Nonabelian Fourier transform for unipotent representations

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1 Unipotent representations for finite groups of Lie type

1.1 Frobenius maps

Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a connected reductive group over k. The structure of G can be understood to a large extent by looking at its maximal connected solvable subgroups of G, denoted as Borel subgroups. If we fix some Borel subgroup B, any maximal torus T in B is also a maximal torus in G, and it determines a set of roots $\Phi = \Phi(G, T) \subset X(T)$. The choice of the Borel B containing T corresponds to a choice of positive roots Φ^+ and therefore of an integral basis $\Delta \subseteq \Phi^+$. Moreover, the subgroups $B, N := N_G(T)$ satisfy the axioms of a BN-pair, as described by Tits whose corresponding Weyl group is $W = N/T = \langle w_{\alpha_i} \mid \alpha_i \in \Delta \rangle$.

Let $F : G \to G$ be a Frobenius map and let G^F be the fixed points under the Frobenius map. One can show that G contains F-stable Borel subgroups, and that inside any F-stable Borel there are F-stable maximal tori. Thus, we may assume that the Borel subgroup B and maximal torus T fixed in the previous paragraph are F-stable. Under these assumptions, the Frobenius map acts on the simple roots by permuting the corresponding the root spaces. Thus, F corresponds to some permutation ρ of Δ satisfying

$$F(\mathcal{X}_\alpha) = \mathcal{X}_{\rho(\alpha)} \quad \text{for all } \alpha \in \Delta.$$

Moreover, one can easily check that ρ is in fact a symmetry of the Dynkin diagram, and these can be completely classified. For each orbit $J \subseteq \Delta$ of ρ, let $w_J \in W_J = \{w_{\alpha_i} \mid \alpha_i \in J\}$ be the unique element such that $w_J(J) = -J$. Moreover, it satisfies that $w_J^2 = 1$. In then follows that the group G^F has a natural BN-pair given by the groups B^F and N^F, whose Weyl group is

$$N^F/T^F = (N/T)^F = W^F = \langle w_J \mid J \subseteq \Delta \text{ is an orbit of } \rho \rangle.$$

Any F-stable Borel subgroup contains an F-stable maximal torus, but the converse might not be true. Any F-stable maximal torus that is contained in an F-stable Borel subgroup is called *maximally split*, and since any two F-stable Borel are conjugate under G^F, any two F-stable maximally split tori are also conjugate under G^F. In fact, one can easily determine the G^F-conjugacy classes of F-stable maximal tori by looking at the Weyl group. To state this result, we first introduce the notion of F-conjugacy classes in W. Given two $w_1, w_2 \in W$, we say that they are F-conjugate if there is some $x \in W$ such that $F(x)w_1x^{-1} = w_2$. Note that if F acts on W trivially, then the F-conjugacy classes are the standard conjugacy classes.

Lemma 1.1. *There is a bijection between*

$$\{G^F\text{-conjugacy classes of } F\text{-stable maximal tori}\} \longrightarrow \{F\text{-conjugacy classes of } W\}$$
$$T' = {}^gT \longmapsto \pi(g^{-1}F(g))$$

From now, we will write T_1 for a maximally split F-stable maximal torus and T_w for any F-stable torus obtained from T_1 by conjugating by some element $g \in G$ such that $\pi(g^{-1}F(g)) = w$. By the previous result, these objects are uniquely defined up to G^F-conjugation.

1.2 Deligne–Lusztig characters and unipotent representations

In their groundbreaking paper from 1976, Deligne and Lusztig attached to each pair (T, θ) of F-stable maximal torus T and character θ of T^F, a virtual character $R_{T,\theta}$ of the group G^F. These virtual characters were constructed using the action of G^F on certain ℓ-adic cohomology groups associated to certain Deligne–Lusztig varieties. We shall not consider the explicit definition of the characters, but we will rather recall without proof some important properties.

1. If the pair (T', θ') is obtained from (T, θ) by conjugation on some element of G^F, then $R_{T,\theta} = R_{T',\theta'}$.
2. If T_1 is a maximally split torus inside some F-stable Borel B, then $R_{T_1,\theta} = \theta_{B^F}^{G^F}$, where $\theta_{B^F}^{G^F}$ is the character of the parabolically induced representation $\mathrm{Ind}_{B^F}^{G^F}\theta$.
3. $R_{T,\theta}(u)$ is independent of θ if u is unipotent. We write $Q_T(u)$ for this common value.
4. The orthogonality relations $(R_{T,\theta}, R_{T',\theta'}) = |\{w \in W(T, T')^F \mid {}^w\theta' = \theta\}|$ hold. In particular, if T, T' are not G^F-conjugate, then $(R_{T,\theta}, R_{T',\theta'}) = 0$.
5. If (T, θ) is in general position, then one of $\pm R_{T,\theta}$ is an irreducible character.
6. If (T, θ) and (T', θ') are not geometrically conjugate, then $R_{T,\theta}$ and $R_{T',\theta'}$ do not share any irreducible component. This is a stronger assumption than not being G^F-conjugate.
7. We have
$$(R_{T,\theta}, 1) = \begin{cases} 1 \text{ if } \theta = 1 \\ 0 \text{ if } \theta \neq 1 \end{cases}$$
8. The dimension of $R_{T,\theta}$ equals
$$R_{T,\theta} = \varepsilon_G\varepsilon_T|G^F : T^F|$$

Let's give a couple of examples for the decomposition of the Deligne–Lusztig characters.

Example 1.2. Suppose first that $G = \mathrm{GL}_2(k)$ and $F = F_q : G \to G$ is the standard Frobenius. Then $G^F = \mathrm{GL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 \neq 0 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ while $T_2^F \cong \mathbb{F}_{q^2}^\times$. Now, if $\theta = \theta_1 \otimes \theta_2$ is a character of T_1^F, then

$$R_{T_1,\theta} = {\theta_{B^F}}^{G^F} = \begin{cases} \overline{\theta} \otimes (1 \oplus \mathrm{St}) & \text{if } \theta_1 = \theta_2 \\ \text{irreducible principal series} & \text{if } \theta_1 \neq \theta_2, \end{cases}$$

where $\overline{\theta}$ is the unique extension of θ to all of $\mathrm{GL}_2(\mathbb{F}_q)$ (this is only possible if $\theta_1 = \theta_2$). On the other hand, suppose that θ' is a character of T_w^F. Then

$$R_{T_w,\theta'} = \begin{cases} \overline{\theta} \otimes (1 \ominus \mathrm{St}) & \text{if } \theta'^q = \theta' \\ \text{irreducible cuspidal} & \text{if } \theta'^q \neq \theta', \end{cases}$$

where $\overline{\theta}$ is the extension of the unique character θ of T_1^F for which (θ, T_1) is geometrically conjugate to (θ', T_w).

Example 1.3. Now suppose that $G = \mathrm{SL}_2(k)$ and $F = F_q : G \to G$ to be the standard Frobenius again. Similarly, $G^F = \mathrm{SL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 = 1 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times$ while $T_2^F \cong C_{q+1}$. If θ is a character of T_1^F then

$$R_{T_1,\theta} = {\theta_{B^F}}^{G^F} = \begin{cases} 1 \oplus \mathrm{St} & \text{if } \theta = 1, \\ R_+(\xi) \oplus R_-(\xi) & \text{if } \theta = \xi = \mathrm{sgn}, \\ \text{irreducible principal series} & \text{if } \theta \neq \theta^{-1}, \end{cases}$$

where $R_+(\xi) \neq R_-(\xi)$ are conjugate under $\mathrm{GL}_2(\mathbb{F}_q)$ and so they have the same dimension $(q+1)/2$. On the other hand, if θ' is a character of T_w^F, then

$$R_{T_w,\theta'} = \begin{cases} 1 \ominus \mathrm{St} & \text{if } \theta' = 1, \\ \ominus R'_+(\xi) \ominus R'_-(\xi) & \text{if } \theta' = \xi = \mathrm{sgn}, \\ \ominus \text{ irreducible cuspidal} & \text{if } \theta' \neq \theta'^{-1}, \end{cases}$$

One can show that for a reductive group G over k with centre Z and semisimple rank l, there are exactly $|Z^F|q^l$ geometric conjugacy classes of pairs (T, θ). Moreover, one can define a geometric conjugacy on the set of irreducible characters of G^F as follows. We say that two characters χ_1, χ_2 are related if there are geometrically conjugate pairs (T, θ) and (T', θ') such that

$$(\chi_1, R_{T,\theta}) \neq 0 \quad \text{and} \quad (\chi_2, R_{T',\theta'}) \neq 0.$$

Clearly, there are then $|Z^F|q^l$ geometric conjugacy classes of characters. We are now ready to give the characterization of a semisimple character and a unipotent one.

Definition 1.4. An irreducible character χ of G^F is called *unipotent* if there is some maximal F-stable torus T of G such that $(R_{T,1}, \chi) \neq 0$. An irreducible character χ of the group G^F is called *semisimple* if

$$\sum_{\substack{u \in G^F \\ u \text{ reg unipotent}}} \chi(u) \neq 0.$$

It is clear from the definitions that unipotent characters form one geometric conjugacy class of irreducible characters, and that if χ is a unipotent character, then $(\chi, R_{T,\theta}) = 0$ for any $\theta \neq 1$. Semisimple characters, on the other hand, have the opposite property. To explain this, we define the class function Ξ to be supported on regular unipotent elements with constant value of $|Z^F|q^l$. By using properties of character duality, one can show that $(\Xi, \Xi) = |Z^F|q^l$ and that $(\Xi, \chi) \in \{-1, 0, 1\}$ for all irreducible characters χ of G^F. Note that this implies that there are exactly $|Z^F|q^l$ semisimple characters. In fact, one can furthermore show that

$$\Xi = \sum_{\kappa} \varepsilon_\kappa \chi_\kappa^{ss} \quad \text{where } \chi_\kappa^{ss} \text{ is irreducible and} \quad \varepsilon_\kappa \chi_\kappa^{ss} = \sum_{\substack{(T,\theta) \in \kappa \\ \text{mod } G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})},$$

where κ runs over the conjugacy classes of pairs (T, θ). These results show that each geometric conjugacy class contains one unique semisimple irreducible character.

1.3 Jordan decomposition for irreducible characters

Finally, we are ready to describe the *Jordan decomposition for characters.* To simplify the discussion, we shall assume that the centre $(Z(G))^F$ of G^F is connected. The idea is that one can completely understand all characters of a finite group of Lie type by understanding its semisimple representations and unipotent representations of its Levi subgroups. To state it, we first recall that there is a natural bijection between geometric conjugacy classes of (T, θ) and conjugacy classes of semisimple elements in the dual group $(G^*)^F$, both sets having size $|Z(G)^F|q^l$.

Definition 1.5. Let (s) be a semisimple conjugacy class of the dual group $(G^*)^{F^*}$. Then the *Lusztig series* $\mathcal{E}(G^F, (s))$ associated to (s) is the set of irreducible characters of G^F appearing in $R_{T,\theta}$ for some pair (T, θ) corresponding to (s).

The Lusztig series $\mathcal{E}(G^F, (s))$ are the geometric conjugacy classes of characters defined in the previous section. If (s) is regular semisimple, then (T, θ) is in general position and $\mathcal{E}(G^F, (s))$ is a singleton. On the other end, the series $\mathcal{E}(G^F, (1))$ contains the unipotent characters.

Theorem 1.6. *Let (s) be a semisimple conjugacy class of $(G^*)^F$ and let H be the dual group of the centralizer $Z_{G^*}(s)$. Then there is a bijection*

$$\mathcal{E}(G^F, (s)) \to \mathcal{E}(H^F, (1)), \quad \chi \mapsto \chi_u,$$

such that for any pair (T,θ) corresponding to $s \in (G^*)^{F^*}$ and any pair (S,ψ) corresponding to $s \in (H^*)^{F^*}$,

$$(\chi, \varepsilon_G \varepsilon_T \cdot R^G_{T,\theta}) = (\chi_u, \varepsilon_H \varepsilon_S \cdot R^H_S(\psi)).$$

In addition, the unique semisimple character $\chi_s \in \mathcal{E}(G^F,(s))$ corresponds to the trivial character of H^F and for any $\chi \in \mathcal{E}(G^F,(s))$, we have that

$$\chi(1) = \chi_s(1)\chi_u(1).$$

To summarize, for any irreducible character χ, there is one unique semisimple character χ_s geometrically conjugate to χ, corresponding to some semisimple conjugacy class of $(G^*)^{F^*}$. One can in fact show that

$$\chi_s(1) = |(G^*)^{F^*} : C^{F^*}|_{p'},$$

where C is the centralizer of s^* in G^*. Finally, there is a natural bijection $\chi \mapsto \chi_u$ between characters in the class containing χ_s and unipotent characters of the dual group of C^{F^*} satisfying

$$\chi(1) = \chi_s(1)\chi_u(1).$$

As it turns out, studying semisimple characters is easy since we have explicit formulas to understand them. So Lusztig turned his attention into understanding unipotent representations of finite groups of Lie type. Lusztig first observed that the study of unipotent characters of G^F can be reduced to the case when G is simple of adjoint type. That's because every unipotent character appears as a component of some $R_{T,1}$, where $R_{T,1}(g) = \mathcal{L}(g, \mathfrak{B}_w)$. But Z^F acts trivially on $\mathfrak{B}_w$, so it lies in the kernel of every unipotent representation. So G can be assumed to be semisimple, and a simple argument shows that G can be further assumed to be simple.

Secondly, following the same approach as Harish–Chandra to classify irreducible characters of G^F, Lusztig showed the following result.

Proposition 1.7. *Let χ be an irreducible character of G^F.*

1. *There is an F-stable parabolic subgroup P of G with F-stable Levi decomposition $P = LN$ and a cuspidal character ϕ of L^F such that $(\chi, {\phi_{P^F}}^{G_F}) \neq 0$.*

2. *Moreover, the pair (P,ϕ) is unique up to G^F-conjugacy.*

3. *The character χ of G^F is unipotent if and only if ϕ is a unipotent character of L^J.*

Proof. The proof of parts 1. and 2. are classical, so we only give a sketch. We fix some F-stable maximal torus T and some integral basis $\Delta \subset \Phi(G,T)$. For each $J \subseteq \Delta$, let $P_J = L_J U_J$ be the standard F-stable parabolic with standard F-stable Levi L_J. Since any parabolic subgroup of G^F is conjugate to some P_J^F, it is enough to prove the assertions for standard parabolics of G^F.

Let V be the G^F representation affording χ and let $\mathcal{J} = \{J \subseteq \Delta \mid (1_{U_J}, \chi|_{U_J}) \neq 0\} = \{J \subseteq \Delta \mid V^{U_J} \neq 0\}$, which is non-empty since $\Delta \in \mathcal{J}$. If $J \in \mathcal{J}$ is minimal with respect to inclusion, we may write $V^{U_J} = \oplus_{i=1}^k U_i$ as a direct sum of irreducible L_J-representations, all of which are cuspidal. The character ϕ afforded by U_1 satisfies the conditions of 1., and part 2. is contained in Carter 9.1.5.

