Nonabelian Fourier transform for unipotent representations

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1 Unipotent representations for finite groups of Lie type

1.1 Frobenius maps

Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a connected reductive group over k. The structure of G can be understood to a large extent by looking at its maximal connected solvable subgroups of G, denoted as Borel subgroups. If we fix some Borel subgroup B, any maximal torus T in B is also a maximal torus in G, and it determines a set of roots $\Phi = \Phi(G, T) \subset X(T)$. The choice of the Borel B containing T corresponds to a choice of positive roots Φ^+ and therefore of an integral basis $\Delta \subseteq \Phi^+$. Moreover, the subgroups $B, N := N_G(T)$ satisfy the axioms of a BN-pair, as described by Tits whose corresponding Weyl group is $W = N/T = \langle w_{\alpha_i} \mid \alpha_i \in \Delta \rangle$.

Let $F : G \to G$ be a Frobenius map and let G^F be the fixed points under the Frobenius map. One can show that G contains F-stable Borel subgroups, and that inside any F-stable Borel there are F-stable maximal tori. Thus, we may assume that the Borel subgroup B and maximal torus T fixed in the previous paragraph are F-stable. Under these assumptions, the Frobenius map acts on the simple roots by permuting the corresponding the root spaces. Thus, F corresponds to some permutation ρ of Δ satisfying

$$F(\mathcal{X}_\alpha) = \mathcal{X}_{\rho(\alpha)} \quad \text{for all } \alpha \in \Delta.$$

Moreover, one can easily check that ρ is in fact a symmetry of the Dynkin diagram, and these can be completely classified. For each orbit $J \subseteq \Delta$ of ρ, let $w_J \in W_J = \{w_{\alpha_i} \mid \alpha_i \in J\}$ be the unique element such that $w_J(J) = -J$. Moreover, it satisfies that $w_J^2 = 1$. In then follows that the group G^F has a natural BN-pair given by the groups B^F and N^F, whose Weyl group is

$$N^F/T^F = (N/T)^F = W^F = \langle w_J \mid J \subseteq \Delta \text{ is an orbit of } \rho \rangle.$$

Any F-stable Borel subgroup contains an F-stable maximal torus, but the converse might not be true. Any F-stable maximal torus that is contained in an F-stable Borel subgroup is called *maximally split*, and since any two F-stable Borel are conjugate under G^F, any two F-stable maximally split tori are also conjugate under G^F. In fact, one can easily determine the G^F-conjugacy classes of F-stable maximal tori by looking at the Weyl group. To state this result, we first introduce the notion of F-conjugacy classes in W. Given two $w_1, w_2 \in W$, we say that they are F-conjugate if there is some $x \in W$ such that $F(x)w_1x^{-1} = w_2$. Note that if F acts on W trivially, then the F-conjugacy classes are the standard conjugacy classes.

Lemma 1.1. *There is a bijection between*

$$\{G^F\text{-conjugacy classes of } F\text{-stable maximal tori}\} \longrightarrow \{F\text{-conjugacy classes of } W\}$$
$$T' = {}^gT \longmapsto \pi(g^{-1}F(g))$$

From now, we will write T_1 for a maximally split F-stable maximal torus and T_w for any F-stable torus obtained from T_1 by conjugating by some element $g \in G$ such that $\pi(g^{-1}F(g)) = w$. By the previous result, these objects are uniquely defined up to G^F-conjugation.

1.2 Deligne–Lusztig characters and unipotent representations

In their groundbreaking paper from 1976, Deligne and Lusztig attached to each pair (T, θ) of F-stable maximal torus T and character θ of T^F, a virtual character $R_{T,\theta}$ of the group G^F. These virtual characters were constructed using the action of G^F on certain ℓ-adic cohomology groups associated to certain Deligne–Lusztig varieties. We shall not consider the explicit definition of the characters, but we will rather recall without proof some important properties.

1. If the pair (T', θ') is obtained from (T, θ) by conjugation on some element of G^F, then $R_{T,\theta} = R_{T',\theta'}$.
2. If T_1 is a maximally split torus inside some F-stable Borel B, then $R_{T_1,\theta} = \theta_{B^F}^{G^F}$, where $\theta_{B^F}^{G^F}$ is the character of the parabolically induced representation $\mathrm{Ind}_{B^F}^{G^F}\theta$.
3. $R_{T,\theta}(u)$ is independent of θ if u is unipotent. We write $Q_T(u)$ for this common value.
4. The orthogonality relations $(R_{T,\theta}, R_{T',\theta'}) = |\{w \in W(T, T')^F \mid {}^w\theta' = \theta\}|$ hold. In particular, if T, T' are not G^F-conjugate, then $(R_{T,\theta}, R_{T',\theta'}) = 0$.
5. If (T, θ) is in general position, then one of $\pm R_{T,\theta}$ is an irreducible character.
6. If (T, θ) and (T', θ') are not geometrically conjugate, then $R_{T,\theta}$ and $R_{T',\theta'}$ do not share any irreducible component. This is a stronger assumption than not being G^F-conjugate.
7. We have
$$(R_{T,\theta}, 1) = \begin{cases} 1 \text{ if } \theta = 1 \\ 0 \text{ if } \theta \neq 1 \end{cases}$$
8. The dimension of $R_{T,\theta}$ equals
$$R_{T,\theta} = \varepsilon_G\varepsilon_T|G^F : T^F|$$

Let's give a couple of examples for the decomposition of the Deligne–Lusztig characters.

Example 1.2. Suppose first that $G = \mathrm{GL}_2(k)$ and $F = F_q : G \to G$ is the standard Frobenius. Then $G^F = \mathrm{GL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 \neq 0 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ while $T_2^F \cong \mathbb{F}_{q^2}^\times$. Now, if $\theta = \theta_1 \otimes \theta_2$ is a character of T_1^F, then

$$R_{T_1,\theta} = \theta_{B^F}{}^{G^F} = \begin{cases} \overline{\theta} \otimes (1 \oplus \mathrm{St}) & \text{if } \theta_1 = \theta_2 \\ \text{irreducible principal series} & \text{if } \theta_1 \neq \theta_2, \end{cases}$$

where $\overline{\theta}$ is the unique extension of θ to all of $\mathrm{GL}_2(\mathbb{F}_q)$ (this is only possible if $\theta_1 = \theta_2$). On the other hand, suppose that θ' is a character of T_w^F. Then

$$R_{T_w,\theta'} = \begin{cases} \overline{\theta} \otimes (1 \ominus \mathrm{St}) & \text{if } \theta'^q = \theta' \\ \text{irreducible cuspidal} & \text{if } \theta'^q \neq \theta', \end{cases}$$

where $\overline{\theta}$ is the extension of the unique character θ of T_1^F for which (θ, T_1) is geometrically conjugate to (θ', T_w).

Example 1.3. Now suppose that $G = \mathrm{SL}_2(k)$ and $F = F_q : G \to G$ to be the standard Frobenius again. Similarly, $G^F = \mathrm{SL}_2(\mathbb{F}_q)$ and G has two F-stable tori up to G^F conjugation, namely

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^* \right\} \quad \text{and} \quad T_w = \left\{ \begin{pmatrix} a & b \\ ub & a \end{pmatrix} \mid a, b \in k,\ a^2 - ub^2 = 1 \right\},$$

where $u \in \mathbb{F}_q^\times$ is a non-square. Then $T_1^F \cong \mathbb{F}_q^\times$ while $T_2^F \cong C_{q+1}$. If θ is a character of T_1^F then

$$R_{T_1,\theta} = \theta_{B^F}{}^{G^F} = \begin{cases} 1 \oplus \mathrm{St} & \text{if } \theta = 1, \\ R_+(\xi) \oplus R_-(\xi) & \text{if } \theta = \xi = \mathrm{sgn}, \\ \text{irreducible principal series} & \text{if } \theta \neq \theta^{-1}, \end{cases}$$

where $R_+(\xi) \neq R_-(\xi)$ are conjugate under $\mathrm{GL}_2(\mathbb{F}_q)$ and so they have the same dimension $(q+1)/2$. On the other hand, if θ' is a character of T_w^F, then

$$R_{T_w,\theta'} = \begin{cases} 1 \ominus \mathrm{St} & \text{if } \theta' = 1, \\ \ominus R'_+(\xi) \ominus R'_-(\xi) & \text{if } \theta' = \xi = \mathrm{sgn}, \\ \ominus \text{ irreducible cuspidal} & \text{if } \theta' \neq \theta'^{-1}, \end{cases}$$

One can show that for a reductive group G over k with centre Z and semisimple rank l, there are exactly $|Z^F|q^l$ geometric conjugacy classes of pairs (T, θ). Moreover, one can define a geometric conjugacy on the set of irreducible characters of G^F as follows. We say that two characters χ_1, χ_2 are related if there are geometrically conjugate pairs (T, θ) and (T', θ') such that

$$(\chi_1, R_{T,\theta}) \neq 0 \quad \text{and} \quad (\chi_2, R_{T',\theta'}) \neq 0.$$

Clearly, there are then $|Z^F|q^l$ geometric conjugacy classes of characters. We are now ready to give the characterization of a semisimple character and a unipotent one.

