longrightarrow 1,$$

where U_c is the pro-unipotent radical of K_c^+ and $\overline{K}_c^+$ is the group of k-rational points of a (possibly disconnected) reductive group $\overline{\mathbf{K}}_c^+$ over k.

Definition 2.6. A **parahoric subgroup** K_c is the inverse image in K_c^+ of the group $\overline{K}_c$ of k-rational points of the identity component $\overline{\mathbf{K}}_c$ of $\overline{\mathbf{K}}_c^+$. We shall sometimes denote "parahoric subgroup" to the triple $(K_c, U_c, \overline{K}_c)$. If c is open in the building, then $(K_c, U_c, \overline{K}_c)$ is a minimal parahoric subgroup and is called an **Iwahori subgroup**. The standard Iwahori subgroup corresponds to $J = \emptyset \subsetneq S_{\text{aff}}$.

Naturally, two parahoric subgroups are conjugate in G if and only if the corresponding facets of the building are in the same G-orbit. Thus, all Iwahori subgroups are conjugate in G. If $c = c_J$ is a standard facet, then we simply write $(K_J, U_J, \overline{K}_J)$ for its associated parahoric subgroup, and K_J is generated by the standard Iwahori subgroup and J. Thus, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\text{aff}}\} \longleftrightarrow \{G - \text{conjugacy classes of parahoric subgroups } (K, U_K, \overline{K})\}$$

Moreover, if the facet c corresponds to $J \subsetneq S_{\text{aff}}$, then

$$K_c^+/K_c \cong \Omega_J = \mathrm{Stab}_\Omega(J).$$

These results can be directly verified for SL_2, PGL_2 and GL_2 using the examples above.

Example 2.7. Suppose that $G = G_2(F)$. The affine Dynkin diagram of G_2 has no symmetries, so $\Omega = 1$ and the extended affine weyl group $\widetilde{W}$ is a Coxeter group of type $\tilde{G}_2$. Since $S_{\text{aff}} = s_0, s_1, s_2$, there are 7 conjugacy classes of parahoric subgroups, satisfying

$$\overline{K}_{\{s_1,s_2\}} = G_2(k), \quad \overline{K}_{\{s_0,s_1\}} = \mathrm{SL}_3(k), \quad \overline{K}_{\{s_0,s_2\}} = \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \tag{2}$$

$$\text{what about singletons} \quad \overline{K}_\emptyset = T(k) = (k^\times)^2. \tag{3}$$

2.4 Types for Bernstein blocks and Hecke algebras

In Section 2.1 we stated the Bernstein decomposition, a fundamental result in the complex representation theory of p-adic groups. The upshot of this result a priori is clear – one can restrict attention to each block individually and study the irreducible objects in each block instead of the entire category $\mathrm{Rep}(G)$. In this section, we briefly introduce the notion of types and their corresponding Hecke algebra, which help us understand each individual Bernstein block. We give precise results for so-called *depth-zero Bernstein blocks* which will be required later on. We begin with the definition of a *type*.

Definition 2.8. Let $[M, \sigma] \in \mathfrak{J}(G)$ be a pair parametrizing a Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$. A pair (K, ρ) consisting of an open compact subgroup K of G and a smooth irreducible representation ρ of K is called a $[M, \sigma]$-type if, for any $(\pi, V) \in \mathrm{Irr}(G)$, the following two conditions are equivalent:

- The representation (π, V) lies in the Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$.
- The restriction of π to K contains ρ; in other words, $\mathrm{Hom}_K(\rho, \pi|_K) \neq 0$.

Associated to every pair (K, ρ), where K is a compact open subgroup of G and (ρ, W) is an irreducible smooth representation of K, one can associate the *Hecke algebra*

$$\mathcal{H}(G, K, \rho) := \mathrm{End}_G(c\text{-}\mathrm{Ind}_K^G \rho),$$

with composition of functions as the product. Alternatively, one can show that the Hecke algebra $\mathcal{H}(G, K, \rho)$ can be seen as the $\mathbb{C}$-vector space of functions $f : G \to \mathrm{End}_{\mathbb{C}}(W)$ satisfying

- $f(k_1 g k_2) = \rho(k_1) \circ f(g) \circ \rho(k_2)$ for any $k_1, k_2 \in K$ and $g \in G$.
- the support of f is compact,

together with multiplication given by *convolution* defined by

$$(f_1 * f_2)(g) = \sum_{x \in G(F)/K} f_1(x) f_2(x^{-1} g).$$

The importance of the theory of types in studying individual Bernstein blocks is highlighted in the following theorem.

Theorem 2.9. *Let (K, ρ) be an $[M, \sigma]$-type. Then the Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$ is equivalent to the category of right unital $\mathcal{H}(G, K, \rho)$-modules; i.e.*

$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathcal{H}(G, K, \rho) \text{ - } mod.$$

Of course, such a result is only useful if one can

1. construct types for the Bernstein blocks we are interested in,
2. understand the structure of the corresponding Hecke algebras and

3. describe the irreducible unital right modules of the Hecke algebra.

In most cases, one can answer the three questions giving rise to beautiful, deep and interesting mathematics. One of the main aims of this document is to try to answer these questions. Let us answer these questions first in a particularly simple case.

Example 2.10. Suppose that G is a simple group and that $x \in \mathcal{B}(G)$ is a vertex in building (facet of minimal dimension) and let $(K_x, U_x, \overline{K}_x)$ be the corresponding parahoric subgroup. If σ is an irreducible smooth representation of K_x^+ that is trivial on U_x and such that $\sigma|_{K_x}$ is a cuspidal representation of $\overline{K}_x$, then

$$\pi := c\text{-Ind}_{K_x^+}^G \sigma$$

is a supercuspidal representation of G, and (K_x^+, σ) is a $[G, \pi]$-type. Moreover, by Schur's lemma, we know that

$$\mathcal{H}(G, K_x^+, \sigma) = \mathrm{End}_G(c\text{-Ind}_{K_x^+}^G \sigma) = \mathrm{End}_G(\pi) = \mathbb{C}$$

is a 1-dimensional vector space. This implies that $\mathrm{Rep}(G)_{[G,\pi]}$ has a unique irreducible object, which of course is $\pi = c\text{-Ind}_{K_x^+}^G \sigma$. Moreover, π has no non-trivial extensions, so

$$\mathrm{Rep}(G)_{[G,\pi]} = \{\pi, \pi \oplus \pi, \pi \oplus \pi \oplus \pi, \ldots\}.$$

Before we finish this section, we state the answer of the first question for *depth-zero blocks.*

Definition 2.11. An irreducible smooth representation (π, V) of G has *depth-zero* if there is some parahoric subgroup $(K, U_K, \overline{K})$ such that $V^{U_K} \neq 0$.

For the remainder of the section, assume for simplicity that G is simple. Suppose that M is a Levi subgroup of G and that σ is a depth-zero supercuspidal representation σ of M. A well-known result of Moy and Prasad states that there is a vertex $x \in \mathcal{B}(M) \subseteq \mathcal{B}(G)$ with corresponding stabilizer $K_x^{(M)+}$, parahoric subgroup $(K_x^{(M)}, U_x^{(M)}, \overline{K}_x^{(M)})$ and a cuspidal representation $\tilde{\tau}$ of $\overline{K}_x^{(M)} Z(G)$ such that

$$\sigma \cong c\text{-Ind}_{K_x^{(M)} Z(G)}^G \tilde{\tau}.$$

2.5 Parahoric restriction and unipotent representations

Parahoric subgroups are ubiquitous objects in the representation theory of p-adic objects, since it provides a bridge between smooth admissible representations of the p-adic group G and finite dimensional representations of the finite groups of Lie type $\overline{K}_c$ defined in the previous section. In this section, we explore this important connection that we will exploit in a latter chapter.

If $(K, U_K, \overline{K})$ is any parahoric subgroup corresponding to a facet c and (π, V) is a smooth admissible representation of G, the space V^{U_K} of fixed points under the pro-unipotent radical is naturally a representation of $\overline{K}$. We can take this idea one step further and define the *parahoric restriction functor*

$$\mathrm{res}_K : R(G) \longrightarrow \mathbb{C}[\overline{K}]^{\overline{K}}, \quad V \longmapsto \text{(character of) } V^{U_K}, \quad \text{for all } V \in \mathrm{Irr}(G), \tag{4}$$

where $R(G)$ is the $\mathbb{C}$-span of $\mathrm{Irr}(G)$ and $\mathbb{C}[\overline{K}]^{\overline{K}}$ is the space of class functions of $\overline{K}$. This is well-defined since the representations are assumed to be admissible. The existence of such a functor is very powerful – we can then apply the techniques of representation theory of finite groups of Lie such as Deligne-Lusztig induction in the setting of p-adic groups. Let us begin first with a natural definition.

Definition 2.12. Let $(K, U_K, \overline{K})$ be a parahoric subgroup and (τ, E) be a cuspidal representation of $\overline{K}$. Define

$$\mathrm{Irr}(G, K, E) = \{(\pi, V) \in \mathrm{Irr}(G) \mid \text{the } \overline{K}\text{-module } V^{U_K} \text{ contains the } \overline{K}\text{-module } E\}.$$

Definition 2.13. We say that an irreducible representation (π, V) of G is *unipotent* if there is a parahoric subgroup $(K, U_K, \overline{K})$ such that V^{U_K} contains a cuspidal unipotent representation of $\overline{K}$; that is, if $(\pi, V) \in \mathrm{Irr}(G, K, E)$ for some pair (K, E) where E is unipotent. We denote the set of unipotent representations of G by

$$\mathrm{Irr}_{\mathrm{un}}(G) = \bigcup_{\substack{J \subsetneq S_{\mathrm{aff}} \\ E \text{ cusp. unip. } \overline{K}_J\text{-rep}}} \mathrm{Irr}(G, K_J, E)$$

We note that if we replace the p-adic group G for the finite group of Lie type G^F and *parahoric* by *parabolic*, then we recover the definition of a unipotent representation in G^F.

Are all pairs (K, E) as above types of certain Bernstein blocks?

Example 2.14. Let G be a split reductive p-adic group with split maximal torus T.

1. Let $\mathcal{I}$ be an Iwahori subgroup with pro-unipotent radical $\mathcal{I}^+$. Then the reductive quotient $\mathcal{I}/\mathcal{I}^+$ is isomorphic to $T(k)$. Thus, all irreducible representations of $\mathcal{I}/\mathcal{I}^+$ are 1-dimensional and the only unipotent representation is the trivial one. Therefore, the irreducible *Iwahori-spherical* representations

$$\mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) = \{(\pi, V) \in \mathrm{Irr}(G) \mid V^{\mathcal{I}} \neq 0\}$$

are all unipotent, and this set coincides with the set of irreducible subrepresentations of $c\text{-}\mathrm{Ind}_B^G \chi$, where χ is an unramified character of T. Thus, it follows that

$$\{(\pi, V) \in \mathrm{Rep}(G) \mid V \text{ is generated by } V^{\mathcal{I}}\}$$

is the *principal Bernstein block* $\mathrm{Rep}(G)_{[T,1]}$.

2. Let $(K, U_K, \overline{K})$ be a maximal parahoric subgroup corresponding to a vertex of the building associated to G and let (σ, E) be a cuspidal (not necessarily unipotent) representation of $\overline{K}$ viewed as a representation of K by inflation. Then the compactly induced $(\pi, V) := c\text{-}\mathrm{Ind}_K^G(\sigma, E)$ is an irreducible supercuspidal representation and by Frobenius reciprocity

$$(\pi, V) \in \mathrm{Irr}(G, K, E).$$

In fact, as we shall observe later, we have that $\mathrm{Irr}(G, K, E) = \{(\pi, V)\}$ and consequently (potentially mention type theory briefly) the block

$$\mathrm{Rep}(G)_{[G,\pi]} = \{\pi, \pi \oplus \pi, \pi \oplus \pi \oplus \pi, \ldots\}.$$

Remark 2.15. For $n \geq 1$, reductive groups over finite fields of type A_n have no irreducible cuspidal unipotent representations. Therefore, if G is a reductive p-adic group of type A_n and $J \subseteq S_{\text{aff}}$ is non-empty, then $\overline{K}_J$ has no cuspidal unipotent representations. This implies that the set of irreducible unipotent representations of G

$$\mathrm{Irr}_{\mathrm{un}}(G) = \mathrm{Irr}(G, \mathcal{I}, \mathbf{1})$$

coincides with the irreducible Iwahori-spherical representations of G.

The next step is to ensure that unipotent representations behave under the parahoric restriction functor (4). The following two results ensure this is indeed the case.