To prove the last assertion, fix some $J \subseteq \Delta$ and some irreducible character ϕ of L_J. Let (T, θ) be such that $(\phi, R_{T,\theta}^{L_J^F})_{L_J^F} \neq 0$, and let χ be an irreducible component of ${\phi_{P_J^F}}^{G^F}$. Then by Frobenius reciprocity, we have that

$$(\chi|_{P_J^F}, \phi_{P_J^F})_{P_J^F} = (\chi, {\phi_{P_J^F}}^{G^F})_{G^F} \neq 0,$$

and since $\phi_{P_J^F}$ is an irreducible P_J^F representation, we have that

$$(\chi, R_{T,\theta}^{G^F}) = (\chi, {(R_{T,\theta}^{L_J^F})_{P_J^F}}^{G^F})_{G^F} = (\chi|_{P_J^F}, (R_{T,\theta}^{L_J^F})_{P_J^F})_{P_J^F} \neq 0.$$

This calculation, together with the fact that unipotent representations form a geometric conjugacy class yield the last part. □

Therefore, to classify the unipotent characters of G^F, it is enough to determine the cuspidal unipotent representations ϕ of the standard Levi subgroups L_J^F of G^F and then calculate the decomposition of ${\phi_{P_J^F}}^{G^F}$ into irreducible characters. The later task can be achieved by Howlett–Lehrer theory (Carter, §10), while the former was achieved by Lusztig by a case by case analysis. For example, Lusztig showed that if G^F is of classical type, then the number of cuspidal unipotent characters is either 0 or 1.

1.4 Families of unipotent characters

Lusztig further observed that the unipotent characters of G^F naturally form families in a remarkable way. Firstly, he parametrized the principal series unipotent characters with irreducible characters of W by showing that there is a natural bijection

$$\{\text{Irreducible characters of } W\} \longrightarrow \{\text{Irreducible components of } \mathrm{Ind}_{B^F}^{G^F} 1\}$$
$$\phi \longmapsto \chi_\phi.$$

To prove this, we first note that

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1) \cong \mathrm{Hom}_{B^F}(1, \mathrm{Ind}_{B^F}^{G^F} 1|_{B^F}) \cong \bigoplus_{w \in B \backslash G / B} \mathrm{Hom}_{B^F \cap {}^w B^F}(1, 1),$$

where in the first step we have applied Frobenius reciprocity and the Mackey decomposition formula for the second one. By the Bruhat decomposition, W is canonically isomorphic to $B \backslash G / B$ and the borel subgroup B gives a natural choice of simple roots, and therefore of simple reflections $S \subset W$.

If we let $T_w \in \mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ be the image of the identity map in $\mathrm{Hom}_{B^F \cap {}^w B^F}(1, 1)$, then one can prove that $\{T_w : w \in W\}$ is a basis for $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ satisfying

$$\begin{aligned} T_s^2 &= (q-1)T_s + qT_1 \quad &&\text{if } s \in S, \\ T_{w_1} T_{w_2} &= T_{w_1 w_2} \quad &&\text{if } l(w_1 w_2) = l(w_1) + l(w_2). \end{aligned}$$

And therefore, $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ is isomorphic to the coxeter algebra $\mathcal{H}(W, S, q)$ of the pair (W, S) with constant parameter q. It is possible to do a change of variables that give the important isomorphism

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1) \cong \mathbb{C}[W].$$

The algebra $\mathbb{C}[W]$ acts on itself by left multiplication, and the irreducible submodules are precisely the irreducible representations of W. By the isomorphism above, $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F}1)$ also decomposes into a direct sum of irreducible submodules under post composition. If $\mathrm{Ind}_{B^F}^{G^F}1 = \oplus_{i=1}^k V_i^{a_i}$ is a direct sum into G^F irreducible components, then

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F}1) = \bigoplus_{i=1}^k \mathrm{Hom}_{G^F}(V_i, \oplus_{j=1}^k V_j^{a_j})^{a_i},$$

and the modules on the right hand side are precisely the irreducible submodules, each one corresponding to one unique irreducible component of $\mathrm{Ind}_{B^F}^{G^F}1$. Thus, irreducible characters of W parametrize principal series cuspidal characters of G^F.

Example 1.8. Let G be a reductive group of type A_l. Then G^F has no cuspidal unipotent representations. Consequently, all unipotent representations of G^F are in the principal series. By the above discussion, this means that the irreducible characters of W completely parametrize all unipotent representations of G^F. Explicitly, given some irreducible character ϕ of W,

$$\chi_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1}$$

For $G = \mathrm{GL}_2(k)$ or $\mathrm{SL}_2(k)$, $\chi_1 = 1$ and $\chi_{\mathrm{sgn}} = \mathrm{St}$.

In general, however, finite groups of Lie type do have cuspidal unipotent characters, and the virtual characters

$$R_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1}$$

as defined above are not irreducible. Lusztig then divided unipotent representations of G^F into families by the rule that two unipotent characters appearing in the same R_ϕ are in the same family and then extending by transitivity. Similarly, we can define an equivalence relation on the irreducible characters of W by the rule that two characters ϕ_1, ϕ_2 are related if R_{ϕ_1} and R_{ϕ_2} share an irreducible component. It is clear that there is a bijection between families of unipotent characters of G^F and families of characters of W. Remarkably, Lusztig proved that these families can be parametrized in the following manner.

Theorem 1.9. *For each family of unipotent representations $\mathcal{F}$ there is a group $\Gamma = \Gamma_{\mathcal{F}} \in \{1, C_2 \times \cdots \times C_2, S_3, S_4, S_5\}$ and a bijection*

$$\begin{aligned} M(\Gamma) &\longrightarrow \mathcal{F} \\ (x,\sigma) &\longmapsto \chi^{\mathcal{F}}_{(x,\sigma)} \end{aligned}$$

satisfying that

$$(\chi^{\mathcal{F}}_{(x,\sigma)}, R_\phi) = \begin{cases} \{(x,\sigma),(y,\tau)\} & \text{if } \chi_\phi = \chi^{\mathcal{F}}_{(y,\tau)} \in \mathcal{F}, \\ 0 & \text{if } \chi_\phi \notin \mathcal{F}. \end{cases}$$

Since $R_{T_1,1} = \sum_{\phi\in\hat{W}} R_\phi$, it follows that for any family $\mathcal{F}$, $(\chi^{\mathcal{F}}_{(1,1)}, R_{T_1,1}) > 0$, so $\chi^{\mathcal{F}}_{(1,1)} = \chi_\phi$ for some character ϕ of W. Characters arising this way are called *special characters* of W and they have distinct characterizations. They are the distinguished elements of the families of characters of W as described above. The upshot of this discussion is that families of unipotent characters can be parametrized by special characters of the Weyl group.

Example 1.10. Let $G = G_2(k)$ and let $F = F_q : G \to G$ be the standard Frobenius. Then $G^F = G_2(\mathbb{F}_q)$, whose Weyl group W is isomorphic to D_{12}. Following Carter, we label the six irreducible representations by $\phi_{1,0}, \phi'_{1,3}, \phi''_{1,3}, \phi_{1,6}, \phi_{2,1}, \phi_{2,2}$, where the first subindex gives the dimension and $\phi_{1,0} = 1$ and $\phi_{1,6} = \det$. The special characters are $\phi_{1,0}, \phi_{1,6}, \phi_{2,1}$ and the families are

$$(\phi_{1,0}), (\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}), (\phi_{1,6}).$$

On the other hand, G_2 has 10 unipotent characters, 6 of which are principal series and 4 are cuspidal. The principal series way can have the same labels as the irreducible characters of W, while the unipotent cuspidal are labelled by $G_2[-1], G_2[\theta], G_2[\theta^2], G_2[1]$. They fall into three families, parametrized as follows.

Description in terms of cuspidal characters	Degree	Pair (x, σ)
$\phi_{1.0}$	1	
$\phi_{2.1}$	$\frac{1}{6}q\Phi_2{}^2\Phi_3$	$(1, 1)$
$G_2[1]$	$\frac{1}{6}q\Phi_1{}^2\Phi_6$	$(1, \varepsilon)$
$\phi_{2.2}$	$\frac{1}{2}q\Phi_2{}^2\Phi_6$	$(g_2, 1)$
$\phi_{1.3}'$	$\frac{1}{3}q\Phi_3\Phi_6$	$(1, r)$
$\phi_{1.3}''$	$\frac{1}{3}q\Phi_3\Phi_6$	$(g_3, 1)$
$G_2[-1]$	$\frac{1}{2}q\Phi_1{}^2\Phi_3$	(g_2, ε)
$G_2[\theta]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ)
$G_2[\theta^2]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ^2)
$\phi_{1.6}$	q^6	

Finally, Lusztig defined a nonabelian Fourier transform on the set of irreducible characters of G^F. For each family $\mathcal{F}$ of unipotent characters parametrized by the group Γ, he considered the $|M(\Gamma)| \times |M(\Gamma)|$ matrix, whose $((x, \sigma), (y, \tau))$ entry is the value

$$\{(x, \sigma), (y, \tau)\} = \frac{1}{|C_\Gamma(x)||C_\Gamma(y)|} \sum_{\substack{g \in \Gamma \\ xgyg^{-1} = gyg^{-1}x}} \sigma(gyg^{-1})\overline{\tau(g^{-1}xg)}.$$

Then he proved that this matrix is Hermitian and that it squares to the identity. It therefore induces an involution on the space $\mathbb{C}[G^F]^{G^F}_{\mathcal{F}}$ of class functions spanned by the characters in $\mathcal{F}$, where we take the natural basis $\{\chi^{\mathcal{F}}_{(x,\sigma)} \mid (x, \sigma) \in M(\Gamma)\}$. Combining for each family, this gives an involution

$$R : \mathbb{C}_{un}[G^F]^{G^F} \longrightarrow \mathbb{C}_{un}[G^F]^{G^F}$$

on the space $\mathbb{C}_{un}[G^F]^{G^F}$ of class functions spanned by unipotent characters. This forces, for example, that $R(\chi_\phi) = R_\phi$ for all characters of W. The involution R transforms unipotent characters into *unipotent almost characters* that also satisfy the orthogonality relations and have a geometrical significance. By this we mean that every almost character agrees up to a scalar with a characteristic function associated to an F-stable character sheaf on G^F (see Shoji's article).

Example 1.11. If G^F is of type A_l, then the unipotent characters coincide with the almost characters.

Example 1.12. If G^F is of type G_2, then R fixes the characters $\phi_{1,0}$ and $\phi_{1,6}$ but transforms the third family according to the Fourier transform matrix

If $\Gamma \cong S_3$ the 8×8 Fourier transform matrix is:

$$
\begin{array}{c}
\\
(1,1) \\ (1,r) \\ (1,\varepsilon) \\ (g_2,1) \\ (g_2,\varepsilon) \\ (g_3,1) \\ (g_3,\theta) \\ (g_3,\theta^2)
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
(1,1) & (1,r) & (1,\varepsilon) & (g_2,1) & (g_2,\varepsilon) & (g_3,1) & (g_3,\theta) & (g_3,\theta^2)
\end{array}\\
\begin{bmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\end{array}
$$

The almost characters satisfy certain *stability properties*. Search what do they exactly mean by this. The aim of the next chapter is to discuss a lift of this map for p-adic groups.

2 Structure theory and representations of p-adic groups

2.1 The Bernstein decomposition

Let F be a nonarchimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k of cardinality q, a power of a prime p. Let $\mathbf{G}$ be a connected, almost simple, split algebraic group over F and let $G = \mathbf{G}(F)$. We denote by $\mathrm{Rep}(G)$ the set of smooth admissible complex representations of G. We begin this chapter by discussing a fundamental result that instrumental in the study of the category $\mathrm{Rep}(G)$. The starting point is the following well-known fact.

Proposition 2.1. *Let (π, V) be an irreducible smooth representation of G. Then there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal representation (σ, M) of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$. Moreover, if P' is another parabolic subgroup with Levi subgroup M and supercuspidal representation (σ', W') such that $\pi \hookrightarrow \mathrm{Ind}_{P'}^{G'} \sigma'$, then there exists $g \in G$ such that $M' = gMg^{-1}$ and $\sigma' \cong {}^g\sigma$.*

Given an irreducible smooth representation (π, V), we denote the G-conjugacy class of $(M, (\sigma, W))$ as above the *supercuspidal support* of (π, V).

The Bernstein decomposition naturally arises when we study whether two irreducible representations with distinct supercuspidal support can have non-trivial extensions between them. This is indeed possible, but only to a very limited extent.

Lemma 2.2. *Let $(\pi, V), (\pi', V')$ be two irreducible representations with supercuspidal support (M, σ), (M', σ'), respectively. If there is a non-trivial extension between V and V', then there exists $g \in G$ and an unramified character χ of $M'(F)$ such that $M' = gMg^{-1}$ and $\sigma' \cong {}^g\sigma \otimes \chi$.*

If the conclusion of the lemma is satisfied, then we say that the pairs (M, σ) and (M', σ') are *inertially equivalent*, and we denote the equivalence by $\sim$ and the inertial equivalence class by $[M, \sigma]_G$. Finally, we let $\mathfrak{I}(G)$ be the set of inertial equivalence classes.

If $[M, \sigma] \in \mathfrak{I}(G)$, then we denote $\mathrm{Rep}(G)_{[M,\sigma]}$ the full subcategory of $\mathrm{Rep}(G)$ whose objects are representations (π, V) satisfying that all for any irreducible subquotient π' of π, there is a parabolic subgroup P' with Levi subgroup M' and supercuspidal representation σ' of M' such that $\pi' \hookrightarrow \mathrm{Ind}_{P'}^{G'}$ and $(M', \sigma') \in [M, \sigma]_G$.

These results are summarized in the following theorem.

Theorem 2.3 (Bernstein decomposition)**.** *We have an equivalence of categories*

$$\mathrm{Rep}(G) \cong \prod_{[M,\sigma] \in \mathfrak{I}(G)} \mathrm{Rep}(G)_{[M,\sigma]} \tag{1}$$

and each full subcategory $\mathrm{Rep}(G)_{[M,\sigma]}$ is indecomposable.

2.2 The apartment of a split maximal torus

Before continuing with representation theoretic aspects of p-adic groups, we first shift towards a more structural focus. As in the previous section, let $\mathbf{G}$ be a connected, almost simple, split algebraic group over F and let

$G = \mathbf{G}(F)$. Let T be a split maximal torus of G over F, and let $X^*(T)$ (resp. $X_*(T)$) be its character (resp. cocharacter) lattice and let

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

the natural perfect pairing between characters and cocharacters of T. Let $\Phi(G,T) \subset X^*(T)$ be the set of roots associated to T, with the corresponding set of coroots $\Phi^\vee(G,T) \subset X_*(T)$. We recall from the previous chapter that a choice of a Borel subgroup B of G containing T is equivalent to the choice of simple roots $\Delta(G,T) = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(G,T)$, which we fix throughout. In addition, the group B together with the normalizer $N := N_G(T)$ form a BN-pair with corresponding Weyl group $W = N(F)/T(F)$.

A natural object arising in the representation theory of G is the apartment $\mathcal{A}(G,T) := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, a real vector space containing all coroots. Moreover, $\mathcal{A}(G,T)$ has the structure of a simplicial complex given by the hyperplanes

$$H_{\alpha,n} = \{x \in \mathcal{A}(G,T) \mid \langle \alpha, x \rangle = n\}, \quad \text{for each } \alpha \in \Phi(G,T)^+ \text{ and } n \in \mathbb{Z}.$$

Whenever the torus T is clear from context, we will omit it from the notation. The complexes on the apartment are called *facets*, and the facets of largest dimension (equivalently, they are open in the apartment) are called *alcoves*. Our choice of simple roots Δ determines a canonical alcove

$$\mathcal{C}_0 = \{x \in \mathcal{A} \mid \langle \alpha, x \rangle > 0, \alpha \in \Delta \ \text{ and } \langle \alpha_0, x \rangle < 1\},$$

commonly referred to as the *fundamental alcove*.

