Nonabelian Fourier transform for unipotent representations

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1 Introduction

Conjecture 1.1. Let G be a simple p-adic group. Then the diagram

$$\begin{tikzcd}
\mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[r, "\mathrm{FT}^\vee_{\mathrm{ell}}"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] \\
\bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}},
\end{tikzcd}$$

commutes, up to certain some roots of unity.

2 Unipotent representations for finite groups of Lie type

Despite the fact that the statement of 1.1 is at heart a result about the representation theory of a p-adic group G, in this first section we restrict our attention to structural and representation theoretic results of finite groups of Lie type. The aim is to understand Lusztig's non-abelian fourier transform $\mathrm{FT}^K : R_{\mathrm{un}}(\overline{K}) \longrightarrow R_{\mathrm{un}}(\overline{K})$ on the space of unipotent representations of the reductive parahoric quotient $\overline{K}$ appearing at the bottom of the diagram in 1.1. To achieve this, we first need to take a step back and look at general theory of finite groups of Lie type and the classification of their representations. Most of the material in this first section can be found in the book by Carter, a great reference on the subject.

To introduce finite groups of Lie type, we first need to briefly look at Frobenius maps. Throughout this chapter, as it is common in the literature, G will denote an algebraic group over an algebraically closed field of positive characteristic. In later sections, G will always denote a p-adic group!

2.1 Frobenius maps

Let k be an algebraically closed field of characteristic $p > 0$ and let G be a connected linear algebraic group over k, which we assume in addition to be reductive. A well-known fact states that G is isomorphic to a closed subgroup of $\mathrm{GL}_n(k)$ for some n. The structure of G can be understood to a large extent by looking at its maximal connected solvable subgroups of G, denoted as Borel subgroups. If we fix some Borel subgroup B, any maximal torus T in B is also a maximal torus in G, and it determines a set of roots $\Phi = \Phi(G, T) \subset X(T)$. The choice of the Borel B containing T corresponds to a choice of positive roots Φ^+ and therefore of an integral basis $\Delta \subseteq \Phi^+$. Moreover, the subgroups $B, N := N_G(T)$ satisfy the axioms of a BN-pair, as described by Tits, whose corresponding Weyl group is $W = N/T = \langle w_{\alpha_i} \mid \alpha_i \in \Delta \rangle$, generated by the reflections along the simple roots in Δ.

For some power $q = p^e, e \geq 1$, consider the map $F_q : \mathrm{GL}_n(k) \to \mathrm{GL}_n(K),\ (a_{ij}) \mapsto (a_{ij}^q)$, which is known to be a rational group homomorphism. A homomorphism $F : G \to G$ is called a *standard Frobenius map* if there exists an embedding $i : G \hookrightarrow \mathrm{GL}_n(k)$ and some $q = p^e$ such that

$$i(F(g)) = F_q(i(g)) \quad \text{for all } g \in G.$$

More generally, a homomorphism $F : G \to G$ is a *Frobenius map* if some power of F is a standard Frobenius map. If $F : G \to G$ is a Frobenius map, we are interested in structure and representation theory of the **finite** group G^F of fixed points under the Frobenius map. Many properties are inherited from G, but some change significantly.

One can show that G contains F-stable Borel subgroups, and that inside any F-stable Borel there are F-stable maximal tori. Thus, we may assume that the Borel subgroup B and maximal torus T fixed in the previous paragraph are F-stable. Under these assumptions, the Frobenius map acts on the simple roots by permuting the corresponding the root spaces. Thus, F corresponds to some permutation ρ of Δ satisfying

$$F(\mathcal{X}_\alpha) = \mathcal{X}_{\rho(\alpha)} \quad \text{for all } \alpha \in \Delta.$$

Moreover, one can easily check that ρ is in fact a symmetry of the Dynkin diagram, and these can be completely classified. When ρ is trivial we say that G^F is a *split* group and *non-split* otherwise. In this document, we will mainly focus on split groups.

For each orbit $J \subseteq \Delta$ of ρ, let $w_J \in W_J = \{w_{\alpha_i} \mid \alpha_i \in J\}$ be the unique element such that $w_J(J) = -J$. It is the longest element of W_J and it satisfies that $w_J^2 = 1$. In then follows that the group G^F has a natural BN-pair given by the groups B^F and N^F, whose Weyl group is

$$N^F/T^F = (N/T)^F = W^F = \langle w_J \mid J \subseteq \Delta \text{ is an orbit of } \rho \rangle.$$

When G^F is split, ρ acts trivially and all orbits are singletons, so $W^F \cong W$.

Example 2.1. Let $G = \mathrm{GL}_n(k)$ and let $F : G \to G$ be a Frobenius map. If $F = F_q$, then ρ is trivial and $G^F = \mathrm{GL}_n(\mathbb{F}_q)$. Alternatively, consider the map $\gamma : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ given by $A \mapsto Q_n^{-1}(A^T)^{-1}Q_n$, where Q_n is the matrix with 1 in the anti-diagonal and 0 everywhere else. Then the map $F' := F_q \circ \gamma$ squares to F_{q^2} so it is a Frobenius map. Then we obtain the non-split group

$$\mathrm{GL}_n(k)^{F'} = \mathrm{GU}_n(\mathbb{F}_q) = \{g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) | \bar{A}^T Q_n A = Q_n\}.$$

It is a key structural fact of G^F than any two F-stable Borel subgroups in G are conjugate under G^F. However, it is not the case that any two F-stable maximal tori in G are G^F-conjugate. A classification of these classes is a fundamental problem in the structure theory of G^F, with a beautiful solution. While it is true that any F-stable Borel subgroup contains an F-stable maximal torus, the converse might not be true. Any F-stable maximal torus that is contained in an F-stable Borel subgroup is called *maximally split*, and it can be shown that any two F-stable maximally split tori are also conjugate under G^F.

Remarkably, one can easily determine the G^F-conjugacy classes of F-stable maximal tori by looking at the Weyl group. To state this result, we first introduce the notion of F-conjugacy classes in W. Given two $w_1, w_2 \in W$, we say that they are F-conjugate if there is some $x \in W$ such that $F(x)w_1x^{-1} = w_2$. Note that if F acts on W trivially, then the F-conjugacy classes are the standard conjugacy classes.

Proposition 2.2. *There is a bijection between*

$$\begin{aligned} \{G^F\text{-conjugacy classes of } F\text{-stable maximal tori}\} &\longrightarrow \{F\text{-conjugacy classes of } W\} \\ T' = {}^gT &\longmapsto \pi(g^{-1}F(g)) \end{aligned}$$

From now, we will write T_1 for a maximally split F-stable maximal torus and T_w for any F-stable torus obtained from T_1 by conjugating by some element $g \in G$ such that $\pi(g^{-1}F(g)) = w$. By the previous result, these objects are uniquely defined up to G^F-conjugation.

2.2 Deligne–Lusztig characters and unipotent representations

In their groundbreaking paper from 1976, Deligne and Lusztig attached to each pair (T, θ) of F-stable maximal torus T and character θ of T^F, a virtual character $R_{T,\theta}$ of the group G^F. These virtual characters were constructed using the action of G^F on certain ℓ-adic cohomology groups associated to certain Deligne–Lusztig varieties. We shall not consider the explicit definition of the characters, but we will rather recall without proof some important properties.

1. If the pair (T', θ') is obtained from (T, θ) by conjugation on some element of G^F in the natural way, then $R_{T,\theta} = R_{T',\theta'}$.

2. If T_1 is a maximally split torus inside some F-stable Borel B, then $R_{T_1,\theta} = \theta_{B^F}^{G^F}$, where $\theta_{B^F}^{G^F}$ is the character of the parabolically induced representation $\mathrm{Ind}_{B^F}^{G^F}\theta$.

3. The orthogonality relations: For any two pairs (T, θ) and (T', θ'), we have that
$$(R_{T,\theta}, R_{T',\theta'}) = |\{w \in W(T, T')^F \mid {}^w\theta' = \theta\}|.$$
In particular, if T, T' are not G^F-conjugate, then $(R_{T,\theta}, R_{T',\theta'}) = 0$.

4. We say that (T, θ) is in general position if $\mathrm{Stab}_{W(T)^F}(\theta)$ is trivial. Thus, if (T, θ) is in general position, the orthogonality relations imply that one of $\pm R_{T,\theta}$ is an irreducible character.

5. For any F-stable maximal torus T,
$$(R_{T,\theta}, 1) = \begin{cases} 1 \text{ if } \theta = 1 \\ 0 \text{ if } \theta \neq 1. \end{cases}$$

6. The dimension of $R_{T,\theta}$ equals
$$\dim R_{T,\theta} = \varepsilon_G \varepsilon_T |G^F : T^F|, \quad \varepsilon_G \varepsilon_T \in \{\pm 1\}.$$

It is possible to define an action of G on the pairs (T, θ) extending the action of G^F. It acts on the tori in the obvious way, but care is required to define the action on the characters θ. We say that two pairs in the same G-orbit are *geometrically conjugate*. For example, all $(T, \mathbf{1})$ are geometrically conjugate. Similarly, we say that two characters χ_1, χ_2 are *geometrically conjugate* if there are geometrically conjugate pairs (T, θ) and (T', θ') such that
$$(\chi_1, R_{T,\theta}) \neq 0 \quad \text{and} \quad (\chi_2, R_{T',\theta'}) \neq 0.$$

7. If (T, θ) and (T', θ') are not geometrically conjugate, then $R_{T,\theta}$ and $R_{T',\theta'}$ do not share any irreducible component in common.

8. If G has centre Z and semisimple rank l, there are $|Z^F|q^l$ geometric conjugacy classes of pairs (T, θ).

Let's give a couple of examples for the decomposition of the Deligne–Lusztig characters, where the properties above can be easily verified.

Example 2.3. Suppose first that $G = \mathrm{GL}_2(k)$ and $F = F_q : G \to G$ is the standard Frobenius. Then $G^F = \mathrm{GL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 \neq 0 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ while $T_2^F \cong \mathbb{F}_{q^2}^\times$. Now, if $\theta = \theta_1 \otimes \theta_2$ is a character of T_1^F, then

$$R_{T_1,\theta} = {\theta_{B^F}}^{G^F} = \begin{cases} \overline{\theta} \otimes (1 \oplus \mathrm{St}) & \text{if } \theta_1 = \theta_2 \\ \text{irreducible principal series} & \text{if } \theta_1 \neq \theta_2, \end{cases}$$

where $\overline{\theta}$ is the unique extension of θ to all of $\mathrm{GL}_2(\mathbb{F}_q)$ (this is only possible if $\theta_1 = \theta_2$). On the other hand, suppose that θ' is a character of T_w^F. Then

$$R_{T_w,\theta'} = \begin{cases} \overline{\theta} \otimes (1 \ominus \mathrm{St}) & \text{if } \theta'^q = \theta' \\ \text{irreducible cuspidal} & \text{if } \theta'^q \neq \theta', \end{cases}$$

where $\overline{\theta}$ is the extension of the unique character θ of T_1^F for which (θ, T_1) is geometrically conjugate to (θ', T_w).

Example 2.4. Now suppose that $G = \mathrm{SL}_2(k)$ and $F = F_q : G \to G$ to be the standard Frobenius again. Similarly, $G^F = \mathrm{SL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 = 1 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times$ while $T_2^F \cong C_{q+1}$. If θ is a character of T_1^F then

$$R_{T_1,\theta} = {\theta_{B^F}}^{G^F} = \begin{cases} 1 \oplus \mathrm{St} & \text{if } \theta = 1, \\ R_+(\xi) \oplus R_-(\xi) & \text{if } \theta = \xi = \mathrm{sgn}, \\ \text{irreducible principal series} & \text{if } \theta \neq \theta^{-1}, \end{cases}$$

where $R_+(\xi) \neq R_-(\xi)$ are conjugate under $\mathrm{GL}_2(\mathbb{F}_q)$ and so they have the same dimension $(q+1)/2$. On the other hand, if θ' is a character of T_w^F, then

$$R_{T_w,\theta'} = \begin{cases} 1 \ominus \mathrm{St} & \text{if } \theta' = 1, \\ \ominus R'_+(\xi) \ominus R'_-(\xi) & \text{if } \theta' = \xi = \mathrm{sgn}, \\ \ominus \text{ irreducible cuspidal} & \text{if } \theta' \neq \theta'^{-1}, \end{cases}$$

Amongst all geometric conjugacy classes of characters of G^F, there is a distinguished class that will take most of our attention.

Definition 2.5. An irreducible character χ of G^F is called *unipotent* if there is some maximal F-stable torus T of G such that $(R_{T,1}, \chi) \neq 0$. The set of unipotent characters of G^F form a geometric conjugacy class of characters.

We remark that if χ is a unipotent character of G^F, then $(\chi, R_{T,\theta}) = 0$ for any (T, θ) with $\theta \neq 1$. This is an immediate consequence of property 7. above. Lusztig realized that a classification of unipotent characters of G^F and certain subgroups (also finite groups of Lie type) was enough to understand all irreducible characters. Firstly, he observed that the study of unipotent characters of G^F can be reduced to the case when G is simple of adjoint type. Secondly, following the same approach as Harish–Chandra, he noticed that it was enough to classify *cuspidal* unipotent characters. There are many equivalent definitions for this notion; the idea is that χ is not obtained by parabolically inducing any unipotent character of a proper parabolic subgroup. The precise statement the following.

Proposition 2.6. *Let χ be an irreducible character of G^F.*

1. *There is an F-stable parabolic subgroup P of G with F-stable Levi decomposition $P = LN$ and a cuspidal character ϕ of L^F such that $(\chi, {\phi_{P^F}}^{G_F}) \neq 0$.*
2. *Moreover, the pair (P, ϕ) is unique up to G^F-conjugacy.*
3. *The character χ of G^F is unipotent if and only if ϕ is a unipotent character of L^J.*

Therefore, to classify the unipotent characters of G^F, it is enough to determine the cuspidal unipotent representations ϕ of the standard Levi subgroups L_J^F of G^F and then calculate the decomposition of ${\phi_{P_J^F}}^{G^F}$ into irreducible characters. The later task can be achieved by Howlett–Lehrer theory (Carter, §10), while the former was achieved by Lusztig in a case by case analysis. For example, he proved that if G^F has classical type, then it has either 0 or 1 cuspidal unipotent characters (if the type $A_n, n \geq 1$, there are no cuspidal unipotent characters). In exceptional types, one can find more exotic behaviour. In type G_2 we find 4 cuspidal unipotent characters and in F_4 we find 7.

2.3 Families of unipotent characters

Lusztig further observed that the unipotent characters of G^F naturally form families in a remarkable way. Firstly, he parametrized the principal series characters with irreducible characters of W by showing that there is a natural bijection

$$\{\text{Irreducible characters of } W\} \longrightarrow \{\text{Irreducible components of } \mathrm{Ind}_{B^F}^{G^F} 1\} \tag{1}$$

$$\phi \longmapsto \chi_\phi. \tag{2}$$

This fact can be elegantly seen as follows. The choice of Borel subgroup B containing the torus determines a set of simple roots, and thus a set of simple reflections S generating W. Now consider the endomorphism algebra $\mathcal{H}(T^F, 1) := \mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$. Using some Mackey theory, one can show that $\mathcal{H}(T^F, 1)$ has natural basis $\{T_w, w \in W\}$ satisfying the multiplication rules

$$T_s^2 = (q-1)T_s + qT_1 \quad \text{if } s \in S,$$
$$T_{w_1}T_{w_2} = T_{w_1 w_2} \quad \text{if } l(w_1 w_2) = l(w_1) + l(w_2).$$

Thus, $\mathcal{H}(T^F, 1)$ is isomorphic to the Coxeter algebra $\mathcal{H}(W, S, q)$ with constant parameter q. This algebra is a deformation of the group algebra $\mathbb{C}[W]$ (by setting $q = 1$). Using Tit's deformation theorem, it follows that

$$\mathcal{H}(T^F, 1) \cong \mathbb{C}[W],$$

from which the above bijection can be easily deduced. We can also deduce the nice formula

$$(\chi_\phi, \mathrm{Ind}_{B^F}^{G^F} 1) = \dim \phi.$$

Example 2.7. Let $G = G_2(k)$ and let $F = F_q : G \to G$ be the standard Frobenius. Then $G^F = G_2(\mathbb{F}_q)$, whose Weyl group W is isomorphic to D_{12}. Following Carter, we label the six irreducible representations by $\phi_{1,0}, \phi'_{1,3}, \phi''_{1,3}, \phi_{1,6}, \phi_{2,1}, \phi_{2,2}$, where the first subindex gives the dimension and, for example, $\phi_{1,0} = 1$ and $\phi_{1,6} = \det$. Moreover, the Weyl group has 6 conjugacy classes often labelled by $e, a, b, ab, (ab)^2$ and $(ab)^3$, where a and b are short and long reflections, respectively. Then

$$\begin{aligned}
R_{T_1,1} &= \phi_{1,0} + \phi_{1,6} + \phi'_{1,3} + \phi''_{1,3} + 2\phi_{2,1} + 2\phi_{2,2}, \\
R_{T_a,1} &= \phi_{1,0} - \phi_{1,6} + \phi'_{1,3} - \phi''_{1,3}, \\
R_{T_b,1} &= \phi_{1,0} - \phi_{1,6} - \phi'_{1,3} + \phi''_{1,3}, \\
R_{T_{ab},1} &= \phi_{1,0} + \phi_{1,6} - \phi_{2,1} && +G_2[-1] + G_2[\theta] + G_2[\theta^2], \\
R_{T_{(ab)^2},1} &= \phi_{1,0} + \phi_{1,6} - \phi_{2,2} && +G_2[1] - G_2[\theta] - G_2[\theta^2], \\
R_{T_{(ab)^3},1} &= \phi_{1,0} + \phi_{1,6} - \phi'_{1,3} - \phi''_{1,3} && -2G_2[1] - 2G_2[-1],
\end{aligned}$$

where $G_2[1], G_2[-1], G_2[\theta], G_2[\theta^2]$ are the commonly used labels for the 4 cuspidal unipotent characters.

For type A_l-groups, one can explicitly describe the representations χ_ϕ in terms of Deligne–Lusztig characters.

Example 2.8. Let G be a reductive group of type A_l. No standard Levi subgroup L^F of G^F has a cuspidal unipotent representation other than the maximally split torus since they all have type $A_m, m \leq l$. Consequently, all unipotent representations of G^F are in the principal series. This means that irreducible characters of W parametrize unipotent characters of G^F. Explicitly, given some irreducible character ϕ of W,

$$\chi_\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w) R_{T_w,1}$$

For $G = \mathrm{GL}_2(k)$ or $\mathrm{SL}_2(k)$, $\chi_1 = 1$ and $\chi_{\mathrm{sgn}} = \mathrm{St}$.

In general, however, finite groups of Lie type do have cuspidal unipotent characters, and the virtual characters

$$R_\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w) R_{T_w,1}$$

as defined above are not irreducible, and are denoted *unipotent almost characters.* Lusztig then divided irreducible characters of W into equivalence classes (called *families*) by the rule that two characters ϕ_1 and ϕ_2 are equivalent if R_{ϕ_1} and R_{ϕ_2} share an irreducible component. Similarly, he extended this notion of families to unipotent characters of G^F by the rule that two unipotent characters appearing in the same R_ϕ are in the same family and then extending by transitivity. Tracing through the definitions, it is not hard to show that there is a bijection between families of characters of W and families of unipotent characters of G^F. Remarkably, Lusztig proved the latter families can be parametrized in the following manner.

Theorem 2.9. *For each family of unipotent representations $\mathcal{F}$ of G^F there is a group $\Gamma_{\mathcal{F}} \in \{1, C_2 \times \cdots \times C_2, S_3, S_4, S_5\}$ and a bijection*

$$\begin{aligned} M(\Gamma_{\mathcal{F}}) &\longrightarrow \mathcal{F} \\ (x,\sigma) &\longmapsto \chi^{\mathcal{F}}_{(x,\sigma)} \end{aligned}$$

satisfying

$$(\chi^{\mathcal{F}}_{(x,\sigma)}, R_\phi) = \begin{cases} \{(x,\sigma),(y,\tau)\} & \text{if } \chi_\phi = \chi^{\mathcal{F}}_{(y,\tau)} \in \mathcal{F}, \\ 0 & \text{if } \chi_\phi \notin \mathcal{F}, \end{cases}$$

where

$$\{(x,\sigma),(y,\tau)\} = \frac{1}{|C_\Gamma(x)||C_\Gamma(y)|} \sum_{\substack{g\in\Gamma \\ xgyg^{-1}=gyg^{-1}x}} \sigma(gyg^{-1})\overline{\tau(g^{-1}xg)}.$$

Since $R_{T_1,1} = \sum_{\phi\in\hat{W}} R_\phi$, it follows that for any family $\mathcal{F}$, $(\chi^{\mathcal{F}}_{(1,1)}, R_{T_1,1}) > 0$, so $\chi^{\mathcal{F}}_{(1,1)} = \chi_\phi$ for some character ϕ of W. Characters arising this way are called *special characters* of W and they have distinct characterizations. They are the distinguished elements of the families of characters of W as described above. The upshot of this discussion is that families of unipotent characters can be parametrized by special characters of the Weyl group.

Example 2.10. Consider again the group $G^F = G_2(\mathbb{F}_q)$. Following the notation in Example 2.7, the special characters of $W \cong D_{12}$ are $\phi_{1,0}, \phi_{1,6}, \phi_{2,1}$ whose corresponding families are

$$(\phi_{1,0}), (\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}), (\phi_{1,6}).$$

On the other hand, the 10 unipotent characters of G^F fall into three families, parametrized as follows.