Proposition 2.16. *Let (π, V) be an irreducible admissible representation of G. If there is some parahoric subgroup $(K, U_K, \overline{K})$ such that V^{U_K} contains a (potentially non-cuspidal) unipotent representation of $\overline{K}$, then (π, V) is a unipotent representation of G.*

Proof. By assumption, V^{U_K} contains a unipotent irreducible representation (σ, W) of $\overline{K}$ so $\mathrm{Hom}_K(W, V^{U_K}) \neq 0$. By Proposition 1.7, there is some standard parabolic subgroup $\overline{P} = \overline{U_P} \cdot \overline{L_P}$ of $\overline{K}$ and cuspidal unipotent representation (τ, E) of $\overline{L_P}$ such that

$$\mathrm{Hom}_K(W, \mathrm{Ind}_{\overline{P}}^{\overline{K}} E) \neq 0,$$

where we view the $\overline{K}$ representations as inflated K representations, trivial on U_K. By the classification of parahoric subgroups in G, it follows that $\overline{P} = H/U_K$, where $(H, U_H, \overline{H})$ is another parahoric subgroup contained in K. Moreover, we have the inclusions $U_K \subseteq U_H \subseteq H \subseteq K$ and therefore $\overline{U_P} = U_H/U_K$ and $\overline{L_P} = \overline{H} = H/U_H$. Since induction and inflation are commuting operations, it follows that

$$\mathrm{Inf}_{\overline{K}}^{K} \mathrm{Ind}_{\overline{P}}^{\overline{K}} E \cong \mathrm{Ind}_{H}^{K} \mathrm{Inf}_{\overline{H}}^{H} E$$

and hence $\mathrm{Hom}_K(W, \mathrm{Ind}_H^K E) \neq 0$. Since W is irreducible and K is compact, it also follows that

$$\mathrm{Hom}_K(\mathrm{Ind}_H^K E, V^{U_K}) = \mathrm{Hom}_H(E, V^{U_K}) \neq 0.$$

Since the representation E is trivial on U_H, the image of any H-equivariant map $E \to V^{U_K}$ lies inside V^{U_H}. Thus,

$$\mathrm{Hom}_H(E, V^{U_H}) = \mathrm{Hom}_H(E, V^{U_K}) \neq 0,$$

and this concludes the proof. $\square$

Conversely, we would like to show that for any irreducible unipotent representation (π, V) of G, the irreducible $\overline{K}$-submodules of V^{U_K} are all unipotent, for any parahoric subgroup $(K, U_K, \overline{K})$. This is a direct corollary of the following theorem.

Theorem 2.17. *Suppose $I \subsetneq S_{\text{aff}}$ and that V^{U_I} contains the cuspidal unipotent representation σ of $\overline{K}_I$. If $J \subsetneq S_{\text{aff}}$ with $V^{U_J} \neq 0$, and J is minimal with respect to this property, then there is $\omega \in \Omega$ such that $I = \omega J$, and V^{U_J} consists of copies of σ^ω. Moreover, if G is exceptional, then $J = I$.*

Proof. See Moy-Prasad for a complete account for general cuspidal representations and not necessarily unipotent and Reeder's paper for a sketch in the unipotent setting. ☐

Corollary 2.18. *Let (π, V) be a unipotent representation of G and let $(H, U_H, \overline{H})$ be a parahoric subgroup. Then the $\overline{H}$-irreducible components of V^{U_H} are all unipotent.*

Proof. Since (π, V) is unipotent, $(\pi, V) \in \mathrm{Irr}(G, K_J, E)$ for some $J \subsetneq S_{\mathrm{aff}}$ and cuspidal unipotent representation E of $\overline{K}_J$. Let (τ, W) be a $\overline{H}$-irreducible component of π^{U_H}. By conjugating if necessary, we may assume that $(H, U_H, \overline{H})$ is a standard parahoric subgroup. Analogously to the proof of Proposition 2.16, there is some $I \subsetneq S_{\mathrm{aff}}$ such that $(K_I, U_I, \overline{K}_I)$ is contained in K and τ is a subrepresentation of $\mathrm{Ind}_{\overline{K}_I}^{\overline{H}} \sigma$. By Theorem 2.17, I is the same Ω-orbit as J and σ is cuspidal unipotent. By Proposition 1.7, this implies that (τ, W) is also unipotent. ☐

Corollary 2.19. *For any two pairs (K, E), (K', E') of a parahoric subgroup and a cuspidal unipotent representation of the reductive quotient,* $\mathrm{Irr}(G, K, E)$ *and* $\mathrm{Irr}(G, K', E')$ *are either disjoint or equal.*

Analogously to the construction of $R(G)$, we define $R_{\mathrm{un}}(G)$ to be the $\mathbb{C}$-span of the irreducible unipotent representations $\mathrm{Irr}_{\mathrm{un}}(G)$. Lemma 2.16 and Theorem 2.17 implies that for each parahoric subgroup $(K, U_K, \overline{K})$ there is a well-defined *restriction function*

$$\mathrm{res}_{\mathrm{un}}^{K} : R_{\mathrm{un}}(G) \longrightarrow \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}}, \quad V \longmapsto (\text{character of})\ V^{U_K}, \quad \text{for all } V \in \mathrm{Irr}(G).$$

It is also convenient to consider simultaneously all such functions for all conjugacy classes of maximal parahoric subgroups, so we define $\mathrm{res}_{\mathrm{un}}^{\mathrm{par}} = (\mathrm{res}_{\mathrm{un}}^{K})_K$.

2.6 Parahoric restriction for unipotent supercuspidal representations

Let G be the simple p-adic group over F. In this section, we investigate the parahoric restriction of supercuspidal unipotent representations of G (if any) with respect to maximal parahoric subgroups. A well-known result of Moy and Prasad states that any supercuspidal unipotent representation (π, V) of G is obtained by compactly inducting an irreducible smooth representation (ρ, E) of K_x^+, where $x \in \mathcal{B}(G)$ is a vertex, such that $\rho|_{K_x}$ is the inflation of a cuspidal representation of $\overline{K}_x$. By conjugating if necessary, we may assume that x lies in the closure of the fundamental alcove $\mathcal{C}_0$. Explicitly,

$$\pi \cong c\text{-}\mathrm{Ind}_{K_x^+}^{G} \rho,$$

so by Frobenius reciprocity we have that

$$\mathrm{Hom}_{K_x}(\rho|_{K_x}, \pi^{U_x}) \supseteq \text{this should be equality } \mathrm{Hom}_{K_x^+}(\rho, \pi^{U_x}) = \mathrm{Hom}_{K_x^+}(\rho, \pi) = \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_{K_x^+}^{G} \rho, \pi) \cong \mathbb{C},$$

so $(\pi, V) \in \mathrm{Irr}(G, K_x, E)$. If $J = \{\alpha \in S_\alpha \mid \langle \alpha, x \rangle = 0\}$, then $K_x = K_J$ and by cuspidality J is a minimal subset of S_{aff}, up to the action of Ω, such that $\pi^{U_J} \neq 0$. Now let $I \subsetneq S_{\mathrm{aff}}$ be another subset such that $V^{U_I} \neq 0$. If π^{U_I} contains an irreducible cuspidal representation of $\overline{K}_I$ then I is also minimal with respect to $V^{U_I} \neq 0$ and by

Theorem 2.17, I and J are in the same Ω-orbit. If π^{U_I} does not contain any irreducible cuspidal representation, then by 1.7, there is some $J' \subset I$ such that $\pi^{U_{J'}}$ contains a cuspidal representation of $\overline{K}_{J'}$ so J and J' lie in the same Ω-orbit, but this is a contradiction since K_J is a maximal parahoric subgroup of G. We have thus shown:

Lemma 2.20. *Let (π, V) be a supercuspidal unipotent representation of G. Then there is one unique Ω-orbit $[J]$ of subsets of S_{aff}, all of which are maximal such that $\pi^{U_I} \neq 0$ if and only if $I \in [J]$.*

Suppose G has type G_2 with simple reflections $S_{\text{aff}} = \{s_0, s_1, s_2\}$. We note that $\Omega = \{1\}$ so Ω-orbits are all singletons. By combining Example 2.7 and Remark 2.15, given $J \subsetneq S_{\text{aff}}$, the reductive quotient $\overline{K}_J$ has cuspidal unipotent representations if and only if $J = J_0 := \{s_1, s_2\}$ or $J = \emptyset$.

In the first case, $K_0 := K_{J_0}$ is the stabilizer of the origin in the apartment $\mathcal{A}(G,T)$ and $\overline{K_0} = G_2(\mathbb{F}_q)$ has 4 cuspidal unipotent representations labelled $G_2[1], G_2[-1], G_2[\theta]$ and $G_2[\theta^2]$, where θ is a primitive third root of unity. For any of these representations σ, Example 2.14 shows that the compactly induced representation $\pi = c\text{-Ind}_{K_0}^G \sigma$ is irreducible and supercuspidal, and

$$\mathrm{Irr}(G, K_0, \sigma) = \{\pi\}.$$

In the second case, $K_\emptyset = \mathcal{I}$ is the standard Iwahori subgroup (stabilizer of the fundamental alcove) and the only cuspidal unipotent representation of I/U_I is the trivial character. Therefore,

$$\mathrm{Irr}_{\text{un}}(G) = \mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) \bigcup \{c\text{-Ind}_{K_0}^G G_2[1], c\text{-Ind}_{K_0}^G G_2[-1], c\text{-Ind}_{K_0}^G G_2[\theta], c\text{-Ind}_{K_0}^G G_2[\theta^2]\}.$$

In the next section, we shall describe a natural way to parametrize this family. We shall now investigate the parahoric restriction of these representations with respect to the *maximal parahoric subgroups* K_0, $K_1 := K_{\{\alpha_0,\alpha_2\}}$ and $K_2 := K_{\{\alpha_0,\alpha_1\}}$.

Firstly, consider the case $\pi = c\text{-Ind}_{K_0}^G \sigma$ for a cuspidal unipotent representation σ of $G_2(\mathbb{F}_q)$. By Frobenius reciprocity, it follows that $\pi^{U_{K_0}} = \sigma \neq 0$ and therefore by Theorem 2.17, the set $J_0 = \{\alpha_1, \alpha_2\}$ is minimal with respect to the property that $V^{U_J} \neq 0$. Suppose for a contradiction that $V^{U_{J_i}} \neq 0$ for $i = 1$ or $i = 2$, where $J_1 := \{\alpha_0, \alpha_2\}$ and $J_2 := \{\alpha_0, \alpha_1\}$. Since J_1 or J_2 cannot be minimal with respect to the same property, then $V^{U_{\{\alpha_0\}}} \neq 0$. But $\overline{K}_{\{\alpha_0\}}$ has no cuspidal unipotent representations, so $V^{K_\emptyset} = V^I \neq 0$, a contradiction to Corollary 2.19.

2.7 The Langlands parametrization of unipotent representations

In this section, we give an overview on the Langlands parametrization of unipotent representations achieved by Lusztig in his celebrated paper of 1995. Firstly, we briefly discuss the results of Kazhdan–Lusztig on the parametrization of Iwahori-spherical representation when G is a p-adic reductive group of *adjoint* type. Throughout, let $G^\vee$ be complex dual group of G.

We recall that the irreducible Iwahori-spherical representations are in bijection with the irreducible modules of $\mathcal{H}_{\mathcal{I}} = \mathcal{H}(G, \mathcal{I}, \mathbf{1})$. Let $\mathcal{B}$ be the variety of Borel subgroups of $G^\vee$ and let

$$\mathcal{Z} = \{(B, u, B') \in \mathcal{B} \times G^\vee \times \mathcal{B} : u \in B \cap B' \text{ unipotent}\}$$

be the Steinberg variety of G, playing a main role in the representation theory of $\mathcal{H}_{\mathcal{I}}$. Importantly, $G^\vee \times \mathbb{C}^\times$ acts on $\mathcal{Z}$ by

$$(g, \lambda)(B, u, B') = (gBg^{-1}, gu^{\lambda^{-1}}g^{-1}, gB'g^{-1}).$$

This action gives rise to the K-group $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$, which is naturally a $\mathbb{C}[z, z^{-1}]$-module and satisfies

$$K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z}) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C}_q \cong \mathcal{H}(G, \mathcal{I}, q). \tag{5}$$

Thus, we want to construct the $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-modules and then specialize to $\mathcal{H}$-modules via (5). This is performed most naturally with Borel-Moore homology.

Let $t \in G^\vee$ be semisimple and let $u \in G^\vee$ be unipotent such that $tut^{-1} = u^q$ and let $\mathcal{B}^{t,u} \subset \mathcal{B}$ be the subvariety of Borel subgroups containing t and u. Then it turns out that $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ is naturally a $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-module, usually reducible. Since these constructions are compatible with conjugation by elements of $G^\vee$, the group $Z_{G^\vee}(t, u)$ acts on $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ by $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-intertwiners. In fact, the neutral component of $Z_{G^\vee}(t, u)$ acts trivially, so we may regard it as an action of the component group $\pi_0(Z_{G^\vee}(t, u))$. This action can be used to decompose $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ as follows:

For each irreducible representation ρ of $\pi_0(Z_{G^\vee}(t, u))$ appearing in $H_*(\mathcal{B}^{t,u}, \mathbb{C})$, the space

$$K_{t,u,\rho} := \mathrm{Hom}_{\pi_0(Z_{G^\vee}(t,u))}(\rho, H_*(\mathcal{B}^{t,u}, \mathbb{C}))$$

is a nonzero $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-module, called standard. The data (t, u, ρ) are called *Kazhdan–Lusztig triples* for $(G^\vee, q)$.

