Quadratic Relations in Hecke Algebras

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May 29, 2025

1 Hecke algebras in positive characteristic

When the field C of values of the representation has positive characteristic, then many of the previous results do not hold. For instance, the category $\mathcal{R}_C(G)$ does not have a nice block decomposition similar to the Bernstein decomposition for complex smooth representations. As a consequence, there is no analogous theory of types for positive characteristic. Nevertheless, for the setting of depth-zero representations, a bijection is still expected between simple right $\mathcal{H}(G, I, \rho_\chi)$-modules and irreducible C-representations (π, V) of $G(F)$ such that the ρ_χ-isotypic part of $\pi|_I$ is non-zero.

This last point suggests that studying the structure of $\mathcal{H}(G, I, \rho_\chi)$ yields relevant information about the structure of $\mathcal{R}_C(G)$. Fortunately, most steps in the proof of the theorem still hold if we replace $\mathbb{C}$ for an algebraically closed field of positive characteristic other than p. It is still true that the elements $\varphi_w, w \in \tilde{W}_\chi$ still form a basis for $\mathcal{H}(G, I, \rho_\chi)$, that $\varphi_{w_1 w_2} = \varphi_{w_1} * \varphi_{w_2}$ whenever $l_\chi(w_1 w_2) = l_\chi(w_1) + l_\chi(w_2)$ and that the elements $\varphi_s, s \in S_{\chi,\mathrm{aff}}$ form a two dimensional subalgebra inside $\mathcal{H}(G, I, \chi_\rho)$. However, the calculations for the coefficients for the quadratic relations rely crucially on semisimplicity of certain representations and character theory, both of which fail for small positive characteristic.

The aim of this document is prove in a direct, explicit way, that the same quadratic relations hold even when the characteristic is small when G is a reductive group of type A_n.

The first part of the proof can be easily deduced from the following fact.

Lemma 1.1. *Let $w \in \tilde{W}_\chi$. Then*

$$[I : I \cap wIw^{-1}] = q^{l(w)}.$$

Lemma 1.2. *Let $s \in S_{\chi,\mathrm{aff}}$. Then*

$$\varphi_s^2(1) = q.$$

Proof. This is a direct computation. Indeed,

$$\varphi_s^2(1) = \int_{IsI} \varphi_s(h)\varphi_s(h^{-1})dh = \chi(s^2)^{-1}\varphi_s(s)^2[IsI : I] = q^{1-l(s)}[I : I \cap sIs^{-1}] = q.$$

□

Then, the hard part of the proof is to show that $\varphi_s^2(s) = (q-1)\varphi_s(s) = (q-1)q^{1-l(s)}\check{\chi}(s)$ if $s \in S_{\chi,\mathrm{aff}}$. The first step for proving this result consists in

1.1 Quadratic relation for a canonical example

We now prove the quadratic relation for a family of reflections containing one reflection for each possible length. We remark that any $s \in S_{\chi,\mathrm{aff}}$ is a reflection and, as such, $l(s)$ will always be odd.

Let $n \geq 1$ be a positive integer and let $G = \mathrm{GL}_{n+1}$. Choose a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ with longest root $\alpha_0 = \alpha_1 + \cdots + \alpha_n$. In this case, the Weyl group $W = N(F)/T(F)$ is generated by $S = \{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$ and the extended Weyl group $\tilde{W} = N(F)/T(\mathcal{O})$ decomposes as a semidirect product of the affine Weyl group $W_{\mathrm{aff}} = N(F)|_{\det \in \mathcal{O}^\times}/T(\mathcal{O})$ and the alcove stabilizer $\Omega = \langle \sigma \rangle \cong \mathbb{Z}$, where

$$\rho = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\varpi & & & 0 \end{pmatrix}.$$

As discussed above, the affine Weyl group is a coxeter group, where the simple reflections are given by

$$S_{\mathrm{aff}} = \{w_{\alpha_1}(1), \ldots, w_{\alpha_n}(1), w_{\alpha_0}(\varpi^{-1})\}.$$

Let $\chi = \chi_0 \otimes \chi_1 \otimes \cdots \otimes \chi_n$ be a character of the torus, where each χ_i is a depth-zero character of $\mathcal{O}^\times$. Assume that $\chi_0 = \chi_n$ and with all others χ_i distinct characters. Under these assumptions,

$$N_\chi = \begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix} \bigsqcup \begin{pmatrix} & & & * \\ & * & & \\ & & \ddots & \\ & & & * & \\ * & & & \end{pmatrix},$$

and in particular

$$S_{\chi,\mathrm{aff}} = \left\{ w_{\alpha_0}(1) = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ 1 & & & \end{pmatrix}, w_{\alpha_0}(\varpi^{-1}) = \begin{pmatrix} & & & \varpi^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ -\varpi & & & \end{pmatrix} \right\}$$

Lemma 1.3. *Under the circumstances discussed above, we have that* $l(w_{\alpha_0}(\varpi^{-1})) = 1$ *and* $l(w_{\alpha_0}(1)) = 2n-1$.

Proof. The first part of the statement is immediate since $w_{\alpha_0} \in S_{\mathrm{aff}}$ is a simple reflection. For the second part, we first note that $w_{\alpha_0}(1)$ is naturally an element of the Weyl group W, generated by $S = \{w_{\alpha_1}, \ldots, w_{\alpha_n}\}$, and therefore $l(\alpha_0(1))$ coincides with its length as an element of W. Thus

$$l(w_{\alpha_0}) = |\{\alpha \in \Phi^+ : w_{\alpha_0}(\alpha) \in \Phi^-\}|.$$

Now, if $\alpha \in \Phi^+$, then $w_{\alpha_0}(\alpha) = \alpha - \langle \alpha, \check{\alpha}_0 \rangle \alpha_0 \in \Phi^-$if and only if $\langle \alpha \check{\alpha}_0 \rangle > 0$. Since $\langle \alpha_i, \check{\alpha}_0 \rangle = 1$ if $i = 1, n$ and 0 otherwise. Consequently,

$$\{\alpha \in \Phi^+ : w_{\alpha_0}(\alpha) \in \Phi^-\} = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_0, \ldots, \alpha_{n-1} + \alpha_n, \alpha_n\},$$

which has size $2n - 1$ as desired. $\square$

The proof of the quadratic equation for $s_0 = w_{\alpha_0}(\varpi^{-1})$ is straightforward. For the sake of completeness, and as means of example, we give a full proof.