Another important property of the apartment is that it carries a natural action of the group N satisfying

- For any $\alpha \in \Phi$ and $\lambda \in F$, the element $\check{\alpha}(\lambda) \in T \subset N$ acts on $\mathcal{A}$ by a translation $-\nu_p(\lambda)\check{\alpha}$.
- The centre of G acts faithfully and fixes every alcove. maybe important to explain this better?
- For any $\alpha \in \Phi$, the element $w_\alpha(1) \in N$ acts on $\mathcal{A}$ by a reflection along $H_{\alpha,0}$. This coincides with the natural action of W on $\mathcal{A}$.

This action preserves the simplicial structure of the apartment and is transitive on the set of alcoves of $\mathcal{A}$. Moreover, the kernel of this action is $T(\mathcal{O})$ and therefore the *extended Weyl group*

$$\widetilde{W} := N(F)/T(\mathcal{O}) \cong W \ltimes X_*(T)$$

acts faithfully on the apartment $\mathcal{A}$ and transitively on the set of alcoves. We denote by $w_{\alpha,n}$ the unique element in $\widetilde{W}$ acting on $\mathcal{A}$ by a reflection on the hyperplane $H_{\alpha,n}$.

In general, however, this action is not simple on the set of alcoves and the group $\Omega = \{w \in \widetilde{W} \mid w(\mathcal{C}_0) = \mathcal{C}_0\}$ is non-empty. These groups fit in a **splitting** short exact sequence

$$1 \longrightarrow W_{\text{aff}} \longrightarrow \widetilde{W} \longrightarrow \Omega \longrightarrow 1,$$

where $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter group generated by the simple reflections $s_0 := w_{\alpha_0,1}$, $s_i = w_{\alpha_i,0}$, $i = 1, \dots, r$ along the walls of the fundamental alcove $\mathcal{C}_0$ and acting simply transitively on the set of alcoves of $\mathcal{A}$. The group

W_{aff} is the *affine Weyl group* associated to the group G. The Weyl groups W, $\widetilde{W}$ and W_{aff} are independent of T, up to isomorphism.

Example 2.4. 1. Let $G = \mathrm{SL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = t^2 \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \frac{\alpha}{2}\mathbb{Z}$ and $X_*(T) = \check{\alpha}\mathbb{Z}$. Moreover, we have that

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) T \quad \text{and} \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right),\ t \in F^\times.$$

The apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. It is easy to check that $\Omega = \{1\}$ so that $\widetilde{W} = W_{\text{aff}}$ is generated by $s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ -\varpi & 0 \end{smallmatrix}\right)$ and $s_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

2. Let $G = \mathrm{PGL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) = t \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \alpha\mathbb{Z}$ and $X_*(T) = \frac{\check{\alpha}}{2}\mathbb{Z}$. Similarly,

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) T, \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right),\ t \in F^\times$$

and the apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. This time, however, $\Omega = \{1, \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)\}$ is non-trivial, and

$$W_{\text{aff}} = \left\langle s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{smallmatrix}\right), s_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle = \{w \in \widetilde{W} \mid \nu(\det(w)) \text{ is even}\}$$

is an index 2 normal subgroup of $\widetilde{W}$.

3. Let $G = \mathrm{GL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & s \end{smallmatrix}\right) = ts^{-1} \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \quad \text{for any } t \in F^\times.$$

In this case, $\Omega \cong \mathbb{Z}$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ and therefore $W_{\text{aff}} = \langle s_0, s_1 \rangle = \{w \in \widetilde{W} \mid \det(w) \in \mathcal{O}^\times\}$ is a normal subgroup of $\widetilde{W}$ of infinite index.

Some of the behaviour observed in the previous example holds in much greater generality. For example, Ω is an abelian group, and it has finite order if and only if G is a simple group. In that case, Ω is in bijection with the centre of complex dual group $G^\vee(\mathbb{C})$ of G. In particular, if G is simply connected, then Ω is trivial, while Ω has the largest size within the isogeny class when G is adjoint. On the other hand, W_{aff} only depends on the isogeny class, and therefore only on the root system of G.

2.3 The Bruhat-Tits building and parahoric subgroups

It is possible to push idea further and construct the Bruhat-Tits building $\mathcal{B}(G)$, a polysimplial space associated to G that contains $\mathcal{A}(G,T)$ for any F-split maximal torus. This is achieved by gluing together the apartments of all F-split maximal tori of G and then gluing them according to some equivalence relation. An important property of the building $\mathcal{B}(G)$ is that it carries a G-action satisfying the following:

1. It extends the action of $N_G(T)$ on $\mathcal{A}(G,T)$ for each F-split maximal torus T.

2. The stabilizer of $\mathcal{A}(G,T)$ is $N_G(T)$ for each F-split maximal torus T.

3. The stabilizer of any facet c of the building is a (maybe disconnected) open compact subgroup of G.

4. The action is strongly transitive on the set $\{(\mathcal{C},\mathcal{A}) \mid \mathcal{C}$ is an alcove inside the apartment $\mathcal{A}\}$.

5. For any pair $(\mathcal{C},\mathcal{A})$ as above, its stabilizer acts on $\mathcal{B}(G)$ as the group Ω. In other words

$$\mathrm{Stab}_G(\mathcal{C},\mathcal{A})/T(\mathcal{O}) = (N\cap \mathrm{Stab}_G(\mathcal{C}))/T(\mathcal{O}) = \mathrm{Stab}_N(\mathcal{C})/T(\mathcal{O}) \cong \Omega$$

Example 2.5. The Bruhat-Tits building of $G = \mathrm{SL}_2(\mathbb{Q}_p)$ or $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ is an infinite tree all of whose vertex have degree $p+1$. Each infinite line inside the building is an apartment corresponding to a distinct F-split maximal torus of G. Consider the apartment $\mathcal{A}(G,T)$, where T is the group of diagonal matrices, and let $\Delta = \{\alpha\}$ be the simple root as above. Then $\mathcal{C}_0$ is the segment between the vertices 0 and $\check{\alpha}/2$.

If $G = \mathrm{SL}_2(\mathbb{Q}_p)$, then

$$K_1 := \mathrm{Stab}(0) = \mathrm{SL}_2(\mathbb{Z}_p),\ K_2 := \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},\ \text{and } \mathcal{I} := \mathrm{Stab}(\mathcal{C}_0) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

are all connected, open compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ not conjugate to each other. K_0 and K_1 are the unique maximal compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ – in particular, the stabilizer of any vertex of the building is conjugate to either K_0 or K_1. The subgroup $\mathcal{I}$ is called the **Iwahori subgroup**, it is conjugate to the stabilizer of any facet in the building and is of fundamental importance in the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

On the other hand, if $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, then

$$K_1 := \mathrm{Stab}(0) = \mathrm{PGL}_2(\mathbb{Z}_p),\ K_2 := \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}_{\det\in\mathbb{Z}_p^\times}$$

are both connected open compact subgroups and conjugate in $\mathrm{PGL}_2(F)$ by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, so $\mathrm{PGL}_2(F)$ has one unique *connected* maximal compact subgroup up to conjugacy. Correspondingly,

$$\mathrm{Stab}(\mathcal{C}_0) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det\in\mathbb{Z}_p^\times} \bigsqcup \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det\in\mathbb{Z}_p^\times}$$

is a disconnected open compact subgroup, whose identity component is the Iwahori subgroup.

The example above suggests that the connected components of the stabilizers of facets in the building depend in a subtle way on the group Ω. This is indeed the case, and we discuss this connection now. Since $\Omega = \mathrm{Stab}_G(\mathcal{C}_0)$ and $\mathcal{C}_0$ is bounded by hyperplanes corresponding to S_{aff}, there is a natural homomorphism of groups

$$\Omega \longrightarrow \mathrm{Aut}(S_{\mathrm{aff}}).$$

Moreover, all permutations of S_{aff} induced by Ω can be easily seen to preserve the affine Dynkin diagram associated to S_{aff}, and if G is simple of adjoint type, then all such automorphisms of S_{aff} are induced by Ω. This greatly restricts the size of Ω.

Next, fix some *proper* subset $J \subset S_{\text{aff}}$ and consider the *standard* facet

$$c_J = \{x \in \overline{\mathcal{C}_0} \mid \langle \alpha, x \rangle \in \mathbb{Z} \text{ for } \alpha \in J \text{ and } \langle \alpha, x \rangle \notin \mathbb{Z} \text{ for } \alpha \in S_{\text{aff}} - J\}.$$

Two facets c_{J_1} and c_{J_2} are conjugate under the action of G if and only if J_1 and J_2 lie in the same Ω-orbit. Moreover, any facet c in the building is conjugate to c_J for some proper subset $J \subset S_{\text{aff}}$. In other words, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\text{aff}}\} \longleftrightarrow \{G - \text{orbits of facets } c \text{ in the BT-building}\}$$

For any facet c of the building of G, we let $K_c^+ := \text{Stab}_G(c)$ be the stabilizer of c. There is a short exact sequence

$$1 \longrightarrow U_c \longrightarrow K_c^+ \longrightarrow \overline{K}_c^+ \longrightarrow 1,$$

where U_c is the pro-unipotent radical of K_c^+ and $\overline{K}_c^+$ is the group of k-rational points of a (possibly disconnected) reductive group $\overline{\mathbf{K}}_c^+$ over k.

Definition 2.6. A **parahoric subgroup** K_c is the inverse image in K_c^+ of the group $\overline{K}_c$ of k-rational points of the identity component $\overline{\mathbf{K}}_c$ of $\overline{\mathbf{K}}_c^+$. We shall sometimes denote "parahoric subgroup" to the triple $(K_c, U_c, \overline{K}_c)$. If c is open in the building, then $(K_c, U_c, \overline{K}_c)$ is a minimal parahoric subgroup and is called an **Iwahori subgroup**. The standard Iwahori subgroup corresponds to $J = \emptyset \subsetneq S_{\text{aff}}$.

Naturally, two parahoric subgroups are conjugate in G if and only if the corresponding facets of the building are in the same G-orbit. Thus, all Iwahori subgroups are conjugate in G. If $c = c_J$ is a standard facet, then we simply write $(K_J, U_J, \overline{K}_J)$ for its associated parahoric subgroup, and K_J is generated by the standard Iwahori subgroup and J. Thus, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\text{aff}}\} \longleftrightarrow \{G - \text{conjugacy classes of parahoric subgroups } (K, U_K, \overline{K})\}$$

Moreover, if the facet c corresponds to $J \subsetneq S_{\text{aff}}$, then

$$K_c^+/K_c \cong \Omega_J = \text{Stab}_\Omega(J).$$

These results can be directly verified for SL_2, PGL_2 and GL_2 using the examples above.

Example 2.7. Suppose that $G = G_2(F)$. The affine Dynkin diagram of G_2 has no symmetries, so $\Omega = 1$ and the extended affine weyl group $\widetilde{W}$ is a Coxeter group of type $\tilde{G}_2$. Since $S_{\text{aff}} = s_0, s_1, s_2$, there are 7 conjugacy classes of parahoric subgroups, satisfying

$$\overline{K}_{\{s_1,s_2\}} = G_2(k), \quad \overline{K}_{\{s_0,s_1\}} = \text{SL}_3(k), \quad \overline{K}_{\{s_0,s_2\}} = \text{SL}_2(k) \times \text{SL}_2(k) \tag{2}$$

$$\text{what about singletons} \quad \overline{K}_\emptyset = T(k) = (k^\times)^2. \tag{3}$$

2.4 Types for Bernstein blocks and Hecke algebras

In Section 2.1 we stated the Bernstein decomposition, a fundamental result in the complex representation theory of p-adic groups. The upshot of this result a priori is clear – one can restrict attention to each block individually and study the irreducible objects in each block instead of the entire category $\mathrm{Rep}(G)$. In this section, we briefly introduce the notion of types and their corresponding Hecke algebra, which help us understand each individual Bernstein block. We give precise results for so-called *depth-zero Bernstein blocks* which will be required later on. We begin with the definition of a *type*.

Definition 2.8. Let $[M, \sigma] \in \mathfrak{J}(G)$ be a pair parametrizing a Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$. A pair (K, ρ) consisting of an open compact subgroup K of G and a smooth irreducible representation ρ of K is called a $[M, \sigma]$-type if, for any $(\pi, V) \in \mathrm{Irr}(G)$, the following two conditions are equivalent:

- The representation (π, V) lies in the Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$.
- The restriction of π to K contains ρ; in other words, $\mathrm{Hom}_K(\rho, \pi|_K) \neq 0$.

Associated to every pair (K, ρ), where K is a compact open subgroup of G and (ρ, W) is an irreducible smooth representation of K, one can associate the *Hecke algebra*

$$\mathcal{H}(G, K, \rho) := \mathrm{End}_G(c\text{-}\mathrm{Ind}_K^G \rho),$$

with composition of functions as the product. Alternatively, one can show that the Hecke algebra $\mathcal{H}(G, K, \rho)$ can be seen as the $\mathbb{C}$-vector space of functions $f : G \to \mathrm{End}_{\mathbb{C}}(W)$ satisfying

- $f(k_1 g k_2) = \rho(k_1) \circ f(g) \circ \rho(k_2)$ for any $k_1, k_2 \in K$ and $g \in G$.
- the support of f is compact,

together with multiplication given by *convolution* defined by

$$(f_1 * f_2)(g) = \sum_{x \in G(F)/K} f_1(x) f_2(x^{-1} g).$$

The importance of the theory of types in studying individual Bernstein blocks is highlighted in the following theorem.

Theorem 2.9. *Let (K, ρ) be an $[M, \sigma]$-type. Then the Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$ is equivalent to the category of right unital $\mathcal{H}(G, K, \rho)$-modules; i.e.*

$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathcal{H}(G, K, \rho) \text{ - } mod.$$

Of course, such a result is only useful if one can

1. construct types for the Bernstein blocks we are interested in,
2. understand the structure of the corresponding Hecke algebras and

3. describe the irreducible unital right modules of the Hecke algebra.

In most cases, one can answer the three questions giving rise to beautiful, deep and interesting mathematics. One of the main aims of this document is to try to answer these questions. Let us answer these questions first in a particularly simple case.

Example 2.10. Suppose that G is a simple group and that $x \in \mathcal{B}(G)$ is a vertex in building (facet of minimal dimension) and let $(K_x, U_x, \overline{K}_x)$ be the corresponding parahoric subgroup. If σ is an irreducible smooth representation of K_x^+ that is trivial on U_x and such that $\sigma|_{K_x}$ is a cuspidal representation of $\overline{K}_x$, then

$$\pi := c\text{-Ind}_{K_x^+}^G \sigma$$

is a supercuspidal representation of G, and (K_x^+, σ) is a $[G, \pi]$-type. Moreover, by Schur's lemma, we know that

$$\mathcal{H}(G, K_x^+, \sigma) = \mathrm{End}_G(c\text{-Ind}_{K_x^+}^G \sigma) = \mathrm{End}_G(\pi) = \mathbb{C}$$

is a 1-dimensional vector space. This implies that $\mathrm{Rep}(G)_{[G,\pi]}$ has a unique irreducible object, which of course is $\pi = c\text{-Ind}_{K_x^+}^G \sigma$. Moreover, π has no non-trivial extensions, so

$$\mathrm{Rep}(G)_{[G,\pi]} = \{\pi, \pi \oplus \pi, \pi \oplus \pi \oplus \pi, \ldots\}.$$

Before we finish this section, we state the answer of the first question for *depth-zero blocks.*

Definition 2.11. An irreducible smooth representation (π, V) of G has *depth-zero* if there is some parahoric subgroup $(K, U_K, \overline{K})$ such that $V^{U_K} \neq 0$.