Theorem 2.21. *Under the assumption that $G^\vee$ is simply connected, we have that*

1. *For each Kazhdan–Lusztig triple (t, u, ρ), the $\mathcal{H}$-module $K_{t,u,\rho}$ has a unique irreducible quotient $L_{t,u,\rho}$.*

2. *Every irreducible $\mathcal{H}$-module if of the form $L_{t,u,\rho}$ for some Kazhdan–Lusztig triple.*

3. *If (t', u', ρ') is another triple, then $L_{t,u,\rho} \cong L_{t',u',\rho'}$ if and only if there is some $g \in G$ such that $t' = gtg^{-1}$, $u' = gug^{-1}$ and $\rho' = \rho \circ \mathrm{Ad}(g^{-1})$.*

The above theorem is a major result and has many interesting consequences. However, the definition of a Kazhdan–Lusztig triple is slightly awkward since the pair (t, u) does not commute, and consequently the classification of these triples up to G-conjugacy seems hard. Thankfully, this situation can be remedied by considering *Kazhdan–Lusztig triples for* $(G^\vee, 1)$. These are defined analogously to the Kazhdan–Lusztig triples for (G, q) but replacing 1 for q throughout. In particular, the semisimple and unipotent part do commute.

Lemma 2.22. *Let G be a p-adic reductive group over a field F of residue cardinality q and let $G^\vee$ be its complex dual. There exists a bijection*

$$\begin{aligned}\{\textit{Kazhdan–Lusztig triples for } (G,1)\}/G &\longleftrightarrow \mathrm{Irr}(\mathcal{H}(G,\mathcal{I},q)) \\ (t,u,\rho) \qquad &\longmapsto \quad L_{t_q,u,\rho_q},\end{aligned}$$

where and (t_q, u, ρ_q) are obtained from (t, u, ρ) in a prescribed way.

We recall that Kazhdan–Lusztig triples for $(G^\vee, 1)$ are defined to be tuples (t, u, ρ) such that ρ is *an irreducible character of* $\pi_0(Z_{G^\vee}(tu))$ *appearing in* $H_*(\mathcal{B}(t, u), \mathbb{C})$. This begs the question: if ρ does not satisfy this condition, does the triple (t, u, ρ) parametrize a (not Iwahori-spherical) representation of G?

This question was studied and completely resolved by Lusztig in his celebrated paper of 1995. He showed that, in order to get a bijection with all pairs (t, u, ρ) without technical conditions on ρ, one needs to consider a wider family of representations. Firstly, one needs to consider not only representations of G, but also of all of its *pure inner twists*. We let $\mathrm{InnT}^p(G)$ be the set of pure inner twists of G. A well known result states that there is a canonical bijection between the sets

$$\mathrm{InnT}^p(G) \longleftrightarrow H^1(F, \mathbf{G}^*) \longleftrightarrow \mathrm{Irr}(Z_{G^\vee}), \tag{6}$$

$$G' \longmapsto \zeta_{G'} \tag{7}$$

For instance, if G is a simply connected p-adic group, then $Z_{G^\vee} = \{1\}$ and therefore G has no pure inner twists other than itself. Secondly, one needs to consider all unipotent representations, and not just the Iwahori-spherical. The following theorem contains this information.

Theorem 2.23 (The arithmetic-geometric correspondence)**.** *There is an explicit bijection between the sets*

$$\bigcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \longleftrightarrow \mathcal{T}(\sqrt{q}) \longleftrightarrow \mathcal{T}(1),$$

where $\mathcal{T}(v_0)$ is set containing all triples (s, u, ρ) such that

- *$t \in G^\vee$ is semisimple,*
- *$u \in G^\vee$ is unipotent satisfying $tut^{-1} = u^{v_0^2}$,*
- *ρ is an irreducible representation of the group of components of the centralizer group $Z_{G^\vee}(t, u)$.*

For the remaining of the section, we explain how this result fits within the modern framework of the local Langlands correspondence. Let W_F be the Weyl group of the field F with inertia subgroup I_F. Moreover, we set $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$.

Under the assumption that $\mathbf{G}$ is a split group, we have the following important definition.

Definition 2.24. A *Langlands parameter* (or L-parameter) for G is a continuous morphism $\varphi : W_F' \to G^\vee$, where $G^\vee$ denotes the $\mathbb{C}$-points of the dual group of $\mathbf{G}$, and $\varphi((w,1))$ is semisimple for each $w \in W_F$.

In its simplest form, the Local Langlands correspondence (LLC) conjectures the existence of a finite to one map between isomorphism classes of smooth admissible complex representations of G and conjugacy classes of Langlands parameters of G satisfying certain nice properties. Using Theorem 2.23, we will see that the the unipotent representations of G and its pure inner twists correspond to the following Langlands parameters.

Definition 2.25. An L-parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ is called *unipotent* if $\varphi(w,1) = 1$ for any element w of the inertia subgroup I_F of W_F. Such parameters are sometimes called *unramified* Langlands parameters and we denote this set by $\Phi_{\mathrm{un}}(G^\vee)$.

Remark 2.26. For any L-parameter $\varphi : W_F' \to G^\vee$, define the commuting elements $u_\varphi = \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ and $s_\varphi = \varphi(\mathrm{Frob}, \mathrm{Id})$. An application of the Jacobson–Morozov theorem implies that an L-parameter is determined by u_φ and $\varphi|_{W_F}$ up to $G^\vee$-conjugacy. If the L-parameter is, in addition, unipotent, then $\varphi|_{W_F}$ is determined by s_φ. Thus, unipotent L-parameters are parametrized by $G^\vee$ conjugacy classes of pairs (u,s) where $u \in G^\vee$ is unipotent, $s \in G^\vee$ is semisimple and they commute. But this is the same as conjugacy classes of elements of $G^\vee$ (by using the Jordan decomposition). This should be reminiscent of the parametrization of Iwahori-spherical representations in Lemma 2.22.

However, under the LLC correspondence, unramified L-parameters do not parametrize unipotent representations, but rather L-packets of unipotent representations. To get a one to one correspondence, we need to introduce refinements of the L-parameters. Given an L-parameter φ, a natural object of interest is the component group A_φ of centralizer $Z_{G^\vee}(\varphi)$ of the image of φ inside $G^\vee$. We remark that when φ is unipotent, it is determined by the commuting elements s_φ and u_φ and therefore $Z_{G^\vee}(\varphi) = Z_{G^\vee}(s_\varphi u_\varphi)$. This object is completely analogous to the centralizer $Z_{G^\vee}(t,u)$, considered by Kazhdan and Lusztig in the setting of representations of Hecke algebras.

Definition 2.27. An *enhanced pure Langlands parameter* is a pair (φ, ϕ), where $\varphi : W_F' \to G^\vee$ is an L-parameter and ϕ is an irreducible representation of A_φ.

Let us introduce some important notation. Define

$$\Phi^p_{\mathrm{e,un}}(G^\vee) = G^\vee \backslash \{(\varphi, \phi) \mid \varphi \text{ unipotent}, \phi \in \widehat{A_\varphi}\},$$

which by the previous paragraph is in natural bijection with the set

$$G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\},$$

where A_x is the component group of $Z_{G^\vee}(x)$.

In this setting the Local Langlands conjecture predicts a natural bijection

$$\begin{array}{rcccl} \mathrm{LLC}^p_{\mathrm{un}} : G^\vee\backslash\{(x,\phi) \mid x\in G^\vee, \phi\in\widehat{A_x}\} & \longleftrightarrow & \Phi^p_{\mathrm{e,un}}(G^\vee) & \longleftrightarrow & \bigsqcup_{G'\in\mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \\ (x,\phi) & & \longmapsto & & \pi(x,\phi), \end{array}$$

where G' runs over the classes of *pure* inner twists of G.

We distinguish between the distinct pure inner twists by looking at characters of $Z_{G^\vee}$. By (6), each pure inner twist G' naturally corresponds to some character $\zeta_{G'}$ of $Z_{G^\vee}$. Similarly, for any pure enhanced L-parameter (φ,ϕ), the representation ϕ induces a character ζ_ϕ on $Z_{G^\vee_{sc}}$. We say that a pair (φ,ϕ) is G'-relevant if $\zeta_\phi=\zeta_{G'}$, in which case $\pi(x_\varphi,\phi)\in\mathrm{Irr}_{\mathrm{un}}(G')$ if φ is unipotent, and we denote the set of G'-relevant pure enhanced unipotent L-parameters by $\Phi^p_{e,\mathrm{un}}(G)$. It is then clear that

$$\Phi_{e,\mathrm{un}}(G^\vee) = \bigsqcup_{G'\in\mathrm{InnT}(G)} \Phi_{e,\mathrm{un}}(G'),$$

and the LLC predicts that $\Phi_{e,\mathrm{un}}(G')$ parametrizes the set $\mathrm{Irr}_{\mathrm{un}}(G')$ for each $G'\in\mathrm{InnT}(G)$.

Example 2.28. If $\mathbf{G}$ is a simple split *simply connected* algebraic group, then $H^1(F,\mathbf{G}^*)=1$ and therefore there is only one class of pure inner forms of G, namely G itself. Correspondingly, $G^\vee=G^\vee_{\mathrm{ad}}$ and $Z_{G^\vee}$ is trivial. Therefore, the above discussion gives a bijection

$$\mathrm{LLC}^p_{\mathrm{un}} : G^\vee\backslash\{(x,\phi) \mid x\in G^\vee, \phi\in\widehat{A_x}\} \longleftrightarrow \Phi^p_{\mathrm{e,un}}(G^\vee) \longleftrightarrow \mathrm{Irr}_{\mathrm{un}}(G^*).$$

Example 2.29. If $\mathbf{G}$ is a simple split *adjoint* algebraic group, then $H^1(F,\mathbf{G}^*)=H^1(F,\mathrm{Inn}(\mathbf{G}^*))$ so for each inner twist there is one unique pure inner twist. Therefore, from the previous discussion, unipotent enhanced L-parameters are in bijection with the set

$$G^\vee\backslash\{(x,\phi) \mid x\in G^\vee, \phi\in\widehat{A_x}\}, \quad \text{where} \quad A_x=Z_{G^\vee}(x)/Z_{G^\vee}(x)^0,$$

and we have a one-to-one correspondence

$$G^\vee\backslash\{(x,\phi) \mid x\in G^\vee, \phi\in\widehat{A_x}\} \longleftrightarrow \bigsqcup_{G'\in\mathrm{InnT}(G)} \mathrm{Irr}_{\mathrm{un}}(G'), \quad (x,\phi)\longmapsto\pi(x,\phi)$$

2.8 Unipotent conjugacy classes of complex simple groups

In the previous paragraph we stated the unramified local Langlands correspondence, which reduces the classification of unipotent representations of G to the classification of conjugacy classes or $G^\vee$ and the structure of the component group of their centralizer. To understand these, one first studies the classification of unipotent conjugacy classes of $G^\vee$, an interesting problem on its own right that uncovers rich structure inside $G^\vee$.

Define $\mathcal{U}$ to be the set of unipotent elements of $G^\vee$. This can be seen to be a closed irreducible subvariety of $G^\vee$ of dimension $\dim G^\vee - \mathrm{rk}G^\vee$. If $u\in\mathcal{U}$ is a unipotent element, its conjugacy class $C(u)\subset H$ is the orbit of u under the conjugation action of $G^\vee$ on itself. Standard results in the structure theory of unipotent elements inside complex reductive groups state that $G^\vee$ has finitely many conjugacy classes of unipotent elements, and that each class C is a locally closed subvariety of $G^\vee$. Moreover, its closure $\overline{C}$ is the union of (finitely many) unipotent conjugacy classes. In particular, there is one unique unipotent conjugacy class C_{reg} of maximal dimension such that C_{reg} is open and $\overline{C_{\text{reg}}} = \mathcal{U}$. Such unipotent elements are called *regular*, and $\dim Z_{G^\vee}(u) = \text{rk}G^\vee$ for any $u \in C_{\text{reg}}$. The boundary of C_{reg} has dimension $\dim G^\vee - \text{rk}G^\vee - 2$ and contains a unique dense unipotent conjugacy class C_{subreg} of *subregular* unipotent elements such that

$$\overline{C_{subreg}} = \overline{C_{\text{reg}}} - C_{\text{reg}} = \mathcal{U} - C_{\text{reg}}.$$

Similarly, $\dim_{Z_{G^\vee}}(u) = \text{rk}G^\vee + 2$ for any $u \in C_{subreg}$. At the other end, there is the trivial class consisting of $\{1\}$, and this is the only closed conjugacy class. There is one further "canonical orbit", the set of *minimal* unipotent elements $C_{\min}$, with the property that they are contained in the closure of every unipotent conjugacy class except for $\{1\}$.

