Quadratic Relations in Hecke Algebras

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1 Depth-zero Hecke algebras for principal series blocks

Let $\mathcal{R}_{[M,\sigma]}(G)$ be a Bernstein block. If M is the maximal torus T (or a conjugate of T), then $\sigma = \tilde{\chi}$ is just a character and the $\mathcal{R}_{[T,\tilde{\chi}]}(G)$ is called a principal series block. These are the ones we aim to understand.

We remark that the block is completely determined the values of $\tilde{\chi}$ on $T(\mathcal{O})$, which has a natural filtration of open compact subgroups

$$T(\mathcal{O}) \supset T(1+\varpi\mathcal{O}) \supset T(1+\varpi^2\mathcal{O}) \supset T(1+\varpi^3\mathcal{O}) \supset \cdots$$

forming a basis for the topology at the identity. This motivates the following definition:

Definition 1.1. The *depth* of χ is the smallest integer r such that χ is trivial on $T(1+\varpi^{r+1}\mathcal{O})$.

In particular, if χ has depth-zero, then χ is trivial on $T(1+\varpi\mathcal{O})$ and therefore it factors through the quotient $T(\mathcal{O}) \to T(k_F)$. We shall abuse notation and denote the resulting character $T(k_F) \to \mathbb{C}^\times$ also as χ. Fortunately, for such blocks, the construction of a type is completely explicit.

Recall that the Iwahori subgroup I of G is defined as

$$I = \langle T(\mathcal{O}); \mathfrak{X}_\alpha(\mathcal{O}), \mathfrak{X}_{-\alpha}(\varpi\mathcal{O}) \mid \alpha \in \Phi^+\rangle$$

with its pro-p unipotent radical

$$I^+ = \langle T(1+\varpi\mathcal{O}); \mathfrak{X}_\alpha(\mathcal{O}), \mathfrak{X}_{-\alpha}(\varpi\mathcal{O}) \mid \alpha \in \Phi^+\rangle.$$

In particular, there is an isomorphism

$$T(\mathcal{O})/T(\mathcal{O}) \cap I^+ \cong I/I^+$$

and therefore χ determines a character ρ_χ of I that is trivial on I^+. Explicitly, I has an Iwahori decomposition

$$I = (I \cap \bar{U}) \cdot T(\mathcal{O}) \cdot (I \cap U),$$

and ρ_χ extends χ in $T(\mathcal{O})$ and is trivial on $I \cap \bar{U}$ and $I \cap U$.

Theorem 1.2. *Suppose that χ is a depth-zero character of $T(\mathcal{O})$. Then (I, ρ_χ) is a $[T, \tilde{\chi}]$-type, where $\tilde{\chi}$ is any extension of χ to $T(F)$.*

This theorem motivates us to study the Hecke algebra $\mathcal{H}(G, I, \rho_\chi)$, where ρ_χ is a character of I arising from a depth-zero character of $T(\mathcal{O})$ as described above.

When $C = \mathbb{C}$ has characteristic 0, one can explicitly describe the structure of $\mathcal{H}(G, I, \rho_\chi)$ in terms of a few objects associated to χ. To state the theorem, we first need some preliminaries. Firstly, we let $N := N_G(T)$ be the normalizer of the torus, and we let $W = N(F)/T(F)$ and $\widetilde{W} = N(F)/T(\mathcal{O})$ be the Weyl group and the extended Weyl group of G respectively. The Weyl group is a coxeter group generated by a set of simple reflections $S = \{s_\alpha \mid \alpha \in \Pi\}$, where $\Pi \subset \Phi$ is any fixed integral basis. There is also a canonical isomorphism of abelian groups $X_*(T) = \mathrm{Hom}_F(\mathbb{G}_m, T) \longrightarrow T(F)/T(\mathcal{O})$ given by $\lambda \mapsto \lambda(\varpi^{-1})$.

There is a canonical surjective homomorphism $\widetilde{W} \to W$ with kernel $X_*(T)$ and a non-canonical section. Therefore there is a non-canonical isomorphism $\widetilde{W} = X_*(T) \rtimes W$. The extended Weyl group $\widetilde{W}$ comes with an action on the apartment $\mathcal{A} = \mathcal{A}(G, T, F) = X_*(T) \otimes \mathbb{R}$ that respects the hyperplane structure and therefore permutes the alcoves. The apartment is effectively a vector space over $\mathbb{R}$ where the origin is the only point fixed by every $s_\alpha,\ \alpha \in \Phi$. The group $X_*(T)$ acts naturally by translations and W is the group of transformations of $\widetilde{W}$ fixing the origin. Inside the apartment, we distinguish the Weyl chamber and fundamental alcove of Φ which are, respectively,

$$\mathcal{C} = \{v \in \mathcal{A} \mid 0 < \langle \alpha, v\rangle, \forall \alpha \in \Phi^+\} \quad \text{and}$$
$$\mathcal{D}_0 = \{v \in \mathcal{A} \mid 0 < \langle \alpha, v\rangle < 1, \forall \alpha \in \Phi^+\}.$$

The extended Weyl group $\widetilde{W}$ contains a subgroup

$$W_{\text{aff}} = \mathbb{Z}\Phi^\vee \rtimes W = \langle s_a \mid a = \alpha + k, \alpha \in \Phi, k \in \mathbb{Z} \text{ is an affine root}\rangle,$$

denoted as the affine Weyl group, which acts simply transitively on the set of alcoves. In particular, W_{aff} is also a Coxeter group generated by the simple reflections $S_{\text{aff}} = S \cup \{s_0\}$ where s_0 is the reflection along the hyperplane $P_{\alpha_0,1} = \{v \in \mathcal{A} \mid \langle \alpha_0, v\rangle = 1\}$. These correspond to reflections along the walls of the fundamental alcove $\mathcal{D}_0$. It then follows that $\widetilde{W} = W_{\text{aff}} \rtimes \Omega$, where $\Omega = \mathrm{Stab}_{\widetilde{W}}(\mathcal{D}_0)$. As coxeter groups, both W and W_{aff} come equipped with a length function. Since W is a parabolic subgroup of W_{aff}, it follows that the length function of both groups coincide on elements of W. Moreover, one can extend the length function uniquely to $l : \widetilde{W} \to \mathbb{Z}^{\geq 0}$ in such a way that $\Omega = \{w \in \widetilde{W} : l(w) = 0\}$. In fact, one can prove that $l(w)$ equals the number of affine hyperplanes separating $\mathcal{D}_0$ to $w \cdot \mathcal{D}_0$.

We now turn out attention to the character χ and define analogous associated groups

$$N_\chi = \{n \in N(F) \mid {}^n\chi = \chi\}$$
$$\widetilde{W}_\chi = \{w \in \widetilde{W} \mid {}^w\chi = \chi\}$$
$$W_\chi = \{w \in W \mid {}^w\chi = \chi\}$$

with surjective group homomorphisms $N_\chi \to \widetilde{W}_\chi \to W_\chi$. We remark that nor $\widetilde{W}_\chi$ nor W_χ are coxeter groups in general, and therefore are not Weyl groups of subroot systems of Φ. However, we may define

$$\Phi_\chi = \{\alpha \in \Phi : \chi \circ \alpha^\vee|_{\mathcal{O}^\times} = 1\},$$

which is clearly a subroot system of Φ. Naturally, we let $\Phi_\chi^\vee = \{\alpha^\vee \mid \alpha \in \Phi_\chi\}$ and $\Phi_{\chi,\mathrm{aff}} = \{\alpha + k \mid \alpha \in \Phi_\chi, k \in \mathbb{Z}\}$ be the associated coroot system and affine root system to Φ_χ.

We also consider the Weyl group W_χ°, affine Weyl group $W_{\chi,\mathrm{aff}}$, Weyl chamber $\mathcal{C}_\chi$ and fundamental alcove $\mathcal{D}_{\chi,0}$ associated to Φ_χ. Explicitly,

$$\begin{aligned} W_\chi^\circ &= \langle s_\alpha \mid \alpha \in \Phi_\chi \rangle, \\ W_{\chi,\mathrm{aff}} &= \langle s_a \mid a \in \Phi_{\chi,\mathrm{aff}} \rangle, \\ \mathcal{C}_\chi &= \{v \in \mathcal{A}(G,T,F) \mid 0 < \alpha(v), \forall \alpha \in \Phi_\chi^+\}, \\ \mathcal{D}_{\chi,0} &= \{v \in \mathcal{A}(G,T,F) \mid 0 < \alpha(v) < 1, \forall \alpha \in \Phi_\chi^+\}. \end{aligned}$$

Analogously to the previous case, W_χ° is a coxeter group generated by a set of simple reflections $S_\chi = \{s_\alpha : \alpha \in \Pi_\chi\}$ along the walls of the Weyl chamber $\mathcal{C}_\chi$, where $\Pi_\chi \subset \Phi_\chi$ is an integral basis. Similarly, $W_{\chi,\mathrm{aff}}$ is a coxeter group generated by a set of simple reflections $S_{\chi,\mathrm{aff}} = \{s_a : a \in \Pi_{\chi,\mathrm{aff}}\}$ along the walls of the fundamental alcove $\mathcal{D}_{\chi,0}$, where $\Pi_{\chi,\mathrm{aff}} \subset \Phi_{\chi,\mathrm{aff}}$ is a set of minimal affine roots. Moreover, W_χ° acts simply transitively on the set of Weyl chambers of Φ_χ and $W_{\chi,\mathrm{aff}}$ acts simply transitively on the set of alcoves of Φ_χ. Finally, $\widetilde{W}_\chi$ also decomposes as a semi-direct product $W_{\chi,\mathrm{aff}} \rtimes \Omega_\chi$, where

$$\Omega_\chi = \{w \in \widetilde{W}_\chi \mid w\mathcal{D}_{\chi,0} = \mathcal{D}_{\chi,0}\}.$$

Finally, the length function of $W_{\chi,\mathrm{aff}}$ as a coxeter group agrees with that of W_χ and can be extended uniquely to a function $l_\chi : \widetilde{W}_\chi \to \mathbb{Z}^{\geq 0}$ such that $\Omega_\chi = \{w \in \widetilde{W}_\chi \mid l_\chi(w) = 0\}$. In fact, $l_\chi(w)$ is the number of affine hyperplanes of Φ_χ separating $\mathcal{D}_{\chi,0}$ to $w \cdot \mathcal{D}_{\chi,0}$.

