Local Langlands for GL_2

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1 Locally Profinite Groups and Smooth Representations

The aim of this first section is to motivate the notions of locally profinite groups and their smooth representations. Such groups arise in nature from taking the points of reductive groups over non-Archimedean local fields. We begin this section by briefly recalling some basic facts about these fields and linear groups associated to them. For the sake of brevity, we will omit proofs. For more detail, the reader can consult, for example, [Gou20].

1.1 Local Fields and Locally Profinite Groups

We begin by recalling some basic objects from algebraic number theory. Given a field F, a *discrete valuation* on F is a surjective function $\nu : F \to \mathbb{Z} \cup \{\infty\}$ satisfying the conditions

1. $\nu(xy) = \nu(x) + \nu(y)$ for any $x, y \in F$
2. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for any $x, y \in F$.
3. $\nu(x) = \infty$ if and only if $x = 0$.

Any discrete valuation ν induces an absolute value on F given by the formula

$$|x| = c^{\nu(x)}$$

for any $c \in (0, 1)$, and therefore it also induces a topology on F. This topology is independent of the choice of c. One easily checks that this absolute value satisfies $|x + y| \leq \max\{|x|, |y|\}$ for any $x, y \in F$. Absolute values with this property are called *non-Archimedean*.

A field F with an absolute value $|\cdot|$ induced by a discrete valuation ν is the fraction field of the *valuation ring*

$$R := \{x \in F : v(x) \geq 0\} = \{x \in F : |x| \leq 1\},$$

which contains a unique maximal ideal

$$\mathfrak{p} := \{x \in F : v(x) > 0\} = \{x \in F : |x| < 1\},$$

the *valuation ideal* or the *ring of integers of* F. The valuation ideal is principal, and it is generated by any $\varpi \in F$ with $\nu(\varpi) = 1$. Such an element is called a *uniformiser* of F. Finally, the *residue field* κ of F is the quotient $R/\mathfrak{p}$. This motivates the following important definition.

Definition 1.1. A field F is a *non-Archimedean local field* if it is complete with respect to a topology induced by a discrete valuation and the residue field is finite.

Remark 1.2. When the residue field is finite, it is conventional to define the absolute value on F by $|x| = q^{-\nu(x)}$, where $q = |\kappa|$. From here onwards, we will follow this convention.

Remark 1.3. Local fields are ubiquitous in number theory. They arise as completions of number fields at non-Archimedean places in characteristic 0, or as completions of finite extensions of $\mathbb{F}_p(t)$ at non-Archimedean places in positive characteristic.

Let us now discuss important aspects of the topology on F and R induced by the discrete valuation ν. We have already seen that R is a local ring with maximal ideal $\mathfrak{p}$ and therefore $U_F := R \setminus \mathfrak{p}$ is the set of units of R. The ideals

$$\mathfrak{p}^n = \{x \in F : \nu(x) \geq n\} = \{x \in F : |x| \leq q^{-n}\} = \varpi^n R, \quad n \in \mathbb{Z}$$

are a complete set of fractional ideals of R and, since the valuation is assumed to be discrete, they are also open subsets of F. Therefore, they are a fundamental system of neighbourhoods of the identity. A direct consequence of this fact implies that F (and therefore R) are totally disconnected topological rings.

Furthermore, the ring R is a closed subring of F, which is assumed to be complete. Hence, R is also complete, and a stardard topological argument shows that R is in fact compact. This proves that R (and therefore any $\mathfrak{p}^n$) is in fact a profinite group, and we have a topological isomorphism

$$R \longrightarrow \varprojlim_{n \geq 1} R/\mathfrak{p}^n \quad x \mapsto (x \pmod{\mathfrak{p}^n})_{n \geq 1}$$

where the maps implicit in the right hand side are the obvious ones.

However, F itself is clearly not compact, and therefore it is not profinite. Nevertheless, F has the important property that any open neighbourhood of the identity contains an open compact (and therefore profinite) subgroup - some $\mathfrak{p}^n$ for a sufficiently large n.

We are now ready to give the main definition of this section, which encapsulates this last property in greater generality.

Definition 1.4. A topological group G (which we always assume to be Hausdorff) is a *locally profinite group* if every open neighbourhood of the identity contains a compact open subgroup.