Definition 1.4. An irreducible character χ of G^F is called *unipotent* if there is some maximal F-stable torus T of G such that $(R_{T,1}, \chi) \neq 0$. An irreducible character χ of the group G^F is called *semisimple* if

$$\sum_{\substack{u \in G^F \\ u \text{ reg unipotent}}} \chi(u) \neq 0.$$

It is clear from the definitions that unipotent characters form one geometric conjugacy class of irreducible characters, and that if χ is a unipotent character, then $(\chi, R_{T,\theta}) = 0$ for any $\theta \neq 1$. Semisimple characters, on the other hand, have the opposite property. To explain this, we define the class function Ξ to be supported on regular unipotent elements with constant value of $|Z^F|q^l$. By using properties of character duality, one can show that $(\Xi, \Xi) = |Z^F|q^l$ and that $(\Xi, \chi) \in \{-1, 0, 1\}$ for all irreducible characters χ of G^F. Note that this implies that there are exactly $|Z^F|q^l$ semisimple characters. In fact, one can furthermore show that

$$\Xi = \sum_{\kappa} \varepsilon_\kappa \chi_\kappa^{ss} \quad \text{where } \chi_\kappa^{ss} \text{ is irreducible and} \quad \varepsilon_\kappa \chi_\kappa^{ss} = \sum_{\substack{(T,\theta) \in \kappa \\ \text{mod } G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})},$$

where κ runs over the conjugacy classes of pairs (T, θ). These results show that each geometric conjugacy class contains one unique semisimple irreducible character.

1.3 Jordan decomposition for irreducible characters

Finally, we are ready to describe the *Jordan decomposition for characters.* To simplify the discussion, we shall assume that the centre $(Z(G))^F$ of G^F is connected. The idea is that one can completely understand all characters of a finite group of Lie type by understanding its semisimple representations and unipotent representations of its Levi subgroups. To state it, we first recall that there is a natural bijection between geometric conjugacy classes of (T, θ) and conjugacy classes of semisimple elements in the dual group $(G^*)^F$, both sets having size $|Z(G)^F|q^l$.

Definition 1.5. Let (s) be a semisimple conjugacy class of the dual group $(G^*)^{F^*}$. Then the *Lusztig series* $\mathcal{E}(G^F, (s))$ associated to (s) is the set of irreducible characters of G^F appearing in $R_{T,\theta}$ for some pair (T, θ) corresponding to (s).

The Lusztig series $\mathcal{E}(G^F, (s))$ are the geometric conjugacy classes of characters defined in the previous section. If (s) is regular semisimple, then (T, θ) is in general position and $\mathcal{E}(G^F, (s))$ is a singleton. On the other end, the series $\mathcal{E}(G^F, (1))$ contains the unipotent characters.

Theorem 1.6. *Let (s) be a semisimple conjugacy class of $(G^*)^F$ and let H be the dual group of the centralizer $Z_{G^*}(s)$. Then there is a bijection*

$$\mathcal{E}(G^F, (s)) \to \mathcal{E}(H^F, (1)), \quad \chi \mapsto \chi_u,$$

such that for any pair (T, θ) corresponding to $s \in (G^*)^{F^*}$ and any pair (S, ψ) corresponding to $s \in (H^*)^{F^*}$,

$$(\chi, \varepsilon_G \varepsilon_T \cdot R^G_{T,\theta}) = (\chi_u, \varepsilon_H \varepsilon_S \cdot R^H_S(\psi)).$$

In addition, the unique semisimple character $\chi_s \in \mathcal{E}(G^F, (s))$ corresponds to the trivial character of H^F and for any $\chi \in \mathcal{E}(G^F, (s))$, we have that

$$\chi(1) = \chi_s(1)\chi_u(1).$$

To summarize, for any irreducible character χ, there is one unique semisimple character χ_s geometrically conjugate to χ, corresponding to some semisimple conjugacy class of $(G^*)^{F^*}$. One can in fact show that

$$\chi_s(1) = |(G^*)^{F^*} : C^{F^*}|_{p'},$$

where C is the centralizer of s^* in G^*. Finally, there is a natural bijection $\chi \mapsto \chi_u$ between characters in the class containing χ_s and unipotent characters of the dual group of C^{F^*} satisfying

$$\chi(1) = \chi_s(1)\chi_u(1).$$

As it turns out, studying semisimple characters is easy since we have explicit formulas to understand them. So Lusztig turned his attention into understanding unipotent representations of finite groups of Lie type. Lusztig first observed that the study of unipotent characters of G^F can be reduced to the case when G is simple of adjoint type. That's because every unipotent character appears as a component of some $R_{T,1}$, where $R_{T,1}(g) = \mathcal{L}(g, \mathfrak{B}_w)$. But Z^F acts trivially on $\mathfrak{B}_w$, so it lies in the kernel of every unipotent representation. So G can be assumed to be semisimple, and a simple argument shows that G can be further assumed to be simple.

Secondly, following the same approach as Harish–Chandra to classify irreducible characters of G^F, Lusztig showed the following result.

Proposition 1.7. *Let χ be an irreducible character of G^F.*

1. *There is an F-stable parabolic subgroup P of G with F-stable Levi decomposition $P = LN$ and a cuspidal character ϕ of L^F such that $(\chi, {\phi_{P^F}}^{G_F}) \neq 0$.*

2. *Moreover, the pair (P, ϕ) is unique up to G^F-conjugacy.*

3. *The character χ of G^F is unipotent if and only if ϕ is a unipotent character of L^J.*

Proof. The proof of parts 1. and 2. are classical, so we only give a sketch. We fix some F-stable maximal torus T and some integral basis $\Delta \subset \Phi(G, T)$. For each $J \subseteq \Delta$, let $P_J = L_J U_J$ be the standard F-stable parabolic with standard F-stable Levi L_J. Since any parabolic subgroup of G^F is conjugate to some P^F_J, it is enough to prove the assertions for standard parabolics of G^F.

Let V be the G^F representation affording χ and let $\mathcal{J} = \{J \subseteq \Delta \mid (1_{U_J}, \chi|_{U_J}) \neq 0\} = \{J \subseteq \Delta \mid V^{U_J} \neq 0\}$, which is non-empty since $\Delta \in \mathcal{J}$. If $J \in \mathcal{J}$ is minimal with respect to inclusion, we may write $V^{U_J} = \oplus_{i=1}^k U_i$ as a direct sum of irreducible L_J-representations, all of which are cuspidal. The character ϕ afforded by U_1 satisfies the conditions of 1., and part 2. is contained in Carter 9.1.5.