Lemma 1.3. *We have that $W_\chi^\circ \subseteq W_\chi$ and $W_{\chi,\mathrm{aff}} \subseteq \widetilde{W}_\chi$. If $G = \mathrm{GL}_n$, then $W_\chi^\circ = W_\chi$*

Proof. For the first part, it is enough to prove that if $\alpha \in \Phi_\chi$, then ${}^{s_\alpha}\chi = \chi$. For this, we note that $T(\mathcal{O})$ is an abelian group generated by the elements $\{\check{\beta}(\lambda) | \beta \in \Phi, \lambda \in \mathcal{O}^\times\}$. Since

$$s_\alpha^{-1} \check{\beta}(\lambda) s_\alpha \check{\beta}(\lambda)^{-1} = s_\alpha \check{(\beta)}(\lambda) \check{\beta}(\lambda)^{-1} = s_{\check{\alpha}}(\check{\beta})(\lambda)\check{\beta}(\lambda)^{-1} = \check{\beta}(\lambda)\check{\alpha}(\lambda^{-\langle \alpha, \check{\beta}\rangle})\check{\beta}(\lambda) = \check{\alpha}(\lambda^{-\langle \alpha, \check{\beta}\rangle}),$$

it follows that ${}^{s_\alpha}\chi(\check{\beta}(\lambda)) = \chi(\check{\beta}(\lambda))$ for all $\beta \in \Phi$ and $\lambda \in \mathcal{O}^\times$ and therefore ${}^{s_\alpha}\chi = \chi$ as required.

If $G = \mathrm{GL}_n$, then $\chi = \chi_1 \otimes \cdots \otimes \chi_n$ where each χ_i is the inflation of a character of $k_F^\times$ to $\mathcal{O}^\times$ trivial on $1 + \varpi\mathcal{O}$. Moreover, W acts by conjugation on $T(F)$ and permutes the n diagonal entries. One can easily see that $W = \mathrm{Sym}\{1, \ldots, n\}$ and that the reflections $\{s_\alpha, \alpha \in \Phi^+\}$ correspond to transpositions τ_α in $\mathrm{Sym}\{1, \ldots, n\}$ in a canonical way. Moreover, if $\tau_\alpha, \alpha \in \Phi^+$ permutes the i-th and j-th diagonal entries with $i < j$, then $\check{\alpha}(\lambda)$ is a diagonal matrix with

$$\check{\alpha}(\lambda)_{k,k} = \begin{cases} \lambda & \text{if } k = i, \\ \lambda^{-1} & \text{if } k = j, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, $\tau_\alpha \in W_\chi$ if and only if $\chi_i = \chi_j$, which is equivalent to $\chi(\check{\alpha}(\lambda)) = 1$ for all $\lambda \in \mathcal{O}^\times$. If $\sigma \in W = \mathrm{Sym}\{1, \ldots, n\}$ is an arbitrary element, then one can easily see that $\sigma \in W_\chi$ if and only if $\chi_i = \chi_{\sigma(i)}$

for each $i = 1, \dots, n$. Thus, if we express σ as a product of disjoint cycles, then $\sigma \in W_\chi$ if and only if all the cycles also lie in W_χ, so we may assume that σ is a cycle itself. Then we can express σ as a product of transposition, all of which permute elements not fixed by σ. If $\sigma \in W_\chi$, then $\chi_i = \chi_{\sigma(i)}$ and each transposition lies in W_χ too and thus in W_χ° by the above argument. This immediately implies that $\sigma \in W_\chi^\circ$ as desired. $\square$

We remark that the latter part of the Lemma breaks down even for $G = \mathrm{SL}_n$. It is still true that $W = \mathrm{Sym}\{1, \dots, n\}$ and that a transposition lies in W_χ if and only if it lies in $W_\chi^\times$, but it can now happen for a cycle to lie in W_χ which is not a product of transpositions in W_χ°. Explicitly, if $G = \mathrm{SL}_n$ with $n \geq 3$ and $n \mid q-1$, then one can choose a character χ_1 of $k_F^\times$ inflated to $\mathcal{O}^\times$ of order n. Then the character

$$\chi(a_1, \dots, a_n) = \chi_1(a_1)\chi_1^2(a_2)\cdots\chi_1^n(a_n)$$

satisfies that $W_\chi^\circ = \{1\}$ but the cycle $(12\cdots n) \in W_\chi$ since $a_1 \cdots a_n = 1$.

If $G = \mathrm{Sp}_4$ for example, the situation is even worse. Let $\chi = \chi_1 \otimes \chi_2$ be a character of the torus, where

$$\chi\begin{pmatrix} s & & & \\ & t & & \\ & & s^{-1} & \\ & & & t^{-1}\end{pmatrix} = \chi_1(s)\chi_2(t). \tag{1}$$

If $\chi_1 = \chi_2 = \mathrm{sgn}$, then $W_\chi = W$ but Φ_χ only contains the short roots, and therefore $W_\chi^\circ \cong C_2 \times C_2$ for any q.

Lemma 1.4. *If $W_\chi^\circ = W_\chi$, then $\widetilde{W}_\chi$ is the extended Weyl group associated to Φ_χ.*

Before we state the main theorem, we first need to define the Hecke algebra $\mathcal{H}(W, S, q_S)$ associated to a coxeter pair (W, S) and a parameter set $q_S : S \to \mathbb{Q}^{\geq 0}$. This is a $\mathbb{C}$-algebra with basis $\{T_w : w \in W\}$ and relations

$$\begin{aligned} T_{w_1 w_2} &= T_{w_1} T_{w_2}, && \text{if } l(w_1 w_2) = l(w_1) + l(w_2), \\ T_s^2 &= (q_s - 1)T_s + qT - 1, && \text{if } s \in S. \end{aligned}$$

Definition 1.5. Let χ be a depth-zero character of $T(\mathcal{O})$. Then we define

$$\mathcal{H}_\chi := \mathcal{H}(W_{\chi,\mathrm{aff}}, S_{\chi,\mathrm{aff}}, q)\widetilde{\otimes}\mathbb{C}[\Omega_\chi],$$

where the twisted tensor product is the usual tensor product in the underlying vector spaces, but where multiplication is given by

$$(T_{w_1} \otimes e_{\sigma_1})(T_{w_2} \otimes e_{\sigma_2}) = T_{w_1} T_{\sigma_1 w_2 \sigma_1^{-1}} \otimes e_{\sigma_1 \sigma_2}$$

We are now ready to state the main theorem. To state it, we fix an extension $\check{\chi} : N(F) \to \mathbb{C}^\times$ of χ and we choose a section $\widetilde{W}_\chi \to N_\chi$, $w \mapsto n_w$ such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$, which they both exist. We abuse notation and write $[IwI]_{\check{\chi}} \in \mathcal{H}(G, I, \rho_\chi)$ for the function supported on In_wI and having value $\check{\chi}(n_w)$ on n_w. Since the double cosset In_wI is independent of the lift, we shall simply write IwI. However, we remark that the function $[IwI]_{\check{\chi}}$ does depend on the lift n_w.

Theorem 1.6. *Let χ be a character of $T(\mathcal{O})$ and let $\check{\chi} : N(F) \to \mathbb{C}^\times$ be an extension of χ. Then there is a support-preserving algebra isomorphism*

$$\mathcal{H}(G, I, \rho_\chi) \longrightarrow \mathcal{H}_\chi$$

such that for all $w \in \widetilde{W}_\chi$ with $w = w'\sigma$, $w' \in W_{\chi,\mathrm{aff}}, \sigma \in \Omega_\chi$, it sends $q^{-l(w)/2}[IwI]_{\check{\chi}}$ to $q^{-l_\chi(w)/2}(T_{w'} \otimes e_\sigma)$.

To prove the theorem, the first observation is that the support of $\mathcal{H}(G, I, \rho_\chi)$ is the intertwiner set

$$\mathrm{Supp}\mathcal{H}(G, I, \rho_\chi) = I_G(\rho_\chi) = IN_\chi I = \bigsqcup_{w \in \widetilde{W}_\chi} IwI$$

and that therefore $\{[IwI]_{\check{\chi}} \mid w \in \widetilde{W}_\chi\}$ is a $\mathbb{C}$-basis for $\mathcal{H}(G, I, \rho_\chi)$.

Secondly, we may define, for each $w \in \widetilde{W}_\chi$, the element

$$\varphi_w := q^{(l_\chi(w) - l(w))/2} [IwI]_{\check{\chi}},$$

corresponding to $T_{w'} \otimes e_\sigma$ under the isomorphism. Then the proof of the theorem is a direct consequence of the following result.

Proposition 1.7. *The elements $\varphi_w, w \in \widetilde{W}_\chi$ satisfy the following relations:*

1. *(Additive relations) $\varphi_{w_1 w_2} = \varphi_{w_1} * \varphi_{w_2}$, if $l_\chi(w_1 w_2) = l_\chi(w_1) + l_\chi(w_2)$.*

2. *(Quadratic relations) $\varphi_s^2 = (q-1)\varphi_s + q\varphi_1$, if $s \in S_{\chi,\mathrm{aff}}$.*

2 Hecke algebras in positive characteristic

When the field C of values of the representation has positive characteristic, then many of the previous results do not hold. For instance, the category $\mathcal{R}_C(G)$ does not have a nice block decomposition similar to the Bernstein decomposition for complex smooth representations. As a consequence, there is no analogous theory of types for positive characteristic. Nevertheless, for the setting of depth-zero representations, a bijection is still expected between simple right $\mathcal{H}(G, I, \rho_\chi)$-modules and irreducible C-representations (π, V) of $G(F)$ such that the ρ_χ-isotypic part of $\pi|_I$ is non-zero.