In this report we will be interested in studying the representation theory of many important groups and rings related to the local field F. The notion of a locally profinite group is an abstract one, but it has the great advantage of accomodating many important groups and rings associated to non-Archimedean local fields and their representation theory.

Examples 1.5. (1) Trivially, any group equipped with the discrete topology is profinite, where $\{e\}$ is the fundamental neighbourhood.

(2) In the preceding discussion, we have shown that the local field F is a locally profinite group, where $\mathfrak{p}^n$ for $n \geq 1$ is a fundamental system of open compact subgroups. We remark that F satisfies the rather special property of being the union of its open compact subgroups.

(3) The multiplicative group $F^\times$ is also a locally profinite group, where the congruence unit groups $U_F^n = 1 + \mathfrak{p}^{n+1}$ for $n \geq 1$ is a fundamental system of open compact subgroups. Unlike F, the group $F^\times$ is not the union of its open compact subgroups.

(4) Given $m \geq 1$ an integer, the additive group $F^m = F \times \cdots \times F$ is also a locally profinite group endowed with the product topology. A fundamental system of open compact subgroups is given by $\mathfrak{p}^n \times \cdots \times \mathfrak{p}^n$ for $n \geq 0$. More generally, any product of locally profinite groups is locally profinite.

(5) The matrix ring $M_m(F)$ is also locally profinite since it is isomorphic to F^{m^2} as additive groups. The open compact subgroups $\mathfrak{p}^n M_m(R)$ are a fundamental system of neighbourhood of the identity.

(6) The group $\mathrm{GL}_m(F)$ of invertible matrices is an open subset of $M_m(F)$ since $\det : M_m(F) \to F$ is continuous and $F^\times$ is an open subset of F. Furthermore, mutiplication by a matrix $A \in M_m(F)$ and inversion of matrices are continuous maps in $M_m(F)$, and therefore $\mathrm{GL}_m(F)$ is also a topological group. The subgroups

$$K = \mathrm{GL}_m(R), \quad K_n = 1 + \mathfrak{p}^{n+1} M_m(R), \quad n \geq 0,$$

are compact open, and a fundamental system of neighbourhoods of the identity.

(7) Let G be a locally profinite group and $H \leq G$ be a closed subgroup. Then H is also a locally profinite group. If in addition H is assumed to be normal in G, then G/H is locally profinite.

We give some further insight into the terminology used. It is an easy exercise to prove that a profinite group is compact and locally profinite. Rather strikingly, the converse also holds. That is, if K is a compact locally profinite group, then

$$K \longrightarrow \varprojlim_N K/N$$

is a topological isomorphism, where N ranges over the normal open subgroups. Since K is compact and N is open, K/N must be finite and discrete, showing that K is profinite.

1.2 Abstract Representations of Groups

Before discussing the representation theory of locally profinite groups, we first review some general results and constructions of representations of arbitrary groups G. We begin by recalling the notion of a representation.

Definition 1.6. A *representation* of a group G over a field k is a pair (π, V) where V is a k-vector space and $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism. We say that $\dim V$ is the *dimension* of the representation.

Equivalently, a representation of G is a k-vector space V equipped with a k-linear G-action. Whenever the representation is clear from the context, we will omit π from the notation and write $g \cdot v$ for $\pi(g)v$.

Throughout this document we will mostly be interested in complex representations, so from now on we will assume that $k = \mathbb{C}$ unless otherwise stated.

We say that $U \leq V$ is a *G-subspace* if U is closed under the G-action; i.e., if $g \cdot U \subseteq U$ for every $g \in G$. When this happens, both U and V/U are naturally G-representations. We say that that a representation (π, V) is *irreducible* (or *simple*) if V has no non-trivial G-subspaces. These are the building blocks of more complicated representations, and thus we are often interested in classifying them.

Definition 1.7. A representation (π, V) of a group G is *semisimple* if it is the direct sum of simple subrepresentations.

Remark 1.8. If G is a finite group, Maschke's Theorem shows that all finite dimensional complex representations of G are semisimple. As a consequence, one can show that any complex irreducible representation of G is finite dimensional, appearing as a subrepresentation of the regular representation $\mathbb{C}G$.

As we shall see later in this chapter, continuous finite-dimensional representations of profinite groups also share these properties. However, it is easy to construct representations of locally profinite groups which are continuous yet not semisimple. For example,

$$\begin{aligned} \phi : \mathbb{Z} &\longrightarrow \mathrm{GL}_2(\mathbb{C}) \\ n &\mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \end{aligned}$$

has a single one-dimensional invariant subspace. One can also construct irreducible representations that are infinite dimensional; we will meet some examples in Section 3.