To prove the last assertion, fix some $J \subseteq \Delta$ and some irreducible character ϕ of L_J. Let (T, θ) be such that $(\phi, R_{T,\theta}^{L_J^F})_{L_J^F} \neq 0$, and let χ be an irreducible component of ${\phi_{P_J^F}}^{G^F}$. Then by Frobenius reciprocity, we have that

$$(\chi|_{P_J^F}, \phi_{P_J^F})_{P_J^F} = (\chi, {\phi_{P_J^F}}^{G^F})_{G^F} \neq 0,$$

and since $\phi_{P_J^F}$ is an irreducible P_J^F representation, we have that

$$(\chi, R_{T,\theta}^{G^F}) = (\chi, {(R_{T,\theta}^{L_J^F})_{P_J^F}}^{G^F})_{G^F} = (\chi|_{P_J^F}, (R_{T,\theta}^{L_J^F})_{P_J^F})_{P_J^F} \neq 0.$$

This calculation, together with the fact that unipotent representations form a geometric conjugacy class yield the last part. $\square$

Therefore, to classify the unipotent characters of G^F, it is enough to determine the cuspidal unipotent representations ϕ of the standard Levi subgroups L_J^F of G^F and then calculate the decomposition of ${\phi_{P_J^F}}^{G^F}$ into irreducible characters. The later task can be achieved by Howlett–Lehrer theory (Carter, §10), while the former was achieved by Lusztig by a case by case analysis. For example, Lusztig showed that if G^F is of classical type, then the number of cuspidal unipotent characters is either 0 or 1.

1.4 Families of unipotent characters

Lusztig further observed that the unipotent characters of G^F naturally form families in a remarkable way. Firstly, he parametrized the principal series unipotent characters with irreducible characters of W by showing that there is a natural bijection

$$\{\text{Irreducible characters of } W\} \longrightarrow \{\text{Irreducible components of } \mathrm{Ind}_{B^F}^{G^F} 1\}$$
$$\phi \longmapsto \chi_\phi.$$

To prove this, we first note that

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1) \cong \mathrm{Hom}_{B^F}(1, \mathrm{Ind}_{B^F}^{G^F} 1|_{B^F}) \cong \bigoplus_{w \in B\backslash G/B} \mathrm{Hom}_{B^F \cap {}^w B^F}(1,1),$$

where in the first step we have applied Frobenius reciprocity and the Mackey decomposition formula for the second one. By the Bruhat decomposition, W is canonically isomorphic to $B\backslash G/B$ and the borel subgroup B gives a natural choice of simple roots, and therefore of simple reflections $S \subset W$.

If we let $T_w \in \mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ be the image of the identity map in $\mathrm{Hom}_{B^F \cap {}^w B^F}(1,1)$, then one can prove that $\{T_w : w \in W\}$ is a basis for $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ satisfying

$$\begin{aligned} T_s^2 &= (q-1)T_s + qT_1 \quad && \text{if } s \in S, \\ T_{w_1} T_{w_2} &= T_{w_1 w_2} \quad && \text{if } l(w_1 w_2) = l(w_1) + l(w_2). \end{aligned}$$

And therefore, $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1)$ is isomorphic to the coxeter algebra $\mathcal{H}(W, S, q)$ of the pair (W, S) with constant parameter q. It is possible to do a change of variables that give the important isomorphism

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F} 1) \cong \mathbb{C}[W].$$

The algebra $\mathbb{C}[W]$ acts on itself by left multiplication, and the irreducible submodules are precisely the irreducible representations of W. By the isomorphism above, $\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F}1)$ also decomposes into a direct sum of irreducible submodules under post composition. If $\mathrm{Ind}_{B^F}^{G^F}1 = \oplus_{i=1}^k V_i^{a_i}$ is a direct sum into G^F irreducible components, then

$$\mathrm{End}(\mathrm{Ind}_{B^F}^{G^F}1) = \bigoplus_{i=1}^{k} \mathrm{Hom}_{G^F}(V_i, \oplus_{j=1}^k V_j^{a_j})^{a_i},$$

and the modules on the right hand side are precisely the irreducible submodules, each one corresponding to one unique irreducible component of $\mathrm{Ind}_{B^F}^{G^F}1$. Thus, irreducible characters of W parametrize principal series cuspidal characters of G^F.

Example 1.8. Let G be a reductive group of type A_l. Then G^F has no cuspidal unipotent representations. Consequently, all unipotent representations of G^F are in the principal series. By the above discussion, this means that the irreducible characters of W completely parametrize all unipotent representations of G^F. Explicitly, given some irreducible character ϕ of W,

$$\chi_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1}$$

For $G = \mathrm{GL}_2(k)$ or $\mathrm{SL}_2(k)$, $\chi_1 = 1$ and $\chi_{\mathrm{sgn}} = \mathrm{St}$.

In general, however, finite groups of Lie type do have cuspidal unipotent characters, and the virtual characters

$$R_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w)R_{T_w,1}$$

as defined above are not irreducible. Lusztig then divided unipotent representations of G^F into families by the rule that two unipotent characters appearing in the same R_ϕ are in the same family and then extending by transitivity. Similarly, we can define an equivalence relation on the irreducible characters of W by the rule that two characters ϕ_1, ϕ_2 are related if R_{ϕ_1} and R_{ϕ_2} share an irreducible component. It is clear that there is a bijection between families of unipotent characters of G^F and families of characters of W. Remarkably, Lusztig proved that these families can be parametrized in the following manner.

Theorem 1.9. *For each family of unipotent representations $\mathcal{F}$ there is a group $\Gamma = \Gamma_{\mathcal{F}} \in \{1, C_2\times\cdots\times C_2, S_3, S_4, S_5\}$ and a bijection*

$$\begin{aligned} M(\Gamma) &\longrightarrow \mathcal{F} \\ (x,\sigma) &\longmapsto \chi^{\mathcal{F}}_{(x,\sigma)} \end{aligned}$$

satisfying that

$$(\chi^{\mathcal{F}}_{(x,\sigma)}, R_\phi) = \begin{cases} \{(x,\sigma),(y,\tau)\} & \text{if } \chi_\phi = \chi^{\mathcal{F}}_{(y,\tau)} \in \mathcal{F}, \\ 0 & \text{if } \chi_\phi \notin \mathcal{F}. \end{cases}$$

Since $R_{T_1,1} = \sum_{\phi\in\hat{W}} R_\phi$, it follows that for any family $\mathcal{F}$, $(\chi^{\mathcal{F}}_{(1,1)}, R_{T_1,1}) > 0$, so $\chi^{\mathcal{F}}_{(1,1)} = \chi_\phi$ for some character ϕ of W. Characters arising this way are called *special characters* of W and they have distinct characterizations. They are the distinguished elements of the families of characters of W as described above. The upshot of this discussion is that families of unipotent characters can be parametrized by special characters of the Weyl group.

Example 1.10. Let $G = G_2(k)$ and let $F = F_q : G \to G$ be the standard Frobenius. Then $G^F = G_2(\mathbb{F}_q)$, whose Weyl group W is isomorphic to D_{12}. Following Carter, we label the six irreducible representations by $\phi_{1,0}, \phi'_{1,3}, \phi''_{1,3}, \phi_{1,6}, \phi_{2,1}, \phi_{2,2}$, where the first subindex gives the dimension and $\phi_{1,0} = 1$ and $\phi_{1,6} = \det$. The special characters are $\phi_{1,0}, \phi_{1,6}, \phi_{2,1}$ and the families are

$$(\phi_{1,0}), (\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}), (\phi_{1,6}).$$

On the other hand, G_2 has 10 unipotent characters, 6 of which are principal series and 4 are cuspidal. The principal series way can have the same labels as the irreducible characters of W, while the unipotent cuspidal are labelled by $G_2[-1], G_2[\theta], G_2[\theta^2], G_2[1]$. They fall into three families, parametrized as follows.