For the remainder of the section, assume for simplicity that G is simple. Suppose that M is a Levi subgroup of G and that σ is a depth-zero supercuspidal representation σ of M. A well-known result of Moy and Prasad states that there is a vertex $x \in \mathcal{B}(M) \subseteq \mathcal{B}(G)$ with corresponding stabilizer $K_x^{(M)+}$, parahoric subgroup $(K_x^{(M)}, U_x^{(M)}, \overline{K}_x^{(M)})$ and a cuspidal representation $\tilde{\tau}$ of $\overline{K}_x^{(M)} Z(G)$ such that

$$\sigma \cong c\text{-Ind}_{K_x^{(M)} Z(G)}^G \tilde{\tau}.$$

2.5 Parahoric restriction and unipotent representations

Parahoric subgroups are ubiquitous objects in the representation theory of p-adic objects, since it provides a bridge between smooth admissible representations of the p-adic group G and finite dimensional representations of the finite groups of Lie type $\overline{K}_c$ defined in the previous section. In this section, we explore this important connection that we will exploit in a latter chapter.

If $(K, U_K, \overline{K})$ is any parahoric subgroup corresponding to a facet c and (π, V) is a smooth admissible representation of G, the space V^{U_K} of fixed points under the pro-unipotent radical is naturally a representation of $\overline{K}$. We can take this idea one step further and define the *parahoric restriction functor*

$$\mathrm{res}_K : R(G) \longrightarrow \mathbb{C}[\overline{K}]^{\overline{K}}, \quad V \longmapsto (\text{character of}) \ V^{U_K}, \quad \text{for all } V \in \mathrm{Irr}(G), \tag{4}$$

where $R(G)$ is the $\mathbb{C}$-span of $\mathrm{Irr}(G)$ and $\mathbb{C}[\overline{K}]^{\overline{K}}$ is the space of class functions of $\overline{K}$. This is well-defined since the representations are assumed to be admissible. The existence of such a functor is very powerful – we can then apply the techniques of representation theory of finite groups of Lie such as Deligne-Lusztig induction in the setting of p-adic groups. Let us begin first with a natural definition.

Definition 2.12. Let $(K, U_K, \overline{K})$ be a parahoric subgroup and (τ, E) be a cuspidal representation of $\overline{K}$. Define

$$\mathrm{Irr}(G, K, E) = \{(\pi, V) \in \mathrm{Irr}(G) \mid \text{the } \overline{K}\text{-module } V^{U_K} \text{ contains the } \overline{K}\text{-module } E\}.$$

Definition 2.13. We say that an irreducible representation (π, V) of G is *unipotent* if there is a parahoric subgroup $(K, U_K, \overline{K})$ such that V^{U_K} contains a cuspidal unipotent representation of $\overline{K}$; that is, if $(\pi, V) \in \mathrm{Irr}(G, K, E)$ for some pair (K, E) where E is unipotent. We denote the set of unipotent representations of G by

$$\mathrm{Irr}_{\mathrm{un}}(G) = \bigcup_{\substack{J \subsetneq S_{\mathrm{aff}} \\ E \text{ cusp. unip. } \overline{K}_J\text{-rep}}} \mathrm{Irr}(G, K_J, E)$$

We note that if we replace the p-adic group G for the finite group of Lie type G^F and *parahoric* by *parabolic*, then we recover the definition of a unipotent representation in G^F.

Are all pairs (K, E) as above types of certain Bernstein blocks?

Example 2.14. Let G be a split reductive p-adic group with split maximal torus T.

1. Let $\mathcal{I}$ be an Iwahori subgroup with pro-unipotent radical $\mathcal{I}^+$. Then the reductive quotient $\mathcal{I}/\mathcal{I}^+$ is isomorphic to $T(k)$. Thus, all irreducible representations of $\mathcal{I}/\mathcal{I}^+$ are 1-dimensional and the only unipotent representation is the trivial one. Therefore, the irreducible *Iwahori-spherical* representations

$$\mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) = \{(\pi, V) \in \mathrm{Irr}(G) \mid V^{\mathcal{I}} \neq 0\}$$

are all unipotent, and this set coincides with the set of irreducible subrepresentations of $c\text{-}\mathrm{Ind}_B^G \chi$, where χ is an unramified character of T. Thus, it follows that

$$\{(\pi, V) \in \mathrm{Rep}(G) \mid V \text{ is generated by } V^{\mathcal{I}}\}$$

is the *principal Bernstein block* $\mathrm{Rep}(G)_{[T,1]}$.

2. Let $(K, U_K, \overline{K})$ be a maximal parahoric subgroup corresponding to a vertex of the building associated to G and let (σ, E) be a cuspidal (not necessarily unipotent) representation of $\overline{K}$ viewed as a representation of K by inflation. Then the compactly induced $(\pi, V) := c\text{-}\mathrm{Ind}_K^G(\sigma, E)$ is an irreducible supercuspidal representation and by Frobenius reciprocity

$$(\pi, V) \in \mathrm{Irr}(G, K, E).$$

In fact, as we shall observe later, we have that $\mathrm{Irr}(G, K, E) = \{(\pi, V)\}$ and consequently (potentially mention type theory briefly) the block

$$\mathrm{Rep}(G)_{[G,\pi]} = \{\pi, \pi \oplus \pi, \pi \oplus \pi \oplus \pi, \ldots\}.$$

Remark 2.15. For $n \geq 1$, reductive groups over finite fields of type A_n have no irreducible cuspidal unipotent representations. Therefore, if G is a reductive p-adic group of type A_n and $J \subseteq S_{\text{aff}}$ is non-empty, then $\overline{K}_J$ has no cuspidal unipotent representations. This implies that the set of irreducible unipotent representations of G

$$\operatorname{Irr}_{\text{un}}(G) = \operatorname{Irr}(G, \mathcal{I}, \mathbf{1})$$

coincides with the irreducible Iwahori-spherical representations of G.

The next step is to ensure that unipotent representations behave under the parahoric restriction functor (4). The following two results ensure this is indeed the case.

Proposition 2.16. *Let (π, V) be an irreducible admissible representation of G. If there is some parahoric subgroup $(K, U_K, \overline{K})$ such that V^{U_K} contains a (potentially non-cuspidal) unipotent representation of $\overline{K}$, then (π, V) is a unipotent representation of G.*

Proof. By assumption, V^{U_K} contains a unipotent irreducible representation (σ, W) of $\overline{K}$ so $\operatorname{Hom}_K(W, V^{U_K}) \neq 0$. By Proposition 1.7, there is some standard parabolic subgroup $\overline{P} = \overline{U_P} \cdot \overline{L_P}$ of $\overline{K}$ and cuspidal unipotent representation (τ, E) of $\overline{L_P}$ such that

$$\operatorname{Hom}_K(W, \operatorname{Ind}_{\overline{P}}^{\overline{K}} E) \neq 0,$$

where we view the $\overline{K}$ representations as inflated K representations, trivial on U_K. By the classification of parahoric subgroups in G, it follows that $\overline{P} = H/U_K$, where $(H, U_H, \overline{H})$ is another parahoric subgroup contained in K. Moreover, we have the inclusions $U_K \subseteq U_H \subseteq H \subseteq K$ and therefore $\overline{U_P} = U_H/U_K$ and $\overline{L_P} = \overline{H} = H/U_H$. Since induction and inflation are commuting operations, it follows that

$$\operatorname{Inf}_{\overline{K}}^{K} \operatorname{Ind}_{\overline{P}}^{\overline{K}} E \cong \operatorname{Ind}_H^K \operatorname{Inf}_{\overline{H}}^{H} E$$

and hence $\operatorname{Hom}_K(W, \operatorname{Ind}_H^K E) \neq 0$. Since W is irreducible and K is compact, it also follows that

$$\operatorname{Hom}_K(\operatorname{Ind}_H^K E, V^{U_K}) = \operatorname{Hom}_H(E, V^{U_K}) \neq 0.$$

Since the representation E is trivial on U_H, the image of any H-equivariant map $E \to V^{U_K}$ lies inside V^{U_H}. Thus,

$$\operatorname{Hom}_H(E, V^{U_H}) = \operatorname{Hom}_H(E, V^{U_K}) \neq 0,$$

and this concludes the proof. $\square$

Conversely, we would like to show that for any irreducible unipotent representation (π, V) of G, the irreducible $\overline{K}$-submodules of V^{U_K} are all unipotent, for any parahoric subgroup $(K, U_K, \overline{K})$. This is a direct corollary of the following theorem.

Theorem 2.17. *Suppose $I \subsetneq S_{\text{aff}}$ and that V^{U_I} contains the cuspidal unipotent representation σ of $\overline{K}_I$. If $J \subsetneq S_{\text{aff}}$ with $V^{U_J} \neq 0$, and J is minimal with respect to this property, then there is $\omega \in \Omega$ such that $I = \omega J$, and V^{U_J} consists of copies of σ^{ω}. Moreover, if G is exceptional, then $J = I$.*

Proof. See Moy-Prasad for a complete account for general cuspidal representations and not necessarily unipotent and Reeder's paper for a sketch in the unipotent setting. □

Corollary 2.18. *Let (π, V) be a unipotent representation of G and let $(H, U_H, \overline{H})$ be a parahoric subgroup. Then the $\overline{H}$-irreducible components of V^{U_H} are all unipotent.*

Proof. Since (π, V) is unipotent, $(\pi, V) \in \mathrm{Irr}(G, K_J, E)$ for some $J \subsetneq S_{\mathrm{aff}}$ and cuspidal unipotent representation E of $\overline{K}_J$. Let (τ, W) be a $\overline{H}$-irreducible component of π^{U_H}. By conjugating if necessary, we may assume that $(H, U_H, \overline{H})$ is a standard parahoric subgroup. Analogously to the proof of Proposition 2.16, there is some $I \subsetneq S_{\mathrm{aff}}$ such that $(K_I, U_I, \overline{K}_I)$ is contained in K and τ is a subrepresentation of $\mathrm{Ind}_{\overline{K}_I}^{\overline{H}} \sigma$. By Theorem 2.17, I is the same Ω-orbit as J and σ is cuspidal unipotent. By Proposition 1.7, this implies that (τ, W) is also unipotent. □

Corollary 2.19. *For any two pairs (K, E), (K', E') of a parahoric subgroup and a cuspidal unipotent representation of the reductive quotient,* $\mathrm{Irr}(G, K, E)$ *and* $\mathrm{Irr}(G, K', E')$ *are either disjoint or equal.*

Analogously to the construction of $R(G)$, we define $R_{\mathrm{un}}(G)$ to be the $\mathbb{C}$-span of the irreducible unipotent representations $\mathrm{Irr}_{\mathrm{un}}(G)$. Lemma 2.16 and Theorem 2.17 implies that for each parahoric subgroup $(K, U_K, \overline{K})$ there is a well-defined *restriction function*

$$\mathrm{res}_{\mathrm{un}}^{K} : R_{\mathrm{un}}(G) \longrightarrow \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}}, \quad V \longmapsto (\text{character of}) \ V^{U_K}, \quad \text{for all } V \in \mathrm{Irr}(G).$$

It is also convenient to consider simultaneously all such functions for all conjugacy classes of maximal parahoric subgroups, so we define $\mathrm{res}_{\mathrm{un}}^{\mathrm{par}} = (\mathrm{res}_{\mathrm{un}}^{K})_K$.

2.6 Parahoric restriction for unipotent supercuspidal representations

Let G be the simple p-adic group over F. In this section, we investigate the parahoric restriction of supercuspidal unipotent representations of G (if any) with respect to maximal parahoric subgroups. A well-known result of Moy and Prasad states that any supercuspidal unipotent representation (π, V) of G is obtained by compactly inducting an irreducible smooth representation (ρ, E) of K_x^+, where $x \in \mathcal{B}(G)$ is a vertex, such that $\rho|_{K_x}$ is the inflation of a cuspidal representation of $\overline{K}_x$. By conjugating if necessary, we may assume that x lies in the closure of the fundamental alcove $\mathcal{C}_0$. Explicitly,

$$\pi \cong c\text{-}\mathrm{Ind}_{K_x^+}^{G} \rho,$$

so by Frobenius reciprocity we have that

$$\mathrm{Hom}_{K_x}(\rho|_{K_x}, \pi^{U_x}) \supseteq \text{this should be equality } \mathrm{Hom}_{K_x^+}(\rho, \pi^{U_x}) = \mathrm{Hom}_{K_x^+}(\rho, \pi) = \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_{K_x^+}^{G} \rho, \pi) \cong \mathbb{C},$$

so $(\pi, V) \in \mathrm{Irr}(G, K_x, E)$. If $J = \{\alpha \in S_\alpha \mid \langle \alpha, x \rangle = 0\}$, then $K_x = K_J$ and by cuspidality J is a minimal subset of S_{aff}, up to the action of Ω, such that $\pi^{U_J} \neq 0$. Now let $I \subsetneq S_{\mathrm{aff}}$ be another subset such that $V^{U_I} \neq 0$. If π^{U_I} contains an irreducible cuspidal representation of $\overline{K}_I$ then I is also minimal with respect to $V^{U_I} \neq 0$ and by

Theorem 2.17, I and J are in the same Ω-orbit. If π^{U_I} does not contain any irreducible cuspidal representation, then by 1.7, there is some $J' \subset I$ such that $\pi^{U_{J'}}$ contains a cuspidal representation of $\overline{K}_{J'}$ so J and J' lie in the same Ω-orbit, but this is a contradiction since K_J is a maximal parahoric subgroup of G. We have thus shown:

Lemma 2.20. *Let (π, V) be a supercuspidal unipotent representation of G. Then there is one unique Ω-orbit $[J]$ of subsets of S_{aff}, all of which are maximal such that $\pi^{U_I} \neq 0$ if and only if $I \in [J]$.*

Suppose G has type G_2 with simple reflections $S_{\text{aff}} = \{s_0, s_1, s_2\}$. We note that $\Omega = \{1\}$ so Ω-orbits are all singletons. By combining Example 2.7 and Remark 2.15, given $J \subsetneq S_{\text{aff}}$, the reductive quotient $\overline{K}_J$ has cuspidal unipotent representations if and only if $J = J_0 := \{s_1, s_2\}$ or $J = \emptyset$.

In the first case, $K_0 := K_{J_0}$ is the stabilizer of the origin in the apartment $\mathcal{A}(G,T)$ and $\overline{K_0} = G_2(\mathbb{F}_q)$ has 4 cuspidal unipotent representations labelled $G_2[1], G_2[-1], G_2[\theta]$ and $G_2[\theta^2]$, where θ is a primitive third root of unity. For any of these representations σ, Example 2.14 shows that the compactly induced representation $\pi = c\text{-Ind}_{K_0}^G \sigma$ is irreducible and supercuspidal, and

$$\mathrm{Irr}(G, K_0, \sigma) = \{\pi\}.$$

In the second case, $K_\emptyset = \mathcal{I}$ is the standard Iwahori subgroup (stabilizer of the fundamental alcove) and the only cuspidal unipotent representation of I/U_I is the trivial character. Therefore,

$$\mathrm{Irr}_{\text{un}}(G) = \mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) \bigcup \{c\text{-Ind}_{K_0}^G G_2[1], c\text{-Ind}_{K_0}^G G_2[-1], c\text{-Ind}_{K_0}^G G_2[\theta], c\text{-Ind}_{K_0}^G G_2[\theta^2]\}.$$

In the next section, we shall describe a natural way to parametrize this family. We shall now investigate the parahoric restriction of these representations with respect to the *maximal parahoric subgroups* K_0, $K_1 := K_{\{\alpha_0,\alpha_2\}}$ and $K_2 := K_{\{\alpha_0,\alpha_1\}}$.