Naturally, we also define the notion of a morphism of representations.

Definition 1.9. A morphism between two complex representations (π, V), (σ, W) of a group G is a linear map $\phi : V \to W$ compatible with the G action. That is,

$$\phi(\pi(g)v) = \sigma(g)\phi(v) \text{ for all } g \in G,\ v \in V.$$

This turns the set of complex representations of G into a category, denoted $\mathrm{Rep}(G)$, which is an *abelian category.*

We finish this subsection by introducing important constructions and functors between these categories that allow us to obtain new representations from old ones, which we will use heavily later on.

Definition 1.10. Given $(\pi, V) \in \mathrm{Rep}_G$, define the dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$, and denote by

$$\begin{aligned} V^* \times V &\longrightarrow \mathbb{C}, \\ (v^*, v) &\longmapsto \langle v^*, v\rangle, \end{aligned}$$

the canonical evaluation homomorphism. Then V^* carries a natural representation of G defined by

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle.$$

This is the *dual representation* of V, and the functor

$$\begin{aligned} (-)^* : \mathrm{Rep}(G) &\longrightarrow \mathrm{Rep}(G) \\ (\pi, V) &\longrightarrow (\pi^*, V^*) \end{aligned}$$

is an additive and exact contravariant functor.

One can also consider the composition of this functor with itself to obtain the *double dual* (π^{**}, V^{**}). There is a canonical G-homomorphism $\delta : V \to V^{**}$ such that

$$\langle \delta(v), v^*\rangle_{V^*} = \langle v^*, v\rangle_V.$$

When V is finite dimensional, δ is a G-isomorphism. For general representations of locally profinite groups, this is not always the case, but under additional assumptions it is possible to give a precise criterion that determines when δ is bijective ([BH06, Corollary 2.8, Proposition 2.9]).

Definition 1.11. Let $H \leq G$ be groups and let (π, V) and (σ, W) be representations of G and H repectively. The restriction of π to H gives a *restriction* functor

$$\begin{aligned} \mathrm{Res}_H^G : \mathrm{Rep}(G) &\longrightarrow \mathrm{Rep}(H) \\ (\pi, V) &\longmapsto (\pi|_H, V) \end{aligned}$$

On the other hand, given $(\sigma, W) \in \mathrm{Rep}(H)$, one can define the vector space

$$X = \{f : G \to W : f(hg) = \sigma(h)f(g) \text{ for all } h \in H, g \in G\},$$

equipped with the G-action $\Sigma : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(X)$ defined by right translation:

$$\Sigma(g)f : x \longmapsto f(xg), \ \ x, g \in G.$$

This defines the *induction* functor

$$\begin{aligned} \mathrm{Ind}_H^G : \mathrm{Rep}(H) &\longrightarrow \mathrm{Rep}(G) \\ (\pi, V) &\longmapsto (\Sigma, X). \end{aligned}$$

As with the dual functor, both the restriction and induction functors are additive and exact, but are now covariant functors. To simplify notation, we will write $\mathrm{Ind}_H^G\sigma$ instead of $\mathrm{Ind}_H^G(\sigma, W)$, which is the usual convention in the literature.

We remark that one can construct the following canonical H-homomorphisms

$$\begin{aligned} a_\sigma : \mathrm{Ind}_H^G\sigma &\longrightarrow W \\ f &\longmapsto f(1) \end{aligned}$$

and

$$\begin{aligned} a_\sigma^c : W &\longrightarrow \mathrm{Ind}_H^G\sigma, \\ w &\longmapsto f_w \end{aligned}$$

where f_w is supported in H and $f_w(h) = \sigma(h)w$ for $h \in H$. The choice of notation will be understood later. These, in turn, induce the maps

$$\begin{aligned} \Psi : \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\sigma) &\longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G\pi, \sigma), \\ \phi &\longmapsto a_\sigma \circ \phi, \end{aligned}$$

and

$$\begin{aligned} \Psi^c : \mathrm{Hom}_G(\mathrm{Ind}_H^G\sigma, \pi) &\longrightarrow \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G\pi), \\ f &\longmapsto f \circ a_\sigma^c. \end{aligned}$$

When G is a finite group, we have the following result.