This last point suggests that studying the structure of $\mathcal{H}(G, I, \rho_\chi)$ yields relevant information about the structure of $\mathcal{R}_C(G)$. Fortunately, most steps in the proof of the theorem still hold if we replace $\mathbb{C}$ for an algebraically closed field of positive characteristic other than p. It is still true that the elements $\varphi_w, w \in \tilde{W}_\chi$ still form a basis for $\mathcal{H}(G, I, \rho_\chi)$, that $\varphi_{w_1 w_2} = \varphi_{w_1} * \varphi_{w_2}$ whenever $l_\chi(w_1 w_2) = l_\chi(w_1) + l_\chi(w_2)$ and that the elements $\varphi_s, s \in S_{\chi,\text{aff}}$ form a two dimensional subalgebra inside $\mathcal{H}(G, I, \chi_\rho)$. However, the calculations for the coefficients for the quadratic relations rely crucially on semisimplicity of certain representations and character theory, both of which fail for small positive characteristic.

The aim of this document is prove in a direct, explicit way, that the same quadratic relations hold even when the characteristic is small when G is a reductive group of type A_n.

The first part of the proof can be easily deduced from the following fact.

Lemma 2.1. *Let $w \in \tilde{W}_\chi$. Then*

$$[I : I \cap wIw^{-1}] = q^{l(w)}.$$

Lemma 2.2. *Let $s \in S_{\chi,\text{aff}}$. Then*

$$\varphi_s^2(1) = q.$$

Proof. This is a direct computation. Indeed,

$$\varphi_s^2(1) = \int_{IsI} \varphi_s(h)\varphi_s(h^{-1})dh = \chi(s^2)^{-1}\varphi_s(s)^2[IsI : I] = q^{1-l(s)}[I : I \cap sIs^{-1}] = q.$$

□

Then, the hard part of the proof is to show that $\varphi_s^2(s) = (q-1)\varphi_s(s) = (q-1)q^{1-l(s)}\check{\chi}(s)$ if $s \in S_{\chi,\text{aff}}$. To prove this result in general, the first step is to prove it for a family of reflections of arbitrary length, which we complete in the following section.

2.1 Quadratic relation for a canonical example

We now prove the quadratic relation for a family of reflections containing one reflection for each possible length. We remark that any $s \in S_{\chi,\text{aff}}$ is a reflection and, as such, $l(s)$ will always be odd.

Let $n \geq 1$ be a positive integer and let $G = \mathrm{GL}_{n+1}$. Choose a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ with longest root $\alpha_0 = \alpha_1 + \cdots + \alpha_n$. In this case, the Weyl group $W = N(F)/T(F)$ is generated by $S =$

$\{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$ and the extended Weyl group $\tilde{W} = N(F)/T(\mathcal{O})$ decomposes as a semidirect product of the affine Weyl group $W_{\text{aff}} = N(F)|_{\det \in \mathcal{O}^\times}/T(\mathcal{O})$ and the alcove stabilizer $\Omega = \langle \sigma \rangle \cong \mathbb{Z}$, where

$$\rho = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\varpi & & & 0 \end{pmatrix}.$$

As discussed above, the affine Weyl group is a coxeter group, where the simple reflections are given by

$$S_{\text{aff}} = \{w_{\alpha_1}(1), \ldots, w_{\alpha_n}(1), w_{\alpha_0}(\varpi^{-1})\}.$$

Let $\chi = \chi_0 \otimes \chi_1 \otimes \cdots \otimes \chi_n$ be a character of the torus, where each χ_i is a depth-zero character of $\mathcal{O}^\times$. Assume that $\chi_0 = \chi_n$ and with all others χ_i distinct characters. Under these assumptions,

$$N_\chi = \begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix} \bigsqcup \begin{pmatrix} & & & & * \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ * & & & & \end{pmatrix},$$

and in particular

$$S_{\chi,\text{aff}} = \left\{ w_{\alpha_0}(1) = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ 1 & & & \end{pmatrix}, w_{\alpha_0}(\varpi^{-1}) = \begin{pmatrix} & & & \varpi^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ -\varpi & & & \end{pmatrix} \right\}$$

Lemma 2.3. *Under the circumstances discussed above, we have that $l(w_{\alpha_0}(\varpi^{-1})) = 1$ and $l(w_{\alpha_0}(1)) = 2n-1$.*

Proof. The first part of the statement is immediate since $w_{\alpha_0} \in S_{\text{aff}}$ is a simple reflection. For the second part, we first note that $w_{\alpha_0}(1)$ is naturally an element of the Weyl group W, generated by $S = \{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$, and therefore $l(\alpha_0(1))$ coincides with its length as an element of W. Thus

$$l(w_{\alpha_0}) = |\{\alpha \in \Phi^+ : w_{\alpha_0}(\alpha) \in \Phi^-\}|.$$

Now, if $\alpha \in \Phi^+$, then $w_{\alpha_0}(\alpha) = \alpha - \langle \alpha, \check{\alpha}_0 \rangle \alpha_0 \in \Phi^-$if and only if $\langle \alpha \check{\alpha}_0 \rangle > 0$. Since $\langle \alpha_i, \check{\alpha}_0 \rangle = 1$ if $i = 1, n$ and 0 otherwise. Consequently,

$$\{\alpha \in \Phi^+ : w_{\alpha_0}(\alpha) \in \Phi^-\} = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_0, \ldots, \alpha_{n-1} + \alpha_n, \alpha_n\},$$

which has size $2n-1$ as desired. $\square$

The proof of the quadratic equation for $s_0 = w_{\alpha_0}(\varpi^{-1})$ is straightforward. For the sake of completeness, and as means of example, we give a full proof.

Since $l(s_0) = 1$, we consider $\varphi_{s_0} = [Is_0I]_{\check{\chi}}$. Then

$$\varphi_{s_0}^2(s_0) = \int_{G(F)} \varphi_{s_0}(s_0 h)\varphi_{s_0}(h^{-1})dh$$

and the integral is zero unless $h \in Is_0I \cap s_0Is_0I$. To understand this integral, we note that for any $x \in G(F)$, there is a bijection

$$IxI/I \longleftrightarrow I/(I \cap xIx^{-1})$$
$$yxI \longmapsto y(I \cap xIx^{-1})$$

and that

$$I \cap s_0Is_0^{-1} = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times & \ddots & & \mathcal{O} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \ddots & \mathcal{O}^\times & \mathcal{O} \\ \varpi^2\mathcal{O} & \varpi\mathcal{O} & \dots & \varpi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}.$$

Therefore, by fixing a section $t : k_F \to \mathcal{O}, a \mapsto a_t$ of the quotient $\mathcal{O} \to k_F$, we obtain

$$Is_0I = \bigsqcup_{c \in k_F} u_c s_0 I, \quad \text{where} \quad u_c = \begin{pmatrix} 1 & & \\ & 1 & \\ \varpi c_t & & 1 \end{pmatrix}.$$

A left cosset u_cs_0I is contained in s_0Is_0I if and only if there is some $z \in k_F$ such that $s_0u_cs_0u_zc_0 \in I$, and a simple calculation shows that this is the case if and only if $c \neq 0$, in which case $z = c^{-1}$. Hence, $Is_0I \cap s_0Is_0I = \sqcup_{c \in k_F^\times} u_cs_0I$ and moreover

$$\iota_c := s_0u_cs_0u_{1/c}s_0 \in I \quad \text{satisfies} \quad \iota_c \equiv \begin{pmatrix} -c & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1/c \end{pmatrix} \pmod{I^+},$$

so $\rho_\chi(\iota_c) = \chi_0(-c)\chi_n(-1/c) = 1$ since $\chi_0 = \chi_n$. Consequently,

$$\varphi_{s_0}^2(s_0) = \sum_{c \in k_F^\times} \int_{u_cs_0I} \varphi_{s_0}(sh)\varphi_{s_0}(h^{-1})dh = \sum_{c \in k_F^\times} \int_I \varphi_{s_0}(su_csk)\varphi_{s_0}(k^{-1}s_0^{-1}u_c^{-1})dk =$$
$$= \sum_{c \in k_F^\times} \varphi_{s_0}(\iota_cs_0^{-1}u_{1/c})\varphi_{s_0}(s_0^{-1}u_c^{-1}) = \varphi_{s_0}(s_0)^2 \sum_{c \in k_F^\times} \rho_\chi(\iota_c)\rho_\chi(u_{1/c})^{-1}\rho_\chi(u_c)^{-1} = |k_F^\times| = q-1,$$

and this completes the proof that $\varphi_{s_0}^2 = (q-1)\varphi_{s_0} + q\varphi_1$.

Next, we turn out attention towards proving the quadratic relation for $s = w_{\alpha_0}(1) \in S_{\chi,\mathrm{aff}}$. In Lemma 2.3 we proved that $l(s) = 2n-1$ and that

$$\{\alpha \in \Phi^+ : w_{\alpha_0}(\alpha) \in \Phi^-\} = \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_0, \dots, \alpha_{n-1}+\alpha_n, \alpha_n\},$$

which implies that

$$I \cap sIs^{-1} = \begin{pmatrix} \mathcal{O}^\times & \varpi\mathcal{O} & \varpi\mathcal{O} & \cdots & \varpi\mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \dots & \varpi\mathcal{O} & \mathcal{O}^\times & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \dots & \varpi\mathcal{O} & \varpi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}$$

and therefore we can express the double cosset IsI as a disjoint union of left cossets of I as

$$IsI = \bigsqcup_{(\mathbf{a},\mathbf{b},c)\in k_F^{2n-1}} u_{\mathbf{a},\mathbf{b},c}sI \quad \text{where} \quad u_{\mathbf{a},\mathbf{b},c} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} & c \\ 0 & 1 & 0 & \cdots & 0 & b_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & b_2 \\ \vdots & & & \ddots & 1 & b_1 \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

Here, we are abusing notation, and all elements of the matrix are assumed to be the lifts from k_F to $\mathcal{O}$ by the section t fixed above.