Firstly, consider the case $\pi = c\text{-Ind}_{K_0}^G \sigma$ for a cuspidal unipotent representation σ of $G_2(\mathbb{F}_q)$. By Frobenius reciprocity, it follows that $\pi^{U_{K_0}} = \sigma \neq 0$ and therefore by Theorem 2.17, the set $J_0 = \{\alpha_1, \alpha_2\}$ is minimal with respect to the property that $V^{U_J} \neq 0$. Suppose for a contradiction that $V^{U_{J_i}} \neq 0$ for $i = 1$ or $i = 2$, where $J_1 := \{\alpha_0, \alpha_2\}$ and $J_2 := \{\alpha_0, \alpha_1\}$. Since J_1 or J_2 cannot be minimal with respect to the same property, then $V^{U_{\{\alpha_0\}}} \neq 0$. But $\overline{K}_{\{\alpha_0\}}$ has no cuspidal unipotent representations, so $V^{K_\emptyset} = V^I \neq 0$, a contradiction to Corollary 2.19.

2.7 The Langlands parametrization of unipotent representations

In this section, we give an overview on the Langlands parametrization of unipotent representations achieved by Lusztig in his celebrated paper of 1995. Firstly, we briefly discuss the results of Kazhdan–Lusztig on the parametrization of Iwahori-spherical representation when G is a p-adic reductive group of *adjoint* type. Throughout, let $G^\vee$ be complex dual group of G.

We recall that the irreducible Iwahori-spherical representations are in bijection with the irreducible modules of $\mathcal{H}_\mathcal{I} = \mathcal{H}(G, \mathcal{I}, \mathbf{1})$. Let $\mathcal{B}$ be the variety of Borel subgroups of $G^\vee$ and let

$$\mathcal{Z} = \{(B, u, B') \in \mathcal{B} \times G^\vee \times \mathcal{B} : u \in B \cap B' \text{ unipotent}\}$$

be the Steinberg variety of G, playing a main role in the representation theory of $\mathcal{H}_\mathcal{I}$. Importantly, $G^\vee \times \mathbb{C}^\times$ acts on $\mathcal{Z}$ by

$$(g, \lambda)(B, u, B') = (gBg^{-1}, gu^{\lambda^{-1}}g^{-1}, gB'g^{-1}).$$

This action gives rise to the K-group $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$, which is naturally a $\mathbb{C}[z, z^{-1}]$-module and satisfies

$$K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z}) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C}_q \cong \mathcal{H}(G, \mathcal{I}, q). \tag{5}$$

Thus, we want to construct the $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-modules and then specialize to $\mathcal{H}$-modules via (5). This is performed most naturally with Borel-Moore homology.

Let $t \in G^\vee$ be semisimple and let $u \in G^\vee$ be unipotent such that $tut^{-1} = u^q$ and let $\mathcal{B}^{t,u} \subset \mathcal{B}$ be the subvariety of Borel subgroups containing t and u. Then it turns out that $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ is naturally a $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-module, usually reducible. Since these constructions are compatible with conjugation by elements of $G^\vee$, the group $Z_{G^\vee}(t, u)$ acts on $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ by $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-intertwiners. In fact, the neutral component of $Z_{G^\vee}(t, u)$ acts trivially, so we may regard it as an action of the component group $\pi_0(Z_{G^\vee}(t, u))$. This action can be used to decompose $H_*(\mathcal{B}^{t,u}, \mathbb{C})$ as follows:

For each irreducible representation ρ of $\pi_0(Z_{G^\vee}(t, u))$ appearing in $H_*(\mathcal{B}^{t,u}, \mathbb{C})$, the space

$$K_{t,u,\rho} := \mathrm{Hom}_{\pi_0(Z_{G^\vee}(t,u))}(\rho, H_*(\mathcal{B}^{t,u}, \mathbb{C}))$$

is a nonzero $K^{G^\vee \times \mathbb{C}^\times}(\mathcal{Z})$-module, called standard. The data (t, u, ρ) are called *Kazhdan–Lusztig triples* for $(G^\vee, q)$.

Theorem 2.21. *Under the assumption that $G^\vee$ is simply connected, we have that*

1. *For each Kazhdan–Lusztig triple (t, u, ρ), the $\mathcal{H}$-module $K_{t,u,\rho}$ has a unique irreducible quotient $L_{t,u,\rho}$.*

2. *Every irreducible $\mathcal{H}$-module if of the form $L_{t,u,\rho}$ for some Kazhdan–Lusztig triple.*

3. *If (t', u', ρ') is another triple, then $L_{t,u,\rho} \cong L_{t',u',\rho'}$ if and only if there is some $g \in G$ such that $t' = gtg^{-1}$, $u' = gug^{-1}$ and $\rho' = \rho \circ \mathrm{Ad}(g^{-1})$.*

The above theorem is a major result and has many interesting consequences. However, the definition of a Kazhdan–Lusztig triple is slightly awkward since the pair (t, u) does not commute, and consequently the classification of these triples up to G-conjugacy seems hard. Thankfully, this situation can be remedied by considering *Kazhdan–Lusztig triples for* $(G^\vee, 1)$. These are defined analogously to the Kazhdan–Lusztig triples for (G, q) but replacing 1 for q throughout. In particular, the semisimple and unipotent part do commute.

Lemma 2.22. *Let G be a p-adic reductive group over a field F of residue cardinality q and let $G^\vee$ be its complex dual. There exists a bijection*

$$\begin{aligned} \{\textit{Kazhdan–Lusztig triples for } (G,1)\}/G &\longleftrightarrow \mathrm{Irr}(\mathcal{H}(G,\mathcal{I},q)) \\ (t,u,\rho) \qquad &\longmapsto \quad L_{t_q,u,\rho_q}, \end{aligned}$$

where and (t_q, u, ρ_q) are obtained from (t, u, ρ) in a prescribed way.

We recall that Kazhdan–Lusztig triples for $(G^\vee, 1)$ are defined to be tuples (t, u, ρ) such that ρ is *an irreducible character of $\pi_0(Z_{G^\vee}(tu))$ appearing in $H_*(\mathcal{B}(t,u),\mathbb{C})$*. This begs the question: if ρ does not satisfy this condition, does the triple (t, u, ρ) parametrize a (not Iwahori-spherical) representation of G?

This question was studied and completely resolved by Lusztig in his celebrated paper of 1995. He showed that, in order to get a bijection with all pairs (t, u, ρ) without technical conditions on ρ, one needs to consider a wider family of representations. Firstly, one needs to consider not only representations of G, but also of all of its *pure inner twists*. We let $\mathrm{InnT}^p(G)$ be the set of pure inner twists of G. A well known result states that there is a canonical bijection between the sets

$$\mathrm{InnT}^p(G) \longleftrightarrow H^1(F, \mathbf{G}^*) \longleftrightarrow \mathrm{Irr}(Z_{G^\vee}), \tag{6}$$

$$G' \longmapsto \zeta_{G'} \tag{7}$$

For instance, if G is a simply connected p-adic group, then $Z_{G^\vee} = \{1\}$ and therefore G has no pure inner twists other than itself. Secondly, one needs to consider all unipotent representations, and not just the Iwahori-spherical. The following theorem contains this information.

Theorem 2.23 (The arithmetic-geometric correspondence)**.** *There is an explicit bijection between the sets*

$$\bigcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \longleftrightarrow \mathcal{T}(\sqrt{q}) \longleftrightarrow \mathcal{T}(1),$$

where $\mathcal{T}(v_0)$ is set containing all triples (s, u, ρ) such that

- *$t \in G^\vee$ is semisimple,*
- *$u \in G^\vee$ is unipotent satisfying $tut^{-1} = u^{v_0^2}$,*
- *ρ is an irreducible representation of the group of components of the centralizer group $Z_{G^\vee}(t, u)$.*

For the remaining of the section, we explain how this result fits within the modern framework of the local Langlands correspondence. Let W_F be the Weyl group of the field F with inertia subgroup I_F. Moreover, we set $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$.

Under the assumption that $\mathbf{G}$ is a split group, we have the following important definition.

Definition 2.24. A *Langlands parameter* (or L-parameter) for G is a continuous morphism $\varphi : W_F' \to G^\vee$, where $G^\vee$ denotes the $\mathbb{C}$-points of the dual group of $\mathbf{G}$, and $\varphi((w,1))$ is semisimple for each $w \in W_F$.

In its simplest form, the Local Langlands correspondence (LLC) conjectures the existence of a finite to one map between isomorphism classes of smooth admissible complex representations of G and conjugacy classes of Langlands parameters of G satisfying certain nice properties. Using Theorem 2.23, we will see that the the unipotent representations of G and its pure inner twists correspond to the following Langlands parameters.

Definition 2.25. An L-parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ is called *unipotent* if $\varphi(w,1) = 1$ for any element w of the inertia subgroup I_F of W_F. Such parameters are sometimes called *unramified* Langlands parameters and we denote this set by $\Phi_{\mathrm{un}}(G^\vee)$.

Remark 2.26. For any L-parameter $\varphi : W_F' \to G^\vee$, define the commuting elements $u_\varphi = \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ and $s_\varphi = \varphi(\mathrm{Frob}, \mathrm{Id})$. An application of the Jacobson–Morozov theorem implies that an L-parameter is determined by u_φ and $\varphi|_{W_F}$ up to $G^\vee$-conjugacy. If the L-parameter is, in addition, unipotent, then $\varphi|_{W_F}$ is determined by s_φ. Thus, unipotent L-parameters are parametrized by $G^\vee$ conjugacy classes of pairs (u,s) where $u \in G^\vee$ is unipotent, $s \in G^\vee$ is semisimple and they commute. But this is the same as conjugacy classes of elements of $G^\vee$ (by using the Jordan decomposition). This should be reminiscent of the parametrization of Iwahori-spherical representations in Lemma 2.22.

However, under the LLC correspondence, unramified L-parameters do not parametrize unipotent representations, but rather L-packets of unipotent representations. To get a one to one correspondence, we need to introduce refinements of the L-parameters. Given an L-parameter φ, a natural object of interest is the component group A_φ of centralizer $Z_{G^\vee}(\varphi)$ of the image of φ inside $G^\vee$. We remark that when φ is unipotent, it is determined by the commuting elements s_φ and u_φ and therefore $Z_{G^\vee}(\varphi) = Z_{G^\vee}(s_\varphi u_\varphi)$. This object is completely analogous to the centralizer $Z_{G^\vee}(t,u)$, considered by Kazhdan and Lusztig in the setting of representations of Hecke algebras.

Definition 2.27. An *enhanced pure Langlands parameter* is a pair (φ, ϕ), where $\varphi : W_F' \to G^\vee$ is an L-parameter and ϕ is an irreducible representation of A_φ.

Let us introduce some important notation. Define

$$\Phi^p_{\mathrm{e,un}}(G^\vee) = G^\vee \backslash \{(\varphi, \phi) \mid \varphi \text{ unipotent}, \phi \in \widehat{A_\varphi}\},$$

which by the previous paragraph is in natural bijection with the set

$$G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\},$$

where A_x is the component group of $Z_{G^\vee}(x)$.

In this setting the Local Langlands conjecture predicts a natural bijection

$$\begin{array}{rcccc}\mathrm{LLC}^p_{\mathrm{un}}: G^\vee\backslash\{(x,\phi)\mid x\in G^\vee, \phi\in\widehat{A_x}\} & \longleftrightarrow & \Phi^p_{\mathrm{e,un}}(G^\vee) & \longleftrightarrow & \displaystyle\bigsqcup_{G'\in\mathrm{InnT}^p(G)}\mathrm{Irr}_{\mathrm{un}}(G') \\ (x,\phi) & & \longmapsto & & \pi(x,\phi),\end{array}$$

where G' runs over the classes of *pure* inner twists of G.

We distinguish between the distinct pure inner twists by looking at characters of $Z_{G^\vee}$. By (6), each pure inner twist G' naturally corresponds to some character $\zeta_{G'}$ of $Z_{G^\vee}$. Similarly, for any pure enhanced L-parameter (φ,ϕ), the representation ϕ induces a character ζ_ϕ on $Z_{G^\vee_{sc}}$. We say that a pair (φ,ϕ) is G'-relevant if $\zeta_\phi=\zeta_{G'}$, in which case $\pi(x_\varphi,\phi)\in\mathrm{Irr}_{\mathrm{un}}(G')$ if φ is unipotent, and we denote the set of G'-relevant pure enhanced unipotent L-parameters by $\Phi^p_{e,\mathrm{un}}(G)$. It is then clear that

$$\Phi_{e,\mathrm{un}}(G^\vee)=\bigsqcup_{G'\in\mathrm{InnT}(G)}\Phi_{e,\mathrm{un}}(G'),$$

and the LLC predicts that $\Phi_{e,\mathrm{un}}(G')$ parametrizes the set $\mathrm{Irr}_{\mathrm{un}}(G')$ for each $G'\in\mathrm{InnT}(G)$.

Example 2.28. If $\mathbf{G}$ is a simple split *simply connected* algebraic group, then $H^1(F,\mathbf{G}^*)=1$ and therefore there is only one class of pure inner forms of G, namely G itself. Correspondingly, $G^\vee=G^\vee_{\mathrm{ad}}$ and $Z_{G^\vee}$ is trivial. Therefore, the above discussion gives a bijection

$$\mathrm{LLC}^p_{\mathrm{un}}: G^\vee\backslash\{(x,\phi)\mid x\in G^\vee,\phi\in\widehat{A_x}\}\longleftrightarrow\Phi^p_{\mathrm{e,un}}(G^\vee)\longleftrightarrow\mathrm{Irr}_{\mathrm{un}}(G^*).$$

Example 2.29. If $\mathbf{G}$ is a simple split *adjoint* algebraic group, then $H^1(F,\mathbf{G}^*)=H^1(F,\mathrm{Inn}(\mathbf{G}^*))$ so for each inner twist there is one unique pure inner twist. Therefore, from the previous discussion, unipotent enhanced L-parameters are in bijection with the set

$$G^\vee\backslash\{(x,\phi)\mid x\in G^\vee,\phi\in\widehat{A_x}\},\quad\text{where}\quad A_x=Z_{G^\vee}(x)/Z_{G^\vee}(x)^0,$$

and we have a one-to-one correspondence

$$G^\vee\backslash\{(x,\phi)\mid x\in G^\vee,\phi\in\widehat{A_x}\}\longleftrightarrow\bigsqcup_{G'\in\mathrm{InnT}(G)}\mathrm{Irr}_{\mathrm{un}}(G'),\quad(x,\phi)\longmapsto\pi(x,\phi)$$

2.8 Unipotent conjugacy classes of complex simple groups

In the previous paragraph we stated the unramified local Langlands correspondence, which reduces the classification of unipotent representations of G to the classification of conjugacy classes or $G^\vee$ and the structure of the component group of their centralizer. To understand these, one first studies the classification of unipotent conjugacy classes of $G^\vee$, an interesting problem on its own right that uncovers rich structure inside $G^\vee$.