Description in terms of cuspidal characters	Degree	Pair (x, σ)
$\phi_{1,0}$	1	
$\phi_{2,1}$	$\frac{1}{6}q\Phi_2{}^2\Phi_3$	$(1, 1)$
$G_2[1]$	$\frac{1}{6}q\Phi_1{}^2\Phi_6$	$(1, \varepsilon)$
$\phi_{2,2}$	$\frac{1}{2}q\Phi_2{}^2\Phi_6$	$(g_2, 1)$
$\phi'_{1,3}$	$\frac{1}{3}q\Phi_3\Phi_6$	$(1, r)$
$\phi''_{1,3}$	$\frac{1}{3}q\Phi_3\Phi_6$	$(g_3, 1)$
$G_2[-1]$	$\frac{1}{2}q\Phi_1{}^2\Phi_3$	(g_2, ε)
$G_2[\theta]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ)
$G_2[\theta^2]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ^2)
$\phi_{1,6}$	q^6	

Finally, Lusztig used these families to define a *nonabelian Fourier transform* FT_{G^F} on the set of irreducible characters of G^F. This is a linear map on complex vector space $R_{\mathrm{un}}(G^F)$ defined as the $\mathbb{C}$-span of the irreducible unipotent characters of G^F. The families described above induce a direct sum decomposition

$$R_{\mathrm{un}}(G^F) = \bigoplus_{\mathcal{F}} R_{\mathrm{un},\mathcal{F}}(G^F),$$

where $R_{\mathrm{un},\mathcal{F}}(G^F)$ is the $\mathbb{C}$-span of the unipotent representations of G^F in the family $\mathcal{F}$. Lusztig's nonabelian Fourier transform acts on each block as follows. For a family $\mathcal{F}$ with associated group $\Gamma_{\mathcal{F}}$, consider the $|M(\Gamma_{\mathcal{F}})| \times |M(\Gamma_{\mathcal{F}})|$ matrix $\mathcal{M}(\mathcal{F})$ whose $((x,\sigma),(y,\tau))$ entry is the value $\{(x,\sigma),(y,\tau)\}$. This matrix is Hermitian and squares to the identity. The map $\mathrm{FT}_{G^F} : R_{\mathrm{un}}(G^F) \to R_{\mathrm{un}}(G^F)$ then acts on each block $R_{\mathrm{un},\mathcal{F}}(G^F)$ by the matrix $\mathcal{M}(\mathcal{F})$. Since all the matrices square to the identity, FT_{G^F} is an involution.

In particular, Lusztig's nonabelian transform satisfies $\mathrm{FT}_{G^F}(\chi_\phi) = R_\phi$, so it maps the unipotent characters of G^F to almost unipotent characters. The latter ones have very nice geometric properties, which were exploited by Lusztig to prove other remarkable results. We will not pursue this further here, but we will see an explicit example for type G_2.

Example 2.11. If G^F is of type G_2, then FT_{G^F} fixes the characters $\phi_{1,0}$ and $\phi_{1,6}$ but transforms the third family according to the Fourier transform matrix

If $\Gamma \cong S_3$ the 8×8 Fourier transform matrix is:

$$\begin{array}{c|c}
 & \begin{matrix} (1,1) & (1,r) & (1,\varepsilon) & (g_2,1) & (g_2,\varepsilon) & (g_3,1) & (g_3,\theta) & (g_3,\theta^2) \end{matrix} \\
\begin{matrix} (1,1) \\ (1,r) \\ (1,\varepsilon) \\ (g_2,1) \\ (g_2,\varepsilon) \\ (g_3,1) \\ (g_3,\theta) \\ (g_3,\theta^2) \end{matrix} &
\begin{bmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\end{array}.$$

3 Unipotent representations of p-adic groups

Having discussed the classification of unipotent characters of finite groups of Lie type and Lusztig's nonabelian Fourier transform, in this section we explain in detail the remaining maps and objects from Conjecture 1.1. To achieve this, we first need to cover the notion of a parahoric subgroup and of a unipotent representation of a p-adic group Therefore, we now shift our focus completely and turn our attention towards the structural and representation theoretic aspects of p-adic groups.

Throughout, we let F be a nonarchimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k of cardinality q, a power of a prime p. Let $\mathbf{G}$ be a connected, almost simple, split algebraic group over F and let $G = \mathbf{G}(F)$ be the group of F-rational points. In this section, we will first discuss some results about the structure theory of the group G, including the Bruhat–Tits building and the classification of parahoric (or open compact) subgroups of G. These results will be instrumental to study and classify the unipotent representations of G refer to definition inside the category $\mathrm{Rep}(G)$ of smooth admissible complex representations of G, the central object of interest in the local Langlands correspondence.

3.1 The apartment of a split maximal torus

To simplify the exposition, we will assume throughout that G is a split p-adic group over F, even though it is not strictly necessary. For any split maximal torus T of G over F, there is a natural perfect pairing

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z},$$

where $X^*(T) = \mathrm{Hom}(T, F^\times)$ and $X_*(T) = \mathrm{Hom}(F^\times, T)$ are its character and cocharacter lattice, respectively, and the pairing is obtained by composition. Let $\Phi(G,T) \subset X^*(T)$ be the set of roots associated to T, with the corresponding set of coroots $\Phi^\vee(G,T) \subset X_*(T)$. Similarly to the previous section, a choice of a Borel subgroup B of G containing T is equivalent to the choice of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(G,T)$, which we now fix throughout. We also let α_0 be the highest root of $\Phi(G,T)$ with respect to Δ. In addition, the group B together with the normalizer $N := N_G(T)$ form a BN-pair with corresponding Weyl group $W = N(F)/T(F)$.

A natural object arising in the representation theory of G is the apartment $\mathcal{A}(G,T) := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, a real vector space spanned by the simple coroots. Moreover, $\mathcal{A}(G,T)$ has the structure of a simplicial complex given by the hyperplanes

$$H_{\alpha,n} = \{x \in \mathcal{A}(G,T) \mid \langle \alpha, x \rangle = n\}, \quad \text{for each } \alpha \in \Phi(G,T)^+ \text{ and } n \in \mathbb{Z}.$$

Whenever the torus T is clear from context, we will omit it from the notation. The complexes on the apartment are called *facets*, and the facets of largest dimension (equivalently, the open facets) are called *alcoves*. Our choice of simple roots Δ determines a canonical alcove

$$\mathcal{C}_0 = \{x \in \mathcal{A} \mid \langle \alpha, x \rangle > 0, \alpha \in \Delta \ \text{ and } \langle \alpha_0, x \rangle < 1\},$$

commonly referred to as the *fundamental alcove*.

Another important property of the apartment is that it carries a natural action of the group N by affine transformations. To state some of its properties, we define, for each root $\alpha \in \Phi(G,T)$ a map (not a group homomorphism!) $w_\alpha : \mathbb{C}^\times \to N$ given by $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$. The action of N on $\mathcal{A}$ satisfies

- For any $\alpha \in \Phi$ and $\lambda \in F$, the element $\check{\alpha}(\lambda) \in T \subset N$ acts on $\mathcal{A}$ by a translation $-\nu_p(\lambda)\check{\alpha}$.
- For any $\alpha \in \Phi$, the element $w_\alpha(t) \in N$ acts on $\mathcal{A}$ by a reflection along $H_{\alpha,-\nu_p(t)}$. When $t \in \mathcal{O}$, this coincides with the natural action of W on $\mathcal{A}$.
- This action preserves the simplicial structure of the apartment and is transitive on the set of alcoves of $\mathcal{A}$.
- The kernel of this action is $T(\mathcal{O})$.

Therefore the *extended Weyl group*

$$\widetilde{W} := N(F)/T(\mathcal{O}) \cong W \ltimes X_*(T)$$

acts faithfully on the apartment $\mathcal{A}$ and transitively on the set of alcoves. We denote by $w_{\alpha,n}$ the unique element in $\widetilde{W}$ acting on $\mathcal{A}$ by a reflection on the hyperplane $H_{\alpha,n}$.

In general, however, this action is not simple on the set of alcoves and the group $\Omega = \{w \in \widetilde{W} \mid w(\mathcal{C}_0) = \mathcal{C}_0\}$ is non-trivial. These groups fit in a **splitting** short exact sequence

$$1 \longrightarrow W_{\mathrm{aff}} \longrightarrow \widetilde{W} \longrightarrow \Omega \longrightarrow 1,$$

where $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is an affine Coxeter group generated by the simple reflections $\tilde{s}_0 := w_{\alpha_0,1},\ s_i = w_{\alpha_i,0},\ i = 1,\dots,r$ along the walls of the fundamental alcove $\mathcal{C}_0$ and acting simply transitively on the set of alcoves of $\mathcal{A}$. The group W_{aff} is the *affine Weyl group* associated to the group G. The Weyl groups W, $\widetilde{W}$ and W_{aff} are independent of T, up to isomorphism.

Example 3.1. 1. Let $G = \mathrm{SL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = t^2 \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \frac{\alpha}{2}\mathbb{Z}$ and $X_*(T) = \check{\alpha}\mathbb{Z}$. Moreover, we have that

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T \quad \text{and} \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right),\ t \in F^\times.$$

The apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. It is easy to check that $\Omega = \{1\}$ so that $\widetilde{W} = W_{\mathrm{aff}}$ is generated by $s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ -\varpi & 0 \end{smallmatrix}\right)$ and $s_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

2. Let $G = \mathrm{PGL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) = t \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \alpha\mathbb{Z}$ and $X_*(T) = \frac{\check{\alpha}}{2}\mathbb{Z}$. Similarly,

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)T, \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right),\ t \in F^\times$$

and the apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. This time, however, $\Omega = \{1, \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)\}$ is non-trivial, and

$$W_{\mathrm{aff}} = \left\langle s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{smallmatrix}\right), s_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\rangle = \{w \in \widetilde{W} \mid \nu(\det(w)) \text{ is even}\}$$

is an index 2 normal subgroup of $\widetilde{W}$.

3. Let $G = \mathrm{GL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G, T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & s \end{smallmatrix}\right) = ts^{-1} \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \quad \text{for any } t \in F^{\times}.$$

   In this case, $\Omega \cong \mathbb{Z}$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ and therefore $W_{\mathrm{aff}} = \langle s_0, s_1 \rangle = \{w \in \widetilde{W} \mid \det(w) \in \mathcal{O}^{\times}\}$ is a normal subgroup of $\widetilde{W}$ of infinite index.

Some of the behaviour observed in the previous example holds in much greater generality. For example, Ω is an abelian group, and it has finite order if and only if G is a simple group. In that case, Ω is in bijection with the centre of complex dual group $G^{\vee}(\mathbb{C})$ of G and with the length preserving automorphisms of the affine Dynkin diagram (cite Iwahori). In particular, if G is simply connected, then Ω is trivial, while Ω has the largest size within the isogeny class when G is adjoint. On the other hand, W_{aff} only depends on the isogeny class, and therefore only on the root system of G.

3.2 The Bruhat-Tits building and parahoric subgroups

When we work with the apartment $\mathcal{A}(G, T)$, we are making an arbitrary choice for the torus. It is possible, however, to push this idea further and construct the Bruhat-Tits building $\mathcal{B}(G)$, a *canonical* polysimplial space associated to G that contains $\mathcal{A}(G, T)$ for any F-split maximal torus. This is achieved by gluing together the apartments of all F-split maximal tori of G according to some natural equivalence relation. An important property of the building $\mathcal{B}(G)$ is that it carries a G-action satisfying the following:

1. It extends the action of $N_G(T)$ on $\mathcal{A}(G, T)$ for each F-split maximal torus T.

2. The stabilizer of $\mathcal{A}(G, T)$ is $N_G(T)$ for each F-split maximal torus T.

3. The stabilizer of any facet c of the building is a (maybe disconnected) open compact subgroup of G.

4. The action is strongly transitive on the set $\{(\mathcal{C}, \mathcal{A}) \mid \mathcal{C} \text{ is an alcove inside the apartment } \mathcal{A}\}$.

5. For any pair $(\mathcal{C}, \mathcal{A})$ as above, its stabilizer acts on $\mathcal{B}(G)$ as the group Ω. In other words

$$\mathrm{Stab}_G(\mathcal{C}, \mathcal{A})/T(\mathcal{O}) = (N \cap \mathrm{Stab}_G(\mathcal{C}))/T(\mathcal{O}) = \mathrm{Stab}_N(\mathcal{C})/T(\mathcal{O}) \cong \Omega.$$

The study of the action of G on its Bruhat–Tits building $\mathcal{B}(G)$ is an invaluable tool to study the structure of G. Many results about G that are hard to prove directly become transparent when studying the building. One such problem is the classification of open maximal compact subgroups of G.

Given a facet c of the building of G, we are interested in the stabilizer $K_c^+ := \mathrm{Stab}_G(c)$. By the properties listed above, this is a (possibly disconnected) open compact subgroup of G. All maximal open compact subgroups of G arise this way. These stabilizers behave well under the conjugation in G since

$$\mathrm{Stab}_G(g \cdot c) = gK_c^+g^{-1} \quad \text{for all facets } c \subset \mathcal{B}(G) \text{ and } g \in G.$$

Example 3.2. The Bruhat-Tits building of $G = \mathrm{SL}_2(\mathbb{Q}_p)$ or $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ is an infinite tree all of whose vertices have degree $p+1$. Each infinite line inside the building is an apartment corresponding to a distinct F-split maximal torus of G. Consider the apartment $\mathcal{A}(G,T)$, where T is the group of diagonal matrices, and let $\Delta = \{\alpha\}$ be the simple root as above. Then $\mathcal{C}_0$ is the segment between the vertices 0 and $\check{\alpha}/2$.

If $G = \mathrm{SL}_2(\mathbb{Q}_p)$, then

$$K_{\{0\}}^+ = \mathrm{SL}_2(\mathbb{Z}_p),\ K_{\{\check{\alpha}/2\}}^+ = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},\ \text{and } \mathcal{I} := K_{\mathcal{C}_0}^+ = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

are all connected, open compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ not conjugate to each other. Since any vertex in the building lies in the G-orbit of 0 or $\check{\alpha}/2$, K_0 and K_1 are the unique maximal compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ up to conjugation. The subgroup $\mathcal{I}$ is called the **Iwahori subgroup**, it is conjugate to the stabilizer of any facet in the building and is of fundamental importance in the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

On the other hand, if $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, then

$$K_{\{0\}}^+ = \mathrm{Stab}(0) = \mathrm{PGL}_2(\mathbb{Z}_p),\ K_{\{\check{\alpha}/2\}}^+ = \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}_{\det \in \mathbb{Z}_p^\times}$$

are both connected open compact subgroups and conjugate in $\mathrm{PGL}_2(F)$ by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, so $\mathrm{PGL}_2(F)$ has one unique *connected* maximal compact subgroup up to conjugacy. But it has another class of maximal compact subgroups, namely

$$K_{\mathcal{C}_0}^+ = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det \in \mathbb{Z}_p^\times} \bigsqcup \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det \in \mathbb{Z}_p^\times},$$

a disconnected open compact subgroup. We define the Iwahori subgroup $\mathcal{I}$ to be identity connected component of $K_{\mathcal{C}_0}^+$.

The example above suggests that the connected components of the stabilizers of facets in the building depend in a subtle way on the group Ω. This is indeed the case, and we discuss this connection now. Since $\Omega = \mathrm{Stab}_{\widetilde{W}}(\mathcal{C}_0)$ and $\mathcal{C}_0$ is bounded by hyperplanes corresponding to S_{aff}, there is a natural homomorphism

$$\Omega \longrightarrow \mathrm{Aut}(S_{\mathrm{aff}}).$$

All permutations of S_{aff} induced by Ω preserve the affine Dynkin diagram associated to S_{aff}, and if G is simple of adjoint type, then all such automorphisms of S_{aff} are induced by Ω. This greatly restricts the size of Ω.

Next, fix some *proper* subset $J \subset S_{\mathrm{aff}}$ and consider the *standard* facet

$$c_J = \{x \in \overline{\mathcal{C}_0} \mid \langle \alpha, x \rangle \in \mathbb{Z} \text{ for } \alpha \in J \text{ and } \langle \alpha, x \rangle \notin \mathbb{Z} \text{ for } \alpha \in S_{\mathrm{aff}} - J\}.$$

Two facets c_{J_1} and c_{J_2} are conjugate under the action of G if and only if J_1 and J_2 lie in the same Ω-orbit. Moreover, any facet c in the building is conjugate to c_J for some proper subset $J \subset S_{\mathrm{aff}}$. In other words, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\text{aff}}\} \longleftrightarrow \{G - \text{orbits of facets } c \text{ in the BT-building}\},$$

satisfying that for any two $J_1 \subseteq J_2$, their corresponding facets satisfy $c_{J_2} \subset \overline{c_{J_1}}$.

The groups K_c^+ will be of great importance since they bridge the representation theory of p-adic groups with the that of finite groups of Lie type. To explain this connection, we note that there is a short exact sequence

$$1 \longrightarrow U_c \longrightarrow K_c^+ \longrightarrow \overline{K}_c^+ \longrightarrow 1,$$

where U_c is the pro-unipotent radical of K_c^+ and $\overline{K}_c^+$ is the group of k-rational points of a (possibly disconnected) reductive group $\overline{\mathbf{K}}_c^+$ over k. To avoid disconnected quotients, it is convenient to introduce the notion of a *parahoric subgroup.*

Definition 3.3. A **parahoric subgroup** K_c is the inverse image in K_c^+ of the group $\overline{K}_c$ of k-rational points of the identity component $\overline{\mathbf{K}}_c$ of $\overline{\mathbf{K}}_c^+$. We shall sometimes denote "parahoric subgroup" to the triple $(K_c, U_c, \overline{K}_c)$. If $c = c_J$ is a standard facet, then we simply write $(K_J, U_J, \overline{K}_J)$ for its associated parahoric subgroup.

Naturally, two parahoric subgroups are conjugate in G if and only if the corresponding facets of the building are in the same G-orbit. Thus, there is an inclusion-preserving bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\text{aff}}\} \longleftrightarrow \{G - \text{conjugacy classes of parahoric subgroups } (K, U_K, \overline{K})\}$$

In particular, if c is open in the building, then $(K_c, U_c, \overline{K}_c)$ is a minimal parahoric subgroup. These is called an **Iwahori subgroups**, and they are all conjugate in G. The standard Iwahori subgroup $\mathcal{I}$ corresponds to $J = \emptyset \subsetneq S_{\text{aff}}$.

Moreover, if the facet c corresponds to $J \subsetneq S_{\text{aff}}$, then

$$K_c^+/K_c \cong \Omega_J = \mathrm{Stab}_\Omega(J).$$

These results can be directly verified for SL_2, PGL_2 and GL_2 using the examples above.

Example 3.4. Suppose that $G = G_2(F)$. The affine Dynkin diagram of G_2 has no symmetries, so $\Omega = 1$ and the extended affine weyl group $\widetilde{W}$ is a Coxeter group of type $\tilde{G}_2$. Since $S_{\text{aff}} = \{s_0, s_1, s_2\}$, there are 7 conjugacy classes of parahoric subgroups, satisfying

$$\overline{K}_{\{s_1,s_2\}} = G_2(k), \quad \overline{K}_{\{s_0,s_1\}} = \mathrm{SL}_3(k), \quad \overline{K}_{\{s_0,s_2\}} = \mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$$
$$\overline{K}_{\{s_1\}} = \overline{K}_{\{s_2\}} = \overline{K}_{\{s_0\}} = \mathrm{SL}_2(k) \quad \text{and} \quad \overline{K}_\emptyset = T(k) = (k^\times)^2.$$

As we mentioned earlier, parahoric subgroups are of great important for they provide a bridge between smooth admissible representations of G and finite dimensional representations of the finite groups of Lie type $\overline{K}_c$. In the next subsection, we carefully describe this bridge and how it helps us classify certain representations of G.

3.3 Parahoric restriction and unipotent representations

In this section, we finally focus our attention in representation theoretic aspects of the p-adic group G, using the techniques developed in previous ones. The main object of study in the local Langlands correspondence is the category of smooth admissible complex representations of G, denoted by $\mathrm{Rep}(G)$. Let $\mathrm{Irr}(G)$ be the set of irreducible representations in $\mathrm{Rep}(G)$. This is a large, and we are interested in a particular subset of known as *unipotent* irreducible representations of G and denoted by $\mathrm{Irr}_{\mathrm{un}}(G)$, which we now define using parahoric subgroups.

Consider a smooth admissible representation (π, V) of G and let $(K, U_K, \overline{K})$ be a parahoric subgroup corresponding to a facet c. The space V^{U_K} of fixed points under the pro-unipotent radical is naturally a representation of $\overline{K} = K/U_K$. We can take this idea one step further and define the *parahoric restriction functor*

$$\mathrm{res}_K : R(G) \longrightarrow R(\overline{K}), \quad V \longmapsto V^{U_K}, \quad \text{for all } V \in \mathrm{Irr}(G), \tag{3}$$

where $R(G)$ is the $\mathbb{C}$-span of $\mathrm{Irr}(G)$ and $R(\overline{K})$ is the $\mathbb{C}$-span of the irreducible representations of $\overline{K}$. This is well-defined since the representations are assumed to be admissible. The existence of such a functor is very powerful – we can then apply the techniques of representation theory of finite groups of Lie such as Deligne–Lusztig induction in the setting of p-adic groups. Let us begin with a natural definition.

Definition 3.5. Let $(K, U_K, \overline{K})$ be a parahoric subgroup and (τ, E) be a cuspidal representation of $\overline{K}$. Define

$$\mathrm{Irr}(G, K, E) = \{(\pi, V) \in \mathrm{Irr}(G) \mid \text{the } \overline{K}\text{-module } V^{U_K} \text{ contains the } \overline{K}\text{-module } E\}.$$

The definition of a unipotent representation is now straightforward.

Definition 3.6. We say that an irreducible representation (π, V) of G is *unipotent* if there is a parahoric subgroup $(K, U_K, \overline{K})$ such that V^{U_K} contains a cuspidal unipotent representation of $\overline{K}$; that is, if $(\pi, V) \in \mathrm{Irr}(G, K, E)$ for some pair (K, E) where E is unipotent.

Therefore, the set of unipotent representations up to isomorphism is

$$\mathrm{Irr}_{\mathrm{un}}(G) = \bigcup_{\substack{J \subsetneq S_{\mathrm{aff}} \\ E \text{ cusp. unip. } \overline{K}_J\text{-rep}}} \mathrm{Irr}(G, K_J, E).$$

We say that each set $\mathrm{Irr}(G, K_J, E)$ generates a distinct *block* of representations of G.

We note that if we replace the p-adic group G for the finite group of Lie type G^F and *parahoric* by *parabolic*, then we recover the definition of a unipotent representation in G^F.

Example 3.7. Let G be a split reductive p-adic group with split maximal torus T. Let $\mathcal{I}$ be an Iwahori subgroup with pro-unipotent radical $\mathcal{I}^+$. Then the reductive quotient $\mathcal{I}/\mathcal{I}^+$ is isomorphic to $T(k)$. Thus, all irreducible representations of $\mathcal{I}/\mathcal{I}^+$ are 1-dimensional and the only unipotent representation is the trivial one. Therefore, the irreducible *Iwahori-spherical* representations

$$\mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) = \{(\pi, V) \in \mathrm{Irr}(G) \mid V^{\mathcal{I}} \neq 0\}$$

are all unipotent. It is a known fact that this set coincides with the set of irreducible subrepresentations of $c\text{-}\mathrm{Ind}_B^G\chi$, where χ is an unramified character of T. We say that $\mathrm{Irr}(G,\mathcal{I},\mathbf{1})$ generates the *principal block*.