Description in terms of cuspidal characters	Degree	Pair (x, σ)
$\phi_{1,0}$	1	
$\phi_{2,1}$	$\frac{1}{6}q\Phi_2{}^2\Phi_3$	$(1, 1)$
$G_2[1]$	$\frac{1}{6}q\Phi_1{}^2\Phi_6$	$(1, \varepsilon)$
$\phi_{2,2}$	$\frac{1}{2}q\Phi_2{}^2\Phi_6$	$(g_2, 1)$
$\phi_{1,3}'$	$\frac{1}{3}q\Phi_3\Phi_6$	$(1, r)$
$\phi_{1,3}''$	$\frac{1}{3}q\Phi_3\Phi_6$	$(g_3, 1)$
$G_2[-1]$	$\frac{1}{2}q\Phi_1{}^2\Phi_3$	(g_2, ε)
$G_2[\theta]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ)
$G_2[\theta^2]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ^2)
$\phi_{1,6}$	q^6	

Finally, Lusztig defined a nonabelian Fourier transform on the set of irreducible characters of G^F. For each family $\mathcal{F}$ of unipotent characters parametrized by the group Γ, he considered the $|M(\Gamma)| \times |M(\Gamma)|$ matrix, whose $((x,\sigma),(y,\tau))$ entry is the value

$$\{(x,\sigma),(y,\tau)\} = \frac{1}{|C_\Gamma(x)||C_\Gamma(y)|} \sum_{\substack{g \in \Gamma \\ xgyg^{-1} = gyg^{-1}x}} \sigma(gyg^{-1})\overline{\tau(g^{-1}xg)}.$$

Then he proved that this matrix is Hermitian and that it squares to the identity. It therefore induces an involution on the space $\mathbb{C}[G^F]^{G^F}_{\mathcal{F}}$ of class functions spanned by the characters in $\mathcal{F}$, where we take the natural basis $\{\chi^{\mathcal{F}}_{(x,\sigma)} \mid (x,\sigma) \in M(\Gamma)\}$. Combining for each family, this gives an involution

$$R : \mathbb{C}_{un}[G^F]^{G^F} \longrightarrow \mathbb{C}_{un}[G^F]^{G^F}$$

on the space $\mathbb{C}_{un}[G^F]^{G^F}$ of class functions spanned by unipotent characters. This forces, for example, that $R(\chi_\phi) = R_\phi$ for all characters of W. The involution R transforms unipotent characters into *unipotent almost characters* that also satisfy the orthogonality relations and have a geometrical significance. By this we mean that every almost character agrees up to a scalar with a characteristic function associated to an F-stable character sheaf on G^F (see Shoji's article).

Example 1.11. If G^F is of type A_l, then the unipotent characters coincide with the almost characters.

Example 1.12. If G^F is of type G_2, then R fixes the characters $\phi_{1,0}$ and $\phi_{1,6}$ but transforms the third family according to the Fourier transform matrix

If $\Gamma \cong S_3$ the 8×8 Fourier transform matrix is:

$$
\begin{array}{c}
\\
(1,1) \\ (1,r) \\ (1,\varepsilon) \\ (g_2,1) \\ (g_2,\varepsilon) \\ (g_3,1) \\ (g_3,\theta) \\ (g_3,\theta^2)
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
(1,1) & (1,r) & (1,\varepsilon) & (g_2,1) & (g_2,\varepsilon) & (g_3,1) & (g_3,\theta) & (g_3,\theta^2)
\end{array} \\
\begin{bmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\end{array}
$$

The almost characters satisfy certain *stability properties*. Search what do they exactly mean by this. The aim of the next chapter is to discuss a lift of this map for p-adic groups.

2 Structure theory and representations of p-adic groups

2.1 The apartment of a split maximal torus

Let F be a nonarchimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k of cardinality q, a power of a prime p. Let $\mathbf{G}$ be a connected, almost simple, split algebraic group over F and let $G = \mathbf{G}(F)$.

Let T be a split maximal torus of G over F, and let $X^*(T)$ (resp. $X_*(T)$) be its character (resp. cocharacter) lattice and let

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

the natural perfect pairing between characters and cocharacters of T. Let $\Phi(G,T) \subset X^*(T)$ be the set of roots associated to T, with the corresponding set of coroots $\Phi^\vee(G,T) \subset X_*(T)$. We recall from the previous section that a choice of a Borel subgroup B of G containing T is equivalent to the choice of simple roots $\Delta(G,T) = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(G,T)$, which we fix throughout. In addition, the group B together with the normalizer $N := N_G(T)$ form a BN-pair with corresponding Weyl group $W = N(F)/T(F)$.

A natural object arising in the representation theory of G is the apartment $\mathcal{A}(G,T) := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, a real vector space naturally containing all coroots. Moreover, $\mathcal{A}(G,T)$ has the structure of a simplicial complex given by the hyperplanes

$$H_{\alpha,n} = \{x \in \mathcal{A}(G,T) \mid \langle \alpha, x \rangle = n\}, \quad \text{for each } \alpha \in \Phi(G,T)^+ \text{ and } n \in \mathbb{Z}.$$

Whenever the torus T is clear from context, we will omit it from the notation. The complexes on the apartment are called *facets*, and the facets of largest dimension are called *alcoves*. Our choice of simple roots Δ determines a canonical alcove

$$\mathcal{C}_0 = \{x \in \mathcal{A} \mid \langle \alpha, x \rangle > 0, \alpha \in \Delta \ \text{ and } \langle \alpha_0, x \rangle < 1\},$$

commonly referred to as the *fundamental alcove*.

Another important property of the apartment is that it carries a natural action of the group N satisfying

- For any $\alpha \in \Phi$ and $\lambda \in F$, the element $\check{\alpha}(\lambda) \in T \subset N$ acts on $\mathcal{A}$ by a translation $-\nu_p(\lambda)\check{\alpha}$.
- The centre of G acts faithfully and fixes every alcove. maybe important to explain this better?
- For any $\alpha \in \Phi$, the element $w_\alpha(1) \in N$ acts on $\mathcal{A}$ by a reflection along $H_{\alpha,0}$. This coincides with the natural action of W on $\mathcal{A}$.

This action preserves the simplicial structure of the apartment and is transitive on the set of alcoves of $\mathcal{A}$. Moreover, the kernel of this action is $T(\mathcal{O})$ and therefore the *extended Weyl group*

$$\widetilde{W} := N(F)/T(\mathcal{O}) \cong W \ltimes X_*(T)$$

acts faithfully on the apartment $\mathcal{A}$ and transitively on the set of alcoves. We denote by $w_{\alpha,n}$ the unique element in $\widetilde{W}$ acting on $\mathcal{A}$ by a reflection on the hyperplane $H_{\alpha,n}$.

In general, however, this action is not simple on the set of alcoves and the group $\Omega = \{w \in \widetilde{W} \mid w(\mathcal{O}) = \mathcal{O}\}$ is non-empty. These groups fit in a **splitting** short exact sequence

$$1 \longrightarrow W_{\mathrm{aff}} \longrightarrow \widetilde{W} \longrightarrow \Omega \longrightarrow 1,$$

where $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter group generated by the simple reflections $s_0 := w_{\alpha_0,1},\ s_i = w_{\alpha_i,0},\ i = 1,\dots,r$ along the walls of the fundamental alcove $\mathcal{C}_0$ and acting simply transitively on the set of alcoves of $\mathcal{A}$. The group W_{aff} is the *affine Weyl group* associated to the group G. The Weyl groups W, $\widetilde{W}$ and W_{aff} are independent of T, up to isomorphism.

Example 2.1. 1. Let $G = \mathrm{SL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) = t^2 \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \frac{\alpha}{2}\mathbb{Z}$ and $X_*(T) = \check{\alpha}\mathbb{Z}$. Moreover, we have that

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)T \quad \text{and} \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0\end{smallmatrix}\right),\ t \in F^\times.$$

The apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. It is easy to check that $\Omega = \{1\}$ so that $\widetilde{W} = W_{\mathrm{aff}}$ is generated by $s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ -\varpi & 0\end{smallmatrix}\right)$ and $s_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$.

2. Let $G = \mathrm{PGL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right) = t \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) = \left(\begin{smallmatrix} t^2 & 0 \\ 0 & 1\end{smallmatrix}\right) \quad \text{for any } t \in F^\times,$$

so $X^*(T) = \alpha\mathbb{Z}$ and $X_*(T) = \frac{\check{\alpha}}{2}\mathbb{Z}$. Similarly,

$$N = T \cup \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)T, \quad w_\alpha(t) = \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0\end{smallmatrix}\right),\ t \in F^\times$$

and the apartment $\mathcal{A}(\mathrm{SL}_2(F), T)$ is a one-dimensional real vector space whose hyperplanes $H_{\alpha,n}$ are the points $\frac{n}{2}\check{\alpha}$. This time, however, $\Omega = \{1, \left(\begin{smallmatrix} 0 & 1 \\ p & 0\end{smallmatrix}\right)\}$ is non-trivial, and

$$W_{\mathrm{aff}} = \left\langle s_0 = \left(\begin{smallmatrix} 0 & \varpi^{-1} \\ \varpi & 0\end{smallmatrix}\right), s_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)\right\rangle = \{w \in \widetilde{W} \mid \nu(\det(w)) \text{ is even}\}$$

is an index 2 normal subgroup of $\widetilde{W}$.