Define $\mathcal{U}$ to be the set of unipotent elements of $G^\vee$. This can be seen to be a closed irreducible subvariety of $G^\vee$ of dimension $\dim G^\vee-\mathrm{rk}G^\vee$. If $u\in\mathcal{U}$ is a unipotent element, its conjugacy class $C(u)\subset H$ is the orbit of u under the conjugation action of $G^\vee$ on itself. Standard results in the structure theory of unipotent elements inside complex reductive groups state that $G^\vee$ has finitely many conjugacy classes of unipotent elements, and that each class C is a locally closed subvariety of $G^\vee$. Moreover, its closure $\overline{C}$ is the union of (finitely many) unipotent conjugacy classes. In particular, there is one unique unipotent conjugacy class C_{reg} of maximal dimension such that C_{reg} is open and $\overline{C_{\mathrm{reg}}} = \mathcal{U}$. Such unipotent elements are called *regular*, and $\dim Z_{G^\vee}(u) = \mathrm{rk}G^\vee$ for any $u \in C_{\mathrm{reg}}$. The boundary of C_{reg} has dimension $\dim G^\vee - \mathrm{rk}G^\vee - 2$ and contains a unique dense unipotent conjugacy class C_{subreg} of *subregular* unipotent elements such that

$$\overline{C_{subreg}} = \overline{C_{\mathrm{reg}}} - C_{\mathrm{reg}} = \mathcal{U} - C_{\mathrm{reg}}.$$

Similarly, $\dim_{Z_{G^\vee}}(u) = \mathrm{rk}G^\vee + 2$ for any $u \in C_{subreg}$. At the other end, there is the trivial class consisting of $\{1\}$, and this is the only closed conjugacy class. There is one further "canonical orbit", the set of *minimal* unipotent elements $C_{\min}$, with the property that they are contained in the closure of every unipotent conjugacy class except for $\{1\}$.

Beyond these four classes, the structure of $\mathcal{U}$ for a general simple complex group can be complicated. To study it, one can define a partial ordering on the set of unipotent conjugacy classes given by

$$C \leq C' \quad \text{if and only if} \quad \overline{C} \subseteq \overline{C'}.$$

One can then picture this partial order in a diagram, called a *Hasse diagram*, and one has the following generic picture.

orbit	dimension
$\mathcal{O}_{prin}$	$\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g})$
$\mathcal{O}_{subreg}$	$\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g}) - 2$
fuzzy structure	
$\mathcal{O}_{min}$	See (4.3.5)
$\{0\}$	0

Example 2.30. If $\mathbf{G}$ is a simple split algebraic group of type G_2, then $G = \mathbf{G}(F)$ is both adjoint and simply connected and consequently there is a bijection

$$G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \mathrm{Irr}_{\mathrm{un}}(G) = \mathrm{Irr}(G, \mathcal{I}, \mathbf{1}) \bigcup \{c\text{-}\mathrm{Ind}_{K_0}^G G_2[\alpha] \mid \alpha \in \{1, -1, \theta, \theta^2\}\}.$$

Let us indicate which pairs $(x = su, \rho)$ in the left correspond to the 4 unipotent supercuspidal representations of G_2. Since supercuspidal representations are square-integrable (Is this true?), it is enough to look at the regular and subregular unipotent elements.

- Let $u = u_{\mathrm{reg}}$ be the regular unipotent element. In that case $A_u = 1$ and therefore $s = 1$, $A_x = A_u = 1$ and ρ is the trivial representation. The corresponding representation $\pi(u_{\mathrm{reg}}, \mathbf{1})$ is the Steinberg representation.

- Let $u = u_{\mathrm{sr}}$ be the subregular unipotent element. In that case, $A_u = S_3$ so up to conjugacy, $s \in \{1, g_2, g_3\}$ where g_i is a lift of order i from A_u to $Z_{G^\vee}(u)$. Moreover, $A_{ug_2} = S_3$ and $A_{ug_3} = s_2$. The corresponding table gives the required parametrization.

Langlands parameter (u, s, ϕ)	$K_0 = G_2$	$K_1 = A_1 + \widetilde{A}_1$	$K_2 = A_2$
$\pi(G_2, 1, \mathbf{1})$	$\phi_{(1,6)}$	$\epsilon \otimes \epsilon$	ϵ
$\pi(G_2(a_1), 1, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,1)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, r)$	$\phi'_{(1,3)}$	$\epsilon \otimes \mathbf{1}$	ϵ
$\pi(G_2(a_1), g_3, \mathbf{1})$	$\phi_{(1,6)} + \phi''_{(1,3)}$	$\mathbf{1} \otimes \epsilon + \epsilon \otimes \epsilon$	r
$\pi(G_2(a_1), g_2, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,2)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, \epsilon)$	$G_2[1]$	0	0
$\pi(G_2(a_1), g_2, \epsilon)$	$G_2[-1]$	0	0
$\pi(G_2(a_1), g_3, \theta)$	$G_2[\theta]$	0	0
$\pi(G_2(a_1), g_3, \theta^2)$	$G_2[\theta^2]$	0	0

3 Parahoric restriction of unipotent representations

Let G be a connected split simple group *of adjoint type* over a p-adic field F. In the previous section we stated the bijection

$$\begin{array}{ccc} \mathrm{LLC}^p_{\mathrm{un}} : G^\vee\backslash\{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} & \longleftrightarrow & \bigsqcup_{G' \in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G') \\ (x,\phi) & \longmapsto & (\pi_{(x,\phi)}, V_{(x,\phi)}), \end{array}$$

a result that parametrizes unipotent representations of G in terms of the geometry of its complex dual group $G^\vee$ and lies in the heart of the local Langlands correspondence.

In order to verify the conjecture from the introduction for the p-adic group G, one needs to explicitly compute the parahoric restriction maps

$$\mathrm{res}^J_{\mathrm{un}} : R_{\mathrm{un}}(G) \longrightarrow R_{\mathrm{un}}(\overline{K}_J), \quad V \longmapsto V^{U_J}$$

from unipotent representations of G to unipotent representations of $\overline{K}_J$. The aim of this section is to describe a general approach to compute the parahoric restriction maps above. These methods are quite complex and subtle, and involve deep mathematics – the difficulty resides in the fact that the parametrization of the unipotent representations of p-adic groups (given by Langlands parameters) and that of finite groups of Lie type (given by Lusztig's labels) are not related in an obvious way.

To do this, we closely follow the discussion in [Re2000,§4,5,6,8]. Let us fix some subset $J \subsetneq S_{\mathrm{aff}}$ and a unipotent representation $V = V_{(x,\phi)}$ of G, with $x \in G^\vee$ and $\phi \in \widehat{A_x}$. We want to decompose the $\overline{K}_J$-module V^{U_J} as a direct sum of irreducible submodules. More concretely, for each irreducible $\overline{K}_J$-representation χ, we want to calculate the value of

$$\langle \chi, V^{U_J}\rangle_{\overline{K}_J} = \langle \chi, V^{U_J}\rangle_{K_J},$$

where we view the $\overline{K}_J$-representations as K_J-representations trivial on U_J.

From Theorem 2.17 and Corollary 2.19, we deduce that all irreducible $\overline{K}_J$-constituents of V^{U_J} lie in the same series representations. Such representations are in bijection with irreducible representations of some canonical associated reflection group. For example, if V is Iwahori-spherical, then all irreducible $\overline{K}_J$-constituents of V^{U_J} lie in the principal series. These representations are in bijection with irreducible representations of the Weyl group W_J of $\overline{K}_J$ (see Section 1.4), generated by the simple reflections corresponding to the simple affine roots in J. With this fact in mind, the method strategy becomes transparent: starting with the subset $J \subsetneq S_{\mathrm{aff}}$ and the unipotent representation V, we construct a representation over the associated reflection group whose direct sum into irreducible representations is compatible with the decomposition of V^{U_J} into K_J-irreducible representations under the bijection stated above. This method consists in two major reduction steps; firstly, a reduction to Hecke algebras modules and, secondly, a reduction to modules over affine Weyl groups.

For each irreducible K_J-representation χ trivial on U_J, we want to calculate the value of $\langle \chi, V^{U_J}\rangle_{K_J}$. Firstly, by the well-known results of Harish–Chandra, there is some parahoric subgroup $(K, U_K, \overline{K})$ contained in $(K_J, U_J, \overline{K}_J)$ and cuspidal representation σ of K trivial on U_K such that

$$\chi^\sigma := \mathrm{Hom}_K(\sigma, \chi) = \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, \chi) \neq 0.$$

The space χ^σ is a naturally a $\mathcal{H}(K_J, K, \sigma)$-module, where $\mathcal{H}(K_J, K, \sigma) \cong \mathrm{End}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma)$ is the subalgebra of functions of $\mathcal{H}(G, K, \sigma)$ supported on K_J. On the other hand, the vector space

$$V^\sigma := \mathrm{Hom}_K(\sigma, V)$$

is naturally an irreducible $\mathcal{H}(G, K, \sigma)$-module and therefore a (potentially reducible) $\mathcal{H}(K_J, K, \sigma)$-module by restriction.

Lemma 3.1. *[First reduction] The multiplicity of the simple K_J-module χ in V^{U_J} is given by*

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)}.$$

Proof. Since $U_J \subseteq U$, we have that

$$\begin{aligned} V^\sigma = \mathrm{Hom}_K(\sigma, V) = \mathrm{Hom}_K(\sigma, V^{U_J}) &\simeq \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, V^{U_J}) \\ &\simeq \bigoplus_\eta \langle \eta, V^{U_J} \rangle_{K_J} \mathrm{Hom}_{K_J}(\mathrm{Ind}_K^{K_J} \sigma, \eta) \\ &\simeq \bigoplus_\eta \langle \eta, V^{U_J} \rangle_{K_J} \eta^\sigma, \end{aligned}$$

and therefore

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)} = \sum_\eta \langle \chi^\sigma, \eta^\sigma \rangle_{\mathcal{H}(K_J, K, \sigma)} \langle \eta, V^{U_J} \rangle_{K_J} = \langle \chi, V^{U_J} \rangle_{K_J},$$

as desired. ∎

For example, if V is Iwahori-spherical of G, all irreducible components χ of V^{U_J} are principal series representations, so $\chi^I = \mathrm{Hom}_I(\mathbf{1}, \chi) \neq 0$. Lemma 3.1 states that

$$\langle \chi, V^{U_J} \rangle_{K_J} = \langle \chi^{\mathcal{I}}, V^{\mathcal{I}} \rangle_{\mathcal{H}(K_J, \mathcal{I}, \mathbf{1})}.$$

Now let us consider the second reduction step. For this one, we let $R = \mathbb{C}(v, v^{-1})$, where v is an indeterminate. We then define $\mathcal{H}(G, K, \sigma)_v$ to be the *generic Hecke algebra* defined over R with the same generators and relations as $\mathcal{H}(G, K, \sigma)$, but with q replaced by v^2. The upshot of considering this generic Hecke algebra is that by specializing v we can recover

$$\mathcal{H}(G, K, \sigma)_{\sqrt{q}} = \mathcal{H}(G, K, \sigma), \qquad \mathcal{H}(G, K, \sigma)_1 = \mathbb{C}[\tilde{W}(K, \sigma)].$$

When $K = \mathcal{I}$ is the Iwahori subgroup and $\sigma = \mathbf{1}$ is the trivial representation, $\mathcal{H}(G, \mathcal{I}, \mathbf{1}) = \mathbb{C}[\widetilde{W}]$, where $\widetilde{W}$ is the extended affine Weyl group.

Fact: For any simple $\mathcal{H}(G, K, \sigma)$-module E *considered in this document*, there is a $\mathcal{H}(G, K, \sigma)_v$-module E_v such that

$$E \simeq E_v \otimes_R \mathbb{C}, \text{ where } f \in R \text{ acts on } \mathbb{C} \text{ by } f(\sqrt{q}).$$

It then follows that we have a $q = 1$ operation that takes simple modules over Hecke algebras to modules over the corresponding Weyl groups, obtained by setting $v = 1$ in all matrix coefficients of the generic module.

Proposition 3.2 (Second reduction)**.** *Let $J \subsetneq S_{\text{aff}}$ and let $(K, U_K, \overline{K})$ be a parahoric subgroup contained in $(K_J, U_J, \overline{K}_J)$ with cuspidal unipotent $\overline{K}$-representation σ. Then the diagram*

$$\begin{array}{ccc} \mathcal{H}(G,K,\sigma)\text{-}mod & \xrightarrow{q=1} & \tilde{W}(K,\sigma)\text{-}mod \\ {\scriptstyle \mathrm{Res}_{\mathcal{H}(K_J,K,\sigma)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}_{\tilde{W}(K,\sigma)_J}} \\ \mathcal{H}(K_J,K,\sigma)\text{-}mod & \xrightarrow{q=1} & \tilde{W}(K,\sigma)_J\text{-}mod \end{array}$$

is commutative, and the bottom arrow is an isometry with respect to the usual inner product of character. That is, for any irreducible K_J-module χ, we have that

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} = \langle \chi^\sigma_{q=1}, V^\sigma_{q=1} \rangle_{\tilde{W}(K,\sigma)_J}. \tag{8}$$

Proof. □

Combining both reduction steps, we obtain the identity

$$\langle \chi, V^{U_J} \rangle_{\overline{K}_J} = \langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(K_J,K,\sigma)} = \langle \chi^\sigma_{q=1}, V^\sigma_{q=1} \rangle_{\tilde{W}(K,\sigma)_J}$$

Thus, the $\tilde{W}(K,\sigma)_J$-module $V^\sigma_{q=1}$ satisfies the desired properties. By calculating its decomposition into $\tilde{W}(K,\sigma)$-irreducible representation, one can directly obtain the decomposition of V^{U_J} into irreducible $\overline{K}_J$-representations.

At this point, we restrict our attention to the parahoric restriction of Iwahori-spherical representations, for the method we use to calculate the $K(I,\mathbf{1})_J$-module $V^{\mathbf{1}}_{q=1}$ differs significantly from the non Iwahori-spherical case. As stated above, for Iwahori-spherical representations, the reflection subgroup $K(I,\mathbf{1}) \cong \widetilde{W}$ is simply the extended affine Weyl group of G, and $\widetilde{W}_J$ is the subgroup of $\widetilde{W}$ generated by the simple reflections of the roots in J.

3.1 The Iwahori-spherical case and Springer Correspondence

As indicated in the previous paragraph, we now fix an Iwahori-spherical representation $V = V_{(x,\phi)}$ of G and some proper subset $J \subsetneq S_{\text{aff}}$. The two reduction steps described above imply that for any principal series representation χ of $\overline{K}_J$,

$$\langle \chi, V^{U_J} \rangle_{\overline{K}_J} = \langle \chi^{\mathbf{1}}_{q=1}, V^{\mathbf{1}}_{q=1} \rangle_{\widetilde{W}_J}. \tag{9}$$

When $V_{(x,\phi)}$ is Iwahori-spherical, we can explicitly describe the structure of the $\widetilde{W}_J$-module $V^{\mathbf{1}}_{q=1}$ explicitly in terms of the geometry of the dual group $G^\vee$ and the data (x,ϕ). In order to extract this information, Springer theory is the key theoretical tool that we shall require. To explain this, consider the Jordan decomposition $x = us$, with commuting $u \in G^\vee$ unipotent and $s \in G^\vee$ semisimple. Let $\mathcal{B}$ and $\mathcal{B}_s$ be the flag varieties of the complex reductive groups $G^\vee$ and $G^\vee_s := Z_{G^\vee}(s)$, respectively. These are algebraic varieties parametrizing the set of Borel subgroups in $G^\vee$ and $G^\vee_s$, respectively, and admit a rational action by conjugation. Noting that $u \in G^\vee_s$, we consider the stabilizers $\mathcal{B}^x$ and $\mathcal{B}^u_s$, called partial flag varieties. These are algebraic subvarieties parametrizing the set of Borel subgroups of $G^\vee$ containing x and the set of Borel subgroups of $G^\vee_s$ containing u.