Example 3.8. For $n \geq 1$, reductive groups over finite fields of type A_n have no irreducible cuspidal unipotent representations. Therefore, if G is a reductive p-adic group of type A_n and $J \subseteq S_{\mathrm{aff}}$ is non-empty, then $\overline{K}_J$ has no cuspidal unipotent representations. This implies that the set of irreducible unipotent representations of G

$$\mathrm{Irr}_{\mathrm{un}}(G) = \mathrm{Irr}(G,\mathcal{I},\mathbf{1})$$

coincides with the irreducible Iwahori-spherical representations of G.

Example 3.9. Let G be a p-adic group of type G_2. For $J \subsetneq S_{\mathrm{aff}}$, $\overline{K}_J$ has unipotent cuspidal representations only if $J = \emptyset$ and $J = J_0 = \{s_1, s_2\}$. In the first case, we obtain the principal block $\mathrm{Irr}(G,\mathcal{I},\mathbf{1})$. In the second case, $\overline{K}_J$ has four unipotent characters, so

$$\mathrm{Irr}_u n(G) = \mathrm{Irr}(G,\mathcal{I},\mathbf{1}) \sqcup \bigsqcup_{\rho\in\{1,-1,\theta,\theta^2\}} \mathrm{Irr}(G, K_{J_0}, G_2[\rho]).$$

However, these last four blocks are very simple since they have a unique irreducible unipotent representation, namely $\mathrm{Irr}(G, K_{J_0}, G_2[\rho]) = \{c\text{-}\mathrm{Ind}_{K_{J_0}}^G G_2[\rho]\}$ for each $\rho \in \{1,-1,\theta,\theta^2\}$.

The next step is to ensure that unipotent representations behave under the parahoric restriction functor (3). The following two results ensure this is indeed the case.

Proposition 3.10. *Let (π, V) be an irreducible admissible representation of G. If there is some parahoric subgroup $(K, U_K, \overline{K})$ such that V^{U_K} contains a (potentially non-cuspidal) unipotent representation of $\overline{K}$, then (π, V) is a unipotent representation of G.*

Proof. By assumption, V^{U_K} contains a unipotent irreducible representation (σ, W) of $\overline{K}$ so $\mathrm{Hom}_K(W, V^{U_K}) \neq 0$. By Proposition 2.6, there is some standard parabolic subgroup $\overline{P} = \overline{U_P} \cdot \overline{L_P}$ of $\overline{K}$ and cuspidal unipotent representation (τ, E) of $\overline{L_P}$ such that

$$\mathrm{Hom}_K(W, \mathrm{Ind}_{\overline{P}}^{\overline{K}} E) \neq 0,$$

where we view the $\overline{K}$ representations as inflated K representations, trivial on U_K. By the classification of parahoric subgroups in G, it follows that $\overline{P} = H/U_K$, where $(H, U_H, \overline{H})$ is another parahoric subgroup contained in K. Moreover, we have the inclusions $U_K \subseteq U_H \subseteq H \subseteq K$ and therefore $\overline{U_P} = U_H/U_K$ and $\overline{L_P} = \overline{H} = H/U_H$. Since induction and inflation are commuting operations, it follows that

$$\mathrm{Inf}_{\overline{K}}^{K}\mathrm{Ind}_{\overline{P}}^{\overline{K}} E \cong \mathrm{Ind}_H^K \mathrm{Inf}_{\overline{H}}^{H} E$$

and hence $\mathrm{Hom}_K(W, \mathrm{Ind}_H^K E) \neq 0$. Since W is irreducible and K is compact, it also follows that

$$\mathrm{Hom}_K(\mathrm{Ind}_H^K E, V^{U_K}) = \mathrm{Hom}_H(E, V^{U_K}) \neq 0.$$

Since the representation E is trivial on U_H, the image of any H-equivariant map $E \to V^{U_K}$ lies inside V^{U_H}. Thus,

$$\mathrm{Hom}_H(E, V^{U_H}) = \mathrm{Hom}_H(E, V^{U_K}) \neq 0,$$

and this concludes the proof. $\square$

Conversely, we would like to show that for any irreducible unipotent representation (π, V) of G, the irreducible $\overline{K}$-submodules of V^{U_K} are all unipotent, for any parahoric subgroup $(K, U_K, \overline{K})$. This is a direct corollary of the following theorem.

Theorem 3.11. *Suppose $I \subsetneq S_{\text{aff}}$ and that V^{U_I} contains the cuspidal unipotent representation σ of $\overline{K}_I$. If $J \subsetneq S_{\text{aff}}$ with $V^{U_J} \neq 0$, and J is minimal with respect to this property, then there is $\omega \in \Omega$ such that $I = \omega J$, and V^{U_J} consists of copies of σ^ω. Moreover, if G is exceptional, then $J = I$.*

Proof. See Moy-Prasad for a complete account for general cuspidal representations and not necessarily unipotent and Reeder's paper for a sketch in the unipotent setting. $\square$

Corollary 3.12. *Let (π, V) be a unipotent representation of G and let $(H, U_H, \overline{H})$ be a parahoric subgroup. Then the $\overline{H}$-irreducible components of V^{U_H} are all unipotent.*

Proof. Since (π, V) is unipotent, $(\pi, V) \in \operatorname{Irr}(G, K_J, E)$ for some $J \subsetneq S_{\text{aff}}$ and cuspidal unipotent representation E of $\overline{K}_J$. Let (τ, W) be a $\overline{H}$-irreducible component of π^{U_H}. By conjugating if necessary, we may assume that $(H, U_H, \overline{H})$ is a standard parahoric subgroup. Analogously to the proof of Proposition 3.10, there is some $I \subsetneq S_{\text{aff}}$ such that $(K_I, U_I, \overline{K}_I)$ is contained in K and τ is a subrepresentation of $\operatorname{Ind}_{\overline{K}_I}^{\overline{H}} \sigma$. By Theorem 3.11, I is the same Ω-orbit as J and σ is cuspidal unipotent. By Proposition 2.6, this implies that (τ, W) is also unipotent. $\square$

Corollary 3.13. *For any two pairs (K, E), (K', E') of a parahoric subgroup and a cuspidal unipotent representation of the reductive quotient,* $\operatorname{Irr}(G, K, E)$ *and* $\operatorname{Irr}(G, K', E')$ *are either disjoint or equal.*

Analogously to the construction of $R(G)$, we define $R_{\text{un}}(G)$ to be the $\mathbb{C}$-span of the irreducible unipotent representations $\operatorname{Irr}_{\text{un}}(G)$. Lemma 3.10 and Theorem 3.11 implies that for each parahoric subgroup $(K, U_K, \overline{K})$ there is a well-defined *restriction function*

$$\operatorname{res}^K_{\text{un}} : R_{\text{un}}(G) \longrightarrow \mathbb{C}_{\text{un}}[\overline{K}]^{\overline{K}}, \quad V \longmapsto (\text{character of})\ V^{U_K}, \quad \text{for all } V \in \operatorname{Irr}(G).$$

It is also convenient to consider simultaneously all such functions for all conjugacy classes of maximal parahoric subgroups, so we define $\operatorname{res}^{\text{par}}_{\text{un}} = (\operatorname{res}^K_{\text{un}})_K$.

3.4 Parahoric restriction for unipotent supercuspidal representations

Let G be the simple p-adic group over F. In this section, we investigate the parahoric restriction of supercuspidal unipotent representations of G (if any) with respect to maximal parahoric subgroups. A well-known result of Moy and Prasad states that any supercuspidal unipotent representation (π, V) of G is obtained by compactly inducting an irreducible smooth representation (ρ, E) of K_x^+, where $x \in \mathcal{B}(G)$ is a vertex, such that $\rho|_{K_x}$ is the inflation of a cuspidal representation of $\overline{K}_x$. By conjugating if necessary, we may assume that x lies in the closure of the fundamental alcove $\mathcal{C}_0$. Explicitly,

$$\pi \cong c\text{-Ind}_{K_x^+}^G \rho,$$

so by Frobenius reciprocity we have that

$$\mathrm{Hom}_{K_x}(\rho|_{K_x}, \pi^{U_x}) \supseteq \text{this should be equality } \mathrm{Hom}_{K_x^+}(\rho, \pi^{U_x}) = \mathrm{Hom}_{K_x^+}(\rho, \pi) = \mathrm{Hom}_G(c\text{-Ind}_{K_x^+}^G \rho, \pi) \cong \mathbb{C},$$

so $(\pi, V) \in \mathrm{Irr}(G, K_x, E)$. If $J = \{\alpha \in S_\alpha \mid \langle \alpha, x\rangle = 0\}$, then $K_x = K_J$ and by cuspidality J is a minimal subset of S_{aff}, up to the action of Ω, such that $\pi^{U_J} \neq 0$. Now let $I \subsetneq S_{\mathrm{aff}}$ be another subset such that $V^{U_I} \neq 0$. If π^{U_I} contains an irreducible cuspidal representation of $\overline{K}_I$ then I is also minimal with respect to $V^{U_I} \neq 0$ and by Theorem 3.11, I and J are in the same Ω-orbit. If π^{U_I} does not contain any irreducible cuspidal representation, then by 2.6, there is some $J' \subset I$ such that $\pi^{U_{J'}}$ contains a cuspidal representation of $\overline{K}_{J'}$ so J and J' lie in the same Ω-orbit, but this is a contradiction since K_J is a maximal parahoric subgroup of G. We have thus shown:

Lemma 3.14. *Let (π, V) be a supercuspidal unipotent representation of G. Then there is one unique Ω-orbit $[J]$ of subsets of S_{aff}, all of which are maximal such that $\pi^{U_I} \neq 0$ if and only if $I \in [J]$.*

Suppose G has type G_2 with simple reflections $S_{\mathrm{aff}} = \{s_0, s_1, s_2\}$. We note that $\Omega = \{1\}$ so Ω-orbits are all singletons. By combining Example 3.4 and Remark 3.8, given $J \subsetneq S_{\mathrm{aff}}$, the reductive quotient $\overline{K}_J$ has cuspidal unipotent representations if and only if $J = J_0 := \{s_1, s_2\}$ or $J = \emptyset$.

In the first case, $K_0 := K_{J_0}$ is the stabilizer of the origin in the apartment $\mathcal{A}(G,T)$ and $\overline{K_0} = G_2(\mathbb{F}_q)$ has 4 cuspidal unipotent representations labelled $G_2[1], G_2[-1], G_2[\theta]$ and $G_2[\theta^2]$, where θ is a primitive third root of unity. For any of these representations σ, Example 3.7 shows that the compactly induced representation $\pi = c\text{-Ind}_{K_0}^G \sigma$ is irreducible and supercuspidal, and

$$\mathrm{Irr}(G, K_0, \sigma) = \{\pi\}.$$

In the second case, $K_\emptyset = \mathcal{I}$ is the standard Iwahori subgroup (stabilizer of the fundamental alcove) and the only cuspidal unipotent representation of I/U_I is the trivial character. Therefore,

$$\mathrm{Irr}_{\mathrm{un}}(G) = \mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) \bigcup \{c\text{-Ind}_{K_0}^G G_2[1], c\text{-Ind}_{K_0}^G G_2[-1], c\text{-Ind}_{K_0}^G G_2[\theta], c\text{-Ind}_{K_0}^G G_2[\theta^2]\}.$$

In the next section, we shall describe a natural way to parametrize this family. We shall now investigate the parahoric restriction of these representations with respect to the *maximal parahoric subgroups* K_0, $K_1 := K_{\{\alpha_0, \alpha_2\}}$ and $K_2 := K_{\{\alpha_0, \alpha_1\}}$.

Firstly, consider the case $\pi = c\text{-Ind}_{K_0}^{G}\sigma$ for a cuspidal unipotent representation σ of $G_2(\mathbb{F}_q)$. By Frobenius reciprocity, it follows that $\pi^{U_{K_0}} = \sigma \neq 0$ and therefore by Theorem 3.11, the set $J_0 = \{\alpha_1, \alpha_2\}$ is minimal with respect to the property that $V^{U_J} \neq 0$. Suppose for a contradiction that $V^{U_{J_i}} \neq 0$ for $i = 1$ or $i = 2$, where $J_1 := \{\alpha_0, \alpha_2\}$ and $J_2 := \{\alpha_0, \alpha_1\}$. Since J_1 or J_2 cannot be minimal with respect to the same property, then $V^{U_{\{\alpha_0\}}} \neq 0$. But $\overline{K}_{\{\alpha_0\}}$ has no cuspidal unipotent representations, so $V^{K_\emptyset} = V^I \neq 0$, a contradiction to Corollary 3.13.

3.5 The Langlands parametrization of unipotent representations

In this section, we give an overview on the Langlands parametrization of unipotent representations achieved by Lusztig in his celebrated paper of 1995. Firstly, we briefly discuss the results of Kazhdan–Lusztig on the parametrization of Iwahori-spherical representation when G is a p-adic reductive group of *adjoint* type. Throughout, let $G^\vee$ be complex dual group of G.

We recall that the irreducible Iwahori-spherical representations are in bijection with the irreducible modules of $\mathcal{H}_{\mathcal{I}} = \mathcal{H}(G, \mathcal{I}, \mathbf{1})$. Let $\mathcal{B}$ be the variety of Borel subgroups of $G^\vee$ and let

$$\mathcal{Z} = \{(B, u, B') \in \mathcal{B} \times G^\vee \times \mathcal{B} : u \in B \cap B' \text{ unipotent}\}$$

be the Steinberg variety of G, playing a main role in the representation theory of $\mathcal{H}_{\mathcal{I}}$. Importantly, $G^\vee \times \mathbb{C}^\times$ acts on $\mathcal{Z}$ by

$$(g, \lambda)(B, u, B') = (gBg^{-1}, gu^{\lambda^{-1}}g^{-1}, gB'g^{-1}).$$

This action gives rise to the K-group $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$, which is naturally a $\mathbb{C}[z, z^{-1}]$-module and satisfies

$$K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z}) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C}_q \cong \mathcal{H}(G, \mathcal{I}, q). \tag{4}$$

Thus, we want to construct the $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-modules and then specialize to $\mathcal{H}$-modules via (4). This is performed most naturally with Borel-Moore homology.

Let $t \in G^\vee$ be semisimple and let $u \in G^\vee$ be unipotent such that $tut^{-1} = u^q$ and let $\mathcal{B}^{t,u} \subset \mathcal{B}$ be the subvariety of Borel subgroups containing t and u. Then it turns out that $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ is naturally a $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-module, usually reducible. Since these constructions are compatible with conjugation by elements of $G^\vee$, the group $Z_{G^\vee}(t, u)$ acts on $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ by $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-intertwiners. In fact, the neutral component of $Z_{G^\vee}(t, u)$ acts trivially, so we may regard it as an action of the component group $\pi_0(Z_{G^\vee}(t, u))$. This action can be used to decompose $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ as follows:

For each irreducible representation ρ of $\pi_0(Z_{G^\vee}(t, u))$ appearing in $H_*(\mathcal{B}^{t,u}, \mathbb{C})$, the space

$$K_{t,u,\rho} := \mathrm{Hom}_{\pi_0(Z_{G^\vee}(t,u))}(\rho, H_*(\mathcal{B}^{t,u}, \mathbb{C}))$$

is a nonzero $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-module, called standard. The data (t, u, ρ) are called *Kazhdan–Lusztig triples* for $(G^\vee, q)$.

Theorem 3.15. *Under the assumption that $G^\vee$ is simply connected, we have that*

1. *For each Kazhdan–Lusztig triple (t, u, ρ), the $\mathcal{H}$-module $K_{t,u,\rho}$ has a unique irreducible quotient $L_{t,u,\rho}$.*

2. *Every irreducible $\mathcal{H}$-module if of the form $L_{t,u,\rho}$ for some Kazhdan–Lusztig triple.*

3. *If (t', u', ρ') is another triple, then $L_{t,u,\rho} \cong L_{t',u',\rho'}$ if and only if there is some $g \in G$ such that $t' = gtg^{-1}$, $u' = gug^{-1}$ and $\rho' = \rho \circ \mathrm{Ad}(g^{-1})$.*

The above theorem is a major result and has many interesting consequences. However, the definition of a Kazhdan–Lusztig triple is slightly awkward since the pair (t, u) does not commute, and consequently the classification of these triples up to G-conjugacy seems hard. Thankfully, this situation can be remedied by considering *Kazhdan–Lusztig triples for* $(G^\vee, 1)$. These are defined analogously to the Kazhdan–Lusztig triples for (G, q) but replacing 1 for q throughout. In particular, the semisimple and unipotent part do commute.

Lemma 3.16. *Let G be a p-adic reductive group over a field F of residue cardinality q and let $G^\vee$ be its complex dual. There exists a bijection*

$$\begin{aligned}\{\textit{Kazhdan–Lusztig triples for } (G,1)\}/G &\longleftrightarrow \mathrm{Irr}(\mathcal{H}(G,\mathcal{I},q))\\ (t,u,\rho) \qquad &\longmapsto \quad L_{t_q,u,\rho_q},\end{aligned}$$

where and (t_q, u, ρ_q) are obtained from (t, u, ρ) in a prescribed way.

We recall that Kazhdan–Lusztig triples for $(G^\vee, 1)$ are defined to be tuples (t, u, ρ) such that ρ is *an irreducible character of $\pi_0(Z_{G^\vee}(tu))$ appearing in $H_*(\mathcal{B}(t, u), \mathbb{C})$*. This begs the question: if ρ does not satisfy this condition, does the triple (t, u, ρ) parametrize a (not Iwahori-spherical) representation of G?

This question was studied and completely resolved by Lusztig in his celebrated paper of 1995. He showed that, in order to get a bijection with all pairs (t, u, ρ) without technical conditions on ρ, one needs to consider a wider family of representations. Firstly, one needs to consider not only representations of G, but also of all of its *pure inner twists*. We let $\mathrm{InnT}^p(G)$ be the set of pure inner twists of G. A well known result states that there is a canonical bijection between the sets

$$\mathrm{InnT}^p(G) \longleftrightarrow H^1(F, \mathbf{G}^*) \longleftrightarrow \mathrm{Irr}(Z_{G^\vee}), \tag{5}$$

$$G' \longmapsto \zeta_{G'} \tag{6}$$

For instance, if G is a simply connected p-adic group, then $Z_{G^\vee} = \{1\}$ and therefore G has no pure inner twists other than itself. Secondly, one needs to consider all unipotent representations, and not just the Iwahori-spherical. The following theorem contains this information.

Theorem 3.17 (The arithmetic-geometric correspondence)**.** *There is an explicit bijection between the sets*

$$\bigcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \longleftrightarrow \mathcal{T}(\sqrt{q}) \longleftrightarrow \mathcal{T}(1),$$

where $\mathcal{T}(v_0)$ is set containing all triples (s, u, ρ) such that

- *$t \in G^\vee$ is semisimple,*
- *$u \in G^\vee$ is unipotent satisfying $tut^{-1} = u^{v_0^2}$,*
- *ρ is an irreducible representation of the group of components of the centralizer group $Z_{G^\vee}(t, u)$.*

For the remaining of the section, we explain how this result fits within the modern framework of the local Langlands correspondence. Let W_F be the Weyl group of the field F with inertia subgroup I_F. Moreover, we set $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$.

Under the assumption that $\mathbf{G}$ is a split group, we have the following important definition.

Definition 3.18. A *Langlands parameter* (or L-parameter) for G is a continuous morphism $\varphi : W_F' \to G^\vee$, where $G^\vee$ denotes the $\mathbb{C}$-points of the dual group of $\mathbf{G}$, and $\varphi((w, 1))$ is semisimple for each $w \in W_F$.

In its simplest form, the Local Langlands correspondence (LLC) conjectures the existence of a finite to one map between isomorphism classes of smooth admissible complex representations of G and conjugacy classes of Langlands parameters of G satisfying certain nice properties. Using Theorem 3.17, we will see that the the unipotent representations of G and its pure inner twists correspond to the following Langlands parameters.

Definition 3.19. An L-parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ is called *unipotent* if $\varphi(w, 1) = 1$ for any element w of the inertia subgroup I_F of W_F. Such parameters are sometimes called *unramified* Langlands parameters and we denote this set by $\Phi_{\mathrm{un}}(G^\vee)$.

Remark 3.20. For any L-parameter $\varphi : W_F' \to G^\vee$, define the commuting elements $u_\varphi = \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ and $s_\varphi = \varphi(\mathrm{Frob}, \mathrm{Id})$. An application of the Jacobson–Morozov theorem implies that an L-parameter is determined by u_φ and $\varphi|_{W_F}$ up to $G^\vee$-conjugacy. If the L-parameter is, in addition, unipotent, then $\varphi|_{W_F}$ is determined by s_φ. Thus, unipotent L-parameters are parametrized by $G^\vee$ conjugacy classes of pairs (u, s) where $u \in G^\vee$ is unipotent, $s \in G^\vee$ is semisimple and they commute. But this is the same as conjugacy classes of elements of $G^\vee$ (by using the Jordan decomposition). This should be reminiscent of the parametrization of Iwahori-spherical representations in Lemma 3.16.

However, under the LLC correspondence, unramified L-parameters do not parametrize unipotent representations, but rather L-packets of unipotent representations. To get a one to one correspondence, we need to introduce refinements of the L-parameters. Given an L-parameter φ, a natural object of interest is the component group A_φ of centralizer $Z_{G^\vee}(\varphi)$ of the image of φ inside $G^\vee$. We remark that when φ is unipotent, it is determined by the commuting elements s_φ and u_φ and therefore $Z_{G^\vee}(\varphi) = Z_{G^\vee}(s_\varphi u_\varphi)$. This object is completely analogous to the centralizer $Z_{G^\vee}(t, u)$, considered by Kazhdan and Lusztig in the setting of representations of Hecke algebras.

Definition 3.21. An *enhanced pure Langlands parameter* is a pair (φ, ϕ), where $\varphi : W_F' \to G^\vee$ is an L-parameter and ϕ is an irreducible representation of A_φ.

Let us introduce some important notation. Define

$$\Phi^p_{\mathrm{e,un}}(G^\vee) = G^\vee \backslash \{(\varphi, \phi) \mid \varphi \text{ unipotent}, \, \phi \in \widehat{A_\varphi}\},$$

which by the previous paragraph is in natural bijection with the set

$$G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\},$$

where A_x is the component group of $Z_{G^\vee}(x)$.