Beyond these four classes, the structure of $\mathcal{U}$ for a general simple complex group can be complicated. To study it, one can define a partial ordering on the set of unipotent conjugacy classes given by

$$C \leq C' \quad \text{if and only if} \quad \overline{C} \subseteq \overline{C'}.$$

One can then picture this partial order in a diagram, called a *Hasse diagram*, and one has the following generic picture.

orbit	dimension
$\mathcal{O}_{prin}$	$\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})$
|	
$\mathcal{O}_{subreg}$	$\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g}) - 2$
|	
fuzzy structure	
|	
$\mathcal{O}_{min}$	See (4.3.5)
|	
$\{0\}$	0

Example 2.30. If $\mathbf{G}$ is a simple split algebraic group of type G_2, then $G = \mathbf{G}(F)$ is both adjoint and simply connected and consequently there is a bijection

$$G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \text{Irr}_{\text{un}}(G) = \text{Irr}(G, \mathcal{I}, \mathbf{1}) \bigcup \{c\text{-Ind}_{K_0}^G G_2[\alpha] \mid \alpha \in \{1, -1, \theta, \theta^2\}\}.$$

Let us indicate which pairs $(x = su, \rho)$ in the left correspond to the 4 unipotent supercuspidal representations of G_2. Since supercuspidal representations are square-integrable (Is this true?), it is enough to look at the regular and subregular unipotent elements.

- Let $u = u_{\text{reg}}$ be the regular unipotent element. In that case $A_u = 1$ and therefore $s = 1$, $A_x = A_u = 1$ and ρ is the trivial representation. The corresponding representation $\pi(u_{\text{reg}}, \mathbf{1})$ is the Steinberg representation.

- Let $u = u_{\mathrm{sr}}$ be the subregular unipotent element. In that case, $A_u = S_3$ so up to conjugacy, $s \in \{1, g_2, g_3\}$ where g_i is a lift of order i from A_u to $Z_{G^\vee}(u)$. Moreover, $A_{ug_2} = S_3$ and $A_{ug_3} = s_2$. The corresponding table gives the required parametrization.

$\pi(G_2(a_1), 1, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,1)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, r)$	$\phi'_{(1,3)}$	$\epsilon \otimes \mathbf{1}$	ϵ
$\pi(G_2(a_1), g_3, \mathbf{1})$	$\phi_{(1,6)} + \phi''_{(1,3)}$	$\mathbf{1} \otimes \epsilon + \epsilon \otimes \epsilon$	r
$\pi(G_2(a_1), g_2, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,2)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, \epsilon)$	$G_2[1]$	0	0
$\pi(G_2(a_1), g_2, \epsilon)$	$G_2[-1]$	0	0
$\pi(G_2(a_1), g_3, \theta)$	$G_2[\theta]$	0	0
$\pi(G_2(a_1), g_3, \theta^2)$	$G_2[\theta^2]$	0	0

3 Parahoric restriction of Iwahori-spherical representations

Let G be a connected split simple group *of adjoint type* over a p-adic field F. Recall that from the previous section we know that there is a bijection

$$\begin{aligned} \mathrm{LLC}^p_{\mathrm{un}} : G^\vee \backslash \{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} &\longleftrightarrow \bigsqcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \\ (x,\phi) \qquad\qquad &\longmapsto \qquad (\pi(x,\phi), V(x,\phi)). \end{aligned}$$

The aim for this section is to give explicit methods that allow us to compute the $\overline{K}_J$-irreducible modules of V^{U_J} for each maximal $J \subsetneq S_{\mathrm{aff}}$ whenever $V = V(x,\phi)$ is an Iwahori-spherical representation of G. These methods are quite complex and involve deep mathematics – the difficulty resides in the fact that the parametrization of the unipotent representations of p-adic groups and that of finite groups of Lie type are not related in an obvious way.

When the representation $V(x,\phi)$ is Iwahori-spherical, all irreducible $\overline{K}_J$-constituents of $V(x,\phi)^{U_J}$ are principal series representations and can be labelled by the representations of the Weyl group of $\overline{K_J}$. The method we present here consists in two major reduction steps; the first one involves Hecke algebras and the second one affine Weyl groups.

Let us assume first that V is any admissible representation of G and fix some $J \subsetneq S_{\mathrm{aff}}$ such that $V^J \neq 0$. For each irreducible K_J-representation χ trivial on U_J, we want to calculate the value of $\langle \chi, V^{U_J} \rangle_{K_J}$. To do this, we note that there is some parahoric subgroup $(K, U_K, \overline{K})$ contained in $(K_J, U_J, \overline{K}_J)$ and cuspidal representation σ of K trivial on U_K such that

$$\chi^\sigma := \mathrm{Hom}_K(\sigma, \chi) = \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, \chi) \neq 0.$$

Moreover, χ^σ is a naturally a $\mathcal{H}(K_J, K, \sigma)$-module, where $\mathcal{H}(K_J, K, \sigma) \cong \mathrm{End}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma)$ is the subalgebra of functions of $\mathcal{H}(G, K, \sigma)$ supported on K_J. Similarly, the vector space

$$V^\sigma := \mathrm{Hom}_K(\sigma, V)$$

is naturally a $\mathcal{H}(G, K, \sigma)$-module and therefore a $\mathcal{H}(K_J, K, \sigma)$-module by restriction.

Lemma 3.1. *[First reduction] The multiplicity of the simple K_J-module χ in V^{U_J} is given by*

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)}.$$

Proof. Since $U_J \subseteq U$, we have that

$$\begin{aligned} V^\sigma = \mathrm{Hom}_K(\sigma, V) = \mathrm{Hom}_K(\sigma, V^{U_J}) &\simeq \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, V^{U_J}) \\ &\simeq \bigoplus_\eta \langle \eta, V^{U_J} \rangle_{K_J} \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, \eta) \\ &\simeq \bigoplus_\eta \langle \eta, V^{U_J} \rangle_{K_J} \eta^\sigma, \end{aligned}$$

and therefore

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} = \sum_\eta \langle \chi^\sigma, \eta^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} \langle \eta, V^{U_J} \rangle_{K_J} = \langle \chi, V^{U_J} \rangle_{K_J},$$

as desired. □

Example 3.2. Assume that V is an Iwahori-spherical representation of G and let χ be an irreducible K_J-representation trivial on U_J such that

$$\langle \chi, V^{U_J} \rangle_{K_J} \neq 0.$$

Let $(K, U_K, \overline{K})$ and σ be as above, so that

$$\chi^\sigma = \mathrm{Hom}_K(\sigma, \chi) \neq 0.$$

Since σ is irreducible and trivial on U_K it follows by composing both conditions that

$$\langle \sigma, V^{U_K} \rangle_K = \langle \sigma, V^{U_J} \rangle_K \neq 0.$$

However, V is assumed to be Iwahori-spherical, so by Theorem 2.17 we have that $K = \mathcal{I}$ is the Iwahori subgroup and $\sigma = \mathbf{1}$ is the trivial representation. Lemma 3.1 states that

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^{\mathcal{I}}, V^{\mathcal{I}} \rangle_{\mathcal{H}(K_J,\mathcal{I},\mathbf{1})}.$$

Now let us consider the second reduction step. For this one, we let $R = \mathbb{C}(v, v^{-1})$, where v is an indeterminate. We then define $\mathcal{H}(G,K,\sigma)_v$ to be the Hecke algebra defined over R with the same generators and relations as $\mathcal{H}(G,K,\sigma)$, but with q replaced by v^2. The upshot of considering this generic Hecke algebra is that by specializing v we can recover

$$\mathcal{H}(G,K,\sigma)_{\sqrt{q}} = \mathcal{H}(G,K,\sigma), \qquad \mathcal{H}(G,K,\sigma)_1 = \mathbb{C}[\widetilde{W}].$$

Fact: For any simple $\mathcal{H}(G,K,\sigma)$-module E *considered in this document*, there is a $\mathcal{H}(G,K,\sigma)_v$-module E_v such that

$$E \simeq E_v \otimes_R \mathbb{C}, \text{ where } f \in R \text{ acts on } \mathbb{C} \text{ by } f(\sqrt{q}).$$

It then follows that we have a $q = 1$ operation that takes simple modules over Hecke algebras to modules over the corresponding Weyl groups, obtained by setting $v = 1$ in all matrix coefficients of the generic module.

Proposition 3.3 (Second reduction). *Let $J \subsetneq S_{\mathrm{aff}}$ and let $(K, U_K, \overline{K})$ be a parahoric subgroup contained in $(K_J, U_J, \overline{K}_J)$ with cuspidal unipotent $\overline{K}$-representation σ. Then the diagram*

$$\begin{tikzcd}
\mathcal{H}(G,K,\sigma)\text{-}mod \arrow[r, "q=1"] \arrow[d, "\mathrm{Res}_{\mathcal{H}(K_J,K,\sigma)}"'] & \widetilde{W}\text{-}mod \arrow[d, "\mathrm{Res}_{\widetilde{W}_J}"] \\
\mathcal{H}(K_J,K,\sigma)\text{-}mod \arrow[r, "q=1"] & \widetilde{W}_J\text{-}mod
\end{tikzcd}$$

is commutative, and the bottom arrow is an isometry with respect to the usual inner product of character. That is, for any irreducible K_J-module χ, we have that

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} = \langle \chi^\sigma_{q=1}, V^\sigma_{q=1} \rangle_{\widetilde{W}_J}. \tag{8}$$

Need to talk about the connection between $V^{\sigma}_{q=1}$ and cohomology groups of flag varieties. This could also be a good point to talk about the Springer Correspondence!

4 The dual nonabelian Fourier transform for unipotent representation of p-adic groups

It is therefore a natural question to ask whether there exists some function $\mathrm{FT}^\vee : R_{\mathrm{un}}(G) \to R_{\mathrm{un}}(G)$ such that the square

$$\begin{tikzcd} R_{\mathrm{un}}(G) \arrow[r, "\mathrm{FT}^\vee"] \arrow[d, "\mathrm{res}_{\mathrm{un}}^{\mathrm{par}}"] & R_{\mathrm{un}}(G) \arrow[d, "\mathrm{res}_{\mathrm{un}}^{\mathrm{par}}"] \\ \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \end{tikzcd}$$

This question is now unresolved, but partial progress has been achieved. To understand it, we first need to look at the Langlands parametrization of unipotent representations.

4.1 The unipotent elliptic space and the dual nonabelian Fourier transform

With the Langlands parametrization, it is then possible to define a dual Fourier transform on a certain subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$, which we now describe. We first fix some unipotent element $u \in G^\vee$ up to $G^\vee$-conjugacy and we denote Γ_u the reductive part of $Z_{G^\vee}(u)$. We then consider the space of elliptic pairs

$$\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(s,h) \mid s,h \in \Gamma_u \text{ semisimple, } sh = hs \text{ and } Z_{G^\vee}(s,h) \text{ is finite}\}$$

up to Γ_u-conjugacy. Then for each $(s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, we define the virtual representation

$$\Pi(u,s,h) := \sum_{\phi \in \widehat{A_{su}}} \phi(h) \pi(su, \phi).$$

Definition 4.1. The elliptic unipotent representation space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ of G is defined as the $\mathbb{C}$-subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$ spanned by the set $\{\Pi(u,s,h) \mid u \in G^\vee,\ (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}\}$.

On the space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ we can define the *dual* Fourier transform in a natural way.

Definition 4.2. The dual elliptic nonabelian Fourier transform is the linear map satisfying

$$\mathrm{FT}^\vee_{\mathrm{ell}} : \mathcal{R}^p_{\mathrm{un,ell}}(G) \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G) \quad \Pi(u,s,h) \longmapsto \Pi(u,h,s) \quad \text{for all} \quad (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}},\ u \in G^\vee \text{ unipotent.}$$

We are now ready to state the main conjecture of this document.

Conjecture 4.3. Let G be a simple p-adic group. Then the diagram

$$\begin{tikzcd} \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[r, "\mathrm{FT}^\vee_{\mathrm{ell}}"] \arrow[d, "\mathrm{res}_{\mathrm{un}}^{\mathrm{par}}"] & \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[d, "\mathrm{res}_{\mathrm{un}}^{\mathrm{par}}"] \\ \bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}}, \end{tikzcd}$$

commutes, up to certain some roots of unity.

A simple yet important observation is that the conjecture can be verified one *unipotent conjugacy class of* $G^\vee$ *at a time* since the virtual representations $\Pi(u,s,h)$ and the dual elliptic nonabelian Fourier transform preserve the unipotent part of the parametrization. In addition, if G is simply connected, then all maximal open compact subgroups coincide with maximal parahorics and $\mathrm{FT}^{\mathrm{par}}$ fixes each component, so the conjecture can be verified *one maximal parahoric at a time* too.

We first show this is indeed the case when $\mathbf{G}$ is a simple algebraic group of type G_2.