Since $l(s_0) = 1$, we consider $\varphi_{s_0} = [Is_0I]_{\check{X}}$. Then

$$\varphi_{s_0}^2(s_0) = \int_{G(F)} \varphi_{s_0}(s_0 h) \varphi_{s_0}(h^{-1}) dh$$

and the integral is zero unless $h \in Is_0I \cap s_0Is_0I$. To understand this integral, we note that for any $x \in G(F)$, there is a bijection

$$IxI/I \longleftrightarrow I/(I \cap xIx^{-1})$$
$$yxI \longmapsto y(I \cap xIx^{-1})$$

and that

$$I \cap s_0 I s_0^{-1} = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times & \ddots & & \mathcal{O} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \ddots & \mathcal{O}^\times & \mathcal{O} \\ \varpi^2\mathcal{O} & \varpi\mathcal{O} & \ldots & \varpi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}.$$

Therefore, by fixing a section $t : k_F \to \mathcal{O}, a \mapsto a_t$ of the quotient $\mathcal{O} \to k_F$, we obtain

$$Is_0I = \bigsqcup_{c \in k_F} u_c s_0 I, \quad \text{where} \quad u_c = \begin{pmatrix} 1 & & \\ & 1 & \\ \varpi c_t & & 1 \end{pmatrix}.$$

A left cosset $u_c s_0 I$ is contained in $s_0 I s_0 I$ if and only if there is some $z \in k_F$ such that $s_0 u_c s_0 u_z c_0 \in I$, and a simple calculation shows that this is the case if and only if $c \neq 0$, in which case $z = c^{-1}$. Hence, $Is_0I \cap s_0Is_0I = \sqcup_{c \in k_F^\times} u_c s_0 I$ and moreover

$$\iota_c := s_0 u_c s_0 u_{1/c} s_0 \in I \quad \text{satisfies} \quad \iota_c \equiv \begin{pmatrix} -c & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1/c \end{pmatrix} \pmod{I^+},$$

so $\rho_\chi(\iota_c) = \chi_0(-c)\chi_n(-1/c) = 1$ since $\chi_0 = \chi_n$. Consequently,

$$\varphi_{s_0}^2(s_0) = \sum_{c\in k_F^\times} \int_{u_c s_0 I} \varphi_{s_0}(sh)\varphi_{s_0}(h^{-1})dh = \sum_{c\in k_F^\times} \int_I \varphi_{s_0}(su_c sk)\varphi_{s_0}(k^{-1}s_0^{-1}u_c^{-1})dk =$$

$$= \sum_{c\in k_F^\times} \varphi_{s_0}(\iota_c s_0^{-1}u_{1/c})\varphi_{s_0}(s_0^{-1}u_c^{-1}) = \varphi_{s_0}(s_0)^2 \sum_{c\in k_F^\times} \rho_\chi(\iota_c)\rho_\chi(u_{1/c})^{-1}\rho_\chi(u_c)^{-1} = |k_F^\times| = q-1,$$

and this completes the proof that $\varphi_{s_0}^2 = (q-1)\varphi_{s_0} + q\varphi_1$.

Next, we turn out attention towards proving the quadratic relation for $s = w_{\alpha_0}(1) \in S_{\chi,\mathrm{aff}}$. In Lemma 1.3 we proved that $l(s) = 2n-1$ and that

$$\{\alpha \in \Phi^+ : w_{\alpha_0}(\alpha) \in \Phi^-\} = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_0, \ldots, \alpha_{n-1}+\alpha_n, \alpha_n\},$$

which implies that

$$I \cap sIs^{-1} = \begin{pmatrix} \mathcal{O}^\times & \varpi\mathcal{O} & \varpi\mathcal{O} & \cdots & \varpi\mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \ldots & \varpi\mathcal{O} & \mathcal{O}^\times & \varpi\mathcal{O} \\ \varpi\mathcal{O} & \varpi\mathcal{O} & \ldots & \varpi\mathcal{O} & \varpi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}$$

and therefore we can express the double cosset IsI as a disjoint union of left cossets of I as

$$IsI = \bigsqcup_{(\mathbf{a},\mathbf{b},c)\in k_F^{2n-1}} u_{\mathbf{a},\mathbf{b},c}sI \quad \text{where} \quad u_{\mathbf{a},\mathbf{b},c} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} & c \\ 0 & 1 & 0 & \cdots & 0 & b_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & b_2 \\ \vdots & & & \ddots & 1 & b_1 \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

Here, we are abusing notation, and all elements of the matrix are assumed to be the lifts from k_F to $\mathcal{O}$ by the section t fixed above.

Lemma 1.4. *Let $(\mathbf{a},\mathbf{b},c) \in k_F^{2n-1}$. Then there is some $(\mathbf{x},\mathbf{y},z) \in k_F^{2n-1}$ such that*

$$su_{\mathbf{a},\mathbf{b},c}su_{\mathbf{x},\mathbf{y},z}s \in I$$

if and only if

$$c(c - \sum_{i+j=n} a_ib_j) \neq 0 \quad \textit{and} \quad a_ib_j = 0 \textit{ if } i+j<n.$$

When these conditions hold, then

$$\mathbf{x} = \mathbf{a}c^{-1}, \quad \mathbf{y} = \mathbf{b}(c - \sum_{i+j=n} a_ib_j)^{-1} \quad \textit{and} \quad z = (c - \sum_{i+j=n} a_ib_j)^{-1}$$

and

$$\iota_{\mathbf{a},\mathbf{b},c} := s u_{\mathbf{a},\mathbf{b},c} s u_{\mathbf{x},\mathbf{y},z} s \in I \quad \textit{satisfies} \quad \iota_{\mathbf{a},\mathbf{b},c} \equiv \begin{pmatrix} \frac{1}{c-\sum_{i+j=n} a_i b_j} & & & & \\ & \frac{c-a_1 b_{n-1}}{c} & & & \\ & & \ddots & & \\ & & & \frac{c-a_{n-1}b_1}{c} & \\ & & & & c \end{pmatrix} \pmod{I^+}. \tag{1}$$

To simplify notation, we let

$$J_n = \{(\mathbf{a},\mathbf{b},c) \in k_F^{2n-1} \mid c(c - \sum_{i+j=n} a_i b_j) \neq 0 \quad \text{and} \quad a_i b_j = 0 \text{ if } i+j<n\}.$$

Corollary 1.5. *We have that*

$$IsI \cap sIsI = \bigsqcup_{(\mathbf{a},\mathbf{b},c)\in J_n} u_{\mathbf{a},\mathbf{b},c} sI$$