Theorem 1.12 (Frobenius reciprocity)**.** *Let G be a finite group. Then the maps Ψ and Ψ^c are bijections that are functorial in both variables σ and π. In categorical terms, we have the adjunctions*

$$\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G.$$

There is an analogue of Frobenius reciprocity for locally profinite groups; see Theorem 1.36 and Theorem 1.38.

1.3 Characters of Local Fields

Now we turn our attention to the representation theory of a locally profinite group G. The category $\mathrm{Rep}(G)$ of abstract representations introduced in the previous section only takes into consideration the group structure of G, but it completely ignores its topology. In the previous section, we have seen that locally profinite groups have a rather special topology and, as it will become apparent as we develop the theory, this topology is a crucial piece of information associated to the group. Consequently, instead of working with $\mathrm{Rep}(G)$, we will work with a full subcategory of representations that satisfy an additional smoothness condition, which will be denoted as $\mathrm{Smo}(G)$ and its elements called *smooth representations*. This condition, as the name suggests, requires the representation to be continuous with respect to the topology on G and the complex topology of $\mathrm{Aut}_{\mathbb{C}}(V)$. In sections 1.3 and 1.4, we will see that these two notions coincide when the representation is finite-dimensional, but not in general. To motivate this condition, we will first describe the simplest case: one-dimensional representations of a local field F: that is, group homomorphisms $\phi : F \to \mathbb{C}^\times$. Later in this section we will also study the one-dimensional representations of $F^\times$. We now follow closely the development from [BH06, §2], while stressing the motivation for the terms introduced through examples.

Definition 1.13. A *character* of a locally profinite group G is a continuous homomorphism $\psi : G \to \mathbb{C}^\times$.

Characters of a locally profinite group G form an abelian group $\hat{G}$ under multiplication, denoted as the *dual group* of G.

Example 1.14. Let G be a finite group with the discrete topology. Then any one-dimensional representation is a character, and we have the simple description $\hat{G} \cong G^{ab}$. In particular, if G is abelian then $\hat{G} \cong G$.

For general locally profinite results, we have this rather surprising result

Lemma 1.15. *Let G be a locally profinite group and $\psi : G \to \mathbb{C}^\times$ a homomorphism. Then ψ is continuous if and only if $\ker \psi$ is open in G. Furthermore, if G is the union of its compact open subgroups, then*

$$\psi(G) \subseteq \{z \in \mathbb{C}^\times : |z| = 1\} = S^1.$$

Remark 1.16. Characters of locally profinite groups that have image in S^1 are called *unitary.*

Proof. If $\ker \psi = \psi^{-1}(1)$ is open in G, then for any $z \in \mathrm{Im}\,\psi$, the preimage $\psi^{-1}(z) = g \ker \psi$ is also open, for any $g \in G$ satisfying $\psi(g) = z$. Then for any $U \subseteq \mathbb{C}^\times$,

$$\psi^{-1}(U) = \bigcup_{z \in U \cap \mathrm{Im}\,\psi} \psi^{-1}(z),$$

so that ψ is continuous. Conversely, if ψ is continuous, then for any open neighbourhood $\mathcal{N}$ of 1, $\psi^{-1}(\mathcal{N})$ contains an open compact subgroup K of G. But $\mathcal{N}$ can be chosen sufficiently small so that it does not contain any non-trivial subgroup of $\mathbb{C}^\times$. Hence, $\psi(K) = 1$, so $K \subseteq \ker\psi$, and since K is open, so is $\ker\psi$. The last assertion is a direct consequence of the fact that the continuous image of a compact set is compact, and S^1 is the unique maximal compact subgroup of $\mathbb{C}^\times$. $\square$

Example 1.17. The local field F is the union of its open compact subgroups, so all characters of F are unitary. This can also be checked directly as follows. Let $\psi : F \to \mathbb{C}^\times$ be a character. By Lemma 1.15, $\ker\psi$ is open in F and therefore it contains $\mathfrak{p}^N$ for some N large enough. Assume, for example, that ψ is trivial on $R = \mathfrak{p}^0$. We will describe such characters inductively for each $\mathfrak{p}^n, n < 0$. Fix some $n < 0$ and assume that $\psi(\mathfrak{p}^n) \subset S^1$. Then $\psi(\varpi^{n-1})^q = \psi(q\varpi^{n-1}) \in S^1$ since $q\varpi^{n-1} \in \mathfrak{p}^n$ and therefore $\psi(\varpi^{n-1}) \in S^1$. Since any $x \in \mathfrak{p}^{n-1}$ can be expressed uniquely as $x = a\varpi^{n-1} + y$ for $a \in \{0, 1, \ldots, q-1\}$ and $y \in \mathfrak{p}^n$, it follows that $\psi(x) = \psi(\varpi^{n-1})^a\psi(y) \in S^1$, so $\psi(\mathfrak{p}^{n-1}) \subset S^1$.