4.2 Type G_2

Example 4.4. Let $\mathbf{G}$ be a simple algebraic group of type G_2, and let $G = G(F)$. Then G is both simply connected and adjoint so it has no pure inner twists other than itself. In addition, G has three maximal parahoric subgroups of types K_0, K_1 and K_2, with reductive quotients of type G_2, A_2 and $A_1 + \tilde{A}_1$, respectively. Thus, commutativity of the above square is equivalent to the commutativity of

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \Big\downarrow \mathrm{res}^{\mathrm{par}}_{\mathrm{un}} & & \Big\downarrow \mathrm{res}^{K_i}_{\mathrm{un}} \\ \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}} & \xrightarrow{\mathrm{FT}^{K_i}} & \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}}, \end{array}$$

for $i = 0,1,2$. Moreover, if $u \in G^\vee$ is unipotent, then $\mathcal{Y}(\Gamma_u)$ is non-empty if and only if $u = u_{\mathrm{reg}}$ is regular and $\Gamma_u = \{(1,1)\}$, or $u = u_{sr}$ is subregular and $\Gamma_u = \{(1,1),(1,g_2),(1,g_3),(g_2,1),(g_2,g_2),(g_3,1),(g_3,g_3),(g_3,g_3')\}$. Therefore, $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ is 9-dimensional, spanned by

$$\begin{cases} \Pi(u_{\mathrm{reg}},1,1) & = \pi(u_{\mathrm{reg}},\mathbf{1}) \\ \Pi(u_{sr},1,1) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) + 2\pi(u_{sr},\mathbf{r}) \\ \Pi(u_{sr},1,g_2) & = \pi(u_{sr},\mathbf{1}) - \pi(u_{sr},\varepsilon) \\ \Pi(u_{sr},1,g_3) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) - \pi(u_{sr},\mathbf{r}) \\ \Pi(u_{sr},g_2,1) & = \pi(u_{sr}g_2,\mathbf{1}) + \pi(u_{sr}g_2,\varepsilon) \\ \Pi(u_{sr},g_2,g_2) & = \pi(u_{sr}g_2,\mathbf{1}) - \pi(u_{sr}g_2,\varepsilon) \\ \Pi(u_{sr},g_3,1) & = \pi(u_{sr}g_3,\mathbf{1}) + \pi(u_{sr}g_3,\theta) + \pi(u_{sr},\theta^2) \\ \Pi(u_{sr},g_3,g_3) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta^2\pi(u_{sr}g_3,\theta) + \theta\pi(u_{sr}g_3,\theta^2) \\ \Pi(u_{sr},g_3,g_3^{-1}) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta\pi(u_{sr}g_3,\theta) + \theta^2\pi(u_{sr}g_3,\theta^2). \end{cases}$$

When $i = 1,2$ and the finite group $\overline{K_i}$ is of type A_2 or $A_1 + \tilde{A}_1$, then FT^{K_i} is the identity map, and therefore it is enough to show that

$$\mathrm{res}^{K_i}_{\mathrm{un}}(\pi(u,s,h)) = \mathrm{res}^{K_i}_{\mathrm{un}}(\Pi(u,h,s))$$

for all $\Pi(u,s,h)$ spanning $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. This is obvious for all cases except for $\pi(u,s,h) = \pi(u_{sr},1,g_2), \pi(u_{sr},1,g_3)$.

5 The p-adic symplectic group

In this chapter, we verify that Conjecture 4.3 holds when $G = \mathrm{Sp}_{2n}(F)$ for some explicit nontrivial examples. In this case, G is a simply connected simple p-adic group, whose complex dual group is $G^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$. In particular, G has no pure inner twists and all maximal compact subgroups are maximal parahoric subgroups. Following the remark after Conjecture 4.3, we will prove the result for specific unipotent conjugacy classes of $\mathrm{SO}_{2n+1}(\mathbb{C})$ and maximal parahoric subgroups.

Firstly, we note that the conjecture is true if u is a regular unipotent element of $G^\vee$. It is well-known that $\dim Z_{G^\vee}(u) = \mathrm{rk}\mathrm{SO}_{2n+1} = n$ and that $\Gamma_u = Z(G^\vee) = 1$ and consequently $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(1,1)\}$. The representation $\Pi(u,1,1) = \pi(u,1,\mathbf{1})$ is fixed by $\mathrm{FT}^{\mathrm{ell}}$ and moreover the Springer correspondence implies that

$$\mathrm{res}_{K_i}\pi(u,1,\mathbf{1}) = \mathrm{St}_{K_i} \quad \text{for any maximal parahoric } K_i \subset G,$$

and this representation is also fixed by each FT^{K_i}, so the conjecture is verified.

5.1 Structure theory of $\mathrm{SO}_{2n+1}(\mathbb{C})$

To progress any further, we need to understand some structural properties of $G^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$ and representation theoretic aspects of the reductive quotients of maximal parahoric subgroups of $G = \mathrm{Sp}_{2n}(F)$, which we discuss now. We recall that

$$\mathrm{SO}_{2n+1} = \{A \in \mathrm{GL}_{2n+1} \mid AJA^T = J\}, \quad \text{where} \quad J = \begin{pmatrix} 1 & & \\ & & I_n \\ & I_n & \end{pmatrix}$$

The diagonal matrices in $\mathrm{SO}_{2n+1}(\mathbb{C})$ are

$$T_n = \left\{ h(a_1,\dots,a_n) := \begin{pmatrix} 1 &&&&&& \\ & a_1 &&&&& \\ && \ddots &&&& \\ &&& a_n &&& \\ &&&& a_1^{-1} && \\ &&&&& \ddots & \\ &&&&&& a_n^{-1} \end{pmatrix} \mid a_1,\dots,a_n \in \mathbb{C}^\times \right\},$$

and they form a maximal torus in $\mathrm{SO}_{2n+1}(\mathbb{C})$. We fix a set of simple roots $\Delta_n = \{\alpha_1,\alpha_2,\dots,\alpha_n\}$ given by

$$\alpha_i(h(a_1,\dots,a_n)) = \begin{cases} a_i a_{i+1}^{-1} & \text{if } 1 \le i \le n-1, \\ a_n & \text{if } i = n. \end{cases}$$

The simple root α_n is the only short one and the corresponding Dynkin diagram is Insert/draw Dynkin diagram. Using the Killing form, we can embed Δ_n in a n-dimensional Euclidean space V with orthonormal basis $e_1,\dots,e_n$ and such that $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = e_n$.

We also let $\alpha_0 := \alpha_1 + \alpha_2 + \cdots + \alpha_n = e_1$ be the highest short root of $\Phi(B_n)$. This is slightly unconventional, since α_0 is normally the highest root of Φ, but this notation will be very useful. For example, $\check{\alpha}_0$ is the highest root of type C_n so $S_{\text{aff}}(C_n) := \{-\check{\alpha}_0, \check{\alpha}_1, \ldots, \check{\alpha}_n\}$ is the set of affine roots of type C_n with corresponding affine Dynkin diagram Insert/draw affine Dynkin diagram.

In particular, $\alpha_0(h(a_1, \ldots, a_n)) = a_1$ and the corresponding simple coroots are

$$\check{\alpha}_i(t) = \begin{cases} h(t^2, 1, \ldots, 1) & \text{if } i = 0, \\ h(1, \ldots, 1, t, t^{-1}, 1, \ldots, 1) & \text{if } 1 \le i \le n-1, \\ h(1, \ldots, 1, t^2) & \text{if } i = n. \end{cases}$$

The Weyl group $W(B_n) = N(T_n)/T_n$ is isomorphic to the group $S_n \ltimes C_2^n$ and it acts on T_n faithfully by the transformations

$$h(a_1, \ldots, a_n) \longmapsto h(a_{\sigma(1)}^{\pm 1}, \ldots, a_{\sigma(n)}^{\pm 1}), \quad \text{where } \sigma \in S_n.$$

For each $i \in \{0, 1, 2, \ldots, n\}$, let $s_i \in W(B_n)$ be the simple reflection associated to the root α_i. Then

$$W(B_n) = \langle s_1, \ldots, s_{n-1} \text{ (long reflections)}, s_n \text{ (short reflection)} \rangle = \langle s_0, s_1, \ldots, s_{n-1} \rangle,$$

and the simple reflections act by

$$s_i \cdot h(a_1, \ldots, a_n) = \begin{cases} h(a_1^{-1}, a_2, \ldots, a_n) & \text{if } i = 0, \\ h(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) & \text{if } 1 \le i \le n-1, \\ h(a_1, \ldots, a_{n-1}, a_n^{-1}) & \text{if } i = n. \end{cases}$$

Moreover, the Euclidean space $V = \oplus_{i=1}^n e_i$ is the Lie algebra of T_n so admits a natural W-action given by

$$s_i(v) = v - \frac{2(\alpha_i, v)}{(\alpha_i, \alpha_i)}\alpha_i, \quad v \in V \quad \text{so} \quad s_i = \begin{pmatrix} I_{i-1} & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_{n-i-1} \end{pmatrix} \text{ for } 1 \le i \le n-1 \quad \text{and} \quad s_n = \begin{pmatrix} I_{n-1} & \\ & -1 \end{pmatrix}$$

and under this action V becomes the natural reflection representation of $W(B_n)$.

Next, we study the structure of unipotent conjugacy classes of $\mathrm{SO}_{2n+1}(\mathbb{C})$. We recall from Section 2.8 that each complex simple group has three canonical classes of unipotent elements: $C_{\text{reg}}, C_{subreg}$ and $C_{\min}$. For $\mathrm{SO}_{2n+1}(\mathbb{C})$, the unipotent conjugacy classes can be parametrized as follows.

Proposition 5.1. *The conjugacy classes of* $\mathrm{SO}_{2n+1}(\mathbb{C})$ *are parametrized by pairs* (λ, μ) *of permutations such that* $2|\lambda| + |\mu| = 2n+1$ *and* μ *has distinct odd parts and no even parts. The regular orbit is parametrized by* $(\emptyset, 2n+1)$*, the subregular orbit by* $(1, 2n-1)$*, the minimal orbit by* $(21^{n-2}, 1)$ *and the zero orbit by* $(1^n, 1)$.

In addition to this result, one can also find nice representatives for each of these orbits. The results are summarized in the following table (need to reference this well).

In the next subsection, we will prove that Conjecture 4.3 holds for the subregular unipotent orbit containing $u = \prod_{i=1}^{n-1} x_{\alpha_i}(1)$. Before we do this, however, we need to discuss the representation theory of parahoric reductive quotients of $\mathrm{Sp}_{2n}(F)$.

Unipotent	(λ,μ)	Dynkin Diagram	$\dim Z_{G^\vee}(u)$	$\dim \mathcal{B}^u$	Γ_u
$\prod_{i=1}^n x_{\alpha_i}(1)$	$(\emptyset, 2n+1)$		n	0	1
$\prod_{i=2}^n x_{\alpha_i}(1)$	$(1, 2n-1)$		$n+2$	1	$\mathbb{C}^\times \rtimes C_2$
...	...	...	...	...	...
$x_{\alpha_1}(1)$	$(21^{n-2}, 1)$		$2n^2-3n+4$	n^2-2n+2	$B_{n-2}\times A_1$
1	$(1^n, 1)$		$2n^2+n$	n^2	SO_{2n+1}

5.2 Unipotent representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$

We recall from Proposition 1.7 that for any unipotent representation χ of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ there exists a parahoric subgroup $P \subseteq \mathrm{Sp}_{2n}(\mathbb{F}_q)$ with Levi decomposition $P = LN$ and a cuspidal unipotent character ϕ of L such that $\chi \hookrightarrow \mathrm{Ind}_P^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\phi$ and that moreover the pair (P,ϕ) are unique up to conjugacy. In the 90s (check this), Lusztig showed that finite groups of Lie type of type $A_m, m \geq 1$ have no cuspidal unipotent representations, while those of type C_m have a unique cuspidal unipotent representation ϕ_s if $m = s^2+s$ for some $s \geq 0$, and none otherwise.

For each $s \geq 0$ such that $m := s^2+s \leq n$, let $P_s = L_sN_s$ be the standard parabolic of type C_m and let ϕ_s be the unique cuspidal unipotent representation of L_s. The unipotent representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ are the components of the representations

$$\mathrm{Ind}_{P_s}^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\phi_s \quad \text{for all } s \geq 0 \text{ satisfying } s^2+s \leq n.$$

For each choice of s, the components of $\mathrm{Ind}_{P_s}^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\phi_s$ define a *series* of unipotent representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ denoted by $\mathcal{E}(\mathrm{Sp}_{2n}(\mathbb{F}_q), P_s, \phi_s)$. Let us focus first on $s = 0$, corresponding to the principal series representations. In this case, $P_0 = B$ is a Borel subgroup, $L_0 = T$ is a maximal torus and $\phi_0 = \mathbf{1}$, so we aim to characterize the components of $\mathrm{Ind}_B^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\mathbf{1}$. In 1.4, we proved that there is a canonical bijection between principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and irreducible representations of its Weyl group $W(C_n) \cong S_n \ltimes C_2^n$. The Weyl group $W(B_n)$ of $\mathrm{SO}_{2n+1}(\mathbb{C})$ is canonically isomorphic to $W(C_n)$ by swapping between short and long simple reflections. Thus irreducible representations of $W(B_n)$ also parametrize principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. We shall abuse notation and denote both types of representations with the same symbol – it will be clear from context whether we refer to principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ or irreducible representations of $W(B_n)$.