Lemma 2.4. *Let $(\mathbf{a},\mathbf{b},c) \in k_F^{2n-1}$. Then there is some $(\mathbf{x},\mathbf{y},z) \in k_F^{2n-1}$ such that*

$$su_{\mathbf{a},\mathbf{b},c}su_{\mathbf{x},\mathbf{y},z}s \in I$$

if and only if

$$c(c - \sum_{i+j=n} a_ib_j) \neq 0 \quad \textit{and} \quad a_ib_j = 0 \textit{ if } i+j<n.$$

When these conditions hold, then

$$\mathbf{x} = \mathbf{a}c^{-1}, \quad \mathbf{y} = \mathbf{b}(c - \sum_{i+j=n} a_ib_j)^{-1} \quad \textit{and} \quad z = (c - \sum_{i+j=n} a_ib_j)^{-1}$$

and

$$\iota_{\mathbf{a},\mathbf{b},c} := su_{\mathbf{a},\mathbf{b},c}su_{\mathbf{x},\mathbf{y},z}s \in I \quad \textit{satisfies} \quad \iota_{\mathbf{a},\mathbf{b},c} \equiv \begin{pmatrix} \frac{1}{c-\sum_{i+j=n} a_ib_j} & & & & \\ & \frac{c-a_1b_{n-1}}{c} & & & \\ & & \ddots & & \\ & & & \frac{c-a_{n-1}b_1}{c} & \\ & & & & c \end{pmatrix} \pmod{I^+}. \tag{2}$$

To simplify notation, we let

$$J_n = \{(\mathbf{a},\mathbf{b},c) \in k_F^{2n-1} \mid c(c - \sum_{i+j=n} a_i b_j) \neq 0 \quad \text{and} \quad a_i b_j = 0 \text{ if } i+j<n\}.$$

Corollary 2.5. *We have that*

$$IsI \cap sIsI = \bigsqcup_{(\mathbf{a},\mathbf{b},c)\in J_n} u_{\mathbf{a},\mathbf{b},c} sI$$

Now, we let $\varphi_s = q^{(1-l(s))/2}[IsI] - \check{\chi} = q^{1-n}[IsI]_{\check{\chi}}$, and using the previous results, we obtain

$$\varphi_s^2(s) = \int_{IsI\cap sIsI} \varphi_s(sh)\varphi_s(h^{-1})dh = \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \int_{u_{\mathbf{a},\mathbf{b},c}sI} \varphi_s(sh)\varphi_s(h^{-1})dh =$$

$$= \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \int_I \varphi_s(su_{\mathbf{a},\mathbf{b},c}sk)\varphi_s(k^{-1}s^{-1}u_{\mathbf{a},\mathbf{b},c}^{-1})dk = \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \varphi_s(\iota_{\mathbf{a},\mathbf{b},c}s^{-1}u_{\mathbf{x},\mathbf{y},z})\varphi_s(s^{-1}u_{\mathbf{a},\mathbf{b},c}^{-1}) =$$

$$= \rho_\chi(s^{-2})^2 \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c})\varphi_s(s)\rho_\chi(u_{\mathbf{x},\mathbf{y},z})^{-1}\varphi_s(s)\rho_\chi(u_{\mathbf{a},\mathbf{b},c})^{-1} = q^{2-2n} \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}).$$

To calculate $\varphi_s^2(s)$, it is therefore enough to compute the sum

$$R_n(q,\{\chi_0,\dots,\chi_{n-1}\}) = \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}) =$$

$$= \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \chi_0\left(\frac{c}{c-\sum_{i+j=n} a_i b_j}\right)\chi_1\left(\frac{c-a_1 b_{n-1}}{c}\right)\cdots\chi_{n-1}\left(\frac{c-a_{n-1}b_1}{c}\right).$$

Proposition 2.6. *For $n \geq 3$ and $\chi_0 \notin \{\chi_1,\dots,\chi_{n-1}\}$, the sums $R_n(q,\{\chi_0,\dots,\chi_{n-1}\})$ satisfy the recurrence relation*

$$R_n(q,\{\chi_0,\dots,\chi_{n-1}\}) = qR_{n-1}(q,\{\chi_0,\chi_2,\dots,\chi_{n-1}\}) + qR_{n-1}(q,\{\chi_0,\dots,\chi_{n-2}\}) - q^2 R_{n-2}(q,\{\chi_0,\chi_2,\dots,\chi_{n-2}\}).$$

To prove this proposition, we will need the following lemma.

Lemma 2.7. *There is a bijection between the sets $\{(\mathbf{a},\mathbf{b},c)\in J_n : a_1 = 0\}$ and $k_F \times J_{n-1}$ given by*

$$((a_1,\dots,a_{n-1}),(b_1,\dots,b_{n-1}),c) \longmapsto (b_{n-1},(a_2,\dots,a_{n-1}),(b_1,\dots,b_{n-2}),c).$$

Analogously, there is a bijection between the sets $\{(\mathbf{a},\mathbf{b},c)\in J_n : b_1 = 0\}$ and $k_F \times J_{n-1}$.

Proof. This is a direct computation. If $a_1 = 0$, then b_{n-1} is a free variable and if we let $a_i' = a_{i+1}$ for $i = 1,\dots,n-2$, then the condition $c(c-\sum_{i+j=n} a_i b_j) \neq 0$ becomes $c(c-\sum_{i+j=n-1} a_i' b_j) \neq 0$ and the condition $a_i b_j = 0$ for $i+j<n$ becomes $a_i b_j < n-1$, as desired. ◻

Proof of Proposition 2.6. Firstly, we decompose the sum $R_n(q,\{\chi_0,\dots,\chi_{n-1}\})$ into

$$R_n(q,\{\chi_0,\dots,\chi_{n-1}\}) = R_n^{a_1}(q,\{\chi_0,\dots,\chi_{n-1}\}) + R_n^{b_1}(q,\{\chi_0,\dots,\chi_{n-1}\}) - R_n^{a_1,b_1}(q,\{\chi_0,\dots,\chi_{n-1}\}), \tag{3}$$

where

$$R_n^{a_1}(q,\{\chi_0,\dots,\chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n\\ a_1=0}} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}), \quad R_n^{b_1}(q,\{\chi_0,\dots,\chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n\\ b_1=0}} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c})$$

$$\text{and} \quad R_n^{a_1,b_1}(q,\{\chi_0,\dots,\chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n\\ a_1=b_1=0}} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}).$$

Next, we can compute each individual term from (3). If $a_1 = 0$, we note that b_{n-1} is completely free, and if we let $\mathbf{a}' = (a_2, \ldots, a_{n-1})$ and $\mathbf{b}' = (b_1, \ldots, b_{n-2})$, by Lemma 2.7 we obtain

$$\begin{aligned}
R_n^{a_1}(q, \{\chi_0, \ldots, \chi_{n-1}\}) &= \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ a_1=0}} \chi_0\left(\frac{c}{c - \sum_{i+j=n} a_i b_j}\right) \chi_2\left(\frac{c - a_2 b_{n-2}}{c}\right) \cdots \chi_{n-1}\left(\frac{c - a_{n-1}b_1}{c}\right) = \\
&= q \sum_{(\mathbf{a}',\mathbf{b}',c)\in J_{n-1}} \chi_0\left(\frac{c}{c - \sum_{i+j=n-1} a'_i b'_j}\right) \chi_2\left(\frac{c - a'_1 b'_{n-2}}{c}\right) \cdots \chi_{n-1}\left(\frac{c - a'_{n-2}b'_1}{c}\right) = \\
&= qR_{n-1}(q, \{\chi_0, \chi_2 \ldots, \chi_{n-1}\}).
\end{aligned}$$

An analogous calculation shows that $R_n^{b_1}(q, \{\chi_0, \ldots, \chi_{n-1}\}) = qR_{n-1}(q, , \{\chi_0, \ldots, \chi_{n-2}\})$. Finally, applying Lemma 2.7 twice gives a bijection

$$\begin{aligned}
\{(\mathbf{a}, \mathbf{b}, c) \in J_n : a_1 = b_1 = 0\} &\longrightarrow k_F^2 \times J_{n-2} \\
((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1}), c) &\longmapsto (a_{n-1}, b_{n-1}, (a_2, \ldots, a_{n-2}), (b_2, \ldots, b_{n-2}), c)
\end{aligned}$$

and if we let $\mathbf{a}' = (a_2, \ldots, a_{n-2})$ and $\mathbf{b}' = (b_2, \ldots, b_{n-2})$, then

$$\begin{aligned}
R_n^{a_1,b_1}(q, \{\chi_0, \ldots, \chi_{n-1}\}) &= \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ a_1=b_1=0}} \chi_0\left(\frac{c}{c - \sum_{i+j=n} a_i b_j}\right) \chi_2\left(\frac{c - a_2 b_{n-2}}{c}\right) \cdots \chi_{n-2}\left(\frac{c - a_{n-2}b_2}{c}\right) = \\
&= q^2 \sum_{(\mathbf{a}',\mathbf{b}',c)\in J_{n-2}} \chi_0\left(\frac{c}{c - \sum_{i+j=n-2} a'_i b'_j}\right) \chi_2\left(\frac{c - a'_1 b'_{n-3}}{c}\right) \cdots \chi_{n-1}\left(\frac{c - a'_{n-3}b'_1}{c}\right) = \\
&= q^2 R_{n-2}(q, \{\chi_0, \chi_2 \ldots, \chi_{n-2}\}).
\end{aligned}$$

Putting everything together yields the desired recurrence relation. □

Naturally, the next step is to compute these sums for small values of n in order to use the above recurrence relation.