We remark that, for each $n < 0$, there are exactly q choices for $\psi(\varpi^n)$, since it is a qth root of $\psi(q\varpi^n)$ and $q\varpi^n \in \mathfrak{p}^{n+1}$. Once this choice is made ψ is completely determined on $\mathfrak{p}^n$. We have shown that all characters of F trivial on R are constructed this way.

It is also worth mentioning that if the uniformizer is chosen appropiately, the above construction can be made explicit. For example, if $F = \mathbb{Q}_p$ and $\varpi = p$, then $\psi(p^{n-1})^p = \psi(p^n)$ for any character ψ of $\mathbb{Q}_p$ and $n \in \mathbb{Z}$. This means that if ψ is trivial on R (in particular, $\psi(1) = 1$), then ψ is determined by a sequence $(\zeta_1, \zeta_2, \zeta_3, \ldots)$ where ζ_n is a p^nth root of unity and $\zeta_n^p = \zeta_{n-1}$. For example,

$$\begin{aligned} \psi : \mathbb{Q}_p &\longrightarrow \mathbb{C}^\times \\ \sum_{k\geq m} a_k p^k &\longmapsto \begin{cases} 1 \text{ if } m \geq 0, \\ \prod_{k=m}^{-1} e^{2\pi i a_k p^k} \text{ if } m < 0. \end{cases} \end{aligned}$$

is a non-trivial character of $\mathbb{Q}_p$.

From this perspective, it is clear that

$$\{\psi \in \widehat{\mathbb{Q}_p} \text{ trivial on } p^n\mathbb{Z}_p \text{ for some } n \in \mathbb{Z}\} \cong \{\psi \in \widehat{\mathbb{Q}_p} \text{ trivial on } \mathbb{Z}_p\} \cong \varprojlim_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p,$$

where the first isomorphism follows by replacing some character ψ trivial on $p^n\mathbb{Z}_p$ for the character $x \mapsto \psi(p^n x)$, trivial on $\mathbb{Z}_p$. The more general statement

$$\{\psi \in \hat{F} \text{ trivial on } \mathfrak{p}^n \text{ for some } n \in \mathbb{Z}\} \cong R$$

also holds for any local field F, but this takes some more work. We prove this fact in Theorem 2.11 (Additive Duality), together with the important isomorphism $\hat{F} \cong F$.

Example 1.18. In contrast, the multiplicative group $F^\times$ is not the union of its open compact subgroups. For instance, no open compact subgroup contains 2 assuming that $\mathrm{char}\kappa \geq 3$. Moreover, it is not the case that all characters of $F^\times$ are unitary. Indeed, the map $\chi : x \mapsto |x|$ is a character of $F^\times$ since $\ker\chi = R^\times$ is an open subgroup of $F^\times$, and it is not unitary.

This example hints at the fact that the group structure of $\hat{F^\times}$ is quite subtle and we will not cover its description here. The interested reader can find a partial description in [BH06, §1.8]

Before stating Additive Duality, the main result of this section, we need one last definition.

Definition 1.19. Let ψ be a non-trivial character of F. The *level* of ψ is the least integer d such that $\mathfrak{p}^d \subseteq \ker\psi$.

The following is a simple property of the level of a character.

Lemma 1.20. *Let $\psi \in \hat{F}$ be a character of level d and let $a \in F$. Then the map $a\psi : x \mapsto \psi(ax)$ is a character of F, and if $a \neq 0$ then $a\psi$ has level $d - \nu(a)$.*

Proof. The map $a\psi$ is clearly a homomorphism. It is also a character since if $x \in \mathfrak{p}^{d-\nu(a)}$, then $ax \in \mathfrak{p}^d$, so $a\psi(x) = 1$, and therefore $\mathfrak{p}^{d-\nu(a)} \subseteq \ker(a\psi)$ and the kernel of $a\psi$ is open. Furthermore, there is some $y \in \mathfrak{p}^{d-1}$ such that $\psi(y) \neq 1$, and so $a\psi(a^{-1}y) \neq 1$. Since $a^{-1}y \in \mathfrak{p}^{d-1-\nu(a)}$, this indeed shows that the level of $a\psi$ is $d - \nu(a)$. □

We are now ready to give the classfication theorem for $\hat{F}$.