In this setting the Local Langlands conjecture predicts a natural bijection

$$\begin{array}{ccccc} \mathrm{LLC}^p_{\mathrm{un}} : G^\vee \backslash \{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} & \longleftrightarrow & \Phi^p_{\mathrm{e,un}}(G^\vee) \longleftrightarrow & \bigsqcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \\ (x,\phi) & & \longmapsto & \pi(x,\phi), \end{array}$$

where G' runs over the classes of *pure* inner twists of G.

We distinguish between the distinct pure inner twists by looking at characters of $Z_{G^\vee}$. By (5), each pure inner twist G' naturally corresponds to some character $\zeta_{G'}$ of $Z_{G^\vee}$. Similarly, for any pure enhanced L-parameter (φ, ϕ), the representation ϕ induces a character ζ_ϕ on $Z_{G^\vee_{sc}}$. We say that a pair (φ,ϕ) is G'-relevant if $\zeta_\phi = \zeta_{G'}$, in which case $\pi(x_\varphi, \phi) \in \mathrm{Irr}_{\mathrm{un}}(G')$ if φ is unipotent, and we denote the set of G'-relevant pure enhanced unipotent L-parameters by $\Phi^p_{e,\mathrm{un}}(G)$. It is then clear that

$$\Phi_{e,\mathrm{un}}(G^\vee) = \bigsqcup_{G' \in \mathrm{InnT}(G)} \Phi_{e,\mathrm{un}}(G'),$$

and the LLC predicts that $\Phi_{e,\mathrm{un}}(G')$ parametrizes the set $\mathrm{Irr}_{\mathrm{un}}(G')$ for each $G' \in \mathrm{InnT}(G)$.

Example 3.22. If $\mathbf{G}$ is a simple split *simply connected* algebraic group, then $H^1(F, \mathbf{G}^*) = 1$ and therefore there is only one class of pure inner forms of G, namely G itself. Correspondingly, $G^\vee = G^\vee_{\mathrm{ad}}$ and $Z_{G^\vee}$ is trivial. Therefore, the above discussion gives a bijection

$$\mathrm{LLC}^p_{\mathrm{un}} : G^\vee \backslash \{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \Phi^p_{\mathrm{e,un}}(G^\vee) \longleftrightarrow \mathrm{Irr}_{\mathrm{un}}(G^*).$$

Example 3.23. If $\mathbf{G}$ is a simple split *adjoint* algebraic group, then $H^1(F, \mathbf{G}^*) = H^1(F, \mathrm{Inn}(\mathbf{G}^*))$ so for each inner twist there is one unique pure inner twist. Therefore, from the previous discussion, unipotent enhanced L-parameters are in bijection with the set

$$G^\vee \backslash \{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\}, \quad \text{where} \quad A_x = Z_{G^\vee}(x)/Z_{G^\vee}(x)^0,$$

and we have a one-to-one correspondence

$$G^\vee \backslash \{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \bigsqcup_{G' \in \mathrm{InnT}(G)} \mathrm{Irr}_{\mathrm{un}}(G'), \quad (x,\phi) \longmapsto \pi(x,\phi)$$

3.6 Unipotent conjugacy classes of complex simple groups

In the previous paragraph we stated the unramified local Langlands correspondence, which reduces the classification of unipotent representations of G to the classification of conjugacy classes or $G^\vee$ and the structure of the component group of their centralizer. To understand these, one first studies the classification of unipotent conjugacy classes of $G^\vee$, an interesting problem on its own right that uncovers rich structure inside $G^\vee$.

Define $\mathcal{U}$ to be the set of unipotent elements of $G^\vee$. This can be seen to be a closed irreducible subvariety of $G^\vee$ of dimension $\dim G^\vee - \mathrm{rk} G^\vee$. If $u \in \mathcal{U}$ is a unipotent element, its conjugacy class $C(u) \subset H$ is the orbit of u under the conjugation action of $G^\vee$ on itself. Standard results in the structure theory of unipotent elements inside complex reductive groups state that $G^\vee$ has finitely many conjugacy classes of unipotent elements, and that each class C is a locally closed subvariety of $G^\vee$. Moreover, its closure $\overline{C}$ is the union of (finitely many) unipotent conjugacy classes. In particular, there is one unique unipotent conjugacy class C_{reg} of maximal dimension such that C_{reg} is open and $\overline{C_{\text{reg}}} = \mathcal{U}$. Such unipotent elements are called *regular*, and $\dim Z_{G^\vee}(u) = \mathrm{rk} G^\vee$ for any $u \in C_{\text{reg}}$. The boundary of C_{reg} has dimension $\dim G^\vee - \mathrm{rk} G^\vee - 2$ and contains a unique dense unipotent conjugacy class C_{subreg} of *subregular* unipotent elements such that

$$\overline{C_{subreg}} = \overline{C_{\text{reg}}} - C_{\text{reg}} = \mathcal{U} - C_{\text{reg}}.$$

Similarly, $\dim_{Z_{G^\vee}}(u) = \mathrm{rk} G^\vee + 2$ for any $u \in C_{subreg}$. At the other end, there is the trivial class consisting of $\{1\}$, and this is the only closed conjugacy class. There is one further "canonical orbit", the set of *minimal* unipotent elements $C_{\min}$, with the property that they are contained in the closure of every unipotent conjugacy class except for $\{1\}$.

Beyond these four classes, the structure of $\mathcal{U}$ for a general simple complex group can be complicated. To study it, one can define a partial ordering on the set of unipotent conjugacy classes given by

$$C \leq C' \quad \text{if and only if} \quad \overline{C} \subseteq \overline{C'}.$$

One can then picture this partial order in a diagram, called a *Hasse diagram*, and one has the following generic picture.

orbit	dimension
$\mathcal{O}_{prin}$	$\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g})$
$\mathcal{O}_{subreg}$	$\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g}) - 2$
fuzzy structure	
$\mathcal{O}_{min}$	See (4.3.5)
$\{0\}$	0

Example 3.24. If $\mathbf{G}$ is a simple split algebraic group of type G_2, then $G = \mathbf{G}(F)$ is both adjoint and simply connected and consequently there is a bijection

$$G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \mathrm{Irr}_{\mathrm{un}}(G) = \mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) \bigcup \{c\text{-}\mathrm{Ind}_{K_0}^G G_2[\alpha] \mid \alpha \in \{1, -1, \theta, \theta^2\}\}.$$

Let us indicate which pairs $(x = su, \rho)$ in the left correspond to the 4 unipotent supercuspidal representations of G_2. Since supercuspidal representations are square-integrable (Is this true?), it is enough to look at the regular and subregular unipotent elements.

- Let $u = u_{\text{reg}}$ be the regular unipotent element. In that case $A_u = 1$ and therefore $s = 1$, $A_x = A_u = 1$ and ρ is the trivial representation. The corresponding representation $\pi(u_{\text{reg}}, \mathbf{1})$ is the Steinberg representation.

- Let $u = u_{\text{sr}}$ be the subregular unipotent element. In that case, $A_u = S_3$ so up to conjugacy, $s \in \{1, g_2, g_3\}$ where g_i is a lift of order i from A_u to $Z_{G^\vee}(u)$. Moreover, $A_{ug_2} = S_3$ and $A_{ug_3} = s_2$. The corresponding table gives the required parametrization.

Langlands parameter (u, s, ϕ)	$K_0 = G_2$	$K_1 = A_1 + \widetilde{A}_1$	$K_2 = A_2$
$\pi(G_2, 1, \mathbf{1})$	$\phi_{(1,6)}$	$\epsilon \otimes \epsilon$	ϵ
$\pi(G_2(a_1), 1, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,1)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, r)$	$\phi'_{(1,3)}$	$\epsilon \otimes \mathbf{1}$	ϵ
$\pi(G_2(a_1), g_3, \mathbf{1})$	$\phi_{(1,6)} + \phi''_{(1,3)}$	$\mathbf{1} \otimes \epsilon + \epsilon \otimes \epsilon$	r
$\pi(G_2(a_1), g_2, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,2)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, \epsilon)$	$G_2[1]$	0	0
$\pi(G_2(a_1), g_2, \epsilon)$	$G_2[-1]$	0	0
$\pi(G_2(a_1), g_3, \theta)$	$G_2[\theta]$	0	0
$\pi(G_2(a_1), g_3, \theta^2)$	$G_2[\theta^2]$	0	0

4 Parahoric restriction of unipotent representations

Let G be a connected split simple group *of adjoint type* over a p-adic field F. In the previous section we stated the bijection

$$\begin{array}{ccc} \mathrm{LLC}^p_{\mathrm{un}} : G^\vee \backslash \{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} & \longleftrightarrow & \bigsqcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \\ (x,\phi) & \longmapsto & (\pi_{(x,\phi)}, V_{(x,\phi)}), \end{array}$$

a result that parametrizes unipotent representations of G in terms of the geometry of its complex dual group $G^\vee$ and lies in the heart of the local Langlands correspondence.

In order to verify the conjecture from the introduction for the p-adic group G, one needs to explicitly compute the parahoric restriction maps

$$\mathrm{res}^J_{\mathrm{un}} : R_{\mathrm{un}}(G) \longrightarrow R_{\mathrm{un}}(\overline{K}_J), \quad V \longmapsto V^{U_J}$$

from unipotent representations of G to unipotent representations of $\overline{K}_J$. The aim of this section is to describe a general approach to compute the parahoric restriction maps above. These methods are quite complex and subtle, and involve deep mathematics – the difficulty resides in the fact that the parametrization of the unipotent representations of p-adic groups (given by Langlands parameters) and that of finite groups of Lie type (given by Lusztig's labels) are not related in an obvious way.

To do this, we closely follow the discussion in [Re2000,§4,5,6,8]. Let us fix some subset $J \subsetneq S_{\mathrm{aff}}$ and a unipotent representation $V = V_{(x,\phi)}$ of G, with $x \in G^\vee$ and $\phi \in \widehat{A_x}$. We want to decompose the $\overline{K}_J$-module V^{U_J} as a direct sum of irreducible submodules. More concretely, for each irreducible $\overline{K}_J$-representation χ, we want to calculate the value of

$$\langle \chi, V^{U_J} \rangle_{\overline{K}_J} = \langle \chi, V^{U_J} \rangle_{K_J},$$

where we view the $\overline{K}_J$-representations as K_J-representations trivial on U_J.

From Theorem 3.11 and Corollary 3.13, we deduce that all irreducible $\overline{K}_J$-constituents of V^{U_J} lie in the same series representations. Such representations are in bijection with irreducible representations of some canonical associated reflection group. For example, if V is Iwahori-spherical, then all irreducible $\overline{K}_J$-constituents of V^{U_J} lie in the principal series. These representations are in bijection with irreducible representations of the Weyl group W_J of $\overline{K}_J$ (see Section 2.3), generated by the simple reflections corresponding to the simple affine roots in J. With this fact in mind, the method strategy becomes transparent: starting with the subset $J \subsetneq S_{\mathrm{aff}}$ and the unipotent representation V, we construct a representation over the associated reflection group whose direct sum into irreducible representations is compatible with the decomposition of V^{U_J} into K_J-irreducible representations under the bijection stated above. This method consists in two major reduction steps; firstly, a reduction to Hecke algebras modules and, secondly, a reduction to modules over affine Weyl groups.

For each irreducible K_J-representation χ trivial on U_J, we want to calculate the value of $\langle \chi, V^{U_J} \rangle_{K_J}$. Firstly, by the well-known results of Harish–Chandra, there is some parahoric subgroup $(K, U_K, \overline{K})$ contained in $(K_J, U_J, \overline{K}_J)$ and cuspidal representation σ of K trivial on U_K such that

$$\chi^\sigma := \mathrm{Hom}_K(\sigma, \chi) = \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, \chi) \neq 0.$$

The space χ^σ is a naturally a $\mathcal{H}(K_J, K, \sigma)$-module, where $\mathcal{H}(K_J, K, \sigma) \cong \mathrm{End}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma)$ is the subalgebra of functions of $\mathcal{H}(G, K, \sigma)$ supported on K_J. On the other hand, the vector space

$$V^\sigma := \mathrm{Hom}_K(\sigma, V)$$

is naturally an irreducible $\mathcal{H}(G, K, \sigma)$-module and therefore a (potentially reducible) $\mathcal{H}(K_J, K, \sigma)$-module by restriction.

Lemma 4.1. *[First reduction] The multiplicity of the simple K_J-module χ in V^{U_J} is given by*

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)}.$$

Proof. Since $U_J \subseteq U$, we have that

$$\begin{aligned} V^\sigma = \mathrm{Hom}_K(\sigma, V) = \mathrm{Hom}_K(\sigma, V^{U_J}) &\simeq \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, V^{U_J}) \\ &\simeq \bigoplus_\eta \langle \eta, V^{U_J} \rangle_{K_J} \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, \eta) \\ &\simeq \bigoplus_\eta \langle \eta, V^{U_J} \rangle_{K_J} \eta^\sigma, \end{aligned}$$

and therefore

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)} = \sum_\eta \langle \chi^\sigma, \eta^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)} \langle \eta, V^{U_J} \rangle_{K_J} = \langle \chi, V^{U_J} \rangle_{K_J},$$

as desired. □

For example, if V is Iwahori-spherical of G, all irreducible components χ of V^{U_J} are principal series representations, so $\chi^I = \mathrm{Hom}_I(\mathbf{1}, \chi) \neq 0$. Lemma 4.1 states that

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^{\mathcal{I}}, V^{\mathcal{I}} \rangle_{\mathcal{H}(K_J, \mathcal{I}, \mathbf{1})}.$$

Now let us consider the second reduction step. For this one, we let $R = \mathbb{C}(v, v^{-1})$, where v is an indeterminate. We then define $\mathcal{H}(G, K, \sigma)_v$ to be the *generic Hecke algebra* defined over R with the same generators and relations as $\mathcal{H}(G, K, \sigma)$, but with q replaced by v^2. The upshot of considering this generic Hecke algebra is that by specializing v we can recover

$$\mathcal{H}(G, K, \sigma)_{\sqrt{q}} = \mathcal{H}(G, K, \sigma), \qquad \mathcal{H}(G, K, \sigma)_1 = \mathbb{C}[\tilde{W}(K, \sigma)].$$

When $K = \mathcal{I}$ is the Iwahori subgroup and $\sigma = \mathbf{1}$ is the trivial representation, $\mathcal{H}(G, \mathcal{I}, \mathbf{1}) = \mathbb{C}[\widetilde{W}]$, where $\widetilde{W}$ is the extended affine Weyl group.

Fact: For any simple $\mathcal{H}(G, K, \sigma)$-module E *considered in this document*, there is a $\mathcal{H}(G, K, \sigma)_v$-module E_v such that

$$E \simeq E_v \otimes_R \mathbb{C}, \text{ where } f \in R \text{ acts on } \mathbb{C} \text{ by } f(\sqrt{q}).$$

It then follows that we have a $q = 1$ operation that takes simple modules over Hecke algebras to modules over the corresponding Weyl groups, obtained by setting $v = 1$ in all matrix coefficients of the generic module.

Proposition 4.2 (Second reduction). *Let $J \subsetneq S_{\text{aff}}$ and let $(K, U_K, \overline{K})$ be a parahoric subgroup contained in $(K_J, U_J, \overline{K}_J)$ with cuspidal unipotent $\overline{K}$-representation σ. Then the diagram*

$$\begin{array}{ccc} \mathcal{H}(G,K,\sigma)\text{-}mod & \xrightarrow{q=1} & \tilde{W}(K,\sigma)\text{-}mod \\ \Big\downarrow{\scriptstyle \mathrm{Res}_{\mathcal{H}(K_J,K,\sigma)}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{\tilde{W}(K,\sigma)_J}} \\ \mathcal{H}(K_J,K,\sigma)\text{-}mod & \xrightarrow{q=1} & \tilde{W}(K,\sigma)_J\text{-}mod \end{array}$$

is commutative, and the bottom arrow is an isometry with respect to the usual inner product of character. That is, for any irreducible K_J-module χ, we have that

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} = \langle \chi^\sigma_{q=1}, V^\sigma_{q=1} \rangle_{\tilde{W}(K,\sigma)_J}. \tag{7}$$

Proof. □

Combining both reduction steps, we obtain the identity

$$\langle \chi, V^{U_J} \rangle_{\overline{K}_J} = \langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} = \langle \chi^\sigma_{q=1}, V^\sigma_{q=1} \rangle_{\tilde{W}(K,\sigma)_J}$$

Thus, the $\tilde{W}(K,\sigma)_J$-module $V^\sigma_{q=1}$ satisfies the desired properties. By calculating its decomposition into $\tilde{W}(K,\sigma)$-irreducible representation, one can directly obtain the decomposition of V^{U_J} into irreducible $\overline{K}_J$-representations.

At this point, we restrict our attention to the parahoric restriction of Iwahori-spherical representations, for the method we use to calculate the $K(I,\mathbf{1})_J$-module $V^{\mathbf{1}}_{q=1}$ differs significantly from the non Iwahori-spherical case. As stated above, for Iwahori-spherical representations, the reflection subgroup $K(I,\mathbf{1}) \cong \widetilde{W}$ is simply the extended affine Weyl group of G, and $\widetilde{W}_J$ is the subgroup of $\widetilde{W}$ generated by the simple reflections of the roots in J.

4.1 The Iwahori-spherical case and Springer Correspondence

As indicated in the previous paragraph, we now fix an Iwahori-spherical representation $V = V_{(x,\phi)}$ of G and some proper subset $J \subsetneq S_{\text{aff}}$. The two reduction steps described above imply that for any principal series representation χ of $\overline{K}_J$,

$$\langle \chi, V^{U_J} \rangle_{\overline{K}_J} = \langle \chi^{\mathbf{1}}_{q=1}, V^{\mathbf{1}}_{q=1} \rangle_{\widetilde{W}_J}. \tag{8}$$

When $V_{(x,\phi)}$ is Iwahori-spherical, we can explicitly describe the structure of the $\widetilde{W}_J$-module $V^{\mathbf{1}}_{q=1}$ explicitly in terms of the geometry of the dual group $G^\vee$ and the data (x,ϕ). In order to extract this information, Springer theory is the key theoretical tool that we shall require. To explain this, consider the Jordan decomposition $x = us$, with commuting $u \in G^\vee$ unipotent and $s \in G^\vee$ semisimple. Let $\mathcal{B}$ and $\mathcal{B}_s$ be the flag varieties of the complex reductive groups $G^\vee$ and $G^\vee_s := Z_{G^\vee}(s)$, respectively. These are algebraic varieties parametrizing the set of Borel subgroups in $G^\vee$ and $G^\vee_s$, respectively, and admit a rational action by conjugation. Noting that $u \in G^\vee_s$, we consider the stabilizers $\mathcal{B}^x$ and $\mathcal{B}^u_s$, called partial flag varieties. These are algebraic subvarieties parametrizing the set of Borel subgroups of $G^\vee$ containing x and the set of Borel subgroups of $G^\vee_s$ containing u.

The varieties $\mathcal{B}_x$ and $\mathcal{B}^u_s$ admit a natural action of A_x induced by the action of $Z_{G^\vee}(x)$ by conjugation. On the other hand, these varieties do not admit a natural action of the Weyl groups W and W_s of $G^\vee$ and $G^\vee_s$, respectively. To obtain representations over the Weyl groups, one needs to consider the singular cohomology spaces $H^*(\mathcal{B}_x)$ and $H^*(\mathcal{B}^u_s)$. These spaces admit not only an A_x-action inherited from the action on the partial flag varieties, but also a natural representation of Weyl groups W and W_s, respectively. Moreover, both actions commute. These last two statements are at the heart of Springer theory, which we shall use continuously from now on. The mathematics behind these results involve sophisticated geometric machinery, so for the most part we will state the results we need without proof. At the top of Springer theory there is the Springer correspondence, which we now state for convenience.

Theorem 4.3. *Let $G^\vee$ be a complex algebraic reductive group with Weyl group W. For each pair (u, ϕ), where $u \in G^\vee$ is a unipotent element and ϕ is an irreducible character of A_u, the ϕ-isotropic subspace of the top cohomology group $H^{\mathrm{top}}(\mathcal{B}^u)^\phi$ is either trivial or an irreducible W-representation. Moreover, each irreducible character of W arises this way for exactly one pair (u, ϕ) up to $G^\vee$ conjugation. In other words, there is an injection*

$$\mathrm{Irr}(W) \hookrightarrow \{\textit{pairs } (u, \phi)\}/G^\vee.$$

Finally, for any unipotent $u \in G^\vee$, the pair $(u, \mathbf{1})$ lies in the image of the map above.

This result is of great importance for it allows us to extract information from the top cohomology group directly. When $G^\vee$ is a classical group, one can give precise description of the injection in terms of combinatorial data. When $G^\vee$ is of exceptional type, one only has a finite amount of information and the injection above can be found in the literature (See tables from Carter). Another important fact that greatly simplifies calculations is the following:

Theorem 4.4. *The odd-degree cohomology spaces of $\mathcal{B}^x$ vanish, so $H^*(\mathcal{B}^x) = \sum_{i=1}^{\dim \mathcal{B}^x} H^{2i}(\mathcal{B}^x)$.*

Example 4.5. If $G^\vee = \mathrm{GL}_n(\mathbb{C})$, by the Jordan decomposition theorem, unipotent conjugacy classes are parametrized by partitions of n and $A_u = \{1\}$ for any unipotent element u. Moreover, $W \cong S_n$, and its irreducible representations are also labelled by partitions of n. If λ is a partition of n, then the Springer correspondence maps V_λ to $(u_\lambda, \mathbf{1})$, where V_λ is the Specht module of S_n and u_λ is a unipotent element of $\mathrm{GL}_n(\mathbb{C})$ corresponding to λ. In particular, the Springer correspondence for $\mathrm{GL}_n(\mathbb{C})$ is a bijection.

If $G^\vee = \mathrm{SL}_n(\mathbb{C})$, conjugacy classes of unipotent elements are still parametrized by partitions of n. This time, however, A_u may be non-trivial for some unipotent classes. The Springer correspondence is therefore not a bijection, and the image of the injection are all pairs $\{(u, \mathbf{1}), u \in \mathrm{SL}_n(\mathbb{C}) \text{ unipotent}\}$.

When $G^\vee$ has type A_n, one can describe the entire cohomology complex $H^*(\mathcal{B}^u)^{\mathbf{1}}$ explicitly. Suppose that $u = u_\lambda$ for a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n. If $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r} \leq S_n$, then Lusztig and Spaltenstein showed

$$H^*(\mathcal{B}^{u_\lambda}) \cong \mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{1}$$

as graded S_n-modules.