The varieties $\mathcal{B}_x$ and $\mathcal{B}^u_s$ admit a natural action of A_x induced by the action of $Z_{G^\vee}(x)$ by conjugation. On the other hand, these varieties do not admit a natural action of the Weyl groups W and W_s of $G^\vee$ and $G^\vee_s$, respectively. To obtain representations over the Weyl groups, one needs to consider the singular cohomology spaces $H^*(\mathcal{B}_x)$ and $H^*(\mathcal{B}_s^u)$. These spaces admit not only an A_x-action inherited from the action on the partial flag varieties, but also a natural representation of Weyl groups W and W_s, respectively. Moreover, both actions commute. These last two statements are at the heart of Springer theory, which we shall use continuously from now on. The mathematics behind these results involve sophisticated geometric machinery, so for the most part we will state the results we need without proof. At the top of Springer theory there is the Springer correspondence, which we now state for convenience.

Theorem 3.3. *Let $G^\vee$ be a complex algebraic reductive group with Weyl group W. For each pair (u, ϕ), where $u \in G^\vee$ is a unipotent element and ϕ is an irreducible character of A_u, the ϕ-isotropic subspace of the top cohomology group $H^{\mathrm{top}}(\mathcal{B}^u)^\phi$ is either trivial or an irreducible W-representation. Moreover, each irreducible character of W arises this way for exactly one pair (u, ϕ) up to $G^\vee$ conjugation. In other words, there is an injection*

$$\mathrm{Irr}(W) \longleftrightarrow \{\textit{pairs } (u, \phi)\}/G^\vee.$$

Finally, for any unipotent $u \in G^\vee$, the pair $(u, \mathbf{1})$ lies in the image of the map above.

This result is of great importance for it allows us to extract information from the top cohomology group directly. When $G^\vee$ is a classical group, one can give precise description of the injection in terms of combinatorial data. When $G^\vee$ is of exceptional type, one only has a finite amount of information and the injection above can be found in the literature (See tables from Carter). Another important fact that greatly simplifies calculations is the following:

Theorem 3.4. *The odd-degree cohomology spaces of $\mathcal{B}^x$ vanish, so $H^*(\mathcal{B}^x) = \sum_{i=1}^{\dim \mathcal{B}^x} H^{2i}(\mathcal{B}^x)$.*

Example 3.5. If $G^\vee = \mathrm{GL}_n(\mathbb{C})$, by the Jordan decomposition theorem, unipotent conjugacy classes are parametrized by partitions of n and $A_u = \{1\}$ for any unipotent element u. Moreover, $W \cong S_n$, and its irreducible representations are also labelled by partitions of n. If λ is a partition of n, then the Springer correspondence maps V_λ to $(u_\lambda, \mathbf{1})$, where V_λ is the Specht module of S_n and u_λ is a unipotent element of $\mathrm{GL}_n(\mathbb{C})$ corresponding to λ. In particular, the Springer correspondence for $\mathrm{GL}_n(\mathbb{C})$ is a bijection.

If $G^\vee = \mathrm{SL}_n(\mathbb{C})$, conjugacy classes of unipotent elements are still parametrized by partitions of n. This time, however, A_u may be non-trivial for some unipotent classes. The Springer correspondence is therefore not a bijection, and the image of the injection are all pairs $\{(u, \mathbf{1}), u \in \mathrm{SL}_n(\mathbb{C}) \text{ unipotent}\}$.

When $G^\vee$ has type A_n, one can describe the entire cohomology complex $H^*(\mathcal{B}^u)^{\mathbf{1}}$ explicitly. Suppose that $u = u_\lambda$ for a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n. If $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r} \leq S_n$, then Lusztig and Spaltenstein showed

$$H^*(\mathcal{B}^{u_\lambda}) \cong \mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{1}$$

as graded S_n-modules.

Example 3.6. The complex reductive group $G^\vee = G_2(\mathbb{C})$ has Weyl group $W \cong D_6$ and 5 unipotent classes labelled and partially ordered by

$$1 < A_1 < \tilde{A}_1 < G_2(a_1) < G_2.$$

The component groups are all trivial except the subregular unipotent class, whose component group is isomorphic to S_3, with irreducible representations $\mathbf{1}, \varepsilon$ and r (the unique 2-dimensional). The characters of W are labelled in the literature by the symbols $\phi_{1,0}$ (trivial), $\phi_{1,6}$ (sign), $\phi'_{1,3}, \phi''_{1,3}$ (the other two characters), $\phi_{2,1}$ (faithful), $\phi_{2,2}$ (lifted from S_3). Then the Springer correspondence gives the pairing

$$\phi_{1,0} \leftrightarrow (G_2, \mathbf{1}), \quad \phi_{2,1} \leftrightarrow (G_2(a_1), \mathbf{1}), \quad \phi'_{1,3} \leftrightarrow (G_2(a_1), r), \quad \phi_{2,2} \leftrightarrow (\tilde{A}_1, \mathbf{1}), \quad \phi''_{1,3} \leftrightarrow (A_1, \mathbf{1}), \quad \phi_{1,6} \leftrightarrow (1, \mathbf{1}),$$

and the pair $(G_2(a_1), \varepsilon)$ is not in the image of the correspondence.

Having explained the Springer correspondence, we can go back to our initial discussion. We now know that $H^*(\mathcal{B}^x)$ has the structure of a $A_x \times W$-module. We can extend this to a $A_x \times \widetilde{W}$-action as follows. The extended Weyl group can be decomposed as a semidirect product $\widetilde{W} = W \ltimes X$, where X is the character lattice of $T^\vee$. Then there is a natural evaluation pairing $X \to \mathbb{C}^\times$ given by $\mu \mapsto \langle s, \mu \rangle$. The character lattice X then acts on $H^*(\mathcal{B}^x)$ by scalars $\langle s, \cdot \rangle$, and this extends the action as desired. Analogously, the $A_x \times W_s$-action of $H^*(\mathcal{B}^u_s)$ can be extended to a $A_x \times \widetilde{W}_s$-action. The main result we need is due to Lusztig.

Theorem 3.7. *Let $V_{x,\phi}$ be an Iwahori-spherical irreducible representation of G. After letting $q \to 1$ and then taking semisimplification, the simple $\mathcal{H}(G, I, \mathbf{1})$-module $V^{\mathbf{1}}_{x,\phi}$ becomes the $\widetilde{W}$-module $\varepsilon \otimes H^*(\mathcal{B}^x)^\phi$.*

Remark 3.8. An important remark is due at this point. The above theorem relates the structure of $V^{\mathbf{1}}_{(x,\phi),q=1}$ as a $\widetilde{W}$-module with the structure of $H^*(\mathcal{B}^x)^\phi$ as a $\widetilde{W}$-module too. However, the former object is naturally p-adic, while the second is complex analytic. The Weyl groups of G and $G^\vee$ are canonically isomorphic, but not equal, since, for example, the isomorphism swaps short with long roots when the group is not simply laced. It is a standard abuse of notation to denote both Weyl groups with the same symbol, but one needs to be careful during explicit calculations inside which Weyl group one is working. Of course, we are ultimately interested in the p-adic side, but since we will be computing the cohomology groups, we will mostly work inside the complex side. In particular, the reflection s_0 inside W will be a **short** reflection, corresponding to the highest short root. At the very end, we then trace the representations back to the p-adic side. See example with G_2.

This theorem is the key theoretical tool that allows the use of Springer theory to compute parahoric restrictions of unipotent representations of G. However, it is not immediately clear how to use Springer theory yet, since $x \in G^\vee$ need not be a unipotent element. Naturally, the solution comes from relating $H^*(\mathcal{B}^x)$ to $H^*(\mathcal{B}^u_s)$. The result is due to Kato.

Proposition 3.9. *The natural restriction map $H^*(\mathcal{B}^x)^\phi \to H^*(\mathcal{B}^u_s)^\phi$ is $\widetilde{W}_s$-equivariant, and induced an isomorphism of $\widetilde{W}$-modules*

$$H^*(\mathcal{B}^x)^\phi \cong \operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}^u_s)^\phi.$$

By (9), Theorem 3.7 and Proposition 3.9, the restriction of $V_{x,\phi}$ can be determined by computing the restriction

$$\left(\operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}^u_s)^\phi\right)|_{\widetilde{W}_J},$$

where $\widetilde{W}_J$ is the subgroup of $\widetilde{W}$ generated by all simple reflections of the roots in J. Working inside $\widetilde{W}$ is very delicate, so following Reeder's development, we decide to work inside $W = \widetilde{W}_{J_0}$. The downside is the fact that, even though W contains an isomorphic copy of $\widetilde{W}_J$, one needs to carefully track this isomorphism since it can twist the original representation by some character.

More concretely, consider the image W_J of the group $\widetilde{W}_J$ under the natural projection map $\widetilde{W} \twoheadrightarrow W$. This restricts to an isomorphism $\widetilde{W}_J \cong W_J$ with inverse map $\psi_J : W_J \to \widetilde{W}_J$, satisfying $\Psi_J(s_\alpha) = s_\alpha$ if $\alpha \in J, \alpha \neq -\alpha_0$ and $\psi_J(s_0) = \tilde{s}_0 = t_{\alpha_0} s_0$ if $\tilde{s}_0 \in J$. Let ψ_J^* the pullback of representation of $\widetilde{W}_J$ to W_J.

Given a semisimple element $t \in G^\vee$, let $W_{J,t} = W_t \cap W_J$ and define the character

$$\chi_t^J := \chi_t \circ \psi_J : W_{J,t} \longrightarrow \mathbb{C}^\times. \tag{10}$$

Then by Mackey theory we obtain the isomorphism

$$\psi_J^* \left[\left(\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}_s^u)^\phi \right) |_{\widetilde{W}_J} \right] \cong \bigoplus_{w \in W_s \backslash W / W_J} \mathrm{Ind}_{W_{J,s^w}}^{W_J} \chi_{s^w}^J \otimes \left[H^*(\mathcal{B}_s^u)^\phi \right]^w. \tag{11}$$

Springer theory tells us the structure of the cohomology groups, so it remains to understand the groups $W_{J,t}$ and the characters χ_t^J. There is a substantial amount of theory behind these objects, and we will only state the results we will need. Associated to J, there is a complex reductive subgroup $G_J^\vee$ of $G^\vee$, whose roots have integral basis J and Weyl group W_J. Then, there is a semidirect product decomposition

$$W_{J,t} \cong W_{J,t}^\circ \ltimes R_{J,t},$$

where $W_{J,t}^\circ$ is a reflection subgroup of W_J generated by the roots of $G^\vee$ trivial on t. It can then be shown that the character χ_t^J is trivial on $W_{J,t}^\circ$. Moreover, if the maybe complete the character explanation here later.

The following easy lemma shows that if the characters χ_t^J are trivial, then computations are greatly simplified.

Lemma 3.10. *With the same notation as in* (11)*, if the characters $\chi_{s^w}^J$ are trivial for all $w \in W_s \backslash W / W_J$, then*

$$\psi_J^* \left[\left(\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H^*(\mathcal{B}_s^u)^\phi \right) |_{\widetilde{W}_J} \right] \cong \left(\mathrm{Ind}_{W_s}^{W} H^*(\mathcal{B}_s^u)^\phi \right) |_{W_J}$$

Proof. This is an immediate consequence of Mackey's formula applied to the right hand side. □

3.2 Parahoric restriction for type G_2

To finish this chapter, we finish with a worked example of the above methods by computing some parahoric restrictions for the p-adic group $G = G_2(F)$. We will restrict our attention to classes of maximal parahoric subgroups, for the other ones can be easily deduced from these. The group $G_2(F)$ has simple roots $\{-\beta_0, \beta_1, \beta_2\}$ with Dynkin diagram

$$0 - 1 \Longrightarrow 2.$$

Thus $G_2(F)$ has 3 classes of maximal parahoric subgroups K_0, K_1, K_2 corresponding to the subsets of affine simple roots $J_0 = \{\beta_1, \beta_2\}, J_1 = \{-\beta_0, \beta_2\}$ and $J_2 = \{-\beta_0, \beta_1\}$ and with reductive types G_2, $A_1 \times \tilde{A}_1$ and A_2, respectively. We now compute these parahoric restrictions for all representations $\pi(us, \phi)$ of $G = G_2(F)$, where u is a subregular unipotent element of $G_2(\mathbb{C})$ labelled by $G_2(a_1)$. This is an interesting case since u is a distinguished unipotent element and $\Gamma_u = A_u \cong S_3$. There are 8 such representations, given by

$$\pi(G_2(a_1),1,\phi), \phi \in \{\mathbf{1},\varepsilon,r\}, \quad \pi(G_2(a_1),g_2,\phi), \phi \in \{\mathbf{1},\varepsilon\}, \quad \pi(G_2(a_1),g_3,\phi), \phi \in \{\mathbf{1},\theta,\theta^2\}.$$

It turns out that 4 of them are unipotent supercuspidal representations of $G_2(F)$ given by

$$\pi(G_2(a_1),1,\epsilon) = c\text{-Ind}_{K_0}^G G_2[1], \quad \pi(G_2(a_1),g_2,\epsilon) = c\text{-Ind}_{K_0}^G G_2[-1], \quad \pi(G_2(a_1),g_3,\theta^k) = c\text{-Ind}_{K_0}^G G_2[\theta^k], k=1,2,$$

while the remaining 4 are Iwahori-spherical. For the supercuspidal representations, their restrictions can be easily computed. By Frobenius reciprocity, their parahoric restrictions to K_0 is irreducible. The restrictions to K_1 and K_2 are both 0, since all representations of $A_1 \times \tilde{A}_1$ and A_2 are principal series.

Thus, we can focus our attention to Iwahori-spherical representations. The computations are quite lenghty, so we give a complete sketch while omitting some unenlightening steps. We recall that we are working inside the *complex* Weyl group W and consequently s_0 is seen as a short reflection along the highest short root, and we denote by s^0 the reflection along the highest root. We first note that u is a subregular unipotent element in $G^\vee = G_2(\mathbb{C})$, while it is a regular unipotent element in $G^\vee_{g_2} = \mathrm{GL}_2(\mathbb{C})$ and $G^\vee_{g_3} = \mathrm{SL}_3(\mathbb{C})$. Therefore, $\dim \mathcal{B}^u = 1$, while $\dim \mathcal{B}^u_{g_2} = \dim \mathcal{B}^u_{g_3} = 0$. By Springer theory, we obtain that

$$H^*(\mathcal{B}^u)^{\mathbf{1}} = H^0(\mathcal{B}^u)^{\mathbf{1}} + H^2(\mathcal{B}^u)^{\mathbf{1}} = \mathbf{1} + \phi_{2,1}, \quad \text{and} \quad H^*(\mathcal{B}^u)^r = H^0(\mathcal{B}^u)^r + H^2(\mathcal{B}^u)^r = \phi'_{1,3},$$

as W-modules, while $H^0(\mathcal{B}^u_{g_2})$ and $H^0(\mathcal{B}^u_{g_3})$ afford the trivial representations of W_{g_2} and W_{g_3} respectively, and

$$\mathrm{Ind}_{W_{g_2}}^W \mathbf{1} = \phi_{1,0} + \phi''_{1,3} \quad \text{and} \quad \mathrm{Ind}_{W_{g_3}}^W \mathbf{1} = \phi_{1,0} + \phi_{2,2}.$$

To obtain the parahoric restrictions to K_0, we note that all characters $\chi_t^{J_0}$ so it remains to twist by $\varepsilon = \phi_{1,6}$ (swapping $\phi_{1,0} \leftrightarrow \phi_{1,6}$ and $\phi'_{1,3} \leftrightarrow \phi''_{1,3}$) and then by the outer automorphism of W exchanging short and long roots (swapping $\phi'_{1,3} \leftrightarrow \phi''_{1,3}$ only).