Theorem 1.21 (Additive Duality)**.** *Let $\psi \in \hat{F}$ be a character of level d. The map $a \mapsto a\psi$ induces the isomorphisms*

$$F \cong \hat{F} \quad \text{and} \quad R \cong \{\psi \in \hat{F} : \mathfrak{p}^d \subseteq \ker\psi\}.$$

The proof of surjectivity of the theorem requires an inductive step, which relies on the following results.

Lemma 1.22. *Let $\psi \in \hat{F}$ be a character of level d and let $u, u' \in U_F$ be two units of F. Then $u\psi$ coincides with $u'\psi$ on $\mathfrak{p}^{d-n}$ if and only if $u'u^{-1} \in U_F^n$.*

Proof. Let $\alpha = \nu(u - u')$. A simple definition chase shows that $u\psi$ and $u'\psi$ agree on $\mathfrak{p}^{d-n}$ if and only if $\mathfrak{p}^{d-n+\alpha} = (u - u')\mathfrak{p}^{d-n} \subseteq \ker\psi$. By definition of level, this is the case if and only if $\alpha \geq n$; that is, if $u \equiv u' \pmod{\mathfrak{p}^n}$ or equivalently $u'u^{-1} \in U_F^n$. □

Lemma 1.23. *Let $\theta : \mathfrak{p}^n \to \mathbb{C}^\times$ be a character. Then there are exacty q characters Θ of $\mathfrak{p}^{n-1}$ such that $\Theta|_{\mathfrak{p}^n} = \theta$.*

Proof. Since $\hat{\kappa} \cong \kappa$, where κ is the residue field of F, it is enough to construct a bijection between $\mathcal{A} := \{\Theta \in \widehat{\mathfrak{p}^{n-1}} : \Theta|_{\mathfrak{p}^n} = \theta\}$ and $\hat{\kappa}$. Let $\phi = \theta^{-1}$ and let Φ be *any* lift of ϕ as a character of $\mathfrak{p}^{n-1}$. Now given $\Theta \in \mathcal{A}$, the character $\Theta \cdot \Phi$ is trivial on $\mathfrak{p}^n$ and thus it descends to a map

$$\overline{\Theta \cdot \Phi} : \kappa \cong \mathfrak{p}^{n-1}/\mathfrak{p}^n \longrightarrow \mathbb{C}^\times.$$

To construct an inverse to the map $\Theta \mapsto \overline{\Theta \cdot \Phi}$, choose some $\chi \in \hat{\kappa}$, view it as a character of $\mathfrak{p}^{n-1}/\mathfrak{p}^n$ and consider the map $\tilde{\chi} : \mathfrak{p}^{n-1} \to \mathbb{C}^\times$ given by $\tilde{\chi}(u) = \chi(u + \mathfrak{p}^n)$. Then the map $\chi \mapsto \Phi^{-1} \cdot \tilde{\chi}$ is the required inverse map. □

We are now ready for the proof of Additive Duality.

Proof of Theorem 1.21. The map $a \mapsto a\psi$ is clearly a homomorphism. To prove injectivity, suppose that $a \neq b$ but $a\psi = b\psi$. It follows that $x(a-b) \in \ker\psi$ for all $x \in F$. But since $a-b \neq 0$, we have $\ker\psi = F$, contradicting our assumption that ψ is non-trivial.

Let $\theta \in \hat{F}$ be any non-trivial character (if θ were trivial, then $0\psi = \theta$), and let l be the level of θ. By replacing θ with $\varpi^{l-d}\theta$, which has level d, we may assume without loss of generality that θ and ψ have the same level d, and therefore they both agree on $\mathfrak{p}^d$. To show there is some $u \in F$ (in fact, $u \in U_F$ necessarily) such that $u\psi = \theta$, we construct a sequence $\{u_n\}_{n\geq 0}$ inductively such that $u_n\psi|_{\mathfrak{p}^{d-n}} = \theta|_{\mathfrak{p}^{d-n}}$ and $u_{n+1} \equiv u_n \pmod{\mathfrak{p}^n}$. Such a sequence is clearly Cauchy, and since F is complete, it converges to some $u \in U_F$ such that $u \equiv u_n \pmod{\mathfrak{p}^n}$ for all $n \geq 1$ and thus $u\psi$ agrees with θ on $\cup_{n\in\mathbb{Z}} \mathfrak{p}^n = F$, which concludes the proof.