Proposition 2.8. *For any two depth-zero characters χ_0, χ_1 of $\mathcal{O}^\times$ such that $\chi_0 \neq \chi_1$, we have that*

$$R_1(q, \{\chi_0\}) = q - 1 \quad \textit{and} \quad R_2(q, \{\chi_0, \chi_1\}) = (q-1)q. \tag{4}$$

Moreover, the sums $R_n(q) := R_n(q, \{\chi_0, \ldots, \chi_{n-1}\})$ are independent of $\{\chi_0, \ldots, \chi_{n-1}\}$ and they satisfy

$$R_n(q) = (q-1)q^{n-1}.$$

Proof. The values of $R_1(q, \{\chi_0\})$ and $R_2(q, \{\chi_0, \chi_1\})$ can be obtain by direct computation. Indeed,

$$R_1(q, \chi_0) = \sum_{c \in k_F^\times} \chi_0(c/c) = q - 1,$$

and

$$R_2(q,\{\chi_0,\chi_1\}) = \sum_{(a,b,c)\in J_2} \chi_0\left(\frac{c}{c-ab}\right)\chi_1\left(\frac{c-ab}{c}\right) = \sum_{\substack{(a,b,c)\in J_2\\ ab=0}} \chi_0(1)\chi_1(1) + \sum_{\substack{(a,b,c)\in J_2\\ ab\neq 0}} \chi_0\chi_1^{-1}\left(\frac{c}{c-ab}\right) =$$

$$= (2q-1)(q-1) + (q-1)\sum_{\substack{c,d\in k_F^\times\\ c\neq d}} \chi_0\chi_1^{-1}\left(\frac{c}{c-d}\right) = (2q-1)(q-1) + (q-1)^2 \sum_{x\in k_F^\times\setminus\{1\}} \chi_0\chi_1^{-1}(x) =$$

$$= (2q-1)(q-1) - (q-1)^2 = (q-1)q,$$

as desired. This result, together with the recurrence relation from Proposition 2.6, inductively shows that the sums $R_n(q,\{\chi_0,\dots,\chi_{n-1}\})$ are independent of $\{\chi_0,\dots,\chi_{n-1}\}$, which we shall simply write as $R_n(q)$. Combining Proposition 2.6 with (4), we have that $R_n(q)$ satisfies the recurrence relation

$$\begin{cases} R_n(q) = 2qR_{n-1}(q) - q^2R_{n-2}(q) \quad \text{for } n\geq 3,\\ R_1(q) = q-1 \quad \text{and} \quad R_2(q) = (q-1)q, \end{cases}$$

whose solution is $R_n(q) = (q-1)q^{n-1}$. □

Remark 2.9. From Corollary 2.5, one can see that $P_n(q) := [IsI\cap sIsI : I] = |J_n|$ is a constant of interest directly related to $R_n(q)$. In fact, following a similar argument as above, one can show that the sequence $P_n(q)$ satisfies

$$\begin{cases} P_n(q) = 2qR_{n-1}(q) - q^2R_{n-2}(q) \quad \text{for } n\geq 3,\\ P_1(q) = q-1 \quad \text{and} \quad P_2(q) = (q-1)(q^2-q+1), \end{cases}$$

whose solution is $P_n(q) = q^{n-2}(q-1)((n-1)q^2-(2n-3)q+(n-1))$.

Now the proof of the quadratic relation for $s = w_{\alpha_0}(1)$ is immediate. Indeed,

$$\varphi_s^2(s) = q^{2-2n}R_n(q) = (q-1)q^{1-n} = (q-1)\varphi_s(s),$$

which implies that

$$\varphi_s^2 = (q-1)\varphi_s + q\varphi_1,$$

as desired.

3 The quadratic relation in general

Throughout this section, we specify the dimension of the Iwahori subgroup and we denote the Iwahori subgroup of $\mathrm{GL}_n(F)$ by I_n. In this section we show that for any character $\chi = \chi_0 \otimes \ldots \otimes \chi_n$ of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}(F)$ and $s \in S_{\chi,\mathrm{aff}}$, the proof of the quadratic relation for $\varphi_s = q^{(1-l(s))/2}[I_{n+1}sI_{n+1}]_{\check{\chi}}$ can be deduced from the particular case proven in the previous section. Let Φ be the root system for GL_{n+1} with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. For each $w \in \widetilde{W}$, let

$$\widetilde{\Delta}(w) = \{\text{hyperplanes } P \mid P \text{ separates } \mathcal{D}_0 \text{ and } w\mathcal{D}_0\}.$$

Recall from Iwahori-Matsumoto that $l(w) = |\widetilde{\Delta}(w)|$ for all $w \in \widetilde{W}$.

Lemma 3.1. *Let $\alpha = \sum_{k=i}^{j} \alpha_k \in \Phi^+$ for some $i \leq j$. Then*

$$\begin{aligned}\widetilde{\Delta}(w_\alpha(1)) &= \{P_{\beta,0} \mid \beta \in \Phi^+ \cap w_\alpha^{-1}\Phi^-\} \\ &= \{P_{\alpha_i,0}, P_{\alpha_i+\alpha_{i+1},0}, \ldots, P_{\alpha,0}, \ldots, P_{\alpha_{j-1}+\alpha_j,0}, P_{\alpha_j,0}\}.\end{aligned}$$

Proof. Let $x \in \mathcal{D}_0$ and let $\beta \in \Phi^+$. Then

$$\langle \beta, w_\alpha(1)(x)\rangle = \langle \beta, x\rangle - \langle \alpha, x\rangle\langle \beta, \check{\alpha}\rangle = \langle w_\alpha(\beta), x\rangle \in \begin{cases} (0,1) & \text{if } w_\alpha(\beta) \in \Phi^+, \\ (-1,0) & \text{if } w_\alpha(\beta) \in \Phi^-. \end{cases}$$

Hence, $P \in \widetilde{\Delta}(w)$ if and only if $P = P_{\beta,0}$ for some $\beta \in \Phi^+ \cap w_\alpha^{-1}\Phi^{-1}$. Since $w_\alpha(\beta) = \beta - \langle \beta, \check{\alpha}\rangle\alpha$, it follows that $\beta \in \Phi^+ \cap w_\alpha^{-1}\Phi^-$ if and only if $\beta = \alpha$ or $\langle \beta, \check{\alpha}\rangle = 1$ and $\alpha - \beta \in \Phi^+$.

If $\alpha = \alpha_i + \cdots + \alpha_j$ for some $j \geq i$, then a standard calculation shows that $\beta \in \Phi^+$ satisfies the above conditions if and only if $\beta = \alpha_i + \cdots + \alpha_k$ for some $k \leq j$ or $\beta = \alpha_k + \cdots + \alpha_j$ for some $k \geq i$. This concludes the proof. $\square$

Lemma 3.2. *Let $\alpha = \sum_{k=i}^{j} \alpha_k \in \Phi^+$ for some $i \leq j$. Then*

$$\begin{aligned}\widetilde{\Delta}(w_\alpha(\varpi^{-1})) &= \{P_{\alpha_1+\cdots+\alpha_{i-1},0}, \ldots, P_{\alpha_{i-1},0}, P_{\alpha_{j+1},0}, \ldots, P_{\alpha_{j+1}+\cdots+\alpha_n,0}\} \\ &\quad \cup \{P_{\alpha_1+\cdots+\alpha_j,1}, \ldots, P_{\alpha_i+\cdots+\alpha_j,1}, \ldots, P_{\alpha_i+\cdots+\alpha_n,1}\}.\end{aligned}$$

Proof. As in the previous proof, let $x \in \mathcal{D}_0$ and let $\beta \in \Phi^+$. Then,

$$\begin{aligned}\langle \beta, w_\alpha(\varpi^{-1})(x)\rangle &= \langle \beta, x\rangle - \langle \alpha, x\rangle\langle \beta, \check{\alpha}\rangle + \langle \beta, \check{\alpha}\rangle = \langle w_\alpha(\beta), x\rangle + \langle \beta, \check{\alpha}\rangle \\ &\in \begin{cases} (1,2) & \text{if } \beta = \alpha, \text{ or } \langle \beta, \check{\alpha}\rangle = 1 \text{ and } w_\alpha(\beta) \in \Phi^+, \\ (0,1) & \text{if } \langle \beta, \check{\alpha}\rangle = 0, \text{ or } \langle \beta, \check{\alpha}\rangle = 1 \text{ and } w_\alpha(\beta) \in \Phi^-, \\ (-1,0) & \text{if } \langle \beta, \check{\alpha}\rangle = -1. \end{cases}\end{aligned}$$

Thus,

$$\widetilde{\Delta}(w_\alpha(\varpi^{-1})) = \{P_{\alpha,1}\} \cup \{P_{\beta,1} \mid \beta \in \Phi^+, \langle \beta, \check{\alpha}\rangle = 1 \text{ and } w_\alpha(\beta) \in \Phi^+\} \cup \{P_{\beta,0} \mid \beta \in \Phi^+ \text{ and } \langle \beta, \check{\alpha}\rangle = -1\}.$$

If $\alpha = \alpha_i + \cdots + \alpha_j$, then further calculations show that $\beta \in \Phi^+$ satisfies $\langle \beta, \check{\alpha} \rangle = 1$ with $w_\alpha(\beta) = \beta - \alpha \in \Phi^+$ if and only if $\beta = \alpha_k + \cdots + \alpha_j$ for some $k < i$ or $\beta = \alpha_i + \cdots + \alpha_k$ for some $k > j$. Similarly, one can show that $\beta \in \Phi^+$ satisfies $\langle \beta, \check{\alpha} \rangle = -1$ if and only if $\beta = \alpha_k + \cdots + \alpha_{i-1}$ for some $k \leq i-1$ or $\beta = \alpha_{j+1} + \cdots + \alpha_k$ for some $k \geq j+1$. $\square$

Corollary 3.3. *For any $\alpha \in \Phi^+$, we have that*

$$l(w_\alpha(1)) + l(w_\alpha(\varpi^{-1})) = \sum_{\beta \in \Phi^+} |\langle \beta, \check{\alpha} \rangle| = 2n$$

Proof. The first equality is an immediate consequence of $l(w) = \widetilde{\Delta}(w)$ together with Lemmas 3.1 and 3.2. Indeed, if $\langle \beta, \check{\alpha} \rangle = 0$ then no hyperplane arising from β lies in $\widetilde{\Delta}(w_\alpha(1)) \cup \widetilde{\Delta}(w_\alpha(\varpi^{-1}))$; if $\langle \beta, \check{\alpha} \rangle = \pm 1$, then exactly one plane arising from β lies $\widetilde{\Delta}(w_\alpha(1)) \cup \widetilde{\Delta}(w_\alpha(\varpi^{-1}))$; finally, $P_{\alpha,0} \in \widetilde{\Delta}(w_\alpha(1))$ and $P_{\alpha,1} \in \widetilde{\Delta}(w_\alpha(\varpi^{-1}))$.