Now, we let $\varphi_s = q^{(1-l(s))/2}[IsI] - \check{\chi} = q^{1-n}[IsI]_{\check{\chi}}$, and using the previous results, we obtain

$$\varphi_s^2(s) = \int_{IsI\cap sIsI} \varphi_s(sh)\varphi_s(h^{-1})dh = \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \int_{u_{\mathbf{a},\mathbf{b},c}sI} \varphi_s(sh)\varphi_s(h^{-1})dh =$$

$$= \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \int_I \varphi_s(s u_{\mathbf{a},\mathbf{b},c} s k)\varphi_s(k^{-1}s^{-1}u_{\mathbf{a},\mathbf{b},c}^{-1})dk = \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \varphi_s(\iota_{\mathbf{a},\mathbf{b},c} s^{-1} u_{\mathbf{x},\mathbf{y},z})\varphi_s(s^{-1}u_{\mathbf{a},\mathbf{b},c}^{-1}) =$$

$$= \rho_\chi(s^{-2})^2 \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c})\varphi_s(s)\rho_\chi(u_{\mathbf{x},\mathbf{y},z})^{-1}\varphi_s(s)\rho_\chi(u_{\mathbf{a},\mathbf{b},c})^{-1} = q^{2-2n} \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}).$$

To calculate $\varphi_s^2(s)$, it is therefore enough to compute the sum

$$R_n(q, \{\chi_0, \dots, \chi_{n-1}\}) = \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}) =$$

$$= \sum_{(\mathbf{a},\mathbf{b},c)\in J_n} \chi_0\left(\frac{c}{c-\sum_{i+j=n} a_i b_j}\right) \chi_1\left(\frac{c-a_1 b_{n-1}}{c}\right) \cdots \chi_{n-1}\left(\frac{c-a_{n-1}b_1}{c}\right).$$

Proposition 1.6. *For $n \geq 3$ and $\chi_0 \notin \{\chi_1, \dots, \chi_{n-1}\}$, the sums $R_n(q, \{\chi_0, \dots, \chi_{n-1}\})$ satisfy the recurrence relation*

$$R_n(q,\{\chi_0,\dots,\chi_{n-1}\}) = qR_{n-1}(q,\{\chi_0,\chi_2,\dots,\chi_{n-1}\}) + qR_{n-1}(q,\{\chi_0,\dots,\chi_{n-2}\}) - q^2 R_{n-2}(q,\{\chi_0,\chi_2,\dots,\chi_{n-2}\}).$$

To prove this proposition, we will need the following lemma.

Lemma 1.7. *There is a bijection between the sets $\{(\mathbf{a},\mathbf{b},c) \in J_n : a_1 = 0\}$ and $k_F \times J_{n-1}$ given by*

$$((a_1,\dots,a_{n-1}),(b_1,\dots,b_{n-1}),c) \longmapsto (b_{n-1},(a_2,\dots,a_{n-1}),(b_1,\dots,b_{n-2}),c).$$

Analogously, there is a bijection between the sets $\{(\mathbf{a},\mathbf{b},c) \in J_n : b_1 = 0\}$ and $k_F \times J_{n-1}$.

Proof. This is a direct computation. If $a_1 = 0$, then b_{n-1} is a free variable and if we let $a'_i = a_{i+1}$ for $i = 1, \dots, n-2$, then the condition $c(c - \sum_{i+j=n} a_i b_j) \neq 0$ becomes $c(c - \sum_{i+j=n-1} a'_i b_j) \neq 0$ and the condition $a_i b_j = 0$ for $i + j < n$ becomes $a_i b_j < n - 1$, as desired. □

Proof of Proposition 1.6. Firstly, we decompose the sum $R_n(q, \{\chi_0, \dots, \chi_{n-1}\})$ into

$$R_n(q, \{\chi_0, \dots, \chi_{n-1}\}) = R_n^{a_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) + R_n^{b_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) - R_n^{a_1,b_1}(q, \{\chi_0, \dots, \chi_{n-1}\}), \qquad (2)$$

where

$$R_n^{a_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ a_1=0}} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}), \quad R_n^{b_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ b_1=0}} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c})$$

$$\text{and} \quad R_n^{a_1,b_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ a_1=b_1=0}} \rho_\chi(\iota_{\mathbf{a},\mathbf{b},c}).$$

Next, we can compute each individual term from (2). If $a_1 = 0$, we note that b_{n-1} is completely free, and if we let $\mathbf{a}' = (a_2, \dots, a_{n-1})$ and $\mathbf{b}' = (b_1, \dots, b_{n-2})$, by Lemma 1.7 we obtain

$$R_n^{a_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ a_1=0}} \chi_0\left(\frac{c}{c - \sum_{i+j=n} a_i b_j}\right) \chi_2\left(\frac{c - a_2 b_{n-2}}{c}\right) \cdots \chi_{n-1}\left(\frac{c - a_{n-1} b_1}{c}\right) =$$

$$= q \sum_{(\mathbf{a}',\mathbf{b}',c)\in J_{n-1}} \chi_0\left(\frac{c}{c - \sum_{i+j=n-1} a'_i b'_j}\right) \chi_2\left(\frac{c - a'_1 b'_{n-2}}{c}\right) \cdots \chi_{n-1}\left(\frac{c - a'_{n-2} b'_1}{c}\right) = qR_{n-1}(q, \{\chi_0, \chi_2 \dots, \chi_{n-1}\}).$$

An analogous calculation shows that $R_n^{b_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) = qR_{n-1}(q, , \{\chi_0, \dots, \chi_{n-2}\})$. Finally, applying Lemma 1.7 twice gives a bijection

$$\begin{aligned} \{(\mathbf{a}, \mathbf{b}, c) \in J_n : a_1 = b_1 = 0\} &\longrightarrow k_F^2 \times J_{n-2} \\ ((a_1, \dots, a_{n-1}), (b_1, \dots, b_{n-1}), c) &\longmapsto (a_{n-1}, b_{n-1}, (a_2, \dots, a_{n-2}), (b_2, \dots, b_{n-2}), c) \end{aligned}$$

and if we let $\mathbf{a}' = (a_2, \dots, a_{n-2})$ and $\mathbf{b}' = (b_2, \dots, b_{n-2})$, then

$$R_n^{a_1,b_1}(q, \{\chi_0, \dots, \chi_{n-1}\}) = \sum_{\substack{(\mathbf{a},\mathbf{b},c)\in J_n \\ a_1=b_1=0}} \chi_0\left(\frac{c}{c - \sum_{i+j=n} a_i b_j}\right) \chi_2\left(\frac{c - a_2 b_{n-2}}{c}\right) \cdots \chi_{n-2}\left(\frac{c - a_{n-2} b_2}{c}\right) =$$

$$= q^2 \sum_{(\mathbf{a}',\mathbf{b}',c)\in J_{n-2}} \chi_0\left(\frac{c}{c - \sum_{i+j=n-2} a'_i b'_j}\right) \chi_2\left(\frac{c - a'_1 b'_{n-3}}{c}\right) \cdots \chi_{n-1}\left(\frac{c - a'_{n-3} b'_1}{c}\right) = q^2 R_{n-2}(q, \{\chi_0, \chi_2 \dots, \chi_{n-2}\}).$$

Putting everything together yields the desired recurrence relation. $\square$

Naturally, the next step is to compute these sums for small values of n in order to use the above recurrence relation.