3. Let $G = \mathrm{GL}_2(F)$ and T the set of diagonal matrices. Then $\Phi(G,T) = \{\pm\alpha\}$ where

$$\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & s\end{smallmatrix}\right) = ts^{-1} \quad \text{and} \quad \check{\alpha}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) \quad \text{for any } t \in F^\times.$$

In this case, $\Omega \cong \mathbb{Z}$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0\end{smallmatrix}\right)$ and therefore $W_{\mathrm{aff}} = \langle s_0, s_1\rangle = \{w \in \widetilde{W} \mid \det(w) \in \mathcal{O}^\times\}$ is a normal subgroup of $\widetilde{W}$ of infinite index.

Some of the behaviour observed in the previous example holds in much greater generality. For example, Ω is an abelian group, and it has finite order if and only if G is a simple group. In that case, Ω is in bijection with the centre of complex dual group $G^\vee(\mathbb{C})$ of G. In particular, if G is simply connected, then Ω is trivial, while Ω has the largest size within the isogeny class when G is adjoint. On the other hand, W_{aff} only depends on the isogeny class, and therefore only on the root system of G.

2.2 The Bruhat-Tits building and parahoric subgroups

It is possible to push idea further and construct the Bruhat-Tits building $\mathcal{B}(G)$, a polysimplial space associated to G that contains $\mathcal{A}(G,T)$ for any F-split maximal torus. This is achieved by gluing together the apartments of all F-split maximal tori of G and then gluing them according to some equivalence relation. An important property of the building $\mathcal{B}(G)$ is that it carries a G-action satisfying the following:

1. It extends the action of $N_G(T)$ on $\mathcal{A}(G,T)$ for each F-split maximal torus T.
2. The stabilizer of $\mathcal{A}(G,T)$ is $N_G(T)$ for each F-split maximal torus T.
3. The stabilizer of any facet c of the building is a (maybe disconnected) open compact subgroup of G.
4. The action is strongly transitive on the set $\{(\mathcal{C},\mathcal{A}) \mid \mathcal{C}$ is an alcove inside the apartment $\mathcal{A}\}$.
5. For any pair $(\mathcal{C},\mathcal{A})$ as above, its stabilizer acts on $\mathcal{B}(G)$ as the group Ω. In other words

$$\mathrm{Stab}_G(\mathcal{C},\mathcal{A})/T(\mathcal{O}) = (N \cap \mathrm{Stab}_G(\mathcal{C}))/T(\mathcal{O}) = \mathrm{Stab}_N(\mathcal{C})/T(\mathcal{O}) \cong \Omega$$

Example 2.2. The Bruhat-Tits building of $G = \mathrm{SL}_2(\mathbb{Q}_p)$ or $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ is an infinite tree all of whose vertex have degree $p+1$. Each infinite line inside the building is an apartment corresponding to a distinct F-split maximal torus of G. Consider the apartment $\mathcal{A}(G,T)$, where T is the group of diagonal matrices, and let $\Delta = \{\alpha\}$ be the simple root as above. Then $\mathcal{C}_0$ is the segment between the vertices 0 and $\check{\alpha}/2$.

If $G = \mathrm{SL}_2(\mathbb{Q}_p)$, then

$$K_1 := \mathrm{Stab}(0) = \mathrm{SL}_2(\mathbb{Z}_p),\ K_2 := \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},\ \text{and } I := \mathrm{Stab}(\mathcal{C}_0) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

are all connected, open compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ not conjugate to each other. K_0 and K_1 are the unique maximal compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ – in particular, the stabilizer of any vertex of the building is conjugate to either K_0 or K_1. The subgroup I is called the **Iwahori subgroup**, it is conjugate to the stabilizer of any facet in the building and is of fundamental importance in the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

On the other hand, if $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, then

$$K_1 := \mathrm{Stab}(0) = \mathrm{PGL}_2(\mathbb{Z}_p),\ K_2 := \mathrm{Stab}(\check{\alpha}/2) = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}_{\det\in\mathbb{Z}_p^\times}$$

are both connected open compact subgroups and conjugate in $\mathrm{PGL}_2(F)$ by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, so $\mathrm{PGL}_2(F)$ has one unique *connected* maximal compact subgroup up to conjugacy. Correspondingly,

$$\mathrm{Stab}(\mathcal{C}_0) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det\in\mathbb{Z}_p^\times} \bigsqcup \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}_{\det\in\mathbb{Z}_p^\times}$$

is a disconnected open compact subgroup, whose identity component is the Iwahori subgroup.

The example above suggests that the connected components of the stabilizers of facets in the building depend in a subtle way on the group Ω. This is indeed the case, and we discuss this connection now. Since

$\Omega = \mathrm{Stab}_G(\mathcal{C}_0)$ and $\mathcal{C}_0$ is bounded by hyperplanes corresponding to S_{aff}, there is a natural homomorphism of groups

$$\Omega \longrightarrow \mathrm{Aut}(S_{\mathrm{aff}}).$$

Moreover, all permutations of S_{aff} induced by Ω can be easily seen to preserve the affine Dynkin diagram associated to S_{aff}, and if G is simple of adjoint type, then all such automorphisms of S_{aff} are induced by Ω. This greatly restricts the size of Ω.

Next, fix some *proper* subset $J \subset S_{\mathrm{aff}}$ and consider the *standard* facet

$$c_J = \{x \in \overline{\mathcal{C}_0} \mid \langle \alpha, x\rangle \in \mathbb{Z} \text{ for } \alpha \in J \text{ and } \langle \alpha, x\rangle \notin \mathbb{Z} \text{ for } \alpha \in S_{\mathrm{aff}} - J\}.$$

Two facets c_{J_1} and c_{J_2} are conjugate under the action of G if and only if J_1 and J_2 lie in the same Ω-orbit. Moreover, any facet c in the building is conjugate to c_J for some proper subset $J \subset S_{\mathrm{aff}}$. In other words, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\mathrm{aff}}\} \longleftrightarrow \{G - \text{orbits of facets } c \text{ in the BT-building}\}$$

For any facet c of the building of G, we let $P_c^+ := \mathrm{Stab}_G(c)$ be the stabilizer of c. There is a short exact sequence

$$1 \longrightarrow U_c \longrightarrow P_c^+ \longrightarrow M_c^+,$$

where U_c is the pro-unipotent radical of P_c^+ and M_c^+ is the group of k-rational points of a (possibly disconnected) reductive group $\mathbf{M}_c^+$ over k.

Definition 2.3. A **parahoric subgroup** P_c is the inverse image in P_c^+ of the group M_c of k-rational points of the identity component $\mathbf{M}_c$ of $\mathbf{M}_c^+$. We shall sometimes denote "parahoric subgroup" to the triple (U_c, P_c, M_c). If c is open in the building, then (U_c, P_c, M_c) is a minimal parahoric subgroup and is called an **Iwahori subgroup**. The standard Iwahori subgroup corresponds to $J = \emptyset \subsetneq S_{\mathrm{aff}}$.

Naturally, two parahoric subgroups are conjugate in G if and only if the corresponding facets of the building are in the same G-orbit. Thus, all Iwahori subgroups are conjugate in G. If $c = c_J$ is a standard facet, then we simply write (U_J, P_J, M_J) for its associated parahoric subgroup, and P_J is generated by the standard Iwahori subgroup and J. Thus, there is a bijection between

$$\{\Omega - \text{orbits of } J \subsetneq S_{\mathrm{aff}}\} \longleftrightarrow \{G - \text{conjugacy classes of parahoric subgroups } (U_c, P_c, M_c)\}$$

Moreover, if the facet c corresponds to $J \subsetneq S_{\mathrm{aff}}$, then

$$P_c^+/P_c \cong \Omega_J = \mathrm{Stab}_\Omega(J).$$

These results can be directly verified for SL_2, PGL_2 and GL_2 using the examples above.