Thus, it remains to construct the sequence above. To construct u_1 we note that by Lemma 1.23, there are exactly $q-1$ non-trivial characters on $\mathfrak{p}^{d-1}$ that are trivial on $\mathfrak{p}^d$. In addition, by Lemma 1.22, as u ranges over the cossets of U_F/U_F^1, the characters $u\psi|_{\mathfrak{p}^{d-1}}$ are distinct. Since $|U_F/U_F^1| = |\kappa^\times| = q-1$, there is some $u_1 \in U_F$ such that $u_1\psi$ agrees with θ on $\mathfrak{p}^{d-1}$.

Assuming now we have constructed $u_1, \ldots, u_n$ in U_F with the desired conditions, we note that by Lemma 1.23, there are exactly q characters of $\mathfrak{p}^{d-n-1}$ that coincide with $\theta|_{\mathfrak{p}^{d-n}}$ when they are restricted. Again by Lemma 1.22, as α ranges over the cosets of U_F^n/U_F^{n+1} the characters $\alpha u_n\psi$ are distinct on $\mathfrak{p}^{d-n-1}$ but they all coincide on $\mathfrak{p}^{d-n}$. Since $|U_F^n/U_F^{n+1}| = |\kappa| = q$, there is some α_n such that $\alpha_n u_n\psi$ coincides with θ on $\mathfrak{p}^{d-n-1}$. Since $\alpha_n \in U_F^n$, $\alpha_n u_n \equiv u_n \pmod{\mathfrak{p}^n}$. Hence $u_{n+1} := \alpha_n u_n$ has the required properties.

Finally, it follows immediately from the definition of level that. under the above isomorphism, the elements $a \in R$ correspond to the characters $\psi \in \hat{F}$ that are trivial on $\mathfrak{p}^d$. This concludes the proof. $\square$

1.4 Smooth Representations of Locally Profinite Groups

We now turn our attention to representations of arbitrary dimension of locally profinite groups and we introduce the notion of *smooth representations*, which form a full subcategory of $\mathrm{Rep}(G)$. For one-dimensional representations, we imposed a natural continuity condition, and Lemma 1.15 showed that characters have open kernel. This is a remarkable result, since this means that the homomorphism is continuous with respect to **any** topology on $\mathbb{C}^\times$, not just the usual one.

If V is a finite-dimensional representation of a locally profinite group G, the group $\mathrm{GL}_{\mathbb{C}}(V)$ has a natural topology as an open subspace of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. Again, it is a natural requirement that finite dimensional representations should be continous with respect to these topologies. It is a fact, analogous to $\mathbb{C}^\times$, that small neighbourhoods of the identity of $\mathrm{GL}_{\mathbb{C}}(V)$ do not contain any non-trivial subgroups. Therefore, the same reasoning as in Lemma 1.15 shows that continuous finite-dimensional representations of G have open kernel too. That is, the homomorphism is continuous with respect to any topology on $\mathrm{GL}_{\mathbb{C}}(V)$.

However, for infinite-dimensional representations V, equipping $\mathrm{GL}_{\mathbb{C}}(V)$ with a topology is not as straightforward, and the requirement of having an open kernel is too restrictive. Here is where the notion of smooth representation becomes relevant, for which we must first introduce the module of invariants and coinvariants.

Definition 1.24. Let $H \leq G$ be groups and (π, V) a representation of G. We define the H-invariants of V to be

$$V^H := \{v \in V : \pi(h)v = v \text{ for all } h \in H\},$$

and the H-coinvariants to be

$$V_H := V/V(H) \text{ where } V(H) = \mathrm{Span}_{\mathbb{C}}\{v - \pi(h)v : v \in V, h \in H\}.$$

That is, V^H (resp. V_H) is the largest subspace (resp. quotient) on which H acts trivially.