Theorem 5.2. *There is a natural one-to-one correspondence between irreducible representations of $W(B_n)$ and ordered pairs of partitions (or bipartitions) (α,β) with $|\alpha|+|\beta| = n$. Let $\alpha = (\alpha_1,\alpha_2,\dots)$ and $\beta = (\beta_1,\beta_2,\dots)$ with*

$$0 \leq \alpha_1 \leq \alpha_2 \dots \quad 0 \leq \beta_1 \leq \beta_2 \dots,$$

where some of the α_i, β_i may be 0. Let α^*, β^* be the dual partitions of α,β. Then consider a subsystem $\Phi' \subseteq \Phi$ of type

$$D_{\alpha_1^*} + D_{\alpha_2^*} + \cdots + B_{\beta_1^*} + B_{\beta_2^*} + \cdots,$$

where D_1 is the empty root system and $D_2 = A_1 \times A_1$ has only long roots. If W' is the Weyl group of Φ', then the corresponding irreducible representation $\phi_{\alpha,\beta}$ of $W(B_n)$ is the Macdonald representation ${j_{W'}}^W(\varepsilon_{W'})$ obtained from W'. Moreover, the dimension of $\phi_{\alpha,\beta}$ is given by the formula

$$\dim \phi_{\alpha,\beta} = \frac{n!}{\mathrm{hook}(\alpha)\mathrm{hook}(\beta)}.$$

Throughout, we will label the representations of $W(B_n)$ by (α, β) instead of $\phi_{\alpha,\beta}$. The above theorem provides a complete classification of the irreducible representations of $W(B_n)$ and, with some work, provides a way to obtain explicit models of each representation a subrepresentation of $\mathrm{Sym}^N(V^*)$ (homogeneous degree N polynomials on V), where $V = \oplus_{i=1}^n \mathbb{C}e_i$ is the natural reflection representation of W described above and N is the number of positive roots of the associated root subsystem Φ'. To simplify notation, let $\{x_1, \dots, x_n\} \subset V^*$ be the dual basis of $\{e_1, \dots, e_n\}$.

Let us discuss some important examples and models of the above theorem.

1. If $(\alpha, \beta) = (n, \emptyset)$, then $\Phi' = \emptyset$ so $N = 0$ and $(n, \emptyset)$ is the trivial representation.

2. If $(\alpha, \beta) = (\emptyset, n)$, then $\Phi' = \{\text{short roots}\}$ so $N = n$ and $(\emptyset, n)$ is the non-trivial character where long reflections act trivially and whose model is given by

$$\mathrm{Span}_{\mathbb{C}}\{x_1 \cdots x_n\} \leq \mathrm{Sym}^n(V^*).$$

3. If $(\alpha, \beta) = (1^n, \emptyset)$, then $\Phi' = \{\text{long roots}\}$ so $N = n^2 - n$ and $(1^n, \emptyset)$ is the non-trivial character where short reflections act trivially and whose model is given by

$$\mathrm{Span}_{\mathbb{C}}\left\{\prod_{1\leq i<j\leq n}(x_i - x_j) \cdot \prod_{1\leq i<j\leq n}(x_i + x_j)\right\} \leq \mathrm{Sym}^{n^2-n}(V^*).$$

4. If $(\alpha, \beta) = (\emptyset, 1^n)$, then $\Phi' = \Phi$ and $W' = W$. Then $N = n^2$ and $(\emptyset, 1^n)$ is the sign representation whose model is given by

$$\mathrm{Span}_{\mathbb{C}}\left\{x_1 \cdots x_n \cdot \prod_{1\leq i<j\leq n}(x_i - x_j) \cdot \prod_{1\leq i<j\leq n}(x_i + x_j)\right\} \leq \mathrm{Sym}^{n^2}(V^*).$$

5. If $(\alpha, \beta) = (n-1, 1)$, then Φ' has type A_1 so $N = 1$ and $(n-1, 1) = \mathrm{Sym}^1 V^* = V^*$ is the natural reflection representation of W.

6. If $(\alpha, \beta) = ((n-1)1, \emptyset)$, then Φ' has type $A_1 \times A_1$ so $N = 2$ and $((n-1)1, \emptyset)$ is the $n-1$-dimensional representation with model

$$\mathrm{Span}_{\mathbb{C}}\{x_1^2 - x_2^2, \dots, x_{n-1}^2 - x_n^2\}.$$

7. If $(\alpha, \beta) = (n-2, 2)$, then Φ' has type $A_1 \times A_1$ so $N = 2$ too and $((n-2), 2)$ is the $n(n-1)/2$-dimensional representation with model

$$\mathrm{Span}_{\mathbb{C}}\{x_i x_j \mid 1 \leq i < j \leq n\}.$$

In particular, by noting that $\mathbb{C}(x_1^2 + \cdots + x_n^2)$ affords the trivial representation and by counting dimensions, this shows that $\mathrm{Sym}^2(V^*) = (n, \emptyset) + ((n-1)1, \emptyset) + (n-2, 2)$.

In addition to the previous examples, we also provide an explicit description of the effect of twisting by the sign representation $\varepsilon = (\emptyset, 1^n)$.

Lemma 5.3. *Let (α, β) be a representation of $W(B_n)$ labelled as in Theorem 5.2 and let α^*, β^* be the dual partitions. Then*

$$\varepsilon \otimes (\alpha, \beta) = (\emptyset, 1^n) \otimes (\alpha, \beta) = (\beta^*, \alpha^*).$$

In particular, if ϕ_1 and ϕ_2 are two characters of the same family, so are $\varepsilon \otimes \phi_1$ and $\varepsilon \otimes \phi_2$.

Lemma 5.4. *The smallest normal subgroup of $W(B_n) = S_n \ltimes C_2^n$ containing a short reflection is isomorphic to C_2^n. The set of short reflections lie in the kernel of a representation (α, β) if and only if $\beta = \emptyset$, and the bijection*

$$\begin{aligned} \mathrm{Irr}(W(A_{n-1})) &\longrightarrow \mathrm{Irr}(W(B_n)) \\ \alpha &\longmapsto (\alpha, \emptyset) \end{aligned}$$

is the natural inflation map of representations.

Next, we wish to understand, using the previous parametrization, the families of the Weyl group as defined in 1.4. To do this, we need to introduce the notion of *symbols* associated to a pair of partitions (α, β). Choose an appropriate number of zeros as parts of α or β so that α has one more part than β. Then define the symbol of (α, β) to be the array

$$\begin{pmatrix} \alpha_1 & & \alpha_2+1 & & \alpha_3+2 & & \cdots & \alpha_m+(m-1) & & \alpha_{m+1}+m \\ & \beta_1 & & \beta_2+1 & & \beta_3+2 & \cdots & & \beta_m+(m-1) & \end{pmatrix}$$

We consider the equivalence relation on the symbols generated by

$$\begin{pmatrix} 0 & & \lambda_1+1 & & \lambda_2+1 & & \cdots & & \lambda_m+1 & & \lambda_{m+1}+1 \\ & 0 & & \mu_1+1 & & \mu_2+1 & \cdots & \mu_{m-1}+1 & & \mu_m+1 & \end{pmatrix}$$
$$\sim \begin{pmatrix} \lambda_1 & & \lambda_2 & & \cdots & & \lambda_m & & \lambda_{m+1} \\ & \mu_1 & & \mu_2 & \cdots & \mu_{m-1} & & \mu_m & \end{pmatrix}.$$

Thus, each pair (α, β) defines a unique equivalence class of symbols.

Theorem 5.5. *Two characters of $W(B_n)$ lie in the same family if and only if they possess symbols for which the unordered sets $\{\lambda_1, \ldots, \lambda_{m+1}, \mu_1, \ldots, \mu_m\}$ are the same. Moreover, each family contains a unique character whose symbol has the property that*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_m \leq \lambda_{m+1},$$

and these are precisely the special characters of $W(B_n)$.

This concludes the classification of principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. To understand the remaining series, we need to consider more general symbols of the form

$$\begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_b \\ & \mu_1 & & \mu_2 & & \mu_3 & \cdots & \mu_b & \end{pmatrix}$$

where $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \ldots < \mu_b$, $a-b$ is odd and positive and λ_1, μ_1 are not both 0. We define the *defect* of such a symbol as $d = a - b$ and its *rank* as

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j - \left[\left(\frac{a+b-1}{2} \right)^2 \right].$$

Theorem 5.6. *Unipotent characters of* $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ *are parametrized in terms of symbols of the form as above whose rank equals* n*. Such a character is a component of* $\mathrm{Ind}_{P_s}^{\mathrm{Sp}_{2n}(\mathbb{F}_q)} \phi_s$*, where* $d = 2s+1$*. Finally, two unipotent characters lie in the same family of characters if and only if their symbols contain the same entries with the same multiplicities.*

5.3 Springer correspondence of $\mathrm{SO}_{2n-1}(\mathbb{C})$

5.4 The subregular unipotent orbit of $\mathrm{SO}_{2n+1}(\mathbb{C})$

In this section, we prove Conjecture 4.3 for the subregular unipotent element. We recall from Proposition 5.1 that the subregular unipotent orbit is parametrized by the bipartition $(1, 2n-1)$. To simplify notation, let $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ be the highest short root of $\Phi(B_n)$. This is slightly unconventional, since α_0 is normally the highest root of Φ, but this notation will be very useful, and its dual root $\check{\alpha}_0$ is the highest root of $\Phi^\vee$.

Lemma 5.7. *The unipotent element* $u = \exp(e_{\alpha_2} + \cdots + e_{\alpha_{n-1}} + e_{\alpha_n})$ *lies in the subregular orbit and the reductive part of its centralizer is* $\Gamma_u = \Gamma_u^0 \rtimes \langle h_u w_{\alpha^0} \rangle$*, where*

$$\Gamma_u^0 = \{(\alpha^0)^\vee(t) \mid t \in \mathbb{C}^\times\} \quad \textit{and} \quad h_u = \begin{pmatrix} 1 & \\ & -I_{2n} \end{pmatrix} = \begin{cases} \check{\alpha}_1(-1)\check{\alpha}_3(-1)\cdots\check{\alpha}_{n-2}(-1)\check{\alpha}_n(i) & \textit{if } n \textit{ is odd,} \\ \check{\alpha}_1(-1)\check{\alpha}_3(-1)\cdots\check{\alpha}_{n-3}(-1)\check{\alpha}_{n-1}(-1) & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. □

The above result implies that there is an isomorphism

$$\Gamma_u \cong \langle z, \delta \mid z \in \mathbb{C}^\times, \delta^2 = 1, \delta z \delta^{-1} = z^{-1} \rangle \cong C^\times \rtimes C_2$$

given by

$$1 \longleftrightarrow t_0 = (\alpha^0)^\vee(\pm 1) = I_{2n+1}, \quad -1 \longleftrightarrow t_1 = (\alpha^0)^\vee(\pm i) = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & I_{n-1} & & \\ & & & -1 & \\ & & & & I_{n-1} \end{pmatrix},$$

$$\delta \longleftrightarrow t_2 = h_u w_{\alpha^0} = \begin{pmatrix} -1 & & & & \\ & & & 1 & \\ & & -I_{n-1} & & \\ & 1 & & & \\ & & & & -I_{n-1} \end{pmatrix}.$$

In particular, we note that $\Gamma_{ut_0} = \Gamma_{ut_1} = \Gamma_u$ so $A_{ut_0} = A_{ut_1} = C_2$ while $\Gamma_{ut_2} = A_{ut_2} = \{t_0, t_1, t_2, t_1 t_2\} = C_2 \times C_2$. We label its representations by $\mathbf{1} \boxtimes \mathbf{1}$, $\mathbf{1} \boxtimes \varepsilon$, $\varepsilon \boxtimes \mathbf{1}$, $\varepsilon \boxtimes \varepsilon$, where the first term indicates the action of t_1 and the second term indicates the action of t_2.

Lemma 5.8. *The group Γ_u has six elliptic pairs up to Γ_u-conjugacy:*

$$\Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(\pm 1, \delta), (\delta, \pm 1), (\delta, \pm\delta)\}.$$

Proof. All elements in Γ_u are semisimple, and any two $\gamma_1, \gamma_2 \in \Gamma_u$ commute if any only if $\gamma_1, \gamma_2 \in \mathbb{C}^\times$ (in which case their common centralizer is infinite) or $(\gamma_1, \gamma_2) \in \{(\pm 1, z\delta), (z\delta, \pm 1), (z\delta, \pm z\delta) \mid z \in \mathbb{C}^\times\}$. It is easy to see that these pairs have finite centralizer and are $\mathbb{C}^\times$-conjugate to $\{(\pm 1, \delta), (\delta, \pm 1), (\delta, \pm\delta)\}$, respectively. $\square$

Thus, we wish to compute the parahoric restriction with respect to maximal compact subgroups of the virtual representations

$$\Pi(u, 1, \delta) = \pi(ut_0, \mathbf{1}) - \pi(ut_0, \varepsilon), \quad \Pi(u, -1, \delta) = \pi(ut_1, \mathbf{1}) - \pi(ut_1, \varepsilon),$$
$$\Pi(u, \delta, 1) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) + \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) + \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) + \pi(ut_2, \varepsilon \boxtimes \varepsilon),$$
$$\Pi(u, \delta, -1) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) + \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) - \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) - \pi(ut_2, \varepsilon \boxtimes \varepsilon),$$
$$\Pi(u, \delta, \delta) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) - \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) + \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) - \pi(ut_2, \varepsilon \boxtimes \varepsilon),$$
$$\Pi(u, \delta, -\delta) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) - \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) - \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) + \pi(ut_2, \varepsilon \boxtimes \varepsilon).$$

We first compute these restrictions for representations having Iwahori-fixed vectors, whose restrictions to $\overline{K}_J$ are a direct sum of principal series representations. The restrictions of the remaining representations will be calculated afterwards.