Example 2.4. Suppose that $G = G_2(F)$. The affine Dynkin diagram of G_2 has no symmetries, so $\Omega = 1$ and the extended affine weyl group $\widetilde{W}$ is a Coxeter group of type $\tilde{G}_2$. Since $S_{\mathrm{aff}} = s_0, s_1, s_2$, there are 7 conjugacy classes of parahoric subgroups, satisfying

$$M_{\{s_1,s_2\}} = G_2(k), \quad M_{\{s_0,s_1\}} = \mathrm{SL}_3(k), \quad M_{\{s_0,s_2\}} = \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \tag{1}$$

$$\text{what about singletons} \quad M_\emptyset = T(k) = (k^\times)^2. \tag{2}$$

2.3 Parahoric restriction

Parahoric subgroups are ubiquitous objects in the representation theory of p-adic objects, since it provides a bridge between smooth admissible representations of the p-adic group G and finite dimensional representations of the finite groups of Lie type M_c defined in the previous section. In this section, we explore this important connection that we will exploit in a latter chapter.

If (U, P, M) is any parahoric subgroup corresponding to a facet c and (π, V) is a smooth admissible representation of G, the space V^U of fixed points under the pro-unipotent radical is naturally a representation of M. We can take this idea one step further and define the *parahoric restriction functor*

$$\mathrm{res}_c : R(G) \longrightarrow \mathbb{C}[M]^M, \quad V \longmapsto \text{(character of) } V^U, \quad \text{for all } V \in \mathrm{Irr}(G),$$

where $\mathbb{C}[M]^M$ is the space of class functions of M. This is well-defined since the representations are assumed to be admissible. The existence of such a functor is very powerful – for example, one can then apply the theory of Deligne-Lusztig induction to the setting of representation of p-adic groups.

Definition 2.5. We say that an irreducible representation (π, V) of G is *unipotent* if there is a parahoric subgroup (P, U, M) such that V^U contains a cuspidal unipotent representation of M. More generally, a smooth admissible representation of G is unipotent if all irreducible subquotients of π are unipotent.

Are all pairs (P, E) as above types of certain Bernstein blocks?

Example 2.6.

Definition 2.7. Let (U_c, P_c, M_c) be a parahoric subgroup and let (τ, E) be a cuspidal representation of M_c. We then define

$$\mathrm{Irr}(G, P_c, E) = \{(\pi, V) \in \mathrm{Irr}(G) \mid \text{the } M_c\text{-module } V^{U_c} \text{ contains the } M_c\text{-module } E\}$$

Definition 2.8. An irreducible smooth admissible representation (π, V) of G is *unipotent* if there exists some parahoric subgroup K of G such that V^{U_K} contains a cuspidal unipotent representation of the finite group $\overline{K}$.

We note that if we replace G for G^F and *parahoric* by *parabolic*, then we recover the definition of a unipotent representation in G^F.

The following two results are basic to ensure that unipotent representations behave under the restriction functor above.

Proposition 2.9. *Let (π, V) be an irreducible admissible representation of G. If there is some parahoric subgroup K such that V^{U_K} contains a non-cuspidal unipotent representation of $\overline{K} = K/U_K$, then (π, V) is a unipotent representation of G.*

Proof. To prove the first claim, suppose that V^{U_K} contains a unipotent irreducible representation (σ, W) of $\overline{K}$, and therefore $\mathrm{Hom}_K(W, V^{U_K}) \neq 0$. By Proposition **??**, there is some standard parabolic subgroup $\overline{P} = \overline{U_P} \cdot \overline{L_P}$ of $\overline{K}$ and cuspidal unipotent representation (τ, U) of $\overline{L_P}$ such that

$$\mathrm{Hom}_K(W, \mathrm{Ind}_{\overline{P}}^{\overline{K}} U) \neq 0,$$

where we view the $\overline{K}$ representations as inflated K representations, trivial on U_K. By the classification of parahoric subgroups in G, it follows that $\bar{P} = H/U_K$, where H is another parahoric subgroup contained in K. Moreover, we have the inclusions $U_K \subseteq U_H \subseteq H \subseteq K$ and therefore $\overline{U_P} = U_H/U_K$ and $\overline{L_P} = H/U_H$. Since induction and inflation are commuting operations, it follows that the induced $\overline{K}$-representation $\mathrm{Ind}_{\overline{P}}^{\overline{K}} U$ is isomorphic to the K-representation $\mathrm{Ind}_H^K U$, where we view U as the induced H-representation trivial on U_H, and hence $\mathrm{Hom}_K(W, \mathrm{Ind}_H^K U) \neq 0$. Since W is irreducible and K is compact, it also follows that

$$\mathrm{Hom}_K(\mathrm{Ind}_H^K U, V^{U_K}) = \mathrm{Hom}_H(U, V^{U_K}) \neq 0.$$

Since the representation U is trivial on U_H, it follows that the image of H-equivariant map $U \to V^{U_K}$ lies inside V^{U_H}. Thus,

$$\mathrm{Hom}_H(U, V^{U_H}) = \mathrm{Hom}_H(U, V^{U_K}) \neq 0,$$

and this concludes the proof. $\square$

Conversely, we would like to show that for any irreducible unipotent representation (π, V) of G, the irreducible $\overline{P}$-submodules of V^{P^+} are all unipotent. This is a direct corollary of the following theorem.

Theorem 2.10. *Let (π, V_π) be an irreducible admissible representation of G. Suppose that there is some parahoric P such that $V^{P^+} \neq 0$ contains an irreducible cuspidal representation (σ, V_σ) of $\overline{P}$. Then, for any parahoric Q such that $V^{Q^+} \neq 0$, any $\overline{Q}$-submodule of V^{Q^+} contains a cuspidal representation (τ, V_τ) of $\overline{L}$ for some parahoric subgroup $L \subseteq Q$. Moreover, there is some $g \in G$ such that $L = {}^gP$ and $\tau = {}^g\sigma$.*

Proof. From the proof of the previous result implies that any $\overline{Q}$-submodule of V^{Q^+} contains a cuspidal representation τ of $\overline{L}$ for some parahoric subgroup $L \subseteq Q$. Thus, we may assume without loss of generality that $Q = L$ so that the irreducible $\overline{Q}$-submodule (τ, V_τ) of V^{Q^+} is cuspidal. To prove the second statement, we let $E_\tau : V_\pi \to V_\tau$ be a Q-equivariant projection. Then, for any $g \in G$, we have the linear map

$$\varphi_g = E_\tau \circ \pi(g^{-1}) : V_\sigma \longrightarrow V_\tau,$$

and for any $h \in P \cap gQg^{-1}$ and $v \in V_\sigma$, we have that

$$\varphi_g \circ \sigma(h)(v) = E_\tau(\pi(g^{-1}hg)\pi(g^{-1})v) = \tau(g^{-1}hg) \circ E_\tau(\pi(g^{-1}))(v) = \tau(g^{-1}hg) \circ \varphi_g(v).$$

Since π is an irreducible representation of G, there must be some $g \in G$ such that $\pi(g^{-1})V_\sigma \not\subseteq \ker(E_\tau)$, in which case $\varphi_g \neq 0$. The image of $P \cap gQg^{-1}$ inside $\overline{P}$ (resp $\overline{gQg^{-1}}$) is a parabolic subgroup $\overline{P}_{{}^gQ}$ (resp. $\overline{{}^gQ}_P$) $\square$

Need to find proofs for this!!

Example 2.11. The Iwahori subgroup $I = \langle T(\mathcal{O}), \mathcal{X}_\alpha(\mathcal{O}), \mathcal{X}_\alpha(\varpi\mathcal{O}) \mid \alpha \in \Phi^+ \rangle$, with pro-unipotent radical $U_I = \langle T(1 + \varpi\mathcal{O}), \mathcal{X}_\alpha(\mathcal{O}), \mathcal{X}_\alpha(\varpi\mathcal{O}) \mid \alpha \in \Phi^+ \rangle$ and reductive quotient $I/U_I = T(k_F)$ is, up to conjugacy, the unique minimal parahoric subgroup of G. Any smooth admissible representation (π, V) of G with an Iwahori fixed vector is unipotent. Indeed, V^{U_I} contains the trivial representation of $T(k_F)$, its unique unipotent representation.