Proposition 1.8. *For any two depth-zero characters χ_0, χ_1 of $\mathcal{O}^\times$ such that $\chi_0 \neq \chi_1$, we have that*

$$R_1(q, \{\chi_0\}) = q - 1 \quad \text{and} \quad R_2(q, \{\chi_0, \chi_1\}) = (q - 1)q. \qquad (3)$$

Moreover, the sums $R_n(q) := R_n(q, \{\chi_0, \dots, \chi_{n-1}\})$ are independent of $\{\chi_0, \dots, \chi_{n-1}\}$ and they satisfy

$$R_n(q) = (q - 1)q^{n-1}.$$

Proof. The values of $R_1(q,\{\chi_0\})$ and $R_2(q,\{\chi_0,\chi_1\})$ can be obtain by direct computation. Indeed,

$$R_1(q,\chi_0)=\sum_{c\in k_F^\times}\chi_0(c/c)=q-1,$$

and

$$\begin{aligned}
R_2(q,\{\chi_0,\chi_1\}) &= \sum_{(a,b,c)\in J_2}\chi_0\left(\frac{c}{c-ab}\right)\chi_1\left(\frac{c-ab}{c}\right)=\sum_{\substack{(a,b,c)\in J_2\\ ab=0}}\chi_0(1)\chi_1(1)+\sum_{\substack{(a,b,c)\in J_2\\ ab\neq 0}}\chi_0\chi_1^{-1}\left(\frac{c}{c-ab}\right)=\\
&=(2q-1)(q-1)+(q-1)\sum_{\substack{c,d\in k_F^\times\\ c\neq d}}\chi_0\chi_1^{-1}\left(\frac{c}{c-d}\right)=(2q-1)(q-1)+(q-1)^2\sum_{x\in k_F^\times\setminus\{1\}}\chi_0\chi_1^{-1}(x)=\\
&=(2q-1)(q-1)-(q-1)^2=(q-1)q,
\end{aligned}$$

as desired. This result, together with the recurrence relation from Proposition 1.6, inductively shows that the sums $R_n(q,\{\chi_0,\ldots,\chi_{n-1}\})$ are independent of $\{\chi_0,\ldots,\chi_{n-1}\}$, which we shall simply write as $R_n(q)$. Combining Proposition 1.6 with (3), we have that $R_n(q)$ satisfies the recurrence relation

$$\begin{cases} R_n(q)=2qR_{n-1}(q)-q^2R_{n-2}(q) & \text{for } n\geq 3,\\ R_1(q)=q-1 \quad \text{and} \quad R_2(q)=(q-1)q, \end{cases}$$

whose solution is $R_n(q)=(q-1)q^{n-1}$. □

Remark 1.9. From Corollary 1.5, one can see that $P_n(q):=[IsI\cap sIsI:I]=|J_n|$ is a constant of interest directly related to $R_n(q)$. In fact, following a similar argument as above, one can show that the sequence $P_n(q)$ satisfies

$$\begin{cases} P_n(q)=2qR_{n-1}(q)-q^2R_{n-2}(q) & \text{for } n\geq 3,\\ P_1(q)=q-1 \quad \text{and} \quad P_2(q)=(q-1)(q^2-q+1), \end{cases}$$

whose solution is $P_n(q)=q^{n-2}(q-1)((n-1)q^2-(2n-3)q+(n-1))$.

Now the proof of the quadratic relation for $s=w_{\alpha_0}(1)$ is immediate. Indeed,

$$\varphi_s^2(s)=q^{2-2n}R_n(q)=(q-1)q^{1-n}=(q-1)\varphi_s(s),$$

which implies that

$$\varphi_s^2=(q-1)\varphi_s+q\varphi_1,$$

as desired.

2 The quadratic relation in general

In this section we show that for any character $\chi = \chi_0 \otimes \ldots \otimes \chi_n$ of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}(F)$ and $s \in S_{\chi,\mathrm{aff}}$, the proof of the quadratic relation for $\varphi_s = q^{(1-l(s))/2}[IsI]_{\check{\chi}}$ can be deduced from the particular case proven in the previous section. Let Φ be the root system for GL_{n+1} with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. For each $w \in \widetilde{W}$, let

$$\widetilde{\Delta}(w) = \{\text{hyperplanes } P \mid P \text{ separates } \mathcal{D}_0 \text{ and } w\mathcal{D}_0\}.$$

Recall from Iwahori-Matsumoto that $l(w) = |\widetilde{\Delta}(w)|$ for all $w \in \widetilde{W}$.

Lemma 2.1. *Let $\alpha = \sum_{k=i}^{j} \alpha_k \in \Phi^+$ for some $i \leq j$. Then*

$$\begin{aligned}\widetilde{\Delta}(w_\alpha(1)) &= \{P_{\beta,0} \mid \beta \in \Phi^+ \cap w_\alpha^{-1}\Phi^-\} \\ &= \{P_{\alpha_i,0}, P_{\alpha_i+\alpha_{i+1},0}, \ldots, P_{\alpha,0}, \ldots, P_{\alpha_{j-1}+\alpha_j,0}, P_{\alpha_j,0}\}.\end{aligned}$$

Proof. Let $x \in \mathcal{D}_0$ and let $\beta \in \Phi^+$. Then

$$\langle \beta, w_\alpha(1)(x)\rangle = \langle \beta, x\rangle - \langle \alpha, x\rangle\langle \beta, \check{\alpha}\rangle = \langle w_\alpha(\beta), x\rangle \in \begin{cases} (0,1) & \text{if } w_\alpha(\beta) \in \Phi^+, \\ (-1,0) & \text{if } w_\alpha(\beta) \in \Phi^-. \end{cases}$$

Hence, $P \in \widetilde{\Delta}(w)$ if and only if $P = P_{\beta,0}$ for some $\beta \in \Phi^+ \cap w_\alpha^{-1}\Phi^{-1}$. Since $w_\alpha(\beta) = \beta - \langle \beta, \check{\alpha}\rangle\alpha$, it follows that $\beta \in \Phi^+ \cap w_\alpha^{-1}\Phi^-$ if and only if $\beta = \alpha$ or $\langle \beta, \check{\alpha}\rangle = 1$ and $\alpha - \beta \in \Phi^+$.