We recall from Lemma 3.1 and Proposition 3.3 that if $\pi(ut, \rho)$ has Iwahori-fixed vectors and $J \subsetneq S_{\mathrm{aff}} = \{\check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_n, -\check{\alpha}_0\}$, for any irreducible K_J module χ trivial on U_J, we have that

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^{\mathcal{I}}, V^{\mathcal{I}} \rangle_{\mathcal{H}(K_J, \mathcal{I}, \mathbf{1})} = \langle \chi^{\mathcal{I}}_{q=1}, \pi(ut, \rho)^{\mathcal{I}}_{q=1} \rangle_{\widetilde{W}_J}.$$

The operation $\chi \mapsto \chi^{\mathcal{I}}_{q=1}$ corresponds to the aforementioned bijection between principal series representations of $\overline{K}_J$ and irreducible representations of $\widetilde{W}_J$, while

$$\pi(ut, \rho)^{\mathcal{I}}_{q=1} = \varepsilon \otimes \mathrm{Ind}_{\widetilde{W}_t}^{\widetilde{W}} [s \otimes H(\mathcal{B}^u_t)^\rho]$$

as a $\widetilde{W}$-module, up to semisimplification. It is convenient to work inside the Weyl group $W(B_n)$ and there is a well canonical isomorphism $\psi_J : W_J \to \widetilde{W}_J$ for some $W_J \leq W$. Following Reference M.Reeder, we obtain that

$$\psi_J^* \left(\pi(ut, \rho)^{\mathcal{I}}_{q=1}|_{\widetilde{W}_J} \right) = \varepsilon \otimes \bigoplus_{w \in W_t \backslash W / W_J} \mathrm{Ind}^{W_J}_{W_J \cap W_{t^w}} \chi^J_{t^w} \otimes [H(\mathcal{B}^u_t)^\rho]^w \tag{9}$$

where $\chi^J_{t^w}$ is a character of $W_{J,t^w} := W_J \cap W_{t^w}$ satisfying the following properties (we need to describe the properties here):

-

Our aim is to compute this restriction with respect to maximal compact subgroups. Since $\mathrm{Sp}_{2n}(F)$ is simply connected, these coincide with maximal parahoric subgroups and its conjugacy classes are in natural bijection with maximal subsets $J \subsetneq S_{\text{aff}}$. For each $r = 0, \ldots, n$, let $J_r = S_{\text{aff}} \backslash \{\check{\alpha}_r\}$ be the maximal subset of J_{aff}, K_r be the corresponding parahoric subgroup with unipotent radical U_r and reductive quotient $\overline{K}_r$.

Case 1: $t = t_0$. In this case, $Z_{G^\vee}(t_0) = G^\vee$ so $W_{t^0} = W$, $\mathcal{B}^u_{t^0} = \mathcal{B}^u$ and the character $\chi^{J_r}_{t^0}$ is trivial. Hence, the right hand side of (9) becomes

$$\varepsilon \otimes H(\mathcal{B}^u)^\rho|_{W_{J_r}}.$$

The variety $\mathcal{B}^u$ is one-dimensional, so $H(\mathcal{B}^u) = H^0(\mathcal{B}^u) + H^2(\mathcal{B}^u)$. The W-action on $H(\mathcal{B}^u)$ is described by the Springer correspondence. We need to analyse the restrictions to W_{J_r} in a case-by-case basis. The 0-th cohomology group is easy to describe since $H^0(\mathcal{B}^u)^{\mathbf{1}}$ affords the trivial W-representation while $H^0(\mathcal{B}^u)^\varepsilon = 0$.

Lemma 5.9. *The graded complex vector space $H(\mathcal{B}^u) = H^0(\mathcal{B}^u) \oplus H^2(\mathcal{B}^u)$ has a natural action of the group $W(B_n) \times A_u$ preserving the grading. The $W(B_n)$-modules $H^0(\mathcal{B}^u)^{\mathbf{1}}$, $H^2(\mathcal{B}^u)^{\mathbf{1}}$ and $H^2(\mathcal{B}^u)^\varepsilon$ afford the representations labelled by $(n, \emptyset)$, $(n-1, 1)$ and $((n-1)1, \emptyset)$, respectively, while $H^0(\mathcal{B}^u)^\varepsilon = 0$*

From this, we can immediately compute that

$$\pi(ut_0, \mathbf{1})^{\mathcal{I}}_{q=1}|_W = \varepsilon \otimes H(\mathcal{B}^u)^{\mathbf{1}} = \varepsilon \otimes [(n, \emptyset) + (n-1, 1)] = (\emptyset, 1^n) + (1, 1^{n-1}),$$
$$\pi(ut_0, \varepsilon)^{\mathcal{I}}_{q=1}|_W = \varepsilon \otimes H(\mathcal{B}^u)^\varepsilon = \varepsilon \otimes ((n-1)1, \emptyset) = (\emptyset, 21^{n-2}),$$

The analogous computation for $\pi(ut_1, \mathbf{1})^{\mathcal{I}}_{q=1}|_W$ and $\pi(ut_1, \varepsilon)^{\mathcal{I}}_{q=1}|_W$ is significantly more involved. Firstly, we note that

$$Z_{G^\vee}(t_1) = Z_{G^\vee}(t_1)^0 \times \langle w_{\alpha^0} \rangle \quad \text{where} \quad Z_{G^\vee}(t_1)^0 = \langle T, \mathfrak{X}_{\pm\alpha_2}, \ldots, \mathfrak{X}_{\pm\alpha_{n-1}}, \mathfrak{X}_{\pm\alpha_n} \rangle$$

is a connected reductive group of type B_{n-1}. The Weyl group of $Z_{G^\vee}(t_1)$ is

$$W_{t_1} = \langle w_{\alpha_2}, \ldots, w_{\alpha_{n-1}}, w_{\alpha_n} \rangle \times \langle w_{\alpha^0} \rangle \cong W(B_{n-1}) \times C_2$$

and acts on T by the transformations

$$h(a_1, \ldots, a_{n-1}, a_n) \longmapsto h(a_1^{\pm 1}, a_{\tau(2)}^{\pm 1}, \ldots, a_{\tau(n)}^{\pm 1}), \quad \text{where } \tau \in S_{n-1}.$$

Thus $|W_{t_1}| = 2^n(n-1)!$ and the index inside W is $[W : W_{t_1}] = n$. From the descriptions above, $u \in Z_{G^\vee}(t_1)$ and is a regular unipotent element, so $\dim \mathcal{B}^u_{t_1} = 0$. However, since $Z_{G^\vee}(t_1)$ is not connected, the structure of $\mathcal{B}^u_{t_1}$ is more interesting.

Lemma 5.10. *The variety $\mathcal{B}^u_{t_1}$ consists of two points. The two dimensional complex vector space $H^0(\mathcal{B}^u_{t_1})$ has a natural action of $W_{t_1} \times A_{ut_1}$ and $H^0(\mathcal{B}^u_{t_1})^{\mathbf{1}}$, $H^0(\mathcal{B}^u_{t_1})^\varepsilon$ are one-dimensional representations of $W_{t_1} = \langle w_{\alpha_1}, \ldots, w_{\alpha_{n-1}} \rangle \times \langle w_{\alpha^0} \rangle \cong W(B_{n-1}) \times C_2$ affording the characters $(n-1, \emptyset) \otimes \mathbf{1}$, $(n-1, \emptyset) \otimes \varepsilon$, respectively.*

Proof. Since $u \in Z_{G^\vee}(t_1)$ is regular, it is contained in a unique Borel subgroup $B = \langle T, \mathfrak{X}_{\alpha_2}, \ldots, \mathfrak{X}_{\alpha_{n-1}}, \mathfrak{X}_{\alpha_n} \rangle$. We know that $t_2 = h_u w_{\alpha^0} \in Z_{G^\vee}(u) - B$, so $u \in {}^{t_2}B$. but $B \neq {}^{t_2}B$. Since $A_u = C_2$, these are the only two Borel subgroups containing u.

We note that $A_{ut_1} = \{1, t_2\}$ and that t_2 acts on $\mathcal{B}^u_{t_1}$ by permuting both points and therefore

$$H^0(\mathcal{B}^u_{t_1})^{\mathbf{1}} = \mathbb{C}(B + {}^{t_2}B), \quad \text{while} \quad H^0(\mathcal{B}^u_{t_1})^{\varepsilon} = \mathbb{C}(B - {}^{t_2}B)$$

By the classical Springer correspondence of type B_{n-1} applied to the regular orbit containing u, we have that $\langle w_{\alpha_1}, \ldots, w_{\alpha_{n-1}} \rangle$ acts trivially on $H^0(\mathcal{B}^u_{t_1})$. On the other hand, since $t_2 \in N(T_n)$ maps to w_{α^0}, the action of w_{α^0} on $H^0(\mathcal{B}^u_{t_1})$ coincides with that of t_2. Maybe should justify this with the generalized Springer correspondence for disconnected reductive groups. □

From these structural results, we can compute the parahoric restrictions of $\pi(ut_1, \mathbf{1})$ and $\pi(ut_1, \varepsilon)$.

Lemma 5.11. *With the notation as above, we have that, as W-modules,*

$$\pi(ut_1, \mathbf{1})^{\mathcal{I}}_{q=1}|_W = (\emptyset, 1^n) + (\emptyset, 21^{n-2}) \quad \textit{and} \quad \pi(ut_1, \varepsilon)^{\mathcal{I}}_{q=1}|_W = (1, 1^{n-1}).$$

Proof. This is a direct calculation. From (Deleted reference), we know that

$$\pi(ut_1, \mathbf{1})^{\mathcal{I}}_{q=1}|_W = \varepsilon \otimes \mathrm{Ind}^W_{W_{t_1}} H(\mathcal{B}^u_{t_1})^{\mathbf{1}} = \varepsilon \otimes \mathrm{Ind}^W_{W(B_{n-1}) \times C_2} (n-1, \emptyset) \otimes \mathbf{1},$$
$$\pi(ut_1, \varepsilon)^{\mathcal{I}}_{q=1}|_W = \varepsilon \otimes \mathrm{Ind}^W_{W_{t_1}} H(\mathcal{B}^u_{t_1})^{\varepsilon} = \varepsilon \otimes \mathrm{Ind}^W_{W(B_{n-1}) \times C_2} (n-1, \emptyset) \otimes \varepsilon,$$

and thus we analyze each induced representation. Recall that $(n-1, 1)$ is the natural n-dimensional reflection representation of $W(B_n)$ acting on the $\mathbb{C}$-span of $\alpha_1, \ldots, \alpha_n$. The line l spanned by the root $\alpha^0 = \alpha_1 + \ldots + \alpha_n$ is a 1-dimensional $W_{t_1} = \langle w_{\alpha_2}, \ldots, w_{\alpha_n} \rangle \times \langle w_{\alpha^0} \rangle$-stable subspace, where $w_{\alpha_2}, \ldots, w_{\alpha_n}$ act trivially while w_{α^0} acts by ε. Thus

$$\mathrm{Hom}_W(\mathrm{Ind}^W_{W(B_{n-1}) \times C_2} (n-1, \emptyset) \otimes \varepsilon, (n-1, 1)) = \mathrm{Hom}_{W(B_{n-1}) \times C_2}((n-1, \emptyset) \otimes \varepsilon, (n-1, 1)) = \mathbb{C},$$

and since $[W : W_{t_1}] = n = \dim(n-1, 1)$, it follows that $\mathrm{Ind}^W_{W_{t_1}} H(\mathcal{B}^u_{t_1})^{\varepsilon} = (n-1, 1)$. We then twist by ε to obtain

$$\pi(ut_1, \varepsilon)^{\mathcal{I}}_{q=1}|_W = (1, 1^{n-1}).$$

On the other hand, the first statement of the Lemma will follow immediately if we prove that

$$\mathrm{Ind}^W_{W_{t_1}} H(\mathcal{B}^u_{t_1})^{\mathbf{1}} = \varepsilon \otimes \big((\emptyset, 1^n) + (\emptyset, 21^{n-2})\big) = (n, \emptyset) + ((n-1)1, \emptyset). \tag{10}$$

We note that $\dim(n, \emptyset) = 1$ while $\dim((n-1)1, \emptyset) = n-1$, so the dimensions in (10) agree in both sides. Moreover, using the fact that $|W_{t_1} \backslash W / W_{t_1}| = 2$, together with Mackey theory, one can easily show that the above induction decomposes into two irreducible subrepresentations. Since $H(\mathcal{B}^u_{t_1})^{\mathbf{1}}$ is the trivial representation of W_{t_1}, the trivial representation $(n, \emptyset)$ must also appear in the induction, so it remains to show that $((n-1)1, \emptyset)$ is the other component. Similarly to the computation above, it is enough by Frobenius reciprocity to show that $((n-1)1, \emptyset)$ contains a (necessarily unique) 1-dimensional vector subspace on which W_{t_1} acts trivially.