Example 4.6. The complex reductive group $G^\vee = G_2(\mathbb{C})$ has Weyl group $W \cong D_6$ and 5 unipotent classes labelled and partially ordered by

$$1 < A_1 < \tilde{A}_1 < G_2(a_1) < G_2.$$

The component groups are all trivial except the subregular unipotent class, whose component group is isomorphic to S_3, with irreducible representations $\mathbf{1}, \varepsilon$ and r (the unique 2-dimensional). The characters of W are labelled in the literature by the symbols $\phi_{1,0}$ (trivial), $\phi_{1,6}$ (sign), $\phi'_{1,3}, \phi''_{1,3}$ (the other two characters), $\phi_{2,1}$ (faithful), $\phi_{2,2}$ (lifted from S_3). Then the Springer correspondence gives the pairing

$$\phi_{1,0} \leftrightarrow (G_2, \mathbf{1}), \quad \phi_{2,1} \leftrightarrow (G_2(a_1), \mathbf{1}), \quad \phi'_{1,3} \leftrightarrow (G_2(a_1), r), \quad \phi_{2,2} \leftrightarrow (\tilde{A}_1, \mathbf{1}), \quad \phi''_{1,3} \leftrightarrow (A_1, \mathbf{1}), \quad \phi_{1,6} \leftrightarrow (1, \mathbf{1}),$$

and the pair $(G_2(a_1), \varepsilon)$ is not in the image of the correspondence.

Having explained the Springer correspondence, we can go back to our initial discussion. We now know that $H^*(\mathcal{B}^x)$ has the structure of a $A_x \times W$-module. We can extend this to a $A_x \times \widetilde{W}$-action as follows. The extended Weyl group can be decomposed as a semidirect product $\widetilde{W} = W \ltimes X$, where X is the character lattice of $T^\vee$. Then there is a natural evaluation pairing $X \to \mathbb{C}^\times$ given by $\mu \mapsto \langle s, \mu \rangle$. The character lattice X then acts on $H^*(\mathcal{B}^x)$ by scalars $\langle s, \cdot \rangle$, and this extends the action as desired. Analogously, the $A_x \times W_s$-action of $H^*(\mathcal{B}_s^u)$ can be extended to a $A_x \times \widetilde{W}_s$-action. The main result we need is due to Lusztig.

Theorem 4.7. *Let $V_{x,\phi}$ be an Iwahori-spherical irreducible representation of G. After letting $q \to 1$ and then taking semisimplification, the simple $\mathcal{H}(G, I, \mathbf{1})$-module $V_{x,\phi}^{\mathbf{1}}$ becomes the $\widetilde{W}$-module $\varepsilon \otimes H^*(\mathcal{B}^x)^\phi$.*

Remark 4.8. An important remark is due at this point. The above theorem relates the structure of $V^{\mathbf{1}}_{(x,\phi),q=1}$ as a $\widetilde{W}$-module with the structure of $H^*(\mathcal{B}^x)^\phi$ as a $\widetilde{W}$-module too. However, the former object is naturally p-adic, while the second is complex analytic. The Weyl groups of G and $G^\vee$ are canonically isomorphic, but not equal, since, for example, the isomorphism swaps short with long roots when the group is not simply laced. It is a standard abuse of notation to denote both Weyl groups with the same symbol, but one needs to be careful during explicit calculations inside which Weyl group one is working. Of course, we are ultimately interested in the p-adic side, but since we will be computing the cohomology groups, we will mostly work inside the complex side. In particular, the reflection s_0 inside W will be a **short** reflection, corresponding to the highest short root. At the very end, we then trace the representations back to the p-adic side. See example with G_2.

This theorem is the key theoretical tool that allows the use of Springer theory to compute parahoric restrictions of unipotent representations of G. However, it is not immediately clear how to use Springer theory yet, since $x \in G^\vee$ need not be a unipotent element. Naturally, the solution comes from relating $H^*(\mathcal{B}^x)$ to $H^*(\mathcal{B}_s^u)$. The result is due to Kato.

Proposition 4.9. *The natural restriction map $H^*(\mathcal{B}^x)^\phi \to H^*(\mathcal{B}_s^u)^\phi$ is $\widetilde{W}_s$-equivariant, and induced an isomorphism of $\widetilde{W}$-modules*

$$H^*(\mathcal{B}^x)^\phi \cong \operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}_s^u)^\phi.$$

By (8), Theorem 4.7 and Proposition 4.9, the restriction of $V_{x,\phi}$ can be determined by computing the restriction

$$\left(\operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}_s^u)^\phi\right)|_{\widetilde{W}_J},$$

where $\widetilde{W}_J$ is the subgroup of $\widetilde{W}$ generated by all simple reflections of the roots in J. Working inside $\widetilde{W}$ is very delicate, so following Reeder's development, we decide to work inside $W = \widetilde{W}_{J_0}$. The downside is the fact that, even though W contains an isomorphic copy of $\widetilde{W}_J$, one needs to carefully track this isomorphism since it can twist the original representation by some character.

More concretely, consider the image W_J of the group $\widetilde{W}_J$ under the natural projection map $\widetilde{W} \twoheadrightarrow W$. This restricts to an isomorphism $\widetilde{W}_J \cong W_J$ with inverse map $\psi_J : W_J \to \widetilde{W}_J$, satisfying $\Psi_J(s_\alpha) = s_\alpha$ if $\alpha \in J, \alpha \neq -\alpha_0$ and $\psi_J(s_0) = \tilde{s}_0 = t_{\alpha_0} s_0$ if $\tilde{s}_0 \in J$. Let ψ_J^* the pullback of representation of $\widetilde{W}_J$ to W_J.

Given a semisimple element $t \in G^\vee$, let $W_{J,t} = W_t \cap W_J$ and define the character

$$\chi_t^J := \chi_t \circ \psi_J : W_{J,t} \longrightarrow \mathbb{C}^\times. \tag{9}$$

Then by Mackey theory we obtain the isomorphism

$$\psi_J^* \left[\left(\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}_s^u)^\phi \right) |_{\widetilde{W}_J} \right] \cong \bigoplus_{w \in W_s \backslash W / W_J} \mathrm{Ind}_{W_{J,s^w}}^{W_J} \chi_{s^w}^J \otimes \left[H^*(\mathcal{B}_s^u)^\phi \right]^w. \tag{10}$$

Springer theory tells us the structure of the cohomology groups, so it remains to understand the groups $W_{J,t}$ and the characters χ_t^J. There is a substantial amount of theory behind these objects, and we will only state the results we will need. Associated to J, there is a complex reductive subgroup $G_J^\vee$ of $G^\vee$, whose roots have integral basis J and Weyl group W_J. Then, there is a semidirect product decomposition

$$W_{J,t} \cong W_{J,t}^\circ \ltimes R_{J,t},$$

where $W_{J,t}^\circ$ is a reflection subgroup of W_J generated by the roots of $G^\vee$ trivial on t. It can then be shown that the character χ_t^J is trivial on $W_{J,t}^\circ$. Moreover, if the maybe complete the character explanation here later.

The following easy lemma shows that if the characters χ_t^J are trivial, then computations are greatly simplified.

Lemma 4.10. *With the same notation as in* (10), *if the characters $\chi_{s^w}^J$ are trivial for all $w \in W_s \backslash W / W_J$, then*

$$\psi_J^* \left[\left(\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}_s^u)^\phi \right) |_{\widetilde{W}_J} \right] \cong \left(\mathrm{Ind}_{W_s}^{W} H^*(\mathcal{B}_s^u)^\phi \right) |_{W_J}$$

Proof. This is an immediate consequence of Mackey's formula applied to the right hand side. ☐

4.2 Parahoric restriction for type G_2

To finish this chapter, we finish with a worked example of the above methods by computing some parahoric restrictions for the p-adic group $G = G_2(F)$. We will restrict our attention to classes of maximal parahoric subgroups, for the other ones can be easily deduced from these. The group $G_2(F)$ has simple roots $\{-\beta_0, \beta_1, \beta_2\}$ with Dynkin diagram

$$0 - 1 \Longrightarrow 2.$$

Thus $G_2(F)$ has 3 classes of maximal parahoric subgroups K_0, K_1, K_2 corresponding to the subsets of affine simple roots $J_0 = \{\beta_1, \beta_2\}, J_1 = \{-\beta_0, \beta_2\}$ and $J_2 = \{-\beta_0, \beta_1\}$ and with reductive types G_2, $A_1 \times \tilde{A}_1$ and A_2, respectively. We now compute these parahoric restrictions for all representations $\pi(us, \phi)$ of $G = G_2(F)$, where u is a subregular unipotent element of $G_2(\mathbb{C})$ labelled by $G_2(a_1)$. This is an interesting case since u is a distinguished unipotent element and $\Gamma_u = A_u \cong S_3$. There are 8 such representations, given by

$$\pi(G_2(a_1), 1, \phi), \phi \in \{\mathbf{1}, \varepsilon, r\}, \quad \pi(G_2(a_1), g_2, \phi), \phi \in \{\mathbf{1}, \varepsilon\}, \quad \pi(G_2(a_1), g_3, \phi), \phi \in \{\mathbf{1}, \theta, \theta^2\}.$$

It turns out that 4 of them are unipotent supercuspidal representations of $G_2(F)$ given by

$$\pi(G_2(a_1), 1, \epsilon) = c\text{-Ind}_{K_0}^G G_2[1], \quad \pi(G_2(a_1), g_2, \epsilon) = c\text{-Ind}_{K_0}^G G_2[-1], \quad \pi(G_2(a_1), g_3, \theta^k) = c\text{-Ind}_{K_0}^G G_2[\theta^k], k = 1, 2,$$

while the remaining 4 are Iwahori-spherical. For the supercuspidal representations, their restrictions can be easily computed. By Frobenius reciprocity, their parahoric restrictions to K_0 is irreducible. The restrictions to K_1 and K_2 are both 0, since all representations of $A_1 \times \tilde{A}_1$ and A_2 are principal series.

Thus, we can focus our attention to Iwahori-spherical representations. The computations are quite lenghty, so we give a complete sketch while omitting some unenlightening steps. We recall that we are working inside the *complex* Weyl group W and consequently s_0 is seen as a short reflection along the highest short root, and we denote by s^0 the reflection along the highest root. We first note that u is a subregular unipotent element in $G^\vee = G_2(\mathbb{C})$, while it is a regular unipotent element in $G^\vee_{g_2} = \mathrm{GL}_2(\mathbb{C})$ and $G^\vee_{g_3} = \mathrm{SL}_3(\mathbb{C})$. Therefore, $\dim \mathcal{B}^u = 1$, while $\dim \mathcal{B}^u_{g_2} = \dim \mathcal{B}^u_{g_3} = 0$. By Springer theory, we obtain that

$$H^*(\mathcal{B}^u)^{\mathbf{1}} = H^0(\mathcal{B}^u)^{\mathbf{1}} + H^2(\mathcal{B}^u)^{\mathbf{1}} = \mathbf{1} + \phi_{2,1}, \quad \text{and} \quad H^*(\mathcal{B}^u)^r = H^0(\mathcal{B}^u)^r + H^2(\mathcal{B}^u)^r = \phi'_{1,3},$$

as W-modules, while $H^0(\mathcal{B}^u_{g_2})$ and $H^0(\mathcal{B}^u_{g_3})$ afford the trivial representations of W_{g_2} and W_{g_3} respectively, and

$$\mathrm{Ind}_{W_{g_2}}^W \mathbf{1} = \phi_{1,0} + \phi''_{1,3} \quad \text{and} \quad \mathrm{Ind}_{W_{g_3}}^W \mathbf{1} = \phi_{1,0} + \phi_{2,2}.$$

To obtain the parahoric restrictions to K_0, we note that all characters $\chi_t^{J_0}$ so it remains to twist by $\varepsilon = \phi_{1,6}$ (swapping $\phi_{1,0} \leftrightarrow \phi_{1,6}$ and $\phi'_{1,3} \leftrightarrow \phi''_{1,3}$) and then by the outer automorphism of W exchanging short and long roots (swapping $\phi'_{1,3} \leftrightarrow \phi''_{1,3}$ only).

To compute the other two columns of Table 1 we compute the remaining characters χ_t^J. Using their properties above, it is not hard to show that all characters we are considering are trivial except for $\chi_{g_2^w}^{J_1}$ and $\chi_{g_3^w}^{J_2}$, and we can use directly Lemma 4.10 to compute the restrictions. The last cases must be dealt with separately.

- **Restriction of $\pi(G_2(a_1), g_3, 1)$ to K_{J_2}.** In this case, $W_{J_2} = \langle s_0, s_1 \rangle$ (where s_0 is the reflection along the highest short root) and $W_{g_3} = \langle s^0, s_2 \rangle$ (where s^0 is the reflection along the highest root). Then $W_{g_3} \backslash W / W_{J_2} = \{1\}$, $W_{J_2, g_3} \cong C_3$ and $\chi_{g_3}^{J_2}$ is a primitive character. Thus,

$$\mathrm{Ind}_{W_{J_2, g_3}}^{W_{J_2}} \chi_{g_3}^{J_2} = r.$$

- **Restriction of $\pi(G_2(a_1), g_2, 1)$ to K_{J_1}.** In this case, $W_{J_1} = \langle s_0, s_2 \rangle$ and $W_{g_2} = \langle s^0, s_1 \rangle$. Then $W_{g_2} \backslash W / W_{J_1} = \{1, w\}$ has two elements, and $W_{J_1, g_2} \cong C_2$ with $\chi_{g_2}^{J_1}$ non-trivial and $W_{J_1, g_2^w} = W_{J_1}$ with $\chi_{g_2^w}^{J_1}$ the trivial character. Thus,

$$\mathrm{Ind}_{W_{J_1, g_2}}^{W_{J_1}} \chi_{g_2}^{J_1} + \mathrm{Ind}_{W_{J_1, g_2^w}}^{W_{J_1}} \chi_{g_2^w}^{J_1} = \varepsilon \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \varepsilon + \mathbf{1} \boxtimes \mathbf{1}.$$

Langlands parameter (u, s, ϕ)	$K_0 = G_2$	$K_1 = A_1 + \widetilde{A}_1$	$K_2 = A_2$
$\pi(G_2, 1, \mathbf{1})$	$\phi_{(1,6)}$	$\epsilon \otimes \epsilon$	ϵ
$\pi(G_2(a_1), 1, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,1)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, r)$	$\phi'_{(1,3)}$	$\epsilon \otimes \mathbf{1}$	ϵ
$\pi(G_2(a_1), g_3, \mathbf{1})$	$\phi_{(1,6)} + \phi''_{(1,3)}$	$\mathbf{1} \otimes \epsilon + \epsilon \otimes \epsilon$	r
$\pi(G_2(a_1), g_2, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,2)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, \epsilon)$	$G_2[1]$	0	0
$\pi(G_2(a_1), g_2, \epsilon)$	$G_2[-1]$	0	0
$\pi(G_2(a_1), g_3, \theta)$	$G_2[\theta]$	0	0
$\pi(G_2(a_1), g_3, \theta^2)$	$G_2[\theta^2]$	0	0

Figure 1: Restrictions of G_2-representations $\pi(G_2(a_1), s, \phi)$

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5 The dual nonabelian Fourier transform for unipotent representation of p-adic groups

It is therefore a natural question to ask whether there exists some function $\mathrm{FT}^\vee : R_{\mathrm{un}}(G) \to R_{\mathrm{un}}(G)$ such that the square

$$\begin{tikzcd} R_{\mathrm{un}}(G) \arrow[r, "\mathrm{FT}^\vee"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & R_{\mathrm{un}}(G) \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] \\ \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \end{tikzcd}$$

This question is now unresolved, but partial progress has been achieved. To understand it, we first need to look at the Langlands parametrization of unipotent representations.

5.1 The unipotent elliptic space and the dual nonabelian Fourier transform

With the Langlands parametrization, it is then possible to define a dual Fourier transform on a certain subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$, which we now describe. We first fix some unipotent element $u \in G^\vee$ up to $G^\vee$-conjugacy and we denote Γ_u the reductive part of $Z_{G^\vee}(u)$. We then consider the space of elliptic pairs

$$\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(s,h) \mid s,h \in \Gamma_u \text{ semisimple, } sh = hs \text{ and } Z_{G^\vee}(s,h) \text{ is finite}\}$$

up to Γ_u-conjugacy. Then for each $(s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, we define the virtual representation

$$\Pi(u,s,h) := \sum_{\phi \in \widehat{A_{su}}} \phi(h)\pi(su,\phi).$$

Definition 5.1. The elliptic unipotent representation space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ of G is defined as the $\mathbb{C}$-subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$ spanned by the set $\{\Pi(u,s,h) \mid u \in G^\vee,\ (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}\}$.

On the space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ we can define the *dual* Fourier transform in a natural way.

Definition 5.2. The dual elliptic nonabelian Fourier transform is the linear map satisfying

$$\mathrm{FT}^\vee_{\mathrm{ell}} : \mathcal{R}^p_{\mathrm{un,ell}}(G) \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G) \quad \Pi(u,s,h) \longmapsto \Pi(u,h,s) \quad \text{for all} \quad (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}},\ u \in G^\vee \text{ unipotent.}$$

We are now ready to state the main conjecture of this document.

Conjecture 5.3. Let G be a simple p-adic group. Then the diagram

$$\begin{tikzcd} \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[r, "\mathrm{FT}^\vee_{\mathrm{ell}}"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] \\ \bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}}, \end{tikzcd}$$

commutes, up to certain some roots of unity.

A simple yet important observation is that the conjecture can be verified one *unipotent conjugacy class of* $G^\vee$ *at a time* since the virtual representations $\Pi(u,s,h)$ and the dual elliptic nonabelian Fourier transform preserve the unipotent part of the parametrization. In addition, if G is simply connected, then all maximal open compact subgroups coincide with maximal parahorics and $\mathrm{FT}^{\mathrm{par}}$ fixes each component, so the conjecture can be verified *one maximal parahoric at a time* too.

We first show this is indeed the case when $\mathbf{G}$ is a simple algebraic group of type G_2.

5.2 Type G_2

Example 5.4. Let $\mathbf{G}$ be a simple algebraic group of type G_2, and let $G = G(F)$. Then G is both simply connected and adjoint so it has no pure inner twists other than itself. In addition, G has three maximal parahoric subgroups of types K_0, K_1 and K_2, with reductive quotients of type G_2, A_2 and $A_1 + \tilde{A}_1$, respectively. Thus, commutativity of the above square is equivalent to the commutativity of

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \Big\downarrow{\scriptstyle \mathrm{res}^{\mathrm{par}}_{\mathrm{un}}} & & \Big\downarrow{\scriptstyle \mathrm{res}^{K_i}_{\mathrm{un}}} \\ \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}} & \xrightarrow{\mathrm{FT}^{K_i}} & \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}}, \end{array}$$

for $i = 0,1,2$. Moreover, if $u \in G^\vee$ is unipotent, then $\mathcal{Y}(\Gamma_u)$ is non-empty if and only if $u = u_{\mathrm{reg}}$ is regular and $\Gamma_u = \{(1,1)\}$, or $u = u_{sr}$ is subregular and $\Gamma_u = \{(1,1),(1,g_2),(1,g_3),(g_2,1),(g_2,g_2),(g_3,1),(g_3,g_3),(g_3,g_3')\}$. Therefore, $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ is 9-dimensional, spanned by

$$\begin{cases} \Pi(u_{\mathrm{reg}},1,1) & = \pi(u_{\mathrm{reg}},\mathbf{1}) \\ \Pi(u_{sr},1,1) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) + 2\pi(u_{sr},\mathbf{r}) \\ \Pi(u_{sr},1,g_2) & = \pi(u_{sr},\mathbf{1}) - \pi(u_{sr},\varepsilon) \\ \Pi(u_{sr},1,g_3) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) - \pi(u_{sr},\mathbf{r}) \\ \Pi(u_{sr},g_2,1) & = \pi(u_{sr}g_2,\mathbf{1}) + \pi(u_{sr}g_2,\varepsilon) \\ \Pi(u_{sr},g_2,g_2) & = \pi(u_{sr}g_2,\mathbf{1}) - \pi(u_{sr}g_2,\varepsilon) \\ \Pi(u_{sr},g_3,1) & = \pi(u_{sr}g_3,\mathbf{1}) + \pi(u_{sr}g_3,\theta) + \pi(u_{sr},\theta^2) \\ \Pi(u_{sr},g_3,g_3) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta^2\pi(u_{sr}g_3,\theta) + \theta\pi(u_{sr}g_3,\theta^2) \\ \Pi(u_{sr},g_3,g_3^{-1}) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta\pi(u_{sr}g_3,\theta) + \theta^2\pi(u_{sr}g_3,\theta^2). \end{cases}$$

When $i = 1,2$ and the finite group $\overline{K_i}$ is of type A_2 or $A_1 + \tilde{A}_1$, then FT^{K_i} is the identity map, and therefore it is enough to show that

$$\mathrm{res}^{K_i}_{\mathrm{un}}(\pi(u,s,h)) = \mathrm{res}^{K_i}_{\mathrm{un}}(\Pi(u,h,s))$$

for all $\Pi(u,s,h)$ spanning $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. This is obvious for all cases except for $\pi(u,s,h) = \pi(u_{sr},1,g_2), \pi(u_{sr},1,g_3)$.

6 The p-adic symplectic group

In this chapter, we verify that Conjecture 1.1 holds when $G = \mathrm{Sp}_{2n}(F)$ and u is a subregular unipotent element. In this case, G is a simply connected simple p-adic group, whose complex dual group is $G^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$. In particular, G has no pure inner twists and all maximal compact subgroups are maximal parahoric subgroups.

Firstly, we briefly verify that the conjecture is true if u is a regular unipotent element of $G^\vee$. It is well-known that $\dim Z_{G^\vee}(u) = \mathrm{rk}\mathrm{SO}_{2n+1} = n$ and that $\Gamma_u = Z(G^\vee) = 1$ and consequently $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(1,1)\}$. The representation $\Pi(u,1,1) = \pi(u,1,\mathbf{1})$ is fixed by $\mathrm{FT}^{\mathrm{ell}}$ and moreover the Springer correspondence implies that

$$\mathrm{res}_{K_i}\pi(u,1,\mathbf{1}) = \mathrm{St}_{K_i} \quad \text{for any maximal parahoric } K_i \subset G,$$

and this representation is also fixed by each FT^{K_i}, so the conjecture is verified.