If $\alpha = \alpha_i + \cdots + \alpha_j$ for some $j \geq i$, then a standard calculation shows that $\beta \in \Phi^+$ satisfies the above conditions if and only if $\beta = \alpha_i + \cdots + \alpha_k$ for some $k \leq j$ or $\beta = \alpha_k + \cdots + \alpha_j$ for some $k \geq i$. This concludes the proof. $\square$

Lemma 2.2. *Let $\alpha = \sum_{k=i}^{j} \alpha_k \in \Phi^+$ for some $i \leq j$. Then*

$$\begin{aligned}\widetilde{\Delta}(w_\alpha(\varpi^{-1})) &= \{P_{\alpha_1+\cdots+\alpha_{i-1},0}, \ldots, P_{\alpha_{i-1},0}, P_{\alpha_{j+1},0}, \ldots, P_{\alpha_{j+1}+\cdots+\alpha_n,0}\} \\ &\quad \cup \{P_{\alpha_1+\cdots+\alpha_j,1}, \ldots, P_{\alpha_i+\cdots+\alpha_j,1}, \ldots, P_{\alpha_i+\cdots+\alpha_n,1}\}.\end{aligned}$$

Proof. As in the previous proof, let $x \in \mathcal{D}_0$ and let $\beta \in \Phi^+$. Then,

$$\begin{aligned}\langle \beta, w_\alpha(\varpi^{-1})(x)\rangle &= \langle \beta, x\rangle - \langle \alpha, x\rangle\langle \beta, \check{\alpha}\rangle + \langle \beta, \check{\alpha}\rangle = \langle w_\alpha(\beta), x\rangle + \langle \beta, \check{\alpha}\rangle \\ &\in \begin{cases} (1,2) & \text{if } \beta = \alpha, \text{ or } \langle \beta, \check{\alpha}\rangle = 1 \text{ and } w_\alpha(\beta) \in \Phi^+, \\ (0,1) & \text{if } \langle \beta, \check{\alpha}\rangle = 0, \text{ or } \langle \beta, \check{\alpha}\rangle = 1 \text{ and } w_\alpha(\beta) \in \Phi^-, \\ (-1,0) & \text{if } \langle \beta, \check{\alpha}\rangle = -1. \end{cases}\end{aligned}$$

Thus,

$$\widetilde{\Delta}(w_\alpha(\varpi^{-1})) = \{P_{\alpha,1}\} \cup \{P_{\beta,1} \mid \beta \in \Phi^+, \langle \beta, \check{\alpha}\rangle = 1 \text{ and } w_\alpha(\beta) \in \Phi^+\} \cup \{P_{\beta,0} \mid \beta \in \Phi^+ \text{ and } \langle \beta, \check{\alpha}\rangle = -1\}.$$

If $\alpha = \alpha_i + \cdots + \alpha_j$, then further calculations show that $\beta \in \Phi^+$ satisfies $\langle \beta, \check{\alpha}\rangle = 1$ with $w_\alpha(\beta) = \beta - \alpha \in \Phi^+$ if and only if $\beta = \alpha_k + \cdots + \alpha_j$ for some $k < i$ or $\beta = \alpha_i + \cdots + \alpha_k$ for some $k > j$. Similarly, one can show that $\beta \in \Phi^+$ satisfies $\langle \beta, \check{\alpha}\rangle = -1$ if and only if $\beta = \alpha_k + \cdots + \alpha_{i-1}$ for some $k \leq i-1$ or $\beta = \alpha_{j+1} + \cdots + \alpha_k$ for some $k \geq j+1$. $\square$

Corollary 2.3. *For any $\alpha \in \Phi^+$, we have that*

$$l(w_\alpha(1)) + l(w_\alpha(\varpi^{-1})) = \sum_{\beta \in \Phi^+} |\langle \beta, \check{\alpha} \rangle| = 2n$$

Proof. The first equality is an immediate consequence of $l(w) = \widetilde{\Delta}(w)$ together with Lemmas 2.1 and 2.2. Indeed, if $\langle \beta, \check{\alpha} \rangle = 0$ then no hyperplane arising from β lies in $\widetilde{\Delta}(w_\alpha(1)) \cup \widetilde{\Delta}(w_\alpha(\varpi^{-1}))$; if $\langle \beta, \check{\alpha} \rangle = \pm 1$, then exactly one plane arising from β lies $\widetilde{\Delta}(w_\alpha(1)) \cup \widetilde{\Delta}(w_\alpha(\varpi^{-1}))$; finally, $P_{\alpha,0} \in \widetilde{\Delta}(w_\alpha(1))$ and $P_{\alpha,1} \in \widetilde{\Delta}(w_\alpha(\varpi^{-1}))$.

The second part of the lemma is a direct computation: if $\alpha = \alpha_i \cdots \alpha_j$, then there are

- j roots in Φ^+ of the form $\alpha_k + \cdots + \alpha_j$ for $k \leq j$,
- $n+1-i$ roots in Φ^+ of the form $\alpha_i + \cdots + \alpha_k$ for $k \geq i$,
- $i-1$ roots in Φ^+ of the form $\alpha_k + \cdots + \alpha_{i-1}$ for $k \leq i-1$,
- $n+1-(j+1)$ roots in Φ^+ of the form $\alpha_{j+1} + \cdots + \alpha_k$ for $k \geq j+1$.

Giving a total of $2n$ roots, each of which contributes once towards $\sum_{\beta \in \Phi^+} |\langle \beta, \check{\alpha} \rangle|$, as desired. □

2.1 The diagram associated to a character

Having established the preliminary lemmas, we now consider a depth zero character $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}(F)$, where each χ_i is a depth-zero character of $\mathcal{O}^\times$. The character χ induces a partition $\mathcal{P}_\chi$ of the set $\{0, 1, \ldots, n\}$ according to which two elements $i, j \in \{0, 1, \ldots, n\}$ are related if and only if $\chi_i = \chi_j$ as characters of $\mathcal{O}^\times$.

Definition 2.4. Let χ be a depth-zero character of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}(F)$ giving rise to a partition $\mathcal{P}_\chi$ of $\{0, 1, \ldots, n\}$ of size r. Then the diagram $\mathfrak{D}_\chi$ associated to χ is a regular $(n+1)$-gon whose vertices are labelled $\{0, 1, \ldots, n\}$ counterclockwise and have been painted in r different colors according to the partition $\mathcal{P}_\chi$, and having a distinguished edge $e = \{0, n\}$.

Example 2.5. In the previous section we considered the family of characters $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ such that $\chi_0 = \chi_n$ and all the remaining characters are distinct. Then $\mathcal{P}_\chi = \{\{0, n\}, \{1\}, \ldots, \{n-1\}\}$ has size n, and Figure 1 shows the diagram with $n = 4$.

The diagram of a character defined above is useful because it will allow us to determine $S_{\chi,\mathrm{aff}}$ easily. The first step towards this goal is to associate elements of $\widetilde{W}_\chi$ to paths in the diagram.