The second part of the lemma is a direct computation: if $\alpha = \alpha_i \cdots \alpha_j$, then there are

- j roots in Φ^+ of the form $\alpha_k + \cdots + \alpha_j$ for $k \leq j$,
- $n + 1 - i$ roots in Φ^+ of the form $\alpha_i + \cdots + \alpha_k$ for $k \geq i$,
- $i - 1$ roots in Φ^+ of the form $\alpha_k + \cdots + \alpha_{i-1}$ for $k \leq i - 1$,
- $n + 1 - (j + 1)$ roots in Φ^+ of the form $\alpha_{j+1} + \cdots + \alpha_k$ for $k \geq j + 1$.

Giving a total of $2n$ roots, each of which contributes once towards $\sum_{\beta \in \Phi^+} |\langle \beta, \check{\alpha} \rangle|$, as desired. $\square$

3.1 The diagram associated to a character

Having established the preliminary lemmas, we now consider a depth zero character $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}(F)$, where each χ_i is a depth-zero character of $\mathcal{O}^\times$. The character χ induces a partition $\mathcal{P}_\chi$ of the set $\{0, 1, \ldots, n\}$ according to which two elements $i, j \in \{0, 1, \ldots, n\}$ are related if and only if $\chi_i = \chi_j$ as characters of $\mathcal{O}^\times$.

Definition 3.4. Let χ be a depth-zero character of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}(F)$ giving rise to a partition $\mathcal{P}_\chi$ of $\{0, 1, \ldots, n\}$ of size r. Then the diagram $\mathfrak{D}_\chi$ associated to χ is a regular $(n+1)$-gon whose vertices are labelled $\{0, 1, \ldots, n\}$ counterclockwise and have been painted in r different colors according to the partition $\mathcal{P}_\chi$, and having a distinguished edge $e = \{0, n\}$.

Example 3.5. In the previous section we considered the family of characters $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ such that $\chi_0 = \chi_n$ and all the remaining characters are distinct. Then $\mathcal{P}_\chi = \{\{0, n\}, \{1\}, \ldots, \{n-1\}\}$ has size n, and Figure 1 shows the diagram with $n = 4$.

The diagram of a character defined above is useful because it will allow us to determine $S_{\chi,\mathrm{aff}}$ easily. The first step towards this goal is to associate elements of $\widetilde{W}_\chi$ to paths in the diagram.

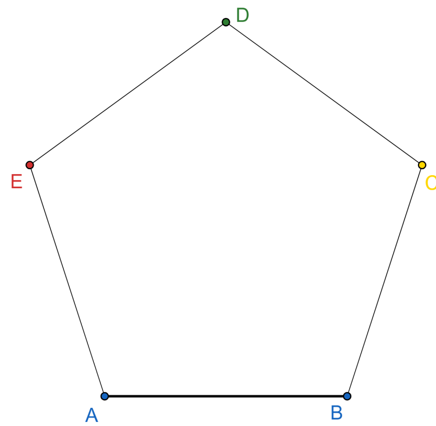

Figure 1: Diagram for $\mathcal{P}_\chi = \{\{0,4\},\{1\},\{2\},\{3\}\}$

Definition 3.6. Let $\mathfrak{D}_\chi$ be the diagram associated to χ and let $i, j \in \{0, 1, \dots, n\}$ be two *distinct* vertices, and we assume that $i < j$. Let

$$\alpha(i,j) = \alpha_{i+1} + \cdots + \alpha_{j-1} + \alpha_j \in \Phi^+$$

be the positive root associated to the pair (i,j). Let $T_{i,j,1}$, (resp. $T_{i,j,0}$) be the trail in $\mathfrak{D}_\chi$ from i to j avoiding (resp. passing through) the distinguished edge $e = \{0, n\}$. Then the associated reflection $w(T_{i,j,k})$ to the path $T_{i,j,k}$ is

$$w(T_{i,j,k}) = \begin{cases} w_{\alpha(i,j)}(1) & \text{if } k = 1, \\ w_{\alpha(i,j)}(\varpi^{-1}) & \text{if } k = 0. \end{cases}$$

Lemma 3.7. *For any path $T_{i,j,k}$, we have that*

$$l(w(T_{i,j,k})) = 2l(T_{i,j,k}) - 1,$$

where the length of a trail is the number of edges on the trail.

Proof. Assume without loss of generality that $i < j$. If $k = 0$, then $l(T_{i,j,0}) = j - i$, while Lemma 3.1 shows that $l(w_{\alpha_{i,j}}(1)) = 2(j-i) + 1$. If $k = 1$, we may use Corollary 3.3 to obtain

$$l(w_{\alpha(i,j)}(\varpi^{-1})) = 2n - l(w_{\alpha(i,j)}(1)) = 2n - 2l(T_{i,j,0}) + 1 = 2(n + 1 - l(T_{i,j,0})) - 1 = 2l(T_{i,j,1}) - 1,$$

and this concludes the lemma. $\square$

For the remainder of this section, we fix a character $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ inducing a partition $\mathcal{P}_\chi = \{X_1, \dots, X_r\}$, and for each $k \in \{1, \dots, r\}$, write $X_k = \{x_{k,1} < \cdots < x_{k,m_k}\}$. In addition, let $\{T_{k,j} \mid j = 1, \dots, m_k\}$ be all the trails joining each consecutive pair of vertices $\{\{x_{k,j}, x_{k,j+1}\} \mid j = 1, \dots, m_k\}$ in X_k. Finally, we denote by

$\pi : \widetilde{W} \to W \cong \mathrm{Sym}\{0, \ldots, n\}$ the unique surjective group homomorphisms satisfying

$$\pi : \widetilde{W} \longrightarrow W \cong \mathrm{Sym}\{0, \ldots, n\}$$
$$w_{\alpha_i}(\lambda) \longmapsto \begin{cases} (i\ i+1) \text{ if } 1 \leq i \leq n, \\ (0\ n) \text{ if } n = 0. \end{cases}$$

This homomorphism has the important property that the action of $w \in \widetilde{W}$ on depth-zero characters permutes the characters on each diagonal entry by $\pi(w) \in \mathrm{Sym}\{0, \ldots, n\}$.

Lemma 3.8. *The root system $\Phi_\chi = \{\alpha \in \Phi \mid \chi \circ \alpha^\vee|_{\mathcal{O}^\times} = 1\}$ of the character χ is given by*

$$\Phi_\chi = \bigoplus_{k=1}^{r} \Phi_\chi^{(k)},$$

where $\Phi_\chi^{(k)}$ is the irreducible root system of type A_{m_k-1} and simple roots given by

$$\beta_{k,j} = \alpha_{x_{k,j}+1} + \cdots + \alpha_{x_{k,j+1}}, \quad 1 \leq j \leq m_k - 1.$$

Proof. Firstly, we note that $\pi(\beta_{k,j}) = (x_{k,j} \ \ x_{k,j+1}) \in \mathrm{Sym}\{0, \ldots, n\}$ and that

$$\check{\beta}_{k,j}(\lambda)_{m,m} = \begin{cases} \lambda & \text{if } m = x_{k,j}, \\ \lambda^{-1} & \text{if } m = x_{k,j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi_{x_{k,j}} = \chi_{x_{k,j+1}}$, it follows that $\beta_{k,j} \in \Phi_\chi$. On the other hand, if $\alpha \in \Phi_\chi \cap \Phi^+$, a similar reasoning shows that $\pi(\alpha)$ is a transposition of two elements $x_{k,j_1}, x_{k,j_2} \in X_k$ for some k and $j_1 < j_2$. But then we have that

$$\alpha = \alpha_{x_{k,j_1}+1} + \cdots + \alpha_{x_{k,j_2}} = \beta_{k,j_1} + \cdots + \beta_{k,j_2-1},$$

which concludes the proof. ☐

Proposition 3.9. *Let χ be a depth-zero character of $T(\mathcal{O})$ and let $\mathfrak{D}_\chi$ be the associated diagram induced by the partition $\mathcal{P}_\chi = \{X_1, \ldots, X_r\}$ of its vertices. Then*

$$S_{\chi,\mathrm{aff}} = \bigcup_{k=1}^{r} \{w(T_{k,1}), \ldots, w(T_{k,m_k})\},$$

where $T_{k,j}$ are the trails constructed above.

Proof. By Lemma 3.8, the partition $\mathcal{P}_\chi = \{X_1, \ldots, X_r\}$ induces a decomposition

$$\Phi_\chi = \bigoplus_{k=1}^{r} \Phi_\chi^{(k)},$$

where $\Phi_\chi^{(k)}$ is an irreducible root system of type A_{m_k-1}. This induces a direct product decomposition

$$W_{\chi,\mathrm{aff}} \cong W_{\chi,\mathrm{aff}}^{(1)} \times \cdots \times W_{\chi,\mathrm{aff}}^{(r)},$$

where

$$W_{\chi,\mathrm{aff}}^{(k)} := \{w \in W_{\chi,\mathrm{aff}} \mid \pi(w)(j) = j \text{ for all } j \notin A_k\}$$

is the affine Weyl group with underlying root system $\Phi_\chi^{(k)}$. Moreover, the direct product implies that the length function of each $W_{\chi,\mathrm{aff}}^{(k)}$ agrees with the length function l_χ on $W_{\chi,\mathrm{aff}}$ and therefore if $S_{\chi,\mathrm{aff},k}$ is the set of simple reflections of $W_{\chi,\mathrm{aff}}^{(k)}$, then

$$S_{\chi,\mathrm{aff}} = \bigcup_{k=1}^{r} S_{\chi,\mathrm{aff},k}.$$

Hence, it suffices to show that

$$S_{\chi,\mathrm{aff},k} = \{w(T_{k,1}), \dots, w(T_{k,m_k})\}.$$

In Lemma 3.8, we showed that the simple roots of $\Phi_\chi^{(k)}$ has simple roots given by

$$\beta_{k,j} = \alpha_{x_{k,j}+1} + \cdots + \alpha_{x_{k,j+1}}, \quad 1 \le j \le m_k - 1.$$

Each simple root gives rise to a simple reflection $w_{\beta_{k,j}}(1) = w(T_{k,j})$ for $1 \le j \le m_k$. The last simple reflection is given by the highest root. Since $\Phi_\chi^{(k)}$ is of type A_{m_k-1}, then the highest root is

$$\beta_{k,0} = \beta_{k,1} + \cdots + \beta_{k,m_k-1} = \alpha_{x_{k,1}+1} + \cdots + \alpha_{x_{k,m_k-1}},$$

and the simple reflection is $w_{\beta_{k,0}}(\varpi^{-1}) = w(T_{k,m_k})$ since T_{k,m_k} crosses the distinguished edge $e = \{0, n\}$ in $\mathfrak{D}_\chi$. Thus,

$$S_{\chi,\mathrm{aff},k} = \{w_{\beta_{k,1}}(1), \dots, w_{\beta_{k,m_k-1}}(1), w_{\beta_{k,0}}(\varpi^{-1})\} = \{w(T_{k,1}), \dots, w(T_{k,m_k})\},$$

as desired. □

3.2 The induced homomorphism on Hecke algebras

Having established a way to easily determine $S_{\chi,\mathrm{aff}}$ from the character χ, in this subsection we construct, for each depth-zero character χ and $s \in S_{\chi,\mathrm{aff}}$, a homomorphism of p-adic groups ψ_s with certain nice properties that induce a homomorphism on Hecke algebras. This homomorphism ψ_s will be our main tool to deduce the quadratic equation for $\varphi_s = q^{(1-l(s))/2}[I_{n+1} s I_{n+1}]_{\check{\chi}}$ from our results in the previous section.