6.1 Structure theory of $\mathrm{SO}_{2n+1}(\mathbb{C})$

To progress any further, we need to understand some structural properties of $G^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$ and representation theoretic aspects of the reductive quotients of maximal parahoric subgroups of $G = \mathrm{Sp}_{2n}(F)$, which we discuss now. We recall that

$$\mathrm{SO}_{2n+1} = \{A \in \mathrm{GL}_{2n+1} \mid AJA^T = J\}, \quad \text{where} \quad J = \begin{pmatrix} 1 & & \\ & & I_n \\ & I_n & \end{pmatrix}$$

The diagonal matrices in $\mathrm{SO}_{2n+1}(\mathbb{C})$ are

$$T_n = \left\{ h(a_1,\dots,a_n) := \begin{pmatrix} 1 & & & & & & \\ & a_1 & & & & & \\ & & \ddots & & & & \\ & & & a_n & & & \\ & & & & a_1^{-1} & & \\ & & & & & \ddots & \\ & & & & & & a_n^{-1} \end{pmatrix} \mid a_1,\dots,a_n \in \mathbb{C}^\times \right\},$$

and they form a maximal torus in $\mathrm{SO}_{2n+1}(\mathbb{C})$. We fix a set of simple roots $\Delta_n = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ given by

$$\alpha_i(h(a_1,\dots,a_n)) = \begin{cases} a_i a_{i+1}^{-1} & \text{if } 1 \le i \le n-1, \\ a_n & \text{if } i = n. \end{cases}$$

The simple root α_n is the only short one and the corresponding Dynkin diagram is Insert/draw Dynkin diagram. Using the Killing form, we can embed Δ_n in a n-dimensional Euclidean space V with orthonormal basis $e_1,\dots,e_n$ and such that $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = e_n$.

We also let $\alpha_0 := \alpha_1 + \alpha_2 + \cdots + \alpha_n = e_1$ be the highest short root of $\Phi(B_n)$. This is slightly unconventional, since α_0 is normally the highest root of Φ, but this notation will be very useful. For example, $\check{\alpha}_0$ is the highest root of type C_n so $S_{\mathrm{aff}}(C_n) := \{-\check{\alpha}_0, \check{\alpha}_1, \ldots, \check{\alpha}_n\}$ is the set of affine roots of type C_n with corresponding affine Dynkin diagram Insert/draw affine Dynkin diagram.

In particular, $\alpha_0(h(a_1, \ldots, a_n)) = a_1$ and the corresponding simple coroots are

$$\check{\alpha}_i(t) = \begin{cases} h(t^2, 1, \ldots, 1) & \text{if } i = 0, \\ h(1, \ldots, 1, t, t^{-1}, 1, \ldots, 1) & \text{if } 1 \le i \le n-1, \\ h(1, \ldots, 1, t^2) & \text{if } i = n. \end{cases}$$

The Weyl group $W(B_n) = N(T_n)/T_n$ is isomorphic to the group $S_n \ltimes C_2^n$ and it acts on T_n faithfully by the transformations

$$h(a_1, \ldots, a_n) \longmapsto h(a_{\sigma(1)}^{\pm 1}, \ldots, a_{\sigma(n)}^{\pm 1}), \quad \text{where } \sigma \in S_n.$$

For each $i \in \{0, 1, 2, \ldots, n\}$, let $s_i \in W(B_n)$ be the simple reflection associated to the root α_i. Then

$$W(B_n) = \langle s_1, \ldots, s_{n-1} \text{ (long reflections)}, s_n \text{ (short reflection)} \rangle = \langle s_0, s_1, \ldots, s_{n-1} \rangle,$$

and the simple reflections act by

$$s_i \cdot h(a_1, \ldots, a_n) = \begin{cases} h(a_1^{-1}, a_2, \ldots, a_n) & \text{if } i = 0, \\ h(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) & \text{if } 1 \le i \le n-1, \\ h(a_1, \ldots, a_{n-1}, a_n^{-1}) & \text{if } i = n. \end{cases}$$

Moreover, the Euclidean space $V = \oplus_{i=1}^n e_i$ is the Lie algebra of T_n so admits a natural W-action given by

$$s_i(v) = v - \frac{2(\alpha_i, v)}{(\alpha_i, \alpha_i)} \alpha_i, \quad v \in V \quad \text{so} \quad s_i = \begin{pmatrix} I_{i-1} & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_{n-i-1} \end{pmatrix} \text{ for } 1 \le i \le n-1 \quad \text{and} \quad s_n = \begin{pmatrix} I_{n-1} & \\ & -1 \end{pmatrix}$$

and under this action V becomes the natural reflection representation of $W(B_n)$.

Next, we study the structure of unipotent conjugacy classes of $\mathrm{SO}_{2n+1}(\mathbb{C})$. We recall from Section 3.6 that each complex simple group has three canonical classes of unipotent elements: $C_{\mathrm{reg}}, C_{subreg}$ and $C_{\min}$. For $\mathrm{SO}_{2n+1}(\mathbb{C})$, the unipotent conjugacy classes can be parametrized as follows.

Proposition 6.1. *The conjugacy classes of $\mathrm{SO}_{2n+1}(\mathbb{C})$ are parametrized by pairs (λ, μ) of permutations such that $2|\lambda| + |\mu| = 2n+1$ and μ has distinct odd parts and no even parts. The regular orbit is parametrized by $(\emptyset, 2n+1)$, the subregular orbit by $(1, 2n-1)$, the minimal orbit by $(21^{n-2}, 1)$ and the zero orbit by $(1^n, 1)$.*

In addition to this result, one can also find nice representatives for each of these orbits. The results are summarized in the following table (need to reference this well).

In the next subsection, we will prove that Conjecture 1.1 holds for the subregular unipotent orbit containing $u = \prod_{i=1}^{n-1} x_{\alpha_i}(1)$. Before we do this, however, we need to discuss the representation theory of parahoric reductive quotients of $\mathrm{Sp}_{2n}(F)$.

Unipotent	(λ,μ)	Dynkin Diagram	$\dim Z_{G^\vee}(u)$	$\dim \mathcal{B}^u$	Γ_u
$\prod_{i=1}^n x_{\alpha_i}(1)$	$(\emptyset, 2n+1)$		n	0	1
$\prod_{i=2}^n x_{\alpha_i}(1)$	$(1, 2n-1)$		$n+2$	1	$\mathbb{C}^\times \rtimes C_2$
...	...	...	...	...	...
$x_{\alpha_1}(1)$	$(21^{n-2}, 1)$		$2n^2-3n+4$	n^2-2n+2	$B_{n-2}\times A_1$
1	$(1^n, 1)$		$2n^2+n$	n^2	SO_{2n+1}

6.2 Unipotent representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$

We recall from Proposition 2.6 that for any unipotent representation χ of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ there exists a parahoric subgroup $P \subseteq \mathrm{Sp}_{2n}(\mathbb{F}_q)$ with Levi decomposition $P = LN$ and a cuspidal unipotent character ϕ of L such that $\chi \hookrightarrow \mathrm{Ind}_P^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\phi$ and that moreover the pair (P,ϕ) are unique up to conjugacy. In the 90s (check this), Lusztig showed that finite groups of Lie type of type $A_m, m\geq 1$ have no cuspidal unipotent representations, while those of type C_m have a unique cuspidal unipotent representation ϕ_s if $m = s^2+s$ for some $s\geq 0$, and none otherwise.

For each $s \geq 0$ such that $m := s^2+s \leq n$, let $P_s = L_sN_s$ be the standard parabolic of type C_m and let ϕ_s be the unique cuspidal unipotent representation of L_s. The unipotent representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ are the components of the representations

$$\mathrm{Ind}_{P_s}^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\phi_s \quad \text{for all } s\geq 0 \text{ satisfying } s^2+s\leq n.$$

For each choice of s, the components of $\mathrm{Ind}_{P_s}^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\phi_s$ define a *series* of unipotent representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ denoted by $\mathcal{E}(\mathrm{Sp}_{2n}(\mathbb{F}_q), P_s, \phi_s)$. Let us focus first on $s=0$, corresponding to the principal series representations. In this case, $P_0 = B$ is a Borel subgroup, $L_0 = T$ is a maximal torus and $\phi_0 = \mathbf{1}$, so we aim to characterize the components of $\mathrm{Ind}_B^{\mathrm{Sp}_{2n}(\mathbb{F}_q)}\mathbf{1}$. In 2.3, we proved that there is a canonical bijection between principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and irreducible representations of its Weyl group $W(C_n)\cong S_n \ltimes C_2^n$. The Weyl group $W(B_n)$ of $\mathrm{SO}_{2n+1}(\mathbb{C})$ is canonically isomorphic to $W(C_n)$ by swapping between short and long simple reflections. Thus irreducible representations of $W(B_n)$ also parametrize principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. We shall abuse notation and denote both types of representations with the same symbol – it will be clear from context whether we refer to principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ or irreducible representations of $W(B_n)$.

Theorem 6.2. *There is a natural one-to-one correspondence between irreducible representations of $W(B_n)$ and ordered pairs of partitions (or bipartitions) (α,β) with $|\alpha|+|\beta| = n$. Let $\alpha = (\alpha_1,\alpha_2,\ldots)$ and $\beta = (\beta_1,\beta_2,\ldots)$ with*

$$0\leq \alpha_1\leq\alpha_2\ldots \quad 0\leq\beta_1\leq\beta_2\ldots,$$

where some of the α_i,β_i may be 0. Let α^*,β^* be the dual partitions of α,β. Then consider a subsystem $\Phi'\subseteq\Phi$ of type

$$D_{\alpha_1^*}+D_{\alpha_2^*}+\cdots+B_{\beta_1^*}+B_{\beta_2^*}+\cdots,$$

where D_1 is the empty root system and $D_2 = A_1 \times A_1$ has only long roots. If W' is the Weyl group of Φ', then the corresponding irreducible representation $\phi_{\alpha,\beta}$ of $W(B_n)$ is the Macdonald representation ${j_{W'}}^W(\varepsilon_{W'})$ obtained from W'. Moreover, the dimension of $\phi_{\alpha,\beta}$ is given by the formula

$$\dim \phi_{\alpha,\beta} = \frac{n!}{\mathrm{hook}(\alpha)\mathrm{hook}(\beta)}.$$

Throughout, we will label the representations of $W(B_n)$ by (α, β) instead of $\phi_{\alpha,\beta}$. The above theorem provides a complete classification of the irreducible representations of $W(B_n)$ and, with some work, provides a way to obtain explicit models of each representation a subrepresentation of $\mathrm{Sym}^N(V^*)$ (homogeneous degree N polynomials on V), where $V = \oplus_{i=1}^n \mathbb{C}e_i$ is the natural reflection representation of W described above and N is the number of positive roots of the associated root subsystem Φ'. To simplify notation, let $\{x_1, \dots, x_n\} \subset V^*$ be the dual basis of $\{e_1, \dots, e_n\}$.

Let us discuss some important examples and models of the above theorem.

1. If $(\alpha, \beta) = (n, \emptyset)$, then $\Phi' = \emptyset$ so $N = 0$ and $(n, \emptyset)$ is the trivial representation.

2. If $(\alpha, \beta) = (\emptyset, n)$, then $\Phi' = \{\text{short roots}\}$ so $N = n$ and $(\emptyset, n)$ is the non-trivial character where long reflections act trivially and whose model is given by
$$\mathrm{Span}_{\mathbb{C}}\{x_1 \cdots x_n\} \leq \mathrm{Sym}^n(V^*).$$

3. If $(\alpha, \beta) = (1^n, \emptyset)$, then $\Phi' = \{\text{long roots}\}$ so $N = n^2 - n$ and $(1^n, \emptyset)$ is the non-trivial character where short reflections act trivially and whose model is given by
$$\mathrm{Span}_{\mathbb{C}}\left\{\prod_{1\leq i<j\leq n}(x_i^2 - x_j^2)\right\} \leq \mathrm{Sym}^{n^2-n}(V^*).$$

4. If $(\alpha, \beta) = (\emptyset, 1^n)$, then $\Phi' = \Phi$ and $W' = W$. Then $N = n^2$ and $(\emptyset, 1^n)$ is the sign representation whose model is given by
$$\mathrm{Span}_{\mathbb{C}}\left\{x_1 \cdots x_n \cdot \prod_{1\leq i<j\leq n}(x_i^2 - x_j^2)\right\} \leq \mathrm{Sym}^{n^2}(V^*).$$

5. If $(\alpha, \beta) = (n-1, 1)$, then Φ' has type A_1 so $N = 1$ and $(n-1, 1) = \mathrm{Sym}^1 V^* = V^*$ is the natural reflection representation of W.

6. If $(\alpha, \beta) = ((n-1)1, \emptyset)$, then Φ' has type $A_1 \times A_1$ so $N = 2$ and $((n-1)1, \emptyset)$ is the $n-1$-dimensional representation with model
$$\mathrm{Span}_{\mathbb{C}}\{x_1^2 - x_2^2, \dots, x_{n-1}^2 - x_n^2\}.$$

7. If $(\alpha, \beta) = (n-2, 2)$, then Φ' has type $A_1 \times A_1$ so $N = 2$ too and $(n-2, 2)$ is the $n(n-1)/2$-dimensional representation with model
$$\mathrm{Span}_{\mathbb{C}}\{x_i x_j \mid 1 \leq i < j \leq n\}.$$

   In particular, by noting that $\mathbb{C}(x_1^2 + \cdots + x_n^2)$ affords the trivial representation and by counting dimensions, this shows that $\mathrm{Sym}^2(V^*) = (n, \emptyset) + ((n-1)1, \emptyset) + (n-2, 2)$.

In addition to the previous examples, we also provide an explicit description of the effect of twisting by the sign representation $\varepsilon = (\emptyset, 1^n)$.

Lemma 6.3. *Let (α, β) be a representation of $W(B_n)$ labelled as in Theorem 6.2 and let α^*, β^* be the dual partitions. Then*

$$\varepsilon \otimes (\alpha, \beta) = (\emptyset, 1^n) \otimes (\alpha, \beta) = (\beta^*, \alpha^*).$$

In particular, if ϕ_1 and ϕ_2 are two characters of the same family, so are $\varepsilon \otimes \phi_1$ and $\varepsilon \otimes \phi_2$.

Lemma 6.4. *The smallest normal subgroup of $W(B_n) = S_n \ltimes C_2^n$ containing a short reflection is isomorphic to C_2^n. The set of short reflections lies in the kernel of a representation (α, β) if and only if $\beta = \emptyset$, and the bijection*

$$\begin{aligned} \mathrm{Irr}(W(A_{n-1})) &\longrightarrow \mathrm{Irr}(W(B_n)) \\ \alpha &\longmapsto (\alpha, \emptyset) \end{aligned}$$

is the natural inflation map of representations.

Next, we wish to understand, using the previous parametrization, the families of the Weyl group as defined in 2.3. To do this, we need to introduce the notion of *symbols* associated to a pair of partitions (α, β). Choose an appropriate number of zeros as parts of α or β so that α has one more part than β. Then define the symbol of (α, β) to be the array

$$\begin{pmatrix} \alpha_1 & & \alpha_2 + 1 & & \alpha_3 + 2 & & \cdots & \alpha_m + (m-1) & & \alpha_{m+1} + m \\ & \beta_1 & & \beta_2 + 1 & & \beta_3 + 2 & \cdots & & \beta_m + (m-1) & \end{pmatrix}$$

We consider the equivalence relation on the symbols generated by

$$\begin{pmatrix} 0 & & \lambda_1 + 1 & & \lambda_2 + 1 & & \cdots & & \lambda_m + 1 & & \lambda_{m+1} + 1 \\ & 0 & & \mu_1 + 1 & & \mu_2 + 1 & \cdots & \mu_{m-1} + 1 & & \mu_m + 1 & \end{pmatrix}$$
$$\sim \begin{pmatrix} \lambda_1 & & \lambda_2 & & \cdots & & \lambda_m & & \lambda_{m+1} \\ & \mu_1 & & \mu_2 & \cdots & \mu_{m-1} & & \mu_m & \end{pmatrix}.$$

Thus, each pair (α, β) defines a unique equivalence class of symbols.

Theorem 6.5. *Two characters of $W(B_n)$ lie in the same family if and only if they possess symbols for which the unordered sets $\{\lambda_1, \ldots, \lambda_{m+1}, \mu_1, \ldots, \mu_m\}$ are the same. Moreover, each family contains a unique character whose symbol has the property that*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_m \leq \lambda_{m+1},$$

and these are precisely the special characters of $W(B_n)$.

This concludes the classification of principal series representations of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. To understand the remaining series, we need to consider more general symbols of the form

$$\begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_b \\ & \mu_1 & & \mu_2 & & \mu_3 & \cdots & \mu_b & \end{pmatrix}$$

where $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \ldots < \mu_b$, $a - b$ is odd and positive and λ_1, μ_1 are not both 0. We define the *defect* of such a symbol as $d = a - b$ and its *rank* as

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j - \left[\left(\frac{a+b-1}{2} \right)^2 \right].$$

Theorem 6.6. *Unipotent characters of* $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ *are parametrized in terms of symbols of the form as above whose rank equals* n. *Such a character is a component of* $\mathrm{Ind}_{P_s}^{\mathrm{Sp}_{2n}(\mathbb{F}_q)} \phi_s$, *where* $d = 2s+1$. *Finally, two unipotent characters lie in the same family of characters if and only if their symbols contain the same entries with the same multiplicities.*

6.3 Springer correspondence of $\mathrm{SO}_{2n-1}(\mathbb{C})$

6.4 The subregular unipotent orbit of $\mathrm{SO}_{2n+1}(\mathbb{C})$

In this section, we prove Conjecture 1.1 for the subregular unipotent element. We recall from Proposition 6.1 that the subregular unipotent orbit is parametrized by the bipartition $(1, 2n-1)$. To simplify notation, let $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ be the highest short root of $\Phi(B_n)$. This is slightly unconventional, since α_0 is normally the highest root of Φ, but this notation will be very useful, and its dual root $\check{\alpha}_0$ is the highest root of $\Phi^\vee$.

Lemma 6.7. *The unipotent element* $u = \exp(e_{\alpha_2} + \cdots + e_{\alpha_{n-1}} + e_{\alpha_n})$ *lies in the subregular orbit and the reductive part of its centralizer is* $\Gamma_u = \Gamma_u^0 \rtimes \langle h_u w_{\alpha^0} \rangle$, *where*

$$\Gamma_u^0 = \{(\alpha^0)^\vee(t) \mid t \in \mathbb{C}^\times\} \quad \textit{and} \quad h_u = \begin{pmatrix} 1 & \\ & -I_{2n} \end{pmatrix} = \begin{cases} \check{\alpha}_1(-1)\check{\alpha}_3(-1)\cdots\check{\alpha}_{n-2}(-1)\check{\alpha}_n(i) & \textit{if } n \textit{ is odd,} \\ \check{\alpha}_1(-1)\check{\alpha}_3(-1)\cdots\check{\alpha}_{n-3}(-1)\check{\alpha}_{n-1}(-1) & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. □

The above result implies that there is an isomorphism

$$\Gamma_u \cong \langle z, \delta \mid z \in \mathbb{C}^\times, \delta^2 = 1, \delta z \delta^{-1} = z^{-1} \rangle \cong C^\times \rtimes C_2$$

given by

$$1 \longleftrightarrow t_0 = (\alpha^0)^\vee(\pm 1) = I_{2n+1}, \quad -1 \longleftrightarrow t_1 = (\alpha^0)^\vee(\pm i) = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & I_{n-1} & & \\ & & & -1 & \\ & & & & I_{n-1} \end{pmatrix},$$

$$\delta \longleftrightarrow t_2 = h_u w_{\alpha^0} = \begin{pmatrix} -1 & & & \\ & & -I_{n-1} & 1 & \\ & 1 & & & \\ & & & & -I_{n-1} \end{pmatrix}.$$

In particular, we note that $\Gamma_{ut_0} = \Gamma_{ut_1} = \Gamma_u$ so $A_{ut_0} = A_{ut_1} = C_2$ while $\Gamma_{ut_2} = A_{ut_2} = \{t_0, t_1, t_2, t_1t_2\} = C_2 \times C_2$. We label its representations by $\mathbf{1} \boxtimes \mathbf{1}$, $\mathbf{1} \boxtimes \varepsilon$, $\varepsilon \boxtimes \mathbf{1}$, $\varepsilon \boxtimes \varepsilon$, where the first term indicates the action of t_1 and the second term indicates the action of t_2.

Lemma 6.8. *The group Γ_u has six elliptic pairs up to Γ_u-conjugacy:*

$$\Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(\pm 1, \delta), (\delta, \pm 1), (\delta, \pm\delta)\}.$$

Proof. All elements in Γ_u are semisimple, and any two $\gamma_1, \gamma_2 \in \Gamma_u$ commute if any only if $\gamma_1, \gamma_2 \in \mathbb{C}^\times$ (in which case their common centralizer is infinite) or $(\gamma_1, \gamma_2) \in \{(\pm 1, z\delta), (z\delta, \pm 1), (z\delta, \pm z\delta) \mid z \in \mathbb{C}^\times\}$. It is easy to see that these pairs have finite centralizer and are $\mathbb{C}^\times$-conjugate to $\{(\pm 1, \delta), (\delta, \pm 1), (\delta, \pm\delta)\}$, respectively. $\square$

Thus, we wish to compute the parahoric restriction with respect to maximal compact subgroups of the virtual representations

$$\Pi(u, 1, \delta) = \pi(ut_0, \mathbf{1}) - \pi(ut_0, \varepsilon), \quad \Pi(u, -1, \delta) = \pi(ut_1, \mathbf{1}) - \pi(ut_1, \varepsilon),$$
$$\Pi(u, \delta, 1) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) + \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) + \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) + \pi(ut_2, \varepsilon \boxtimes \varepsilon),$$
$$\Pi(u, \delta, -1) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) + \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) - \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) - \pi(ut_2, \varepsilon \boxtimes \varepsilon),$$
$$\Pi(u, \delta, \delta) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) - \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) + \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) - \pi(ut_2, \varepsilon \boxtimes \varepsilon),$$
$$\Pi(u, \delta, -\delta) = \pi(ut_2, \mathbf{1} \boxtimes \mathbf{1}) - \pi(ut_2, \mathbf{1} \boxtimes \varepsilon) - \pi(ut_2, \varepsilon \boxtimes \mathbf{1}) + \pi(ut_2, \varepsilon \boxtimes \varepsilon).$$

We first compute these restrictions for representations having Iwahori-fixed vectors, whose restrictions to $\overline{K}_J$ are a direct sum of principal series representations. The restrictions of the remaining representations will be calculated afterwards.