Definition 2.6. Let $\mathfrak{D}_\chi$ be the diagram associated to χ and let $i, j \in \{0, 1, \ldots, n\}$ be two *distinct* vertices, and we assume that $i < j$. Let

$$\alpha_{i,j} = \alpha_{i+1} + \cdots + \alpha_{j-1} + \alpha_j \in \Phi^+$$

be the positive root associated to the pair (i, j). Let $T_{i,j,1}$, (resp. $T_{i,j,0}$) be the trail in $\mathfrak{D}_\chi$ from i to j avoiding (resp. passing through) the distinguished edge $e = \{0, n\}$. Then the associated reflection $w(T_{i,j,k})$ to the path

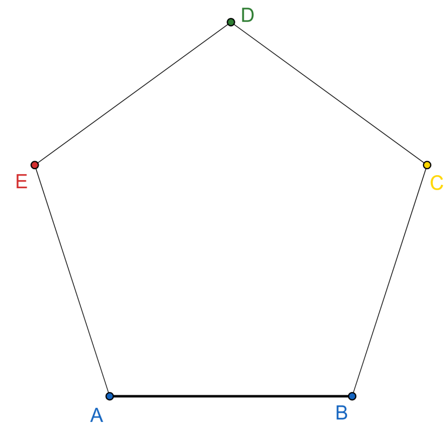

Figure 1: Diagram for $\mathcal{P}_\chi = \{\{0,4\},\{1\},\{2\},\{3\}\}$

$T_{i,j,k}$ is

$$w(T_{i,j,k}) = \begin{cases} w_{\alpha_{i,j}}(1) & \text{if } k = 1, \\ w_{\alpha_{i,j}}(\varpi^{-1}) & \text{if } k = 0. \end{cases}$$

Lemma 2.7. *For any path $T_{i,j,k}$, we have that*

$$l(w(T_{i,j,k})) = 2l(T_{i,j,k}) - 1,$$

where the length of a trail is the number of edges on the trail.

Proof. Assume without loss of generality that $i < j$. If $k = 0$, then $l(T_{i,j,0}) = j - i$, while Lemma 2.1 shows that $l(w_{\alpha_{i,j}}(1)) = 2(j-i)+1$. If $k = 1$, we may use Corollary 2.3 to obtain

$$l(w_{\alpha_{i,j}}(\varpi^{-1})) = 2n - l(w_{\alpha_{i,j}}(1)) = 2n - 2l(T_{i,j,0}) + 1 = 2(n+1-l(T_{i,j,0})) - 1 = 2l(T_{i,j,1}) - 1,$$

and this concludes the lemma. $\square$

Proposition 2.8. *Let χ be a depth-zero character of $T(\mathcal{O})$ and let $\mathfrak{D}_\chi$ be the associated diagram induced by the partition $\mathcal{P}_\chi = \{A_1, \ldots, A_r\}$ of its vertices. For each set $k \in \{1, \ldots, r\}$, write $A_k = \{a_{k,1} < \cdots < a_{k,m_k}\}$ and let $\{T_{k,i} \mid i = 1, \ldots, m_k\}$ be all the paths joining each consecutive pair of vertices in A_k. Then*

$$S_{\chi,\text{aff}} = \bigcup_{k=1}^{r} \{w(T_{k,1}), \ldots, w(T_{k,m_k})\}.$$

Proof. Firstly, we note that if $\mathcal{P}_\chi = \{A_1, \ldots, A_r\}$, which each A_k of size m_k, then

$$\widetilde{W}_\chi \cong \widetilde{W}_{\chi^{(1)}} \times \cdots \times \widetilde{W}_{\chi^{(r)}},$$

where $\chi^{(k)} = \chi_{a_{k,1}} \otimes \cdots \otimes \chi_{a_{k,m_k}}$ is the depth-zero character of GL_{m_k}, and $\widetilde{W}_{\chi^{(k)}}$ is viewed as a subgroup of $\widetilde{W}_\chi$ in the obvious way. Each $\widetilde{W}_{\chi^{(k)}}$ is isomorphic to the extended Weyl group of GL_{m_k}, and if we let $S_{\chi,\text{aff},k}$

be the set of simple reflections of $\widetilde{W}_{\chi^{(k)}}$ viewed as a subgroup of $\widetilde{W}_\chi$, then

$$S_{\chi,\mathrm{aff}} = \bigcup_{k=1}^{r} S_{\chi,\mathrm{aff},k}.$$

It therefore suffices to show that

$$S_{\chi,\mathrm{aff},k} = \{w(T_{k,1}), \ldots, w(T_{k,m_k})\}.$$

still remains to prove this, which is definitely expected to be true! $\square$

2.2 The induced homomorphism on Hecke algebras

Having established a way to easily determine $S_{\chi,\mathrm{aff}}$ from the character χ, in this subsection we construct, for each depth-zero character χ and $s \in S_{\chi,\mathrm{aff}}$, a homomorphism of p-adic groups ψ_s with certain nice properties that induce a homomorphism on Hecke algebras. This homomorphism ψ_s will be our main tool to deduce the quadratic equation for $\varphi_s = q^{(1-l(s))/2}[IsI]_{\check{\chi}}$ from our results in the previous section.

Before we can construct these homomorphisms, we first need to consider the element

$$\rho_n := \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\varpi & & & \end{pmatrix} \in \mathrm{GL}_{n+1}(F),$$

which is the generator of the alcove stabilizer Ω. This element has a natural action on the set of depth-zero characters of $T(\mathcal{O})$ that is compatible with a certain action on the diagrams.

Lemma 2.9. *The action of ρ_n on the space of depth-zero characters of $T(\mathcal{O})$ is compatible with a clockwise rotation of $2\pi/n$ radians on the coloring of the diagrams.*

In other words, if χ is a depth-zero character of $T(\mathcal{O})$, then ${}^{\rho_n}\chi(\cdot) = \chi(\rho_n^{-1} \cdot \rho_n)$ is another depth-zero character. Moreover,

$$\mathfrak{D}_{^{\rho_n}\chi} = r_n(\mathfrak{D}_\chi) \quad \text{and} \quad S_{^{\rho_n}\chi,\mathrm{aff}} = \rho_n S_{\chi,\mathrm{aff}} \rho_n^{-1},$$

where $r_n(\mathfrak{D})$ is obtained by rotating the coloring of $\mathfrak{D}$ clockwise $2\pi/(n+1)$ radians.