Before we can construct these homomorphisms, we first need to consider the element

$$\rho_n := \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\varpi & & & \end{pmatrix} \in \mathrm{GL}_{n+1}(F),$$

which is the generator of the alcove stabilizer Ω. This element has a natural action on the set of depth-zero characters of $T(\mathcal{O})$ that is compatible with a certain action on the diagrams.

Lemma 3.10. *The action of ρ_n on the space of depth-zero characters of $T(\mathcal{O})$ is compatible with a clockwise rotation of $2\pi/n$ radians on the coloring of the diagrams.*

In other words, if χ is a depth-zero character of $T(\mathcal{O})$, then ${}^{\rho_n}\chi(\cdot) = \chi(\rho_n^{-1} \cdot \rho_n)$ is another depth-zero character. Moreover,

$$\mathfrak{D}_{{}^{\rho_n}\chi} = r_n(\mathfrak{D}_\chi) \quad \textit{and} \quad S_{{}^{\rho_n}\chi,\mathrm{aff}} = \rho_n S_{\chi,\mathrm{aff}} \rho_n^{-1},$$

where $r_n(\mathfrak{D})$ is obtained by rotating the coloring of $\mathfrak{D}$ clockwise $2\pi/(n+1)$ radians.

Proof. We first note that

$$\rho_n^{-1} \begin{pmatrix} a_0 & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \rho_n = \begin{pmatrix} a_n & & & \\ & a_0 & & \\ & & \ddots & \\ & & & a_{n-1} \end{pmatrix}. \tag{5}$$

Therefore, if $\chi = \chi_0 \otimes \chi_1 \otimes \cdots \otimes \chi_n$, then ${}^{\rho_n}\chi = \chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0$, and by construction, $\mathfrak{D}_{{}^{\rho_n}\chi} = r_n(\mathfrak{D}_\chi)$. $\square$

Now we are ready to construct a homomorphism ψ_s of p-adic groups associated to $s \in S_{\chi,\mathrm{aff}}$, where χ is some depth-zero character of $T(\mathcal{O}) \subseteq \mathrm{GL}_{n+1}(F)$. By Proposition 3.9, the reflection s corresponds to some trail T_s connecting two distinct vertices i and $j = i + l \pmod{n+1}$ of $\mathfrak{D}_\chi$ **counterclockwise** and with length $l = (1 + l(s))/2 \leq n$. Then define

$$\psi_s : \mathrm{GL}_{l+1}(F) \longrightarrow \mathrm{GL}_{n+1}(F) \tag{6}$$

$$A \longmapsto \rho_n^{n+1-i} \begin{pmatrix} A & \\ & \mathrm{Id}_{n-l} \end{pmatrix} \rho_n^{i-n-1}, \tag{7}$$

which is clearly a continuous homomorphism of p-adic groups. Note that ψ_s is the composition of the two homomorphisms

$$\begin{aligned} \phi_{l,n} : \mathrm{GL}_{l+1}(F) &\longrightarrow \mathrm{GL}_{n+1}(F) \\ A &\longmapsto \begin{pmatrix} A & \\ & \mathrm{Id}_{n-l} \end{pmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_s : \mathrm{GL}_{n+1}(F) &\longrightarrow \mathrm{GL}_{n+1}(F) \\ B &\longmapsto \rho_n^{n+1-i} B \rho_n^{i-n-1}. \end{aligned}$$

We remark that if $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ with $\chi_0 = \chi_n$ and all the other χ_i distinct (as in the previous section) and $s = w_{\alpha_0}(1) \in S_{\chi,\mathrm{aff}}$, then ψ_s is the identity map on $\mathrm{GL}_{n+1}(F)$.

Lemma 3.11. *For any depth-zero character χ of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}$ and any $s \in S_{\chi,\mathrm{aff}}$, the homomorphism $\psi_s : \mathrm{GL}_{l+1}(F) \longrightarrow \mathrm{GL}_{n+1}(F)$ satisfies the following properties:*

1. *$\psi_s^{-1}(T(\mathcal{O})) = T(\mathcal{O})$ and $\psi_s^{-1}(N_{n+1}) = N_{l+1}$, where $N_i = N_{\mathrm{GL}_i}(T(\mathcal{O}))$,*
2. *$\psi_s^{-1}(I_{n+1}) = I_{l+1}$.*

Proof. Both properties are immediate for the homomorphism $\phi_{l,n}$. The map ϕ_s is an isomorphism, and it preserves the $\mathcal{O}$-points of the torus T and the normalizer of the tori N since $\rho_n \in N(F) = N_G(T(\mathcal{O}))$. It also preserves the Iwahori subgroup I_{n+1} since ρ_n is a fundamental alcove stabilizer. This concludes the proof. $\square$

The previous lemma shows that ψ_s induces a homomorphism of extended Weyl groups $\overline{\psi}_s : \widetilde{W}_{l+1} \to \widetilde{W}_{n+1}$. If s corresponds to a trail T_s on $\mathfrak{D}_\chi$ connecting the vertices i and $j = i + l \pmod{n+1}$ in $\mathfrak{D}_\chi$, we consider the character

$$\chi^{(s)} := \chi_i \otimes \chi_{i+1} \otimes \cdots \otimes \chi_{j-1} \otimes \chi_j$$

of $T(\mathcal{O}) \subset \mathrm{GL}_{l+1}(F)$, where subscripts are taken mod $n+1$. The following two results provide two fundamental properties of the character $\chi^{(s)}$.

Lemma 3.12. *The homomorphism $\overline{\psi}_s$ satisfies*

$$\overline{\psi}_s(\widetilde{W}_{\chi^{(s)}}) \subseteq \widetilde{W}_\chi \quad \textit{and} \quad \overline{\psi}_s(w_{\beta_0}(1)) = s,$$

where β_0 is the highest root of GL_{l+1}.

Proof. We note that $\psi_s = \phi_s \circ \phi_{l,n}$ and that both ϕ_s and $\phi_{l,n}$ also induce maps of Weyl groups such that $\overline{\psi}_s = \overline{\phi}_s \circ \overline{\phi}_{l,n}$. By definition of $\phi_{l,n}$, we can see that

$$\overline{\phi}_{l,n}(\widetilde{W}_{\chi^{(s)}}) \subseteq \widetilde{W}_{\chi'}, \quad \text{where} \quad \chi' = \chi^{(s)} \otimes \chi_{j+1} \otimes \cdots \otimes \chi_{i-1} = {}^{\rho_n^i}\chi$$

and that

$$\overline{\phi}_{l,n}(w_{\beta_0}(1)) = w_{\alpha_1+\cdots+\alpha_l}(1),$$

which corresponds to the trail in $\mathfrak{D}_\chi$ connecting 0 and l counterclockwise. On the other hand $\overline{\phi}_s(w) = \rho_n^{n+1-i} w \rho_n^{i-n-1}$ for any $w \in \widetilde{W}_{n+1}$, so

$$\overline{\phi}_s(\widetilde{W}_{\chi'}) = \widetilde{W}_{\rho_n^{-i}\chi'} = \widetilde{W}_\chi.$$

Moreover,

$$\overline{\phi}_s(w_{\alpha_1+\cdots+\alpha_l}(1)) = \rho_n^{n+1-i} w_{\alpha_1+\cdots+\alpha_l}(1) \rho_n^{i-n-1}$$

corresponds to the trail T_s in $\mathfrak{D}_\chi$ connecting i and $j = i + l \pmod{n+1}$ counterclockwise. Thus,

$$\overline{\psi}_s(w_{\beta_0}(1)) = \overline{\phi}_s(\overline{\phi}_{l,n}(w_{\beta_0}(1))) = \overline{\psi}_s(w_{\alpha_1+\cdots+\alpha_l}(1)) = s \qquad \text{and} \qquad \overline{\psi}_s(\widetilde{W}_{\chi^{(s)}}) \subseteq \overline{\phi}_s(\widetilde{W}_{\chi'}) = \widetilde{W}_\chi,$$

as desired. $\square$

Lemma 3.13. *For any $k \in I_{l+1} \subset \mathrm{GL}_{l+1}(F)$,*

$$\rho_\chi(\psi_s(k)) = \rho_{\chi^{(s)}}(k).$$

Proof. Fix some $k \in I_{l+1} \subset \mathrm{GL}_{l+1}(F)$ and let $\chi = \chi_0 \otimes \cdots \otimes \chi_n$. By definition of $\phi_{l,n}$, we have that

$$\rho_\chi(\phi_{l,n}(k)) = \rho_{\chi'}(k),$$

where $\chi' = \chi_0 \otimes \cdots \otimes \chi_l$ is obtained from χ by taking the characters of the first $l+1$ entries. Moreover, since conjugating by ρ_n permutes the diagonal entries according to (5), it follows that for any $h \in I \subset \mathrm{GL}_{n+1}$,

$$\rho_\chi(\rho_n^{-1} h \rho_n) = \rho_{\rho_n\chi}(h), \quad \text{where} \quad {}^{\rho_n}\chi = \chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0.$$

Therefore, by the construction of ϕ_s, we have that

$$\rho_\chi(\phi_s(h)) = \rho_{\rho_n^i \chi}(h), \quad \text{where} \quad {}^{\rho_n^i}\chi = \chi_i \otimes \chi_{i+1} \otimes \cdots \otimes \chi_{i-1}.$$

Putting everything together, we obtain

$$\rho_\chi(\psi_s(k)) = \rho_\chi(\phi_s(\phi_{l,n}(k))) = \rho_{\rho_n^i \chi}(\phi_{l,n}(k)) = \rho_{\chi^{(s)}}(k),$$

as desired. $\square$

We are finally ready to prove the existence of a surjective algebra homomorphism between Hecke algebras.