To prove this last assertion, we need to analyze the construction of $((n-1)1, \emptyset)$ as a Macdonald representation of $W(B_n)$. By Carter 11.4.2, the root system associated to the bipartition $((n-1)1, \emptyset)$ has type $D_2 = A_1 \times A_1$, and by convenience we choose $\Phi' = \{\pm\alpha_1, \pm\alpha_0\}$ with corresponding Weyl group $W' = \langle w_{\alpha_1}, w_{\alpha_0} \rangle \cong C_2^2$. Let V

be the n-dimensional vector space affording the natural reflection representation $(n-1,n)$ of $W(B_n)$. Since Φ' has two positive roots, the representation $((n-1)1,\emptyset)$ is a component of the representation $\mathrm{Sym}^2(V^*)$. By an analogous argument, the irreducible $n(n-1)/2$ - dimensional representation $(n-2,2)$ is also a subrepresentation of $\mathrm{Sym}^2(V^*)$. We now calculate explicitly a basis for each of these representations.

Let us fix an orthonormal basis $\{e_1,\dots,e_n\}$ of V such that $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = e_n$. Then one can calculate that $W(B_n)$ acts on $V^* \cong V$ by the transformations

$$w_{\alpha_i} = \begin{pmatrix} I_{i-1} & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_{n-i-1} \end{pmatrix} \quad \text{for } 1 \le i \le n-1, \quad \text{and} \quad w_{\alpha_n} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\{x_1,\dots,x_n\}$ be the dual basis of $\{e_1,\dots,e_n\}$ in V^* so $\mathrm{Sym}^2(V^*) = \mathrm{Span}_{\mathbb{C}}\{x_ix_j \mid 1 \le i \le j \le n\}$ is the space of degree 2 homogeneous polynomials on V. There is a direct sum decomposition $\mathrm{Sym}^2(V^*) = U_1 \oplus U_2 \oplus U_3$ in $W(B_n)$-invariant subspaces, where

$$U_1 = \mathrm{Span}_{\mathbb{C}}\{\sum_{i=1}^n x_i^2\}, \quad U_2 = \mathrm{Span}_{\mathbb{C}}\{x_i^2 - x_{i+1}^2 \mid 1 \le i \le n-1\}, \quad U_3 = \mathrm{Span}_{\mathbb{C}}\{x_ix_j \mid 1 \le i < j \le n\}$$

have dimensions 1, $n-1$ and $n(n-1)/2$, respectively. In the preceding paragraph we discussed the fact that both $((n-1)1,\emptyset)$ and $(n-2,2)$ are components of $\mathrm{Sym}^2(V^*)$ and it thus follows that the subspaces U_1, U_2 and U_3 afford the representations $(n,\emptyset)$, $((n-1)1,\emptyset)$ and $(n-2,2)$ of $W(B_n)$, respectively.

It is now easy to see from this explicit model of $((n-1)1,\emptyset)$ that $W_{t_1} = \langle w_{\alpha_2},\dots,w_{\alpha_n}, w_{\alpha^0}\rangle$ acts trivially on the 1-dimensional subspace of $\mathrm{Span}_{\mathbb{C}}\{x_i^2 - x_{i+1}^2 \mid 1 \le i \le n-1\}$ spanned by

$$(n-1)x_1^2 - x_2^2 - \cdots - x_{n-1}^2 - x_n^2.$$

This immediately implies that

$$\mathrm{Ind}_{W(B_{n-1})\times C_2}^{W}(n-1,\emptyset) \otimes \mathbf{1} = (n,\emptyset) + ((n-1)1,\emptyset),$$

thus finishing the proof. $\square$

Finally, we need to compute the parahoric restrictions of $\pi(ut_2,)$. To apply similar ideas to the previous case, we first note that t_2 is conjugate to the diagonal matrix $\begin{pmatrix} 1 & \\ & -I_{2n} \end{pmatrix}$, and therefore

$$Z_{G^\vee}(t_2) = Z_{G^\vee}(t_2)^0 \times \langle w_{\alpha_n}\rangle \quad \text{where} \quad Z_{G^\vee}(t_2)^0 = \langle T, \mathfrak{X}_\beta \mid \beta \text{ is a long root}\rangle,$$

where $Z_{G^\vee}(t_2)^0$ is a connected reductive group of type D_n. We can then deduce that $W_{t_2} = W$ and that u is a regular unipotent element in $Z_{G^\vee}(t_2)$, so $\dim \mathcal{B}_{t_2}^u = 0$. The following result gives an explicit description of $H(\mathcal{B}_{t_2}^u) = H^0(\mathcal{B}_{t_2}^u)$ as a W-representation.

Lemma 5.12. *The cohomology space $H(\mathcal{B}_{t_2}^u)$ is 2-dimensional and has a natural action of $W \times A_{ut_2}$ where $A_{ut_2} = \{t_0,t_1,t_2,t_1t_2\}$. It admits a direct sum decomposition*

$$H(\mathcal{B}_{t_2}^u) = H^0(\mathcal{B}_{t_2}^u)^{\mathbf{1}\otimes\mathbf{1}} \oplus H^0(\mathcal{B}_{t_2}^u)^{\varepsilon\otimes\mathbf{1}}$$

into 1-dimensional W-representations, affording the characters $(n,\emptyset)$ and $(\emptyset,n)$, respectively.

From this we immediately get that

$$\pi(ut_2, \mathbf{1}\otimes\mathbf{1}) = \varepsilon \otimes H(\mathcal{B}^u_{t_2})^{\mathbf{1}\otimes\mathbf{1}} = \varepsilon\otimes(n,\emptyset) = (\emptyset, 1^n),$$
$$\pi(ut_2, \varepsilon\otimes\mathbf{1}) = \varepsilon \otimes H(\mathcal{B}^u_{t_2})^{\varepsilon\otimes\mathbf{1}} = \varepsilon\otimes(\emptyset,n) = (1^n, \emptyset),$$

while $\pi(ut_2, \mathbf{1}\otimes\varepsilon)$ and $\pi(ut_2, \varepsilon\otimes\varepsilon)$ are not Iwahori-spherical.

Lemma 5.13. *The representation $\pi(ut_2, \mathbf{1}\otimes\varepsilon)$ has no U_{J_0} fixed vectors, while the parahoric restriction of $\pi(ut_2, \varepsilon\otimes\varepsilon)$ to $\overline{K}_0$ is the defect 1 irreducible $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ representation labelled by the symbol*

$$\begin{pmatrix} 0 & 1 & 2 & \dots & n-2 & n-1 & n \\ & & 1 & \dots & n-2 & & \end{pmatrix}.$$

$\pi(us,\phi)$	$K_0 \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$
$(t_0, \mathbf{1})$	$(\emptyset, 1^n) + (1, 1^{n-1})$
(t_0, ε)	$(\emptyset, 21^{n-2})$
$(t_1, \mathbf{1})$	$(\emptyset, 1^n) + (\emptyset, 21^{n-2})$
(t_1, ε)	$(1, 1^{n-1})$
$(t_2, \mathbf{1}\otimes\mathbf{1})$	$(\emptyset, 1^n)$
$(t_2, \mathbf{1}\otimes\varepsilon)$	0
$(t_2, \varepsilon\otimes\mathbf{1})$	$(1^n, \emptyset)$
$(t_2, \varepsilon\otimes\varepsilon)$	θ_{1,K_0}

e

6 Other interesting results

Lemma 6.1. *Let G be a group and let H be a subgroup. Suppose that G acts on another group N by $G \to \mathrm{Aut}(N)$. Then the following square is commutative.*

$$\begin{array}{ccc} \mathrm{Rep}(H \ltimes N) & \xrightarrow{\mathrm{Ind}_{H\ltimes N}^{G\ltimes N}} & \mathrm{Rep}(G \ltimes N) \\ {\scriptstyle \mathrm{Inf}_H^{H\ltimes N}}\uparrow & & {\scriptstyle \mathrm{Inf}_G^{G\ltimes N}}\uparrow \\ \mathrm{Rep}(H) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{Rep}(G) \end{array}$$

Corollary 6.2. *Let $W(B_n) = \langle w_{\alpha_1}, \ldots, w_{\alpha_n} \rangle$ be the Weyl group of type B_n. There is a unique quotient map $\phi : W(B_n) \twoheadrightarrow W(A_{n-1})$ containing all short reflections in the* $\ker \phi$ *with a natural section that identifies $W(A_{n-1}) \cong \langle w_{\alpha_2}, \ldots, w_{\alpha_n} \rangle$. Finally, let $W' = \langle w_{\alpha_1}, \ldots, w_{\alpha_{n-1}}, w_{\alpha_0} \rangle \cong W(B_{n-1}) \times C_2$. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Rep}(W') & \xrightarrow{\mathrm{Ind}} & \mathrm{Rep}(W(B_n)) \\ {\scriptstyle \mathrm{Inf}}\uparrow & & {\scriptstyle \mathrm{Inf}}\uparrow \\ \mathrm{Rep}(W(A_{n-2})) & \xrightarrow{\mathrm{Ind}} & \mathrm{Rep}(W(A_{n-1})) \end{array}$$

Proof. Apply the previous Lemma with $G = W(A_{n-1}) = \langle w_{\alpha_2}, \ldots, w_{\alpha_n} \rangle \cong S_n$, $H = \langle w_{\alpha_2}, \ldots, w_{\alpha_{n-1}} \rangle \cong S_{n-1}$ and $N = \langle \text{short reflections of } W(B_n) \rangle = \langle\langle w_{\alpha_1} \rangle\rangle \cong C_2^n$ together with the natural action $S_n \to \mathrm{Aut}(C_2^n)$ by permutations on each coordinate. $\square$

The above Corollary is useful to compute inductions if we have a nice description of the inflation maps.

Proposition 6.3. *Let $n \geq 2$. Then the inflation map* $\mathrm{Inf} : \mathrm{Irr}(W(A_{n-1})) \to \mathrm{Irr}(W(B_n))$ *send the irreducible $W(A_{n-1})$-representation labelled by (α) to the irreducible $W(B_n)$-representation labelled by $(\alpha, \emptyset)$, where α is a partition of $[n]$.*

Proof. Fix a partition $\alpha = (\alpha_1, \alpha_2, \ldots)$ of $[n]$ and let $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots)$ be its dual partition. Let Φ' be a subroot system of Φ of type $D_{\alpha_1^*} + D_{\alpha_2^*} + \cdots$. Then we have that Φ' is simply laced, and only consists of long roots so its Weyl group W' is generated by long reflections too.

Idea of the remaining of the proof By taking integral basis of each $D_{\alpha_i^*}$, we can take an appropriate basis $\{x_1, \ldots, x_n\}$ of V^* and disjoint subsets $J_1, \ldots, J_r \subseteq [n]$ such that the sign representation of W' is given by the one dimensional subspace spanned by

$$\prod_{k=1}^{r} \prod_{i,j \in J_k,\ i<j} (x_i^2 - x_j^2).$$

Then it should be able to relate the generated representation to the one obtained from (α) as a $W(A_{n-1})$. $\square$

With these results, we can now reprove Lemma 5.11. By Corollary 6.2,

$$\mathrm{Ind}_{W_{s_1}}^{W(B_n)} \mathbf{1} = Inf_{W(A_{n-1})}^{W(B_n)} \mathrm{Ind}_{W(A_{n-2})}^{W(A_{n-1})} \mathbf{1}$$

A simple computation shows that $\mathrm{Ind}_{W(A_{n-2})}^{W(A_{n-1})}\mathbf{1} = \{f : S_{n-1}\backslash S_n \to \mathbb{C}\}$, which decomposes as a direct sum of the S_n-invariant subspaces

$$U_1 = \{f : S_{n-1}\backslash S_n \to \mathbb{C} \mid f \text{ constant}\}, \quad U_2 = \{f : S_{n-1}\backslash S_n \to \mathbb{C} \mid \sum_{\sigma \in S_{n-1}\backslash S_n} f(\sigma) = 0\},$$

both of which are irreducible and correspond to the S_n-representations labelled by (n) and $((n-1)1)$, respectively. combining this with Proposition 6.3, we get that

$$\mathrm{Ind}_{W_{s_1}}^{W(B_n)}\mathbf{1} = (n, \emptyset) + ((n-1)1, \emptyset)$$