Proof. We first note that

$$\rho_n^{-1} \begin{pmatrix} a_0 & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \rho_n = \begin{pmatrix} a_n & & & \\ & a_0 & & \\ & & \ddots & \\ & & & a_{n-1} \end{pmatrix}. \tag{4}$$

Therefore, if $\chi = \chi_0 \otimes \chi_1 \otimes \cdots \otimes \chi_n$, then ${}^{\rho_n}\chi = \chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0$, and by construction, $\mathfrak{D}_{^{\rho_n}\chi} = r_n(\mathfrak{D}_\chi)$. $\square$

Now we are ready to construct a homomorphism ψ_s of p-adic groups associated to $s \in S_{\chi,\mathrm{aff}}$, where χ is some depth-zero character of $T(\mathcal{O}) \subseteq \mathrm{GL}_{n+1}(F)$. By Proposition 2.8, the reflection s corresponds to some trail

T_s connecting two distinct vertices i and $j = i + l \pmod{n+1}$ of $\mathfrak{D}_\chi$ **counterclockwise** and with length $l = (1 + l(s))/2 \le n$. Then define

$$\psi_s : \mathrm{GL}_{l+1}(F) \longrightarrow \mathrm{GL}_{n+1}(F) \tag{5}$$

$$A \longmapsto \rho_n^{n+1-i} \begin{pmatrix} A & \\ & \mathrm{Id}_{n-l} \end{pmatrix} \rho_n^{i-n-1}, \tag{6}$$

which is clearly a continuous homomorphism of p-adic groups. Note that ψ_s is the composition of the two homomorphisms

$$\begin{aligned} \phi_{l,n} : \mathrm{GL}_{l+1}(F) &\longrightarrow \mathrm{GL}_{n+1}(F) \\ A &\longmapsto \begin{pmatrix} A & \\ & \mathrm{Id}_{n-l} \end{pmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_s : \mathrm{GL}_{n+1}(F) &\longrightarrow \mathrm{GL}_{n+1}(F) \\ B &\longmapsto \rho_n^{n+1-i} B \rho_n^{i-n-1}. \end{aligned}$$

We remark that if $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ with $\chi_0 = \chi_n$ and all the other χ_i distinct (as in the previous section) and $s = w_{\alpha_0}(1) \in S_{\chi,\mathrm{aff}}$, then ψ_s is the identity map on $\mathrm{GL}_{n+1}(F)$.

Lemma 2.10. *For any depth-zero character χ of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}$ and any $s \in S_{\chi,\mathrm{aff}}$, the homomorphism $\psi_s : \mathrm{GL}_{l+1}(F) \longrightarrow \mathrm{GL}_{n+1}(F)$ satisfies the following properties:*

1. *$\psi_s^{-1}(T(\mathcal{O})) = T(\mathcal{O})$ and $\psi_s^{-1}(N) = N$,*
2. *$\psi_s^{-1}(I) = I$.*

Proof. Both properties are immediate for the homomorphism $\phi_{l,n}$. The map ϕ_s is an isomorphism, and it preserves the $\mathcal{O}$-points of the torus T and the normalizer of the tori N since $\rho_n \in N(F) = N_G(T(\mathcal{O}))$. It also preserves the Iwahori subgroup I since ρ_n is a fundamental alcove stabilizer. This concludes the proof. □

The previous lemma shows that ψ_s induces a homomorphism of extended Weyl groups $\overline{\psi}_s : \widetilde{W}_{l+1} \to \widetilde{W}_{n+1}$. If s corresponds to a trail T_s on $\mathfrak{D}_\chi$ connecting the vertices i and $j = i + l \pmod{n+1}$ in $\mathfrak{D}_\chi$, we consider the character

$$\chi^{(s)} := \chi_i \otimes \chi_{i+1} \otimes \cdots \otimes \chi_{j-1} \otimes \chi_j$$

of $T(\mathcal{O}) \subset \mathrm{GL}_{l+1}(F)$, where subscripts are taken mod $n+1$. The following two results provide two fundamental properties of the character $\chi^{(s)}$.

Lemma 2.11. *The homomorphism $\overline{\psi}_s$ satisfies*

$$\overline{\psi}_s(\widetilde{W}_{\chi^{(s)}}) \subseteq \widetilde{W}_\chi \quad \textit{and} \quad \overline{\psi}_s(w_{\beta_0}(1)) = s,$$

where β_0 is the highest root of GL_{l+1}.

Proof. We note that $\psi_s = \phi_s \circ \phi_{l,n}$ and that both ϕ_s and $\phi_{l,n}$ also induce maps of Weyl groups such that $\overline{\psi}_s = \overline{\phi}_s \circ \overline{\phi}_{l,n}$. By definition of $\phi_{l,n}$, we can see that

$$\overline{\phi}_{l,n}(\widetilde{W}_{\chi^{(s)}}) \subseteq \widetilde{W}_{\chi'}, \quad \text{where} \quad \chi' = \chi^{(s)} \otimes \chi_{j+1} \otimes \cdots \otimes \chi_{i-1} = {}^{\rho_n^i}\chi$$

and that

$$\overline{\phi}_{l,n}(w_{\beta_0}(1)) = w_{\alpha_1+\cdots+\alpha_l}(1),$$

which corresponds to the trail in $\mathfrak{D}_\chi$ connecting 0 and l counterclockwise. On the other hand $\overline{\phi}_s(w) = \rho_n^{n+1-i} w \rho_n^{i-n-1}$ for any $w \in \widetilde{W}_{n+1}$, so

$$\overline{\phi}_s(\widetilde{W}_{\chi'}) = \widetilde{W}_{\rho_n^{-i}\chi'} = \widetilde{W}_\chi.$$

Moreover,

$$\overline{\phi}_s(w_{\alpha_1+\cdots+\alpha_l}(1)) = \rho_n^{n+1-i} w_{\alpha_1+\cdots+\alpha_l}(1) \rho_n^{i-n-1}$$

corresponds to the trail T_s in $\mathfrak{D}_\chi$ connecting i and $j = i + l \pmod{n+1}$ counterclockwise. Thus,

$$\overline{\psi}_s(w_{\beta_0}(1)) = \overline{\phi}_s(\overline{\phi}_{l,n}(w_{\beta_0}(1))) = \overline{\psi}_s(w_{\alpha_1+\cdots+\alpha_l}(1)) = s \qquad \text{and} \qquad \overline{\psi}_s(\widetilde{W}_{\chi^{(s)}}) \subseteq \overline{\phi}_s(\widetilde{W}_{\chi'}) = \widetilde{W}_\chi,$$

as desired. $\square$

Lemma 2.12. *For any $k \in I \subset \mathrm{GL}_{l+1}(F)$,*

$$\rho_\chi(\psi_s(k)) = \rho_{\chi^{(s)}}(k).$$

Proof. Fix some $k \in I \subset \mathrm{GL}_{l+1}(F)$ and let $\chi = \chi_0 \otimes \cdots \otimes \chi_n$. By definition of $\phi_{l,n}$, we have that

$$\rho_\chi(\phi_{l,n}(k)) = \rho_{\chi'}(k),$$

where $\chi' = \chi_0 \otimes \cdots \otimes \chi_l$ is obtained from χ by taking the characters of the first $l+1$ entries. Moreover, since conjugating by ρ_n permutes the diagonal entries according to (4), it follows that for any $h \in I \subset \mathrm{GL}_{n+1}$,

$$\rho_\chi(\rho_n^{-1} h \rho_n) = \rho_{{}^{\rho_n}\chi}(h), \quad \text{where} \quad {}^{\rho_n}\chi = \chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0.$$