We recall from Lemma 4.1 and Proposition 4.2 that if $\pi(ut, \rho)$ has Iwahori-fixed vectors and $J \subsetneq S_{\mathrm{aff}} = \{\check{\alpha}_1, \check{\alpha}_2, \dots, \check{\alpha}_n, -\check{\alpha}_0\}$, for any irreducible K_J module χ trivial on U_J, we have that

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^{\mathcal{I}}, V^{\mathcal{I}} \rangle_{\mathcal{H}(K_J, \mathcal{I}, \mathbf{1})} = \langle \chi^{\mathcal{I}}_{q=1}, \pi(ut, \rho)^{\mathcal{I}}_{q=1} \rangle_{\widetilde{W}_J}.$$

The operation $\chi \mapsto \chi^{\mathcal{I}}_{q=1}$ corresponds to the aforementioned bijection between principal series representations of $\overline{K}_J$ and irreducible representations of $\widetilde{W}_J$, while

$$\pi(ut, \rho)^{\mathcal{I}}_{q=1} = \varepsilon \otimes \mathrm{Ind}_{\widetilde{W}_t}^{\widetilde{W}} [s \otimes H(\mathcal{B}^u_t)^\rho]$$

as a $\widetilde{W}$-module, up to semisimplification. It is convenient to work inside the Weyl group $W(B_n)$ and there is a well canonical isomorphism $\psi_J : W_J \to \widetilde{W}_J$ for some $W_J \leq W$. Following Reference M.Reeder, we obtain that

$$\psi_J^* \left(\pi(ut, \rho)^{\mathcal{I}}_{q=1} |_{\widetilde{W}_J} \right) = \varepsilon \otimes \bigoplus_{w \in W_t \backslash W / W_J} \mathrm{Ind}^{W_J}_{W_J \cap W_{t^w}} \chi^J_{t^w} \otimes [H(\mathcal{B}^u_t)^\rho]^w \tag{11}$$

where $\chi^J_{t^w}$ is a character of $W_{J,t^w} := W_J \cap W_{t^w}$ satisfying the following properties:

1. There is a semidirect decomposition $W_{J,t^w} = W^0_{J,t^w} \rtimes R_{J,t^w}$, where W^0_{J,t^w} is generated by the reflections about roots of $G^\vee_J$ which are trivial on t^w and R_{J,t^w} is the stabilizer in W_{J,t^w} of some system of positive roots for the centralizer of t^w of $G^\vee_J$.

2. The character $\chi^J_{t^w}$ is trivial on W^0_{J,t^w}.

3. There is an explicit formula for other elements in R_{J,t^w}.

Our aim is to compute this restriction with respect to maximal compact subgroups. Since $\mathrm{Sp}_{2n}(F)$ is simply connected, these coincide with maximal parahoric subgroups and its conjugacy classes are in natural bijection with maximal subsets $J \subsetneq S_{\mathrm{aff}}$. For each $r = 0, \dots, n$, let $J_r = S_{\mathrm{aff}} \backslash \{\check{\alpha}_r\}$ be the maximal subset of J_{aff}, K_{J_r} be the corresponding parahoric subgroup with unipotent radical U_{J_r} and reductive quotient $\overline{K}_{J_r}$ isomorphic to $\mathrm{Sp}_{2r}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-r)}(\mathbb{F}_q)$. In particular, the Weyl group of $\overline{K}_{J_r}$ inside W is generated by

$$W_{J_r} = \langle s_0, \dots, s_{r-1}\rangle \times \langle s_{r+1}, \dots, s_n\rangle$$

and it is the direct product of two irreducible Weyl groups of type B_r and B_{n-r}.

Case 0: $t = t_0$**.** In this case, $Z_{G^\vee}(t_0) = G^\vee$ so $W_{t^0} = W$, $\mathcal{B}^u_{t^0} = \mathcal{B}^u$ and the character $\chi^{J_r}_{t_0}$ is trivial. Hence, the right hand side of (11) becomes

$$\varepsilon \otimes H(\mathcal{B}^u)^\rho|_{W_{J_r}}.$$

The variety $\mathcal{B}^u$ is one-dimensional, so $H(\mathcal{B}^u) = H^0(\mathcal{B}^u) + H^2(\mathcal{B}^u)$. The W-action on $H(\mathcal{B}^u)$ is described by the Springer correspondence. The 0-th cohomology group is easy to describe since $H^0(\mathcal{B}^u)^{\mathbf{1}}$ affords the trivial W-representation while $H^0(\mathcal{B}^u)^\varepsilon = 0$. On the other hand, the top cohomology groups $H^2(\mathcal{B}^u)^{\mathbf{1}}$ and $H^2(\mathcal{B}^u)^\varepsilon$ afford the W-representations $(n-1,1)$ and $((n-1)1,\emptyset)$, respectively.

The restriction of the 0-th cohomology modules to W_{J_r} is straightforward, so we focus on the restriction of the top cohomology modules to W_{J_r}. Let $V_{(n-1,1)} = \oplus_{i=1}^n \mathbb{C}x_i$ be the natural reflection representation and let $V_{((n-1)1,\emptyset)} = \oplus_{i=1}^{n-1}\mathbb{C}(x_i^2 - x_{i+1}^2)$ be the representation labelled by $((n-1)1,\emptyset)$.

If $r = 0, n$, then $W_{J_r} = W$ so there is nothing to prove. If $1 \le r \le n-1$, then there are direct sum decompositions

$$\begin{aligned}
V_{(n-1,1)} &= \mathrm{Span}\{x_i \mid 1 \le i \le r\} \oplus \mathrm{Span}\{x_i \mid r+1 \le i \le n\},\\
V_{((n-1)1,\emptyset)} &= \mathrm{Span}\{x_i^2 - x_{i+1}^2 \mid 1 \le i \le r-1\} \oplus \mathrm{Span}\{x_i^2 - x_{i+1}^2 \mid r+1 \le i \le n-1\} \oplus \mathrm{Span}\{(n-r)\sum_{i=1}^r x_i^2 - r\sum_{i=r+1}^n x_i^2\}
\end{aligned}$$

into irreducible W_{J_r}-modules, affording the representations $(r-1,1) \boxtimes (n-r,\emptyset)$ and $(r,\emptyset) \boxtimes (n-r-1,1)$ for the first line, and $((r-1)1,\emptyset) \boxtimes (n-r,\emptyset)$, $(r,\emptyset) \boxtimes ((n-r-1)1,\emptyset)$ and $(r,\emptyset) \boxtimes (n-r,\emptyset)$ for the second line. This finishes the first case.

Case 1: $t = t_1$**.** In this case,

$$Z_{G^\vee}(t_1) = Z_{G^\vee}(t_1)^0 \times \langle s_0\rangle \quad \text{where} \quad Z_{G^\vee}(t_1)^0 = \langle T, \mathfrak{X}_{\pm\alpha_2}, \dots, \mathfrak{X}_{\pm\alpha_{n-1}}, \mathfrak{X}_{\pm\alpha_n}\rangle \tag{12}$$

is a connected reductive group of type B_{n-1}. The Weyl group of $Z_{G^\vee}(t_1)$ is

$$W_{t_1} = \langle s_0\rangle \times \langle s_2, \dots, s_{n-1}, s_n\rangle \cong C_2 \times W(B_{n-1}) \cong C_2^n \rtimes S_n$$

and acts on T by the transformations

$$h(a_1, \dots, a_{n-1}, a_n) \longmapsto h(a_1^{\pm 1}, a_{\tau(2)}^{\pm 1}, \dots, a_{\tau(n)}^{\pm 1}), \quad \text{where } \tau \in S_{n-1}.$$

Thus $|W_{t_1}| = 2^n(n-1)!$ and the index inside W is $[W : W_{t_1}] = n$. To understand the right hand side of (11), we first need to understand the double coset space $W_{t_1}\backslash W/W_{J_r}$. If $r = 0$ or $r = n$, then $W_{J_r} \cong W$, so there is only one double coset. In these two cases, the right hand side of (11) simplifies to

$$\varepsilon \otimes \mathrm{Ind}_{W_{t_1}}^{W} \chi_{t_1}^{J_r} \otimes H(\mathcal{B}_{t_1}^{u})^{\rho}.$$

From the properties listed in 6.4, one can easily check that $\chi_{t_1}^{J_0}$ is the trivial character while $\chi_{t_1}^{J_n}$ is trivial on $W_{t_1}^0 = \langle s_2, \ldots, s_n\rangle$, but it takes s_0 to -1. To finish the calculations, we need to study the W_{t_1}-action on the cohomology complex $H(\mathcal{B}_{t_1}^u)$. From (12), $u \in Z_{G^\vee}(t_1)$ and is a regular unipotent element, so $\dim \mathcal{B}_{t_1}^u = 0$. However, since $Z_{G^\vee}(t_1)$ is not connected, the structure of $\mathcal{B}_{t_1}^u$ is more interesting. The following result gives an explicit description of $H^0(\mathcal{B}_{t_1}^u)$ as a W_{t_1}-module.

Lemma 6.9. *The variety $\mathcal{B}_{t_1}^u$ consists of two points. The two dimensional complex vector space $H^0(\mathcal{B}_{t_1}^u)$ has a natural action of $W_{t_1} \times A_{ut_1}$ and it admits a direct sum decomposition*

$$H^0(\mathcal{B}_{t_1}^u) = H^0(\mathcal{B}_{t_1}^u)^{\mathbf{1}} + H^0(\mathcal{B}_{t_1}^u)^{\varepsilon}$$

into 1-dimensional representations of $W_{t_1} \cong C_2 \times W(B_{n-1})$ affording the characters $\mathbf{1} \boxtimes (n-1, \emptyset)$, $\varepsilon \boxtimes (n-1, \emptyset)$, respectively.

It is therefore enough, in order to calculate these restrictions, to decompose the abstract W-representations

$$\mathrm{Ind}_{W_{t_1}}^{W} \mathbf{1} \boxtimes (n-1, \emptyset) \quad \text{and} \quad \mathrm{Ind}_{W_{t_1}}^{W} \varepsilon \boxtimes (n-1, \emptyset)$$

into a direct sum of irreducible representations. Firstly, we note that the trivial representation $(n, \emptyset)$ is certainly an irreducible factor of the first representation. Similarly, the representation $((n-1)1, \emptyset)$, with model given by $\mathrm{Span}\{x_i^2 - x_{i+1}^2 \mid 1 \le i \le n-1\}$ has a one-dimensional subspace spanned by $(n-1)x_1^2 - x_2^2 - \cdots - x_n^2$ on which W_{t_1} acts trivially. By Frobenius reciprocity, it follows that $((n-1)1, \emptyset)$ is another irreducible factor. By counting dimensions, we have that

$$\mathrm{Ind}_{W_{t_1}}^{W} \mathbf{1} \boxtimes (n-1, \emptyset) = (n, \emptyset) + ((n-1)1, \emptyset). \tag{13}$$

To calculate the second induction, we follow a similar argument. The one-dimensional subspace of the natural reflection representation of W spanned by e_1 is W_{t_1} invariant, and it affords the character $\varepsilon \otimes (n-1, \emptyset)$ of W_{t_1}. By Frobenius reciprocity and counting dimensions,

$$\mathrm{Ind}_{W_{t_1}}^{W} \mathbf{1} \boxtimes (n-1, \emptyset) = (n-1, 1). \tag{14}$$

Putting everything together immediately gives the desired parahoric restrictions. See the table at the end

If $1 \le r \le n-1$, the problem is more involved. The natural quotient map $W \twoheadrightarrow S_n$ induces a bijection

$$W_{t_1}\backslash W/W_{J_r} \longrightarrow \mathrm{Stab}_{S_n}(1)\backslash S_n/\mathrm{Stab}_{S_n}(1, \ldots, r) \times \mathrm{Stab}_{S_n}(r+1, \ldots, n),$$

and the action of $\mathrm{Stab}_{S_n}(1, \ldots, r) \times \mathrm{Stab}_{S_n}(r+1, \ldots, n)$ on $\mathrm{Stab}_{S_n}(1)\backslash S_n$ has two orbits:

$$\{\mathrm{Stab}(1), \mathrm{Stab}(1)(1\ 2), \ldots, \mathrm{Stab}(1\ r)\} \quad \text{and} \quad \{\mathrm{Stab}(1)(1\ r+1), \ldots, \mathrm{Stab}(1\ n)\}.$$

For convenience, we choose representatives e and η of $W_{t_1}\backslash W/W_{J_r}$, where η acts on T by the permutation $(1\ n)$. This calculation shows that

$$\bigoplus_{w\in W_{t_1}\backslash W/W_{J_r}} \mathrm{Ind}_{W_{J_r,t_1^w}}^{W_{J_r}} \chi_{t_1^w}^{J_r} \otimes \left[H(\mathcal{B}_{t_1}^u)^\rho\right]^w = \mathrm{Ind}_{W_{J_r,t_1}}^{W_{J_r}} \chi_{t_1}^{J_r} \otimes H(\mathcal{B}_{t_1}^u)^\rho \oplus \mathrm{Ind}_{W_{J_r,t_1^\eta}}^{W_{J_r}} \chi_{t_1^\eta}^{J_r} \otimes \left[H(\mathcal{B}_{t_1}^u)^\rho\right]^\eta \tag{15}$$

To calculate the W_{J_r}-modules above, we first calculate the characters $\chi_{t_1}^{J_r}$ and $\chi_{t_1^\eta}^{J_r}$. Using the properties stated in 6.4, we note that $W_{J_r,t_1} = \langle s_0\rangle \times W_{J_r,t_1}^0$ while $W_{J_r,t_1^\eta} = W_{J_r,t_1^\eta}^0 \times \langle s_n\rangle$, while $\chi_{t_1}^{J_r}(s_0) = -1$ and $\chi_{t_1^\eta}^{J_r}(s_n) = 1$.

We are now ready to calculate the right hand side of (15). If $\rho = \mathbf{1}$, then $H^0(\mathcal{B}_{t_1}^u)^{\mathbf{1}}$ is the trivial module so

$$\mathrm{Ind}_{W_{J_r,t_1}}^{W_{J_r}} \chi_{t_1}^{J_r} = \mathrm{Ind}_{\langle s_0\rangle\times W(B_{r-2})\times W(B_{n-r})}^{W(B_r)\times W(B_{n-r})} \varepsilon \boxtimes (r-2,\emptyset)\boxtimes(n-r,\emptyset) = (r-1,1)\boxtimes(n-r,\emptyset),$$
$$\mathrm{Ind}_{W_{J_r,t_1^\eta}}^{W_{J_r}} \chi_{t_1^\eta}^{J_r} = \mathrm{Ind}_{W(B_r)\times W(B_{n-r-2})\times\langle s_n\rangle}^{W(B_r)\times W(B_{n-r})} (r,\emptyset)\boxtimes(n-r-2,\emptyset)\boxtimes\mathbf{1} = (r,\emptyset)\boxtimes[(n-r,\emptyset)+((n-r-1)1,\emptyset)].$$

If $\rho=\varepsilon$, then s_0 acts on $H^0(\mathcal{B}_{t_1}^u)^\varepsilon$ by -1 while s_n acts on $[H^0(\mathcal{B}_{t_1}^u)^\varepsilon]^\eta$ also by -1. Thus,

$$\mathrm{Ind}_{W_{J_r,t_1}}^{W_{J_r}} \chi_{t_1}^{J_r}\otimes H^0(\mathcal{B}_{t_1}^u)^\varepsilon = \mathrm{Ind}_{\langle s_0\rangle\times W(B_{r-2})\times W(B_{n-r})}^{W(B_r)\times W(B_{n-r})} \mathbf{1}\boxtimes(r-2,\emptyset)\boxtimes(n-r,\emptyset) = [(r,\emptyset)+((r-1)1,\emptyset)]\boxtimes(n-r,\emptyset),$$
$$\mathrm{Ind}_{W_{J_r,t_1^\eta}}^{W_{J_r}} \chi_{t_1^\eta}^{J_r}\otimes[H^0(\mathcal{B}_{t_1}^u)^\varepsilon]^\eta = \mathrm{Ind}_{W(B_r)\times W(B_{n-r-2})\times\langle s_n\rangle}^{W(B_r)\times W(B_{n-r})} (r,\emptyset)\boxtimes(n-r-2,\emptyset)\boxtimes\varepsilon = (r,\emptyset)\boxtimes(n-r-1,1).$$

Remark 6.10. The above computations do not hold if $r=1$ or $r=n-1$, but very similar ideas apply. If $r=1$, all formulas above hold without change except for

$$\mathrm{Ind}_{W_{J_1,t_1}}^{W_{J_1}} \chi_{t_1}^{J_1}\otimes H^0(\mathcal{B}_{t_1}^u)^\varepsilon = (1,\emptyset)\boxtimes(n-1,\emptyset).$$

Similarly, all formulas hold for $r=n-1$ except for

$$\mathrm{Ind}_{W_{J_{n-1},t_1^\eta}}^{W_{J_{n-1}}} \chi_{t_1^\eta}^{J_{n-1}}\otimes[H^0(\mathcal{B}_{t_1}^u)^{\mathbf{1}}]^\eta = (n-1,\emptyset)\boxtimes(1,\emptyset)$$

This gives us all the parahoric restrictions. See table at the end too.

Case 2: $t=t_2$**.** Firstly, we note that t_2 is conjugate to the diagonal matrix $\left(\begin{smallmatrix}1 & \\ & -I_{2n}\end{smallmatrix}\right)$, and therefore

$$Z_{G^\vee}(t_2) = Z_{G^\vee}(t_2)^0\times\langle s_0\rangle \quad\text{where}\quad Z_{G^\vee}(t_2)^0 = \langle T,\mathfrak{X}_\beta \mid \beta \text{ is a long root}\rangle,$$

where $Z_{G^\vee}(t_2)^0$ is a connected reductive group of type D_n. We can then deduce that the Weyl group of $Z_{G^\vee}(t_2)$ is $W_{t_2}=W$ and that u is a regular unipotent element in $Z_{G^\vee}(t_2)$, so $\dim\mathcal{B}_{t_2}^u=0$. Thus, the right hand side of (11) simplifies to

$$\varepsilon\otimes\chi_{t_2}^{J_r}\otimes H^0(\mathcal{B}_{t_2}^u)^\rho|_{W_{J_r}}. \tag{16}$$

By the properties listed in 6.4, the character $\chi_{t_2}^{J_r}$ of $W_{J_r}=\langle s_0,s_1,\dots,s_{r-1}\rangle\times\langle s_{r+1},\dots,s_{n-1},s_n\rangle$ is trivial on the long reflections (these generate W_{J_r,t_2}^0) while $\chi_{t_2}^{J_r}$ sends s_n to 1 and s_0 to -1. Thus, the character $\chi_{t_2}^{J_r}$ is the W_{J_r} representation labelled by $(\emptyset,r)\boxtimes(n-r,\emptyset)$.

Finally, we need an explicit description of the W-action on $H(\mathcal{B}_{t_2}^u)=H^0(\mathcal{B}_{t_2}^u)$.

Lemma 6.11. *The variety $\mathcal{B}_{t_2}^u$ consists of two points. The two dimensional complex vector space $H^0(\mathcal{B}_{t_2}^u)$ has a natural action of $W\times A_{ut_2}$ where $A_{ut_2}=\{t_0,t_1,t_2,t_1t_2\}$. It admits a direct sum decomposition*

$$H(\mathcal{B}_{t_2}^u) = H^0(\mathcal{B}_{t_2}^u)^{\mathbf{1}\otimes\mathbf{1}}\oplus H^0(\mathcal{B}_{t_2}^u)^{\varepsilon\otimes\mathbf{1}}$$

into 1-dimensional W-representations, affording the characters $(n,\emptyset)$ and $(\emptyset,n)$, respectively.