To compute the other two columns of Table 1 we compute the remaining characters χ_t^J. Using their properties above, it is not hard to show that all characters we are considering are trivial except for $\chi_{g_2^w}^{J_1}$ and $\chi_{g_3^w}^{J_2}$, and we can use directly Lemma 3.10 to compute the restrictions. The last cases must be dealt with separately.

- **Restriction of $\pi(G_2(a_1),g_3,1)$ to K_{J_2}.** In this case, $W_{J_2} = \langle s_0, s_1 \rangle$ (where s_0 is the reflection along the highest short root) and $W_{g_3} = \langle s^0, s_2 \rangle$ (where s^0 is the reflection along the highest root). Then $W_{g_3} \backslash W / W_{J_2} = \{1\}$, $W_{J_2,g_3} \cong C_3$ and $\chi_{g_3}^{J_2}$ is a primitive character. Thus,

$$\mathrm{Ind}_{W_{J_2,g_3}}^{W_{J_2}} \chi_{g_3}^{J_2} = r.$$

- **Restriction of $\pi(G_2(a_1),g_2,1)$ to K_{J_1}.** In this case, $W_{J_1} = \langle s_0, s_2 \rangle$ and $W_{g_2} = \langle s^0, s_1 \rangle$. Then $W_{g_2} \backslash W / W_{J_1} = \{1, w\}$ has two elements, and $W_{J_1,g_2} \cong C_2$ with $\chi_{g_2}^{J_1}$ non-trivial and $W_{J_1,g_2^w} = W_{J_1}$ with $\chi_{g_2^w}^{J_1}$ the trivial character. Thus,

$$\mathrm{Ind}_{W_{J_1,g_2}}^{W_{J_1}} \chi_{g_2}^{J_1} + \mathrm{Ind}_{W_{J_1,g_2^w}}^{W_{J_1}} \chi_{g_2^w}^{J_1} = \varepsilon \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \varepsilon + \mathbf{1} \boxtimes \mathbf{1}.$$

Langlands parameter (u, s, ϕ)	$K_0 = G_2$	$K_1 = A_1 + \widetilde{A}_1$	$K_2 = A_2$
$\pi(G_2, 1, \mathbf{1})$	$\phi_{(1,6)}$	$\epsilon \otimes \epsilon$	ϵ
$\pi(G_2(a_1), 1, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,1)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, r)$	$\phi'_{(1,3)}$	$\epsilon \otimes \mathbf{1}$	ϵ
$\pi(G_2(a_1), g_3, \mathbf{1})$	$\phi_{(1,6)} + \phi''_{(1,3)}$	$\mathbf{1} \otimes \epsilon + \epsilon \otimes \epsilon$	r
$\pi(G_2(a_1), g_2, \mathbf{1})$	$\phi_{(1,6)} + \phi_{(2,2)}$	$\epsilon \otimes \epsilon + \epsilon \otimes \mathbf{1} + \mathbf{1} \otimes \epsilon$	$\epsilon + r$
$\pi(G_2(a_1), 1, \epsilon)$	$G_2[1]$	0	0
$\pi(G_2(a_1), g_2, \epsilon)$	$G_2[-1]$	0	0
$\pi(G_2(a_1), g_3, \theta)$	$G_2[\theta]$	0	0
$\pi(G_2(a_1), g_3, \theta^2)$	$G_2[\theta^2]$	0	0

Figure 1: Restrictions of G_2-representations $\pi(G_2(a_1), s, \phi)$

4 The dual nonabelian Fourier transform for unipotent representation of p-adic groups

It is therefore a natural question to ask whether there exists some function $\mathrm{FT}^\vee : R_{\mathrm{un}}(G) \to R_{\mathrm{un}}(G)$ such that the square

$$\begin{array}{ccc} R_{\mathrm{un}}(G) & \xrightarrow{\mathrm{FT}^\vee} & R_{\mathrm{un}}(G) \\ \Big\downarrow{\scriptstyle \mathrm{res}^{\mathrm{par}}_{\mathrm{un}}} & & \Big\downarrow{\scriptstyle \mathrm{res}^{\mathrm{par}}_{\mathrm{un}}} \\ \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} & \xrightarrow{\mathrm{FT}^{\mathrm{par}}} & \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \end{array}$$

This question is now unresolved, but partial progress has been achieved. To understand it, we first need to look at the Langlands parametrization of unipotent representations.

4.1 The unipotent elliptic space and the dual nonabelian Fourier transform

With the Langlands parametrization, it is then possible to define a dual Fourier transform on a certain subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$, which we now describe. We first fix some unipotent element $u \in G^\vee$ up to $G^\vee$-conjugacy and we denote Γ_u the reductive part of $Z_{G^\vee}(u)$. We then consider the space of elliptic pairs

$$\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(s, h) \mid s, h \in \Gamma_u \text{ semisimple, } sh = hs \text{ and } Z_{G^\vee}(s, h) \text{ is finite}\}$$

up to Γ_u-conjugacy. Then for each $(s, h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, we define the virtual representation

$$\Pi(u, s, h) := \sum_{\phi \in \widehat{A_{su}}} \phi(h) \pi(su, \phi).$$

Definition 4.1. The elliptic unipotent representation space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ of G is defined as the $\mathbb{C}$-subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$ spanned by the set $\{\Pi(u, s, h) \mid u \in G^\vee,\ (s, h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}\}$.

On the space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ we can define the *dual* Fourier transform in a natural way.

Definition 4.2. The dual elliptic nonabelian Fourier transform is the linear map satisfying

$$\mathrm{FT}^\vee_{\mathrm{ell}} : \mathcal{R}^p_{\mathrm{un,ell}}(G) \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G) \quad \Pi(u, s, h) \longmapsto \Pi(u, h, s) \quad \text{for all} \quad (s, h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}},\ u \in G^\vee \text{ unipotent.}$$

We are now ready to state the main conjecture of this document.

Conjecture 4.3. Let G be a simple p-adic group. Then the diagram

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\quad\mathrm{FT}^\vee_{\mathrm{ell}}\quad} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \big\downarrow{\scriptstyle \mathrm{res}^{\mathrm{par}}_{\mathrm{un}}} & & \big\downarrow{\scriptstyle \mathrm{res}^{\mathrm{par}}_{\mathrm{un}}} \\ \bigoplus_{G'}\bigoplus_{K'}\mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}} & \xrightarrow{\mathrm{FT}^{\mathrm{par}}} & \bigoplus_{G'}\bigoplus_{K'}\mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}}, \end{array}$$

commutes, up to certain some roots of unity.

A simple yet important observation is that the conjecture can be verified one *unipotent conjugacy class of* $G^\vee$ *at a time* since the virtual representations $\Pi(u,s,h)$ and the dual elliptic nonabelian Fourier transform preserve the unipotent part of the parametrization. In addition, if G is simply connected, then all maximal open compact subgroups coincide with maximal parahorics and $\mathrm{FT}^{\mathrm{par}}$ fixes each component, so the conjecture can be verified *one maximal parahoric at a time* too.

We first show this is indeed the case when $\mathbf{G}$ is a simple algebraic group of type G_2.

4.2 Type G_2

Example 4.4. Let $\mathbf{G}$ be a simple algebraic group of type G_2, and let $G = G(F)$. Then G is both simply connected and adjoint so it has no pure inner twists other than itself. In addition, G has three maximal parahoric subgroups of types K_0, K_1 and K_2, with reductive quotients of type G_2, A_2 and $A_1 + \tilde{A}_1$, respectively. Thus, commutativity of the above square is equivalent to the commutativity of

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \big\downarrow{\scriptstyle \mathrm{res}^{\mathrm{par}}_{\mathrm{un}}} & & \big\downarrow{\scriptstyle \mathrm{res}^{K_i}_{\mathrm{un}}} \\ \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}} & \xrightarrow{\mathrm{FT}^{K_i}} & \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}}, \end{array}$$

for $i = 0,1,2$. Moreover, if $u \in G^\vee$ is unipotent, then $\mathcal{Y}(\Gamma_u)$ is non-empty if and only if $u = u_{\mathrm{reg}}$ is regular and $\Gamma_u = \{(1,1)\}$, or $u = u_{sr}$ is subregular and $\Gamma_u = \{(1,1),(1,g_2),(1,g_3),(g_2,1),(g_2,g_2),(g_3,1),(g_3,g_3),(g_3,g_3')\}$. Therefore, $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ is 9-dimensional, spanned by

$$\begin{cases} \Pi(u_{\mathrm{reg}},1,1) & = \pi(u_{\mathrm{reg}},\mathbf{1}) \\ \Pi(u_{sr},1,1) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) + 2\pi(u_{sr},\mathbf{r}) \\ \Pi(u_{sr},1,g_2) & = \pi(u_{sr},\mathbf{1}) - \pi(u_{sr},\varepsilon) \\ \Pi(u_{sr},1,g_3) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) - \pi(u_{sr},\mathbf{r}) \\ \Pi(u_{sr},g_2,1) & = \pi(u_{sr}g_2,\mathbf{1}) + \pi(u_{sr}g_2,\varepsilon) \\ \Pi(u_{sr},g_2,g_2) & = \pi(u_{sr}g_2,\mathbf{1}) - \pi(u_{sr}g_2,\varepsilon) \\ \Pi(u_{sr},g_3,1) & = \pi(u_{sr}g_3,\mathbf{1}) + \pi(u_{sr}g_3,\theta) + \pi(u_{sr},\theta^2) \\ \Pi(u_{sr},g_3,g_3) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta^2\pi(u_{sr}g_3,\theta) + \theta\pi(u_{sr}g_3,\theta^2) \\ \Pi(u_{sr},g_3,g_3^{-1}) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta\pi(u_{sr}g_3,\theta) + \theta^2\pi(u_{sr}g_3,\theta^2). \end{cases}$$

When $i=1,2$ and the finite group $\overline{K_i}$ is of type A_2 or $A_1+\tilde{A}_1$, then FT^{K_i} is the identity map, and therefore it is enough to show that

$$\mathrm{res}^{K_i}_{\mathrm{un}}(\pi(u,s,h)) = \mathrm{res}^{K_i}_{\mathrm{un}}(\Pi(u,h,s))$$

for all $\Pi(u,s,h)$ spanning $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. This is obvious for all cases except for $\pi(u,s,h)=\pi(u_{sr},1,g_2), \pi(u_{sr},1,g_3)$.

5 The p-adic group of type F_4

In this chapter, we let $G = F_4(F)$ be a simple p-adic group of type F_4, whose extended Dynkin diagram is

$$0 - 4 - 3 \Longrightarrow 2 - 1.$$

Let us also fix a set of simple affine roots $S_{\text{aff}} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_0\}$, where β_1, β_2 are the short simple roots. Since the extended Dynkin diagram has no non-trivial automorphisms, the fundamental group $\Omega(F_4)$ is trivial, and therefore $G = F_4(F)$ is both a simply connected and adjoint p-adic group. This is an important fact that simple groups of type F_4 share with simple groups of type G_2 that simplifies many results and calculations.

Since G is simply connected, it has no pure inner twists other than itself. Moreover, the set of maximal open compact subgroups of G agree with the set of maximal parahoric subgroups of G, and their conjugacy classes can be parametrized by proper maximal subsets $J \subsetneq S_{\text{aff}}$. It is then easy to see that there are five maximal open compact subgroups $\{K_0, K_4, K_3, K_2, K_1\}$ of G up to conjugacy, where K_i corresponds to $J_i := S_{\text{aff}} \setminus \{\alpha_i\}$ for $0 \le i \le 4$. The reductive quotients $\overline{K}_0, \overline{K}_4, \overline{K}_3, \overline{K}_2, \overline{K}_1$ are finite reductive groups of type $F_4, A_1 \times C_3, A_2 \times \tilde{A}_2, A_3 \times \tilde{A}_1, B_4$, respectively. In particular,

$$\mathcal{C}(G)_{\text{cpt,un}} = \bigoplus_{i=0}^{4} R_{\text{un}}(\overline{K}_i)$$

and $\text{FT}^{\text{par}} = (\text{FT}^{K_i})_i$ is Lusztig's Fourier transform on $R_{\text{un}}(\overline{K}_i)$ on each coordinate.

The aim of this chapter is to prove that Conjecture 4.3 holds for G. In other words, we prove that

Theorem 5.1. *Let G be a simple p-adic group of type F_4. Then the diagram*

$$\begin{array}{ccc} \mathcal{R}^p_{\text{un},ell}(G) & \xrightarrow{\text{FT}^\vee_{ell}} & \mathcal{R}^p_{\text{un},ell}(G) \\ \Big\downarrow{\scriptstyle \text{res}^{\text{par}}_{\text{un}}} & & \Big\downarrow{\scriptstyle \text{res}^{\text{par}}_{\text{un}}} \\ \bigoplus_{i=0}^5 R_{\text{un}}(\overline{K}_i) & \xrightarrow{(\text{FT}^{K_i})_i} & \bigoplus_{i=0}^5 R_{\text{un}}(\overline{K}_i), \end{array}$$

commutes.

Firstly, we need to compute the basis $\{\Pi(u, s, h) \mid u \in G^\vee, (s, h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\text{ell}}\}$ of the space $\mathcal{R}^p_{\text{un,ell}}(G)$. Here u is a unipotent element of $G^\vee = F_4(\mathbb{C})$, the complex simple reductive group of type F_4, and $\mathcal{Y}(\Gamma_u)$ is the set of elliptic pairs of Γ_u (see Section 4.1 for the Definitions). Thus, the first step is to classify all such pairs for all unipotent conjugacy classes of $F_4(\mathbb{C})$. Thankfully, this is a well-known classification given in Table 1. In particular, $\mathcal{R}^p_{\text{un,ell}}(G)$ is a 31-dimensional vector space.

The main difficulty in proving Theorem 5.1 lies in the explicit computation of the restriction map

$$\text{res}^{\text{par}}_{\text{un}} : \mathcal{R}^p_{\text{un,ell}}(G) \longrightarrow \bigoplus_{i=0}^{5} R_{\text{un}}(\overline{K}_i).$$

We note that the conjecture can be verified one unipotent class $u \in G^\vee$ and one parahoric subgroup K_i at a time. Thus, the proof of Theorem 5.1 will consist on a case-by-case analysis of the distinct options.

5.1 Restriction to hyperspecial parahoric $K_0 \longrightarrow F_4$

Unipotent	Γ_u^0	A_u	$\lvert\Gamma_u\backslash(\Gamma_u)_{\mathrm{ell}}\rvert$	$\mathcal{F}_u n$	$\Gamma_{\mathcal{F}_u}$
F_4	1	1	1	$\phi_{1,24}$	1
$F_4(a_1)$	1	S_2	4	$\phi_{4,13}$	C_2
$F_4(a_2)$	1	S_2	4	$\phi_{9,10}$	1
B_3	PGL(2)	1	1	$\phi''_{8,9}$	1
$F_4(a_3)$	1	S_4	21	$\phi_{12,4}$	S_4

Table 1: Elliptic pairs for F_4