Therefore, by the construction of ϕ_s, we have that

$$\rho_\chi(\phi_s(h)) = \rho_{{}^{\rho_n^i}\chi}(h), \quad \text{where} \quad {}^{\rho_n^i}\chi = \chi_i \otimes \chi_{i+1} \otimes \cdots \otimes \chi_{i-1}.$$

Putting everything together, we obtain

$$\rho_\chi(\psi_s(k)) = \rho_\chi(\phi_s(\phi_{l,n}(k))) = \rho_{{}^{\rho_n^i}\chi}(\phi_{l,n}(k)) = \rho_{\chi^{(s)}}(k),$$

as desired. $\square$

We are finally ready to prove the existence of a surjective algebra homomorphism between Hecke algebras.

Proposition 2.13. *The homomorphism ψ_s induces a surjective Hecke algebra homomorphism*

$$\Psi_s : \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi) \longrightarrow \mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}}),$$

where

$$\Psi_s(\varphi)(g) = \varphi(\psi_s(g)), \qquad \varphi \in \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi),\ g \in \mathrm{GL}_{l+1}(F).$$

Furthermore, for every $w \in \widetilde{W}_{\chi^{(s)}}$,

$$\Psi_s([I\overline{\psi}_s(w)I]) = [IwI].$$

Proof. For any $\varphi \in \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi)$, we have that $\mathrm{supp}(\Psi_s(\varphi)) = \psi_s^{-1}(\mathrm{supp}(\varphi))$, a compact subset of $\mathrm{GL}_{l+1}(F)$ since ψ_s is a continuous map. Moreover, if $k_1, k_2 \in I \subset \mathrm{GL}_{l+1}(F)$ and $g \in \mathrm{GL}_{l+1}(F)$, then by Lemma 2.12

$$\Psi_s(\varphi)(k_1 g k_2) = \varphi(\psi_s(k_1 g k_2)) = \rho_\chi(\psi_s(k_1))\varphi(\psi_s(g))\rho_\chi(\psi_s(k_2)) = \rho_{\chi^{(s)}}(k_1)\Psi_s(g)\rho_{\chi^{(s)}}(k_2),$$

so $\Psi_s(\varphi) \in \mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}})$.

By Lemma 2.11, $\overline{\psi}_s(w) \in \widetilde{W}_\chi$ for any $w \in \widetilde{W}_{\chi^{(s)}}$ and since ψ_s is an injective homomorphism,

$$\Psi_s\left([I\overline{\psi}_s(w)I]\right)(g) = [I\overline{\psi}_s(w)I](\psi_s(g)) = \begin{cases} [IwI](g) & \text{if } g \in IwI, \\ 0 & \text{otherwise.} \end{cases} = [IwI](g)$$

This also implies that the map Ψ_s is surjective since $\{[IwI] \mid w \in \widetilde{W}_{\chi^{(s)}}\}$ is a C-basis of $\mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}})$. Finally, to show that Ψ_s is an algebra homomorphism, we compute, for each $\varphi_1, \varphi_2 \in \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi)$ and $g \in \mathrm{GL}_{l+1}(F)$,

$$\begin{aligned}\Psi_s(\varphi_1 * \varphi_2)(g) = (\varphi_1 * \varphi_2)(\psi_s(g)) &= \int_{\mathrm{GL}_{n+1}} \varphi_1(\psi_s(g)h)\varphi_2(h^{-1})dh = \int_{\mathrm{GL}_{l+1}} \varphi_1(\psi_s(g)\psi(h))\varphi_2(\psi_s(h)^{-1})dh \\ &= \int_{\mathrm{GL}_{l+1}} \Psi_s(\varphi_1)(gh) * \Psi_s(\varphi_2)(h^{-1})dh = \Psi_s(\varphi_1) * \Psi_s(\varphi_2)(g),\end{aligned}$$

as desired. □

With the previous proposition, we now have all the ingredients to prove the quadratic relation for $\varphi_s = q^{(1-l(s))/2}[IsI]_{\check{\chi}} \in \mathcal{H}(\mathrm{GL}_{n+1})$.

Theorem 2.14. *Let $\chi = \chi_0 \otimes \cdots \otimes \chi_n$ be a depth-zero character of $T(\mathcal{O}) \subset \mathrm{GL}_{n+1}$ and let $s \in S_{\chi,\mathrm{aff}}$. If*

$$\varphi_s = q^{\frac{1-l(s)}{2}}[IsI]_{\check{\chi}} \in \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi),$$

then

$$\varphi_s^2 = (q-1)\varphi_s + q\varphi_1.$$

Proof. Consider the algebra homomorphism $\Psi_s : \mathcal{H}(\mathrm{GL}_{n+1}, I, \rho_\chi) \longrightarrow \mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}})$, and by Proposition 2.13 we have that

$$\begin{aligned}\Psi_s(\varphi_s) = q^{\frac{(1-l(s))}{2}}\Psi_s([IsI]_{\check{\chi}}) &= q^{\frac{(1-l(s))}{2}}\Psi_s([I\overline{\psi}_s(w_{\beta_0}(1))I]_{\check{\chi}}) \\ &= q^{\frac{(1-l(w_{\beta_0}(1)))}{2}}[Iw_{\beta_0}(1)I] = \varphi_{w_{\beta_0}(1)} \in \mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}}).\end{aligned}$$

By the results on the previous section, we know that $\varphi_{w_{\beta_0}(1)}$ satisfies the desired quadratic relation, and since Ψ_s is an algebra homomorphism, it follows that

$$\varphi_s^2 - (q-1)\varphi_s - q\varphi_1 \in \ker \Psi_s \cap \mathrm{Span}_C\{\varphi_1, \varphi_s\}.$$

However, $\Psi_s(\varphi_s) = \varphi_{w_{\beta_0}(1)}$ and $\Psi_s(\varphi_1) = \varphi_1$, which are non-zero and linearly independent in $\mathcal{H}(\mathrm{GL}_{l+1}, I, \rho_{\chi^{(s)}})$. Thus, $\ker \Psi_s \cap \mathrm{Span}_C\{\varphi_1, \varphi_s\} = \{0\}$, yielding

$$\varphi_s^2 = (q-1)\varphi_s + q\varphi_1$$

and this concludes the proof. □