Analogously to the construction of $R(G)$, we define $\mathrm{Irr}_{\mathrm{un}}(G)$ to be the set of irreducible unipotent representations of G, and $R_{\mathrm{un}}(G)$ to be its $\mathbb{C}$-span. Lemma 2.9 implies that there is a well-defined *restriction function*

$$\mathrm{res}^K_{\mathrm{un}} : R_{\mathrm{un}}(G) \longrightarrow \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}}, \quad V \longmapsto (\text{character of) } V^{U_K}, \quad \text{for all } V \in \mathrm{Irr}(G).$$

It is also convenient to consider simultaneously all such functions for all conjugacy classes of maximal parahoric subgroups, so we define $\mathrm{res}^{\mathrm{par}}_{\mathrm{un}} = (\mathrm{res}^K_{\mathrm{un}})_K$. In the previous section, we defined a nonabelian Fourier transform

$$\mathrm{FT}^K : \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \longrightarrow \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}},$$

taking the unipotent characters to the almost characters of $\overline{K}$, having nice geometric properties. We also consider all these maps simultaneously for all conjugacy classes of maximal parahoric subgroups, which we denote as $\mathrm{FT}^{\mathrm{par}} = (\mathrm{FT}^K)_{K \text{ maximal}}$.

It is therefore a natural question to ask whether there exists some function $\mathrm{FT}^\vee : R_{\mathrm{un}}(G) \to R_{\mathrm{un}}(G)$ such that the square

$$\begin{tikzcd} R_{\mathrm{un}}(G) \arrow[r, "\mathrm{FT}^\vee"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & R_{\mathrm{un}}(G) \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] \\ \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_K \mathbb{C}_{\mathrm{un}}[\overline{K}]^{\overline{K}} \end{tikzcd}$$

This question is now unresolved, but partial progress has been achieved. To understand it, we first need to look at the Langlands parametrization of unipotent representations.

2.4 The Langlands parametrization of unipotent representations

For this section, we introduce the Weil group W_F of the field F with inertia subgroup I_F. Moreover, we set $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$. Under the assumption that $\mathbf{G}$ is a split group, we have the following important definition.

Definition 2.12. A *Langlands parameter* (or L-parameter) for G is a continuous morphism $\varphi : W'_F \to G^\vee$, where $G^\vee$ denotes the $\mathbb{C}$-points of the dual group of $\mathbf{G}$, and $\varphi((w,1))$ is semisimple for each $w \in W_F$.

In its simplest form, the Local Langlands correspondence (LLC) conjectures the existence of a finite to one map between isomorphism classes of smooth admissible complex representations of G and conjugacy classes of Langlands parameters of G satisfying certain nice properties. In particular, the LLC conjectures that the unipotent representations of G correspond to the following Langlands parameters.

Definition 2.13. An L-parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ is called *unipotent* if $\varphi(w,1) = 1$ for any element w of the inertia subgroup I_F of W_F. Such parameters are sometimes called *unramified* Langlands parameters.

Remark 2.14. For any L-parameter $\varphi : W'_F \to G^\vee$, define the elements $u_\varphi = \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ and $s_\varphi = \varphi(\mathrm{Frob}, \mathrm{Id})$ that commute. An application of the Jacobson–Morozov theorem implies that an L-parameter is determined by u_φ and $\varphi|_{W_F}$ up to $G^\vee$-conjugacy. If the L-parameter is, in addition, unipotent, then $\varphi|_{W_F}$ is determined by s_φ. Thus, unipotent L-parameters are parametrized by $G^\vee$ conjugacy classes of pairs (u,s) where $u \in G^\vee$ is unipotent, $s \in G^\vee$ is semisimple and they commute. But this is the same as conjugacy classes of elements of $G^\vee$ (by using the Jordan decomposition).

However, this is a finite-to-one correspondence, so it does not parametrize unipotent representations, but rather L-packets of unipotent representations. To get a one to one correspondence, we need to introduce refinements of the L-parameters. Given an L-parameter φ, a natural object of interest is the centralizer $Z_{G^\vee}(\varphi)$ of the image of φ inside $G^\vee$. Now, $G^\vee$ sits within $G^\vee_{\mathrm{sc}} = (G_{\mathrm{ad}})^\vee \twoheadrightarrow G^\vee \twoheadrightarrow G^\vee_{\mathrm{ad}}$. We then denote $Z^1_{G^\vee}(\varphi)$ to be the inverse image of $Z_{G^\vee}(\varphi)$ under the isogeny $G^\vee_{\mathrm{sc}} \twoheadrightarrow G^\vee$. We then denote by A_φ, A^1_φ to be the component group of $Z_{G^\vee}(\varphi)$ and $Z^1_{G^\vee}(\varphi)$, respectively.

Definition 2.15. An enhanced Langlands parameter is a pair (φ, ϕ), where $\varphi : W'_F \to G^\vee$ is an L-parameter and ϕ is an irreducible representation of A^1_φ.

Let us introduce some notation. We denote by $\Phi_{\mathrm{un}}(G^\vee)$ the set of unipotent L-parameters and $\Phi_{e,\mathrm{un}}(G^\vee)$ the set of all *enhanced* unipotent L-parameters. Similarly to L-parameters, an enhanced L-parameter (φ, ϕ) is determined by the triple $(u_\varphi, \varphi|_{W_F}, \phi)$, and if it is also unipotent, then it is determined by the triple $(u_\varphi, s_\varphi, \phi)$. Moreover, if $x_\varphi = s_\varphi u_\varphi$, then $Z_{G^\vee}(\varphi) = Z_{G^\vee}(x_\varphi)$, and therefore there is a canonical bijection

$$\begin{aligned} \Phi_{\mathrm{e,un}}(G^\vee) &\longleftrightarrow G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A^1_x}\}, \\ (\varphi, \phi) &\longmapsto (s_\varphi u_\varphi, \phi) \end{aligned}$$

where A^1_x is the component group of $Z^1_{G^\vee}(x)$.

Then the LLC predicts that there is a natural bijection

$$\begin{array}{ccccc} \mathrm{LLC}_{\mathrm{un}} : G^\vee \backslash \{(x, \phi) \mid x \in G^\vee, \phi \in \widehat{A^1_x}\} & \longleftrightarrow \Phi_{\mathrm{e,un}}(G^\vee) \longleftrightarrow & \displaystyle\bigsqcup_{G' \in \mathrm{InnT}(G)} \mathrm{Irr}_{\mathrm{un}}(G') \\ (x, \phi) & \longmapsto & \pi(x, \phi), \end{array}$$

where G' runs over the classes of inner twists of G. We say that a pair (x, ϕ) is G'-relevant if $\pi(x, \phi) \in \mathrm{Irr}_{\mathrm{un}}(G')$, and this property can be checked explicitly. Firstly, we note that there is a canonical bijection

$$\mathrm{InnT}(G) \longleftrightarrow H^1(F, \mathrm{Inn}(\mathbf{G}^*)) \longleftrightarrow \mathrm{Irr}(Z_{G^\vee_{sc}}), \quad G' \longmapsto \zeta_{G'}.$$

Similarly, for any pair (x, ϕ), the representation ϕ induces a character ζ_ϕ on $Z_{G^\vee_{sc}}$. Then $\pi(x, \phi)$ is G' relevant if and only if $\zeta_\phi = \zeta_{G'}$ and we denote the set of G'-relevant parameters by $\Phi_{e,\mathrm{un}}(G')$.

It is then clear that

$$\Phi_{e,\mathrm{un}}(G^\vee) = \bigsqcup_{G' \in \mathrm{InnT}(G)} \Phi_{e,\mathrm{un}}(G'),$$

and the LLC predicts that $\Phi_{e,\mathrm{un}}(G')$ parametrizes the set $\mathrm{Irr}_{\mathrm{un}}(G')$ for each $G' \in \mathrm{InnT}(G)$.

2.4.1 Pure Langlands parameters

One can also refine L-parameters by considering instead pairs (φ, ϕ), where φ is an L-parameters and ϕ is an irreducible representations of A_φ. We then set

$$\Phi^p_{\mathrm{e,un}}(G^\vee) = G^\vee \backslash \{(\varphi, \phi) \mid \varphi \text{ unipotent}, \ \phi \in \widehat{A_\varphi}\},$$

which is in natural bijection with the set

$$G^\vee\backslash\{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\},$$

where A_x is the component group of $Z_{G^\vee}(x)$.