Proposition 3.14. *The homomorphism ψ_s induces a surjective C-linear map*

$$\Psi_s : \mathcal{H}(\mathrm{GL}_{n+1}, I_{n+1}, \rho_\chi) \longrightarrow \mathcal{H}(\mathrm{GL}_{l+1}, I_{l+1}, \rho_{\chi^{(s)}}),$$

where

$$\Psi_s(\varphi)(g) = \varphi(\psi_s(g)), \qquad \varphi \in \mathcal{H}(\mathrm{GL}_{n+1}, I_{n+1}, \rho_\chi),\ g \in \mathrm{GL}_{l+1}(F).$$

Furthermore, for every $w \in \widetilde{W}_{\chi^{(s)}}$,

$$\Psi_s([I_{n+1}\overline{\psi}_s(w)I_{n+1}]) = [I_{l+1}wI_{l+1}],$$

and the Ψ_s gives an isomorphism between $\mathcal{H}(\mathrm{GL}_{l+1}, I_{l+1}, \rho_{\chi^{(s)}})$ and the subalgebra of $\mathcal{H}(\mathrm{GL}_{n+1}, I_{n+1}, \rho_\chi)$ spanned by $\{[I_{n+1}\overline{\psi}_s(w)I_{n+1}] \mid w \in \widetilde{W}_{\chi^{(s)}}\}$.

Proof. For any $\varphi \in \mathcal{H}(\mathrm{GL}_{n+1}, I_{n+1}, \rho_\chi)$, we have that $\mathrm{supp}(\Psi_s(\varphi)) = \psi_s^{-1}(\mathrm{supp}(\varphi))$, a compact subset of $\mathrm{GL}_{l+1}(F)$ since ψ_s is a continuous map. Moreover, if $k_1, k_2 \in I \subset \mathrm{GL}_{l+1}(F)$ and $g \in \mathrm{GL}_{l+1}(F)$, then by Lemma 3.13

$$\Psi_s(\varphi)(k_1gk_2) = \varphi(\psi_s(k_1gk_2)) = \rho_\chi(\psi_s(k_1))\varphi(\psi_s(g))\rho_\chi(\psi_s(k_2)) = \rho_{\chi^{(s)}}(k_1)\Psi_s(g)\rho_{\chi^{(s)}}(k_2),$$

so $\Psi_s(\varphi) \in \mathcal{H}(\mathrm{GL}_{l+1}, I_{l+1}, \rho_{\chi^{(s)}})$.

Moreover, by Lemma 3.12, $\overline{\psi}_s(w) \in \widetilde{W}_\chi$ for any $w \in \widetilde{W}_{\chi^{(s)}}$ and since ψ_s is an injective homomorphism,

$$\Psi_s\left([I_{n+1}\overline{\psi}_s(w)I_{n+1}]\right)(g) = [I_{n+1}\overline{\psi}_s(w)I_{n+1}](\psi_s(g)) = \begin{cases} [I_{l+1}wI_{l+1}](g) & \text{if } g \in I_{l+1}wI_{l+1}, \\ 0 & \text{otherwise.} \end{cases} = [I_{l+1}wI_{l+1}](g)$$

This also implies that Ψ_s is a surjective C-linear map since $\{[I_{l+1}wI_{l+1}] \mid w \in \widetilde{W}_{\chi^{(s)}}\}$ is a C-basis of $\mathcal{H}(\mathrm{GL}_{l+1}, I_{l+1}, \rho_{\chi^{(s)}})$. Moreover, it is clear that Ψ_s gives an isomorphism of the two spaces as C-vector spaces, so it is an isomorphism as C-algebras if we show that

$$\Psi_s\left([I_{n+1}\overline{\psi}_s(w_1)I_{n+1}] * [I_{n+1}\overline{\psi}_s(w_2)I_{n+1}]\right) = \Psi_s\left([I_{n+1}\overline{\psi}_s(w_1)I_{n+1}]\right) * \Psi_s\left([I_{n+1}\overline{\psi}_s(w_2)I_{n+1}]\right) =$$
$$= [I_{l+1}w_1I_{l+1}] * [I_{l+1}w_2I_{l+1}]$$

for any $w_1, w_2 \in \widetilde{W}_{\chi^{(s)}}$. To simplify notation, let $\varphi_k = [I_{n+1}\overline{\psi}_s(w_k)I_{n+1}]$ and $v_k = \overline{\psi}_s(w_k)$ for $k = 1, 2$. By definition,

$$\Psi_s(\varphi_1 * \varphi_2)(g) = \int_{\psi(g)^{-1}I_{n+1}v_1I_{n+1}\cap I_{n+1}v_2^{-1}I_{n+1}} \varphi_1(\psi_s(g)A)\varphi_2(A^{-1})dA.$$

and a direct computation yields that if

$$\psi_s(g)^{-1}I_{n+1}v_1I_{n+1}\cap I_{n+1}v_2^{-1}I_{n+1} = \sqcup_{j\in J}h_jI_{n+1}, \quad \text{then} \quad g^{-1}I_{l+1}w_1I_{l+1}\cap I_{l+1}w_2^{-1}I_{l+1} = \sqcup_{j\in J}\psi_s^{-1}(h_jI_{n+1}),$$

none of which are empty (one can give an easy justification here; by parabolic subgroups stuff, v_2 and w_2 have the same length in their respective Weyl groups, and therefore each left I_{n+1} cosset of $I_{n+1}v_2^{-1}I_{n+1}$ has an element lying in the image of Ψ_s). Therefore, we may choose each h_j to be in the image of ψ_s, in which case $\psi_s^{-1}(h_jI_{n+1}) = \psi_s^{-1}(h_j)I_{l+1}$. Hence,

$$\begin{aligned}\Psi_s(\varphi_1 * \varphi_2)(g) &= \sum_{j\in J}\int_{h_jI_{n+1}} \varphi_1(\psi_s(g)A)\varphi_2(A^{-1})dA \\ &= \sum_{j\in J}\int_{I_{n+1}} \varphi_1(\psi_s(g)h_jK)\varphi_2(K^{-1}h_j^{-1})dK = \sum_{j\in J}\varphi_1(\psi_s(g)h_j)\varphi_2(h_j^{-1})\end{aligned}$$

while

$$\Psi_s(\varphi_1) * \Psi_s(\varphi_2)(g) = \sum_{j\in J}\int_{\psi_s^{-1}(h_j)I_{l+1}} \varphi_1(\psi_s(gB))\varphi_2(\psi_s(B))dB$$

$$= \sum_{j\in J}\int_{I_{l+1}} \varphi_1(\psi_s(g)h_j\psi_s(K))\varphi_2(\psi_s(K)^{-1}h_j^{-1})dK = \sum_{j\in J}\varphi_1(\psi_s(g)h_j)\varphi_2(h_j^{-1}),$$

and this concludes the proof. □

With the previous proposition, we now have all the ingredients to prove the quadratic relation for $\varphi_s = q^{(1-l(s))/2}[I_{n+1}sI_{n+1}]_{\check{\chi}} \in \mathcal{H}(\mathrm{GL}_{n+1})$.

Theorem 3.15. *Let $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ be a depth-zero character of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}$ and let $s \in S_{\chi,\mathrm{aff}}$. If*

$$\varphi_s = q^{\frac{1-l(s)}{2}}[I_{n+1}sI_{n+1}]_{\check{\chi}} \in \mathcal{H}(\mathrm{GL}_{n+1}, I_{n+1}, \rho_\chi),$$

then

$$\varphi_s^2 = (q-1)\varphi_s + q\varphi_1.$$

Proof. Consider the linear map $\Psi_s : \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi) \longrightarrow \mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}})$, and by Proposition 3.14 we have that

$$\begin{aligned}\Psi_s(\varphi_s) = q^{\frac{(1-l(s))}{2}}\Psi_s([I_{n+1}sI_{n+1}]_{\check{\chi}}) &= q^{\frac{(1-l(s))}{2}}\Psi_s([I_{n+1}\overline{\psi}_s(w_{\beta_0}(1))I_{n+1}]_{\check{\chi}}) \\ &= q^{\frac{(1-l(w_{\beta_0}(1)))}{2}}[I_{l+1}w_{\beta_0}(1)I_{l+1}] = \varphi_{w_{\beta_0}(1)} \in \mathcal{H}(\mathrm{GL}_{l+1}, I_{n+1}, \rho_{\chi^{(s)}}).\end{aligned}$$

By the results on the previous section, we know that $\varphi_{w_{\beta_0}(1)}$ satisfies the desired quadratic relation. Moreover, since $s = \overline{\psi}_s(w_{\beta_0}(1))$, we have that φ_s lies in the subalgebra of $\mathcal{H}(\mathrm{GL}_{n+1}, I_{n+1}, \rho_\chi)$ defined in the previous proposition, isomorphic to $\mathcal{H}(\mathrm{GL}_{l+1}, I_{l+1}, \rho_{\chi^{(s)}})$ under Ψ_s. Therefore, φ_s satisfies the quadratic relation

$$\varphi_s^2 = (q-1)\varphi_s + q\varphi_1$$

and this concludes the proof. □