In particular, the above result implies that $H^0(\mathcal{B}^u_{t_2})^{\mathbf{1}\otimes\mathbf{1}}|_{W_{J_r}}$ is the trivial representation labelled by $(r,\emptyset)\boxtimes(n-r,\emptyset)$ while $H^0(\mathcal{B}^u_{t_2})^{\mathbf{1}\otimes\mathbf{1}}|_{W_{J_r}}$ is the character labelled by $(\emptyset,r)\boxtimes(\emptyset,n-r)$

Lemma 6.12. *The representation $\pi(ut_2, \mathbf{1}\otimes\varepsilon)$ has no U_{J_0} fixed vectors, while the parahoric restriction of $\pi(ut_2, \varepsilon\otimes\varepsilon)$ to $\overline{K}_0$ is the defect 1 irreducible $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ representation labelled by the symbol*

$$\begin{pmatrix} 0 & 1 & 2 & \dots & n-2 & n-1 & n \\ & & 1 & \dots & n-2 & & \end{pmatrix}.$$

$\pi(us, \phi)$	$K_0 \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$
$(t_0, \mathbf{1})$	$(\emptyset, 1^n) + (1, 1^{n-1})$
(t_0, ε)	$(\emptyset, 21^{n-2})$
$(t_1, \mathbf{1})$	$(\emptyset, 1^n) + (\emptyset, 21^{n-2})$
(t_1, ε)	$(1, 1^{n-1})$
$(t_2, \mathbf{1}\otimes\mathbf{1})$	$(\emptyset, 1^n)$
$(t_2, \mathbf{1}\otimes\varepsilon)$	0
$(t_2, \varepsilon\otimes\mathbf{1})$	$(1^n, \emptyset)$
$(t_2, \varepsilon\otimes\varepsilon)$	θ_{1,K_0}

$\pi(us,\phi)$	$K_0 \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$	$K_1 \to \mathrm{Sp}_2(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q)$	$K_r \to \mathrm{Sp}_{2r}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-r)}(\mathbb{F}_q)$	$K_{n-1} \to \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q) \times \mathrm{Sp}_2(\mathbb{F}_q)$	$K_n \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$
$(t_0, \mathbf{1})$	$(\emptyset, 1^n)+$ $(1, 1^{n-1})$	$(\emptyset, 1) \boxtimes (\emptyset, 1^{n-1})+$ $(1, \emptyset) \boxtimes (\emptyset, 1^{n-1}) + (\emptyset, 1) \boxtimes (1, 1^{n-2})$	$(\emptyset, 1^r) \boxtimes (\emptyset, 1^{n-r})+$ $(1, 1^{r-1}) \boxtimes (\emptyset, 1^{n-r}) + (\emptyset, 1^r) \boxtimes (1, 1^{n-r-1})$	$(\emptyset, 1^{n-1}) \boxtimes (\emptyset, 1)+$ $(1, 1^{n-2}) \boxtimes (\emptyset, 1) + (\emptyset, 1^{n-1}) \boxtimes (1, \emptyset)$	$(\emptyset, 1^n)+$ $(1, 1^{n-1})$
(t_0, ε)	$(\emptyset, 21^{n-2})$	$(\emptyset, 1) \boxtimes (\emptyset, 1^{n-1})+$ $(\emptyset, 1) \boxtimes (\emptyset, 21^{n-3})$	$(\emptyset, 1^r) \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes (\emptyset, 21^{n-r-2}) + (\emptyset, 21^{r-2}) \boxtimes (\emptyset, 1^{n-r})$	$(\emptyset, 1^{n-1}) \boxtimes (\emptyset, 1)+$ $(\emptyset, 21^{n-3}) \boxtimes (\emptyset, 1)$	$(\emptyset, 21^{n-2})$
$(t_1, \mathbf{1})$	$(\emptyset, 1^n)+$ $(\emptyset, 21^{n-2})$	$(1, \emptyset) \boxtimes (\emptyset, 1^{n-1})+$ $(\emptyset, 1) \boxtimes [(\emptyset, 1^{n-1}) + (\emptyset, 21^{n-3})]$	$(1, 1^{r-1}) \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes [(\emptyset, 1^{n-r}) + (\emptyset, 21^{r-n-2})]$	$(1, 1^{n-2}) \boxtimes (\emptyset, 1)+$ $(\emptyset, 1^{n-1}) \boxtimes (\emptyset, 1)$	$(1, 1^{n-1})$
(t_1, ε)	$(1, 1^{n-1})$	$(\emptyset, 1) \boxtimes (1, 1^{n-2})+$ $(\emptyset, 1) \boxtimes (\emptyset, 1^{n-1})$	$(\emptyset, 1^r) \boxtimes (1, 1^{n-r-1})+$ $[(\emptyset, 1^r) + (\emptyset, 21^{r-2})] \boxtimes (\emptyset, 1^{n-r})$	$(\emptyset, 1^{n-1}) \boxtimes (1, \emptyset)+$ $[(\emptyset, 1^{n-1}) + (\emptyset, 21^{n-3})] \boxtimes (\emptyset, 1)$	$(\emptyset, 1^n)+$ $(\emptyset, 21^{n-2})$
$(t_2, \mathbf{1} \otimes \mathbf{1})$	$(\emptyset, 1^n)$	$(1, \emptyset) \boxtimes (\emptyset, 1^{n-1})$	$(1^r, \emptyset) \boxtimes (\emptyset, 1^{n-r})$	$(1^{n-1}, \emptyset) \boxtimes (\emptyset, 1)$	$(1^n, \emptyset)$
$(t_2, \mathbf{1} \otimes \varepsilon)$	0	0	$\theta'_r \boxtimes (\emptyset, 1^{n-r})$	$\theta'_{n-1} \boxtimes (\emptyset, 1)$	θ'_n
$(t_2, \varepsilon \otimes \mathbf{1})$	$(1^n, \emptyset)$	$(\emptyset, 1) \boxtimes (1^{n-1}, \emptyset)$	$(\emptyset, 1^r) \boxtimes (1^{n-r}, \emptyset)$	$(\emptyset, 1^{n-r}) \boxtimes (1, \emptyset)$	$(\emptyset, 1^n)$
$(t_2, \varepsilon \otimes \varepsilon)$	θ'_n	$(\emptyset, 1) \boxtimes \theta'_{n-1}$	$(\emptyset, 1^r) \boxtimes \theta'_{n-r}$	0	0

	$K_0 \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$	$K_1 \to \mathrm{Sp}_2(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q)$	$K_r \to \mathrm{Sp}_{2r}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-r)}(\mathbb{F}_q)$	$K_{n-1} \to \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q) \times \mathrm{Sp}_2(\mathbb{F}_q)$	$K_n \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$
$\Pi(u,1,\delta)$	$(\emptyset, 1^n)+$ $(1,1^{n-1}) - (\emptyset, 21^{n-2})$	$(1,\emptyset) \boxtimes (\emptyset, 1^{n-1})+$ $(\emptyset,1) \boxtimes [(1,1^{n-2}) - (\emptyset, 21^{n-3})]$	$[(1,1^{r-1}) - (\emptyset, 21^{r-2})] \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes [(1,1^{n-r-1}) - (\emptyset, 21^{n-r-2})]$	$[(1,1^{n-2}) - (\emptyset, 21^{n-3})] \boxtimes (\emptyset,1)$ $+(\emptyset, 1^{n-1}) \boxtimes (1,\emptyset)$	$(1,1^{n-1}) - (\emptyset, 21^{n-2})$ $+(\emptyset, 1^n)$
$\Pi(u,\delta,1)$	$(\emptyset, 1^n)+$ $(1^n,\emptyset) + \theta'_n$	$(1,\emptyset) \boxtimes (\emptyset, 1^{n-1})+$ $(\emptyset,1) \boxtimes [(1^{n-1},\emptyset) + \theta'_{n-1}]$	$[(1^r,\emptyset) + \theta'_r] \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes [(1^{n-r},\emptyset) + \theta'_{n-r}]$	$[(1^{n-1},\emptyset) + \theta'_{n-1}] \boxtimes (\emptyset,1)$ $+(\emptyset, 1^{n-1}) \boxtimes (1,\emptyset)$	$(1^n,\emptyset) + \theta'_n$ $+(\emptyset, 1^n)$
$\Pi(u,-1,\delta)$	$(\emptyset, 1^n)+$ $(\emptyset, 21^{n-2}) - (1,1^{n-1})$	$(1,\emptyset) \boxtimes (\emptyset, 1^{n-1})+$ $(\emptyset,1) \boxtimes [(\emptyset, 21^{n-3}) - (1,1^{n-2})]$	$[(1,1^{r-1}) - (\emptyset, 21^{r-2})] \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes [(\emptyset, 21^{r-n-2}) - (1,1^{r-n-1})]$	$[(1,1^{n-2}) - (\emptyset, 21^{n-3})] \boxtimes (\emptyset,1)$ $-(\emptyset, 1^{n-1}) \boxtimes (1,\emptyset)$	$(1,1^{n-1}) - (\emptyset, 21^{n-2})$ $-(\emptyset, 1^n)$
$\Pi(u,\delta,-1)$	$(\emptyset, 1^n)-$ $-[\theta'_n + (1^n,\emptyset)]$	$(1,\emptyset) \boxtimes (\emptyset, 1^{n-1})-$ $-(\emptyset,1) \boxtimes [\theta'_{n-1} + (1^{n-1},\emptyset)]$	$[(1^r,\emptyset) + \theta'_r] \boxtimes (\emptyset, 1^{n-r})-$ $-(\emptyset, 1^r) \boxtimes [\theta'_{n-r} - (1^{r-n},\emptyset)]$	$[(1^{n-1},\emptyset) + \theta'_{n-1}] \boxtimes (\emptyset,1)$ $-(\emptyset, 1^{n-1}) \boxtimes (1,\emptyset)$	$(1^n,\emptyset) + \theta'_n$ $-(\emptyset, 1^n)$
$\Pi(u,\delta,\delta)$	$(\emptyset, 1^n)+$ $(1^n,\emptyset) - \theta'_n$	$(1,\emptyset) \boxtimes (\emptyset, 1^n - 1)+$ $(\emptyset,1) \boxtimes [(1^{n-1},\emptyset) - \theta'_{n-1}]$	$[(1^r,\emptyset) - \theta'_r] \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes [(1^{n-r},\emptyset) - \theta'_{n-r}]$	$[(1^{n-1,\emptyset}) - \theta'_{n-1}] \boxtimes (\emptyset,1)$ $+(\emptyset, 1^{n-1}) \boxtimes (1,\emptyset)$	$(1^n,\emptyset) - \theta'_n$ $+(\emptyset, 1^n)$
$\Pi(u,\delta,-\delta)$	$(\emptyset, 1^n)+$ $\theta'_n - (1^n,\emptyset)$	$(1,\emptyset) \boxtimes (\emptyset, 1^n - 1)+$ $(\emptyset,1) \boxtimes [\theta'_{n-1} - (1^{n-1},\emptyset)]$	$[(1^r,\emptyset) - \theta'_r] \boxtimes (\emptyset, 1^{n-r})+$ $(\emptyset, 1^r) \boxtimes [\theta'_{n-r} - (1^{n-r},\emptyset)]$	$[(1^{n-1,\emptyset}) - \theta'_{n-1}] \boxtimes (\emptyset,1)$ $-(\emptyset, 1^{n-1}) \boxtimes (1,\emptyset)$	$(1^n,\emptyset) - \theta'_n$ $-(\emptyset, 1^n)$

$\pi(us,\phi)$	$K_0 \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$	$K_1 \to \mathrm{Sp}_2(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q)$	$K_r \to \mathrm{Sp}_{2r}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-r)}(\mathbb{F}_q)$	$K_{n-1} \to \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q) \times \mathrm{Sp}_2(\mathbb{F}_q)$	$K_n \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$
$(t_0,\mathbf{1})$	$(\emptyset,1^n)+$ $(\mathbf{1},\mathbf{1^{n-1}})$	$(\emptyset,1)\boxtimes(\emptyset,1^{n-1})+$ $(1,\emptyset)\boxtimes(\emptyset,1^{n-1})+(\emptyset,1)\boxtimes(\mathbf{1},\mathbf{1^{n-2}})$	$(\emptyset,1^r)\boxtimes(\emptyset,1^{n-r})+$ $(\mathbf{1},\mathbf{1^{r-1}})\boxtimes(\emptyset,1^{n-r})+(\emptyset,1^r)\boxtimes(\mathbf{1},\mathbf{1^{n-r-1}})$	$(\emptyset,1^{n-1})\boxtimes(\emptyset,1)+$ $(\mathbf{1},\mathbf{1^{n-2}})\boxtimes(\emptyset,1)+(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(\emptyset,1^n)+$ $(\mathbf{1},\mathbf{1^{n-1}})$
(t_0,ε)	$(\emptyset,21^{n-2})$	$(\emptyset,1)\boxtimes(\emptyset,1^{n-1})+$ $(\emptyset,1)\boxtimes(\emptyset,21^{n-3})$	$(\emptyset,1^r)\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,21^{r-2})\boxtimes(\emptyset,1^{n-r})+(\emptyset,1^r)\boxtimes(\emptyset,21^{n-r-2})$	$(\emptyset,1^{n-1})\boxtimes(\emptyset,1)+$ $(\emptyset,21^{n-3})\boxtimes(\emptyset,1)$	$(\emptyset,21^{n-2})$
$(t_1,\mathbf{1})$	$(\emptyset,1^n)+$ $(\emptyset,21^{n-2})$	$(1,\emptyset)\boxtimes(\emptyset,1^{n-1})+$ $(\emptyset,1)\boxtimes[(\emptyset,1^{n-1})+(\emptyset,21^{n-3})]$	$(\mathbf{1},\mathbf{1^{r-1}})\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,1^r)\boxtimes[(\emptyset,1^{n-r})+(\emptyset,21^{r-n-2})]$	$(\mathbf{1},\mathbf{1^{n-2}})\boxtimes(\emptyset,1)+$ $(\emptyset,1^{n-1})\boxtimes(\emptyset,1)$	$(\mathbf{1},\mathbf{1^{n-1}})$
(t_1,ε)	$(\mathbf{1},\mathbf{1^{n-1}})$	$(\emptyset,1)\boxtimes(\mathbf{1},\mathbf{1^{n-2}})+$ $(\emptyset,1)\boxtimes(\emptyset,1^{n-1})$	$(\emptyset,1^r)\boxtimes(\mathbf{1},\mathbf{1^{n-r-1}})+$ $[(\emptyset,1^r)+(\emptyset,21^{r-2})]\boxtimes(\emptyset,1^{n-r})$	$(\emptyset,1^{n-1})\boxtimes(1,\emptyset)+$ $[(\emptyset,1^{n-1})+(\emptyset,21^{n-3})]\boxtimes(\emptyset,1)$	$(\emptyset,1^n)+$ $(\emptyset,21^{n-2})$
$(t_2,\mathbf{1}\otimes\mathbf{1})$	$(\emptyset,1^n)$	$(1,\emptyset)\boxtimes(\emptyset,1^{n-1})$	$(1^r,\emptyset)\boxtimes(\emptyset,1^{n-r})$	$(1^{n-1},\emptyset)\boxtimes(\emptyset,1)$	$(1^n,\emptyset)$
$(t_2,\mathbf{1}\otimes\varepsilon)$	0	0	$\theta'_r\boxtimes(\emptyset,1^{n-r})$	$\theta'_{n-1}\boxtimes(\emptyset,1)$	θ'_n
$(t_2,\varepsilon\otimes\mathbf{1})$	$(1^n,\emptyset)$	$(\emptyset,1)\boxtimes(1^{n-1},\emptyset)$	$(\emptyset,1^r)\boxtimes(1^{n-r},\emptyset)$	$(\emptyset,1^{n-r})\boxtimes(1,\emptyset)$	$(\emptyset,1^n)$
$(t_2,\varepsilon\otimes\varepsilon)$	θ'_n	$(\emptyset,1)\boxtimes\theta'_{n-1}$	$(\emptyset,1^r)\boxtimes\theta'_{n-r}$	0	0

	$K_0 \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$	$K_1 \to \mathrm{Sp}_2(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q)$	$K_r \to \mathrm{Sp}_{2r}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-r)}(\mathbb{F}_q)$	$K_{n-1} \to \mathrm{Sp}_{2(n-1)}(\mathbb{F}_q) \times \mathrm{Sp}_2(\mathbb{F}_q)$	$K_n \to \mathrm{Sp}_{2n}(\mathbb{F}_q)$
$\Pi(u,1,\delta)$	$(\emptyset,1^n)+$ $(\mathbf{1},\mathbf{1^{n-1}})-(\emptyset,21^{n-2})$	$(1,\emptyset)\boxtimes(\emptyset,1^{n-1})+$ $(\emptyset,1)\boxtimes[(\mathbf{1},\mathbf{1^{n-2}})-(\emptyset,21^{n-3})]$	$[(\mathbf{1},\mathbf{1^{r-1}})-(\emptyset,21^{r-2})]\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,1^r)\boxtimes[(\mathbf{1},\mathbf{1^{n-r-1}})-(\emptyset,21^{n-r-2})]$	$[(\mathbf{1},\mathbf{1^{n-2}})-(\emptyset,21^{n-3})]\boxtimes(\emptyset,1)$ $+(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(\mathbf{1},\mathbf{1^{n-1}})-(\emptyset,21^{n-2})$ $+(\emptyset,1^n)$
$\Pi(u,\delta,1)$	$(\emptyset,1^n)+$ $(1^n,\emptyset)+\theta'_n$	$(1,\emptyset)\boxtimes(\emptyset,1^{n-1})+$ $(\emptyset,1)\boxtimes[(1^{n-1},\emptyset)+\theta'_{n-1}]$	$[(1^r,\emptyset)+\theta'_r]\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,1^r)\boxtimes[(1^{n-r},\emptyset)+\theta'_{n-r}]$	$[(1^{n-1},\emptyset)+\theta'_{n-1}]\boxtimes(\emptyset,1)$ $+(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(1^n,\emptyset)+\theta'_n$ $+(\emptyset,1^n)$
$\Pi(u,-1,\delta)$	$(\emptyset,1^n)+$ $(\emptyset,21^{n-2})-(\mathbf{1},\mathbf{1^{n-1}})$	$(1,\emptyset)\boxtimes(\emptyset,1^{n-1})+$ $(\emptyset,1)\boxtimes[(\emptyset,21^{n-3})-(\mathbf{1},\mathbf{1^{n-2}})]$	$[(\mathbf{1},\mathbf{1^{r-1}})-(\emptyset,21^{r-2})]\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,1^r)\boxtimes[(\emptyset,21^{r-n-2})-(\mathbf{1},\mathbf{1^{r-n-1}})]$	$[(\mathbf{1},\mathbf{1^{n-2}})-(\emptyset,21^{n-3})]\boxtimes(\emptyset,1)$ $-(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(\mathbf{1},\mathbf{1^{n-1}})-(\emptyset,21^{n-2})$ $-(\emptyset,1^n)$
$\Pi(u,\delta,-1)$	$(\emptyset,1^n)-$ $-[\theta'_n+(1^n,\emptyset)]$	$(1,\emptyset)\boxtimes(\emptyset,1^{n-1})-$ $-(\emptyset,1)\boxtimes[{\theta_{n-1}}'+(1^{n-1},\emptyset)]$	$[(1^r,\emptyset)+\theta'_r]\boxtimes(\emptyset,1^{n-r})-$ $-(\emptyset,1^r)\boxtimes[\theta'_{n-r}+(1^{r-n},\emptyset)]$	$[(1^{n-1},\emptyset)+\theta'_{n-1}]\boxtimes(\emptyset,1)$ $-(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(1^n,\emptyset)+\theta'_n$ $-(\emptyset,1^n)$
$\Pi(u,\delta,\delta)$	$(\emptyset,1^n)+$ $(1^n,\emptyset)-\theta'_n$	$(1,\emptyset)\boxtimes(\emptyset,1^n-1)+$ $(\emptyset,1)\boxtimes[(1^{n-1},\emptyset)-\theta'_{n-1}]$	$[(1^r,\emptyset)-\theta'_r]\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,1^r)\boxtimes[(1^{n-r},\emptyset)-\theta'_{n-r}]$	$[(1^{n-1},\emptyset)-\theta'_{n-1}]\boxtimes(\emptyset,1)$ $+(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(1^n,\emptyset)-\theta'_n$ $+(\emptyset,1^n)$
$\Pi(u,\delta,-\delta)$	$(\emptyset,1^n)+$ $\theta'_n-(1^n,\emptyset)$	$(1,\emptyset)\boxtimes(\emptyset,1^n-1)+$ $(\emptyset,1)\boxtimes[\theta'_{n-1}-(1^{n-1},\emptyset)]$	$[(1^r,\emptyset)-\theta'_r]\boxtimes(\emptyset,1^{n-r})+$ $(\emptyset,1^r)\boxtimes[\theta'_{n-r}-(1^{n-r},\emptyset)]$	$[(1^{n-1,\emptyset})-\theta'_{n-1}]\boxtimes(\emptyset,1)$ $-(\emptyset,1^{n-1})\boxtimes(1,\emptyset)$	$(1^n,\emptyset)-\theta'_n$ $-(\emptyset,1^n)$

7 The p-adic group of type F_4

In this chapter, we let $G = F_4(F)$ be a simple p-adic group of type F_4, whose extended Dynkin diagram is

$$0 - 4 - 3 \Longrightarrow 2 - 1.$$

Let us also fix a set of simple affine roots $S_{\mathrm{aff}} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_0\}$, where β_1, β_2 are the short simple roots. Since the extended Dynkin diagram has no non-trivial automorphisms, the fundamental group $\Omega(F_4)$ is trivial, and therefore $G = F_4(F)$ is both a simply connected and adjoint p-adic group. This is an important fact that simple groups of type F_4 share with simple groups of type G_2 that simplifies many results and calculations.

Since G is simply connected, it has no pure inner twists other than itself. Moreover, the set of maximal open compact subgroups of G agree with the set of maximal parahoric subgroups of G, and their conjugacy classes can be parametrized by proper maximal subsets $J \subsetneq S_{\mathrm{aff}}$. It is then easy to see that there are five maximal open compact subgroups $\{K_0, K_4, K_3, K_2, K_1\}$ of G up to conjugacy, where K_i corresponds to $J_i := S_{\mathrm{aff}} \setminus \{\alpha_i\}$ for $0 \leq i \leq 4$. The reductive quotients $\overline{K}_0, \overline{K}_4, \overline{K}_3, \overline{K}_2, \overline{K}_1$ are finite reductive groups of type $F_4, A_1 \times C_3, A_2 \times \tilde{A}_2, A_3 \times \tilde{A}_1, B_4$, respectively. In particular,

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{i=0}^{4} R_{\mathrm{un}}(\overline{K}_i)$$

and $\mathrm{FT}^{\mathrm{par}} = (\mathrm{FT}^{K_i})_i$ is Lusztig's Fourier transform on $R_{\mathrm{un}}(\overline{K}_i)$ on each coordinate.

The aim of this chapter is to prove that Conjecture 1.1 holds for G. In other words, we prove that

Theorem 7.1. *Let G be a simple p-adic group of type F_4. Then the diagram*

$$\begin{tikzcd} \mathcal{R}^p_{\mathrm{un},ell}(G) \arrow[r, "\mathrm{FT}^\vee_{ell}"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & \mathcal{R}^p_{\mathrm{un},ell}(G) \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] \\ \bigoplus_{i=0}^5 R_{\mathrm{un}}(\overline{K}_i) \arrow[r, "(\mathrm{FT}^{K_i})_i"] & \bigoplus_{i=0}^5 R_{\mathrm{un}}(\overline{K}_i), \end{tikzcd}$$

commutes.

Firstly, we need to compute the basis $\{\Pi(u, s, h) \mid u \in G^\vee, (s, h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}\}$ of the space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. Here u is a unipotent element of $G^\vee = F_4(\mathbb{C})$, the complex simple reductive group of type F_4, and $\mathcal{Y}(\Gamma_u)$ is the set of elliptic pairs of Γ_u (see Section 5.1 for the Definitions). Thus, the first step is to classify all such pairs for all unipotent conjugacy classes of $F_4(\mathbb{C})$. Thankfully, this is a well-known classification given in Table 1. In particular, $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ is a 31-dimensional vector space.

The main difficulty in proving Theorem 7.1 lies in the explicit computation of the restriction map

$$\mathrm{res}^{\mathrm{par}}_{\mathrm{un}} : \mathcal{R}^p_{\mathrm{un,ell}}(G) \longrightarrow \bigoplus_{i=0}^{5} R_{\mathrm{un}}(\overline{K}_i).$$

We note that the conjecture can be verified one unipotent class $u \in G^\vee$ and one parahoric subgroup K_i at a time. Thus, the proof of Theorem 7.1 will consist on a case-by-case analysis of the distinct options.

7.1 Restriction to hyperspecial parahoric $K_0 \longrightarrow F_4$

Unipotent	Γ_u^0	A_u	$\lvert\Gamma_u\backslash(\Gamma_u)_{\mathrm{ell}}\rvert$	$\mathcal{F}_u n$	$\Gamma_{\mathcal{F}_u}$
F_4	1	1	1	$\phi_{1,24}$	1
$F_4(a_1)$	1	S_2	4	$\phi_{4,13}$	C_2
$F_4(a_2)$	1	S_2	4	$\phi_{9,10}$	1
B_3	PGL(2)	1	1	$\phi''_{8,9}$	1
$F_4(a_3)$	1	S_4	21	$\phi_{12,4}$	S_4

Table 1: Elliptic pairs for F_4