In this setting the Local Langlands conjecture predicts a natural bijection

$$G^\vee\backslash\{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \bigsqcup_{G'\in \mathrm{InnT}^p(G)} \mathrm{Irr}_{\mathrm{un}}(G'), \quad (x,\phi) \longmapsto \pi(x,\phi)$$

where G' now runs over the classes of *pure* inner twists of G. We distinguish between the distinct pure inner twists by looking at characters of $Z_{G^\vee}$.

Example 2.16. If $\mathbf{G}$ is a simple split *simply connected* algebraic group, then $H^1(F,\mathbf{G}^*) = 1$ and therefore there is only one class of pure inner forms of G, namely G itself. Correspondingly, $G^\vee = G^\vee_{\mathrm{ad}}$ and $Z_{G^\vee}$ is trivial. Therefore, the above discussion gives a bijection

$$\mathrm{LLC}^p_{\mathrm{un}} : G^\vee\backslash\{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \Phi^p_{\mathrm{e,un}}(G^\vee) \longleftrightarrow \mathrm{Irr}_{\mathrm{un}}(G^*).$$

Example 2.17. If $\mathbf{G}$ is a simple split *adjoint* algebraic group, then $H^1(F,\mathbf{G}^*) = H^1(F,\mathrm{Inn}(\mathbf{G}^*))$ so for each inner twist there is one unique pure inner twist. Therefore, from the previous discussion, unipotent enhanced L-parameters are in bijection with the set

$$G^\vee\backslash\{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\}, \quad \text{where} \quad A_x = Z_{G^\vee}(x)/Z_{G^\vee}(x)^0,$$

and we have a one-to-one correspondence

$$G^\vee\backslash\{(x,\phi) \mid x \in G^\vee, \phi \in \widehat{A_x}\} \longleftrightarrow \bigsqcup_{G'\in \mathrm{InnT}(G)} \mathrm{Irr}_{\mathrm{un}}(G'), \quad (x,\phi) \longmapsto \pi(x,\phi)$$

3 The dual nonabelian Fourier transform for unipotent representation of p-adic groups

3.1 The unipotent elliptic space and the dual nonabelian Fourier transform

With the Langlands parametrization, it is then possible to define a dual fourier transform on a certain subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$, which we now describe. We first fix some unipotent element $u \in G^\vee$ up to $G^\vee$-conjugacy and we denote Γ_u the reductive part of $Z_{G^\vee}(u)$. We then consider the space of elliptic pairs

$$\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(s,h) \mid s,h \in \Gamma_u \text{ semisimple, } sh = hs \text{ and } Z_{G^\vee}(s,h) \text{ is finite}\}$$

up to Γ_u-conjugacy. Then for each $(s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, we define the virtual representation

$$\pi(u,s,h) := \sum_{\phi \in \widehat{A_{su}}} \phi(h) \pi(su, \phi).$$

Definition 3.1. The elliptic unipotent representation space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ of G is defined as the $\mathbb{C}$-subspace of $\bigoplus_{G' \in \mathrm{InnT}^p(G)} R_{\mathrm{un}}(G')$ spanned by the set $\{\pi(u,s,h) \mid u \in G^\vee,\ (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}\}$.

On the space $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ we can define the *dual* Fourier transform in a natural way.

Definition 3.2. The dual elliptic nonabelian Fourier transform is the linear map satisfying

$$\mathrm{FT}^\vee_{\mathrm{ell}} : \mathcal{R}^p_{\mathrm{un,ell}}(G) \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G) \quad \pi(u,s,h) \longmapsto \pi(u,h,s) \quad \text{for all} \quad (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}},\ u \in G^\vee \text{ unipotent.}$$

With this linear map, we then have the diagram

$$\begin{tikzcd}
\mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[r, "\mathrm{FT}^\vee_{\mathrm{ell}}"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] \\
\bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}} \arrow[r, "\mathrm{FT}^{\mathrm{par}}"] & \bigoplus_{G'} \bigoplus_{K'} \mathbb{C}_{\mathrm{un}}[\overline{K'}]^{\overline{K'}},
\end{tikzcd}$$

and a natural question to ask is whether this square commutes. We first show this is indeed the case when $\mathbf{G}$ is a simple algebraic group of type G_2.

3.2 Type G_2

Example 3.3. Let $\mathbf{G}$ be a simple algebraic group of type G_2, and let $G = \mathbf{G}(F)$. Then G is both simply connected and adjoint so it has no pure inner twists other than itself. In addition, G has three maximal parahoric subgroups of types K_0, K_1 and K_2, with reductive quotients of type G_2, A_2 and $A_1 + \tilde{A}_1$, respectively. Thus, commutativity of the above square is equivalent to the commutativity of

$$\begin{tikzcd}
\mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[r, "\mathrm{FT}^\vee_{\mathrm{ell}}"] \arrow[d, "\mathrm{res}^{\mathrm{par}}_{\mathrm{un}}"] & \mathcal{R}^p_{\mathrm{un,ell}}(G) \arrow[d, "\mathrm{res}^{K_i}_{\mathrm{un}}"] \\
\mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}} \arrow[r, "\mathrm{FT}^{K_i}"] & \mathbb{C}_{\mathrm{un}}[\overline{K_i}]^{\overline{K_i}},
\end{tikzcd}$$

for $i = 0, 1, 2$. Moreover, if $u \in G^\vee$ is unipotent, then $\mathcal{Y}(\Gamma_u)$ is non-empty if and only if $u = u_{\mathrm{reg}}$ is regular and $\Gamma_u = \{(1,1)\}$, or $u = u_{sr}$ is subregular and $\Gamma_u = \{(1,1),(1,g_2),(1,g_3),(g_2,1),(g_2,g_2),(g_3,1),(g_3,g_3),(g_3,g_3')\}$. Therefore, $\mathcal{R}^p_{\mathrm{un,ell}}(G)$ is 9-dimensional, spanned by

$$\begin{cases}
\pi(u_{\mathrm{reg}},1,1) & = \pi(u_{\mathrm{reg}},\mathbf{1}) \\
\pi(u_{sr},1,1) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) + 2\pi(u_{sr},\mathbf{r}) \\
\pi(u_{sr},1,g_2) & = \pi(u_{sr},\mathbf{1}) - \pi(u_{sr},\varepsilon) \\
\pi(u_{sr},1,g_3) & = \pi(u_{sr},\mathbf{1}) + \pi(u_{sr},\varepsilon) - \pi(u_{sr},\mathbf{r}) \\
\pi(u_{sr},g_2,1) & = \pi(u_{sr}g_2,\mathbf{1}) + \pi(u_{sr}g_2,\varepsilon) \\
\pi(u_{sr},g_2,g_2) & = \pi(u_{sr}g_2,\mathbf{1}) - \pi(u_{sr}g_2,\varepsilon) \\
\pi(u_{sr},g_3,1) & = \pi(u_{sr}g_3,\mathbf{1}) + \pi(u_{sr}g_3,\theta) + \pi(u_{sr},\theta^2) \\
\pi(u_{sr},g_3,g_3) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta^2\pi(u_{sr}g_3,\theta) + \theta\pi(u_{sr}g_3,\theta^2) \\
\pi(u_{sr},g_3,g_3^{-1}) & = \pi(u_{sr}g_3,\mathbf{1}) + \theta\pi(u_{sr}g_3,\theta) + \theta^2\pi(u_{sr}g_3,\theta^2).
\end{cases}$$

When $i = 1, 2$ and the finite group $\overline{K_i}$ is of type A_2 or $A_1 + \tilde{A}_1$, then FT^{K_i} is the identity map, and therefore it is enough to show that

$$\mathrm{res}^{K_i}_{\mathrm{un}}(\pi(u,s,h)) = \mathrm{res}^{K_i}_{\mathrm{un}}(\pi(u,h,s))$$

for all $\pi(u,s,h)$ spanning $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. This is obvious for all cases except for $\pi(u,s,h) = \pi(u_{sr},1,g_2), \pi(u_{sr},1,g_3)$