Definition 1.25. A representation V of G is *smooth* if for all $v \in V$ there exists a compact open subgroup $K \leq G$ such that $v \in V^K$. In other words,

$$V = \bigcup_K V^K$$

as we range over all compact open subgroups K of G. We say that V is *admissible* if V^K is finite dimensional for all compact open K.

Smooth representations of G are a full abelian subcategory of $\mathrm{Rep}(G)$ denoted by $\mathrm{Smo}(G)$.

Remark 1.26. If (π, V) is a finite-dimensional smooth representation and $\{v_1, \ldots, v_n\}$ is a $\mathbb{C}$-basis such that $v_i \in V^{K_i}$ for some open compact subgroups K_i, then

$$K := \bigcap_{i=1}^n K_i \subseteq \ker \pi$$

is open and compact too, so the kernel is open. Conversely, if $\ker \pi$ is open, then there is some open compact subgroup K fixing all of V, so in this case smooth and continuous coincide.

As we hinted in Remark 1.8, smooth representations of locally profinite groups have remarkable algebraic structures, and they share many properties with representations of finite groups, particularly if the group is compact (and thus profinite). A direct application of Zorn's Lemma provides the following useful criterion to determine whether a representation is semisimple.

Proposition 1.27. *Let (π, V) be a smooth representation of a locally profinite group G. The following are equivalent:*

1. *V is the sum of its irreducible G-subspaces.*

2. *V is the direct sum of a family of irreducible G-subspaces (i.e. V is semisimple)*

3. *any G-subspace of V has a G-complement in V.*

Proof. [BH06, Lemma 2.2] □

Using this proposition, we can now prove that smooth representations of profinite groups behave in a similar way to those of finite groups. We note that any open compact subgroup K of a locally profinite group G is profinite, and that any smooth G-representation is natually a smooth K-representation by restriction. Therefore, the following results apply for any open compact subgroup of G.

Proposition 1.28. *Let (π, V) be a representation of a profinite group K. If V is irreducible then it is finite dimensional. If V is finite dimensional, then it is semisimple.*

Proof. The first statement is a matter of following the definitons. Fix any non-zero $v \in V$, and suppose $v \in V^{K_0}$ for some open compact $K_0 \leq K$. Then the subspace

$$U = \mathrm{Span}\{\pi(k)v : k \in K\} = \mathrm{Span}\{\pi(k)v : k \in K/K_0\}$$

is clearly a K-subspace and it is also finite dimensional since K_0 is open and K is compact, so $[K : K_0]$ is finite.

To prove the second statement, let v and K_0 be as above. By replacing K_0 by $\cap_{g \in K/K_0} gK_0g^{-1}$ if needed, we may assume that K_0 is normal in K. As above, the subspace

$$W = \mathrm{Span}\{\pi(k)v : k \in K\}$$

is finite dimensional and K_0 acts trivially on it. Thus W factors through a finite dimensional representation of the finite group K/K_0. By Maschke's Theorem, W is then the sum of its irreducible K subspaces. Since v was arbitrary this shows that condition 1. of Proposition 1.27 is satisfied, so V is semisimple.

□

This proposition has important structural results. Let $\hat{K}$ denote the set of equivalence classes of irreducible smooth representations of K. As we shall see, this notation is consistent with $\hat{F}$ since all irreducible smooth representations of F are one-dimensional.

Let (π, V) be a smooth representation of a locally profinite group G and let K be an open compact subgroup. For each $\rho \in \hat{K}$, let V^ρ be the sum of all irreducible K-subspaces of V isomorphic to ρ, the *ρ-isotypic component* of V. In particular, $V^{1_K} = V^K$.

Proposition 1.29. *Let G be a locally profinite group and K a compact open subgroup of G. Let $(\tau, U), (\pi, V)$, $(\sigma, W) \in \mathrm{Smo}(G)$ and $a : U \to V$ and $b : V \to W$ be G-homomorphisms.*

1. *The space V is the sum of the K-isotypic components:*

$$V = \bigoplus_{\rho \in \hat{K}} V^\rho.$$

2. *The following holds:*

$$W^\rho \cap b(V) = b(V^\rho).$$

3. *The sequence*

$$U \xrightarrow{a} V \xrightarrow{b} W$$

is exact if and only if

$$U^K \xrightarrow{a} V^K \xrightarrow{b} W^K$$

is exact for every compact open subgroup K of G.

4. *Denoting by $V(K)$ the span of the elements $v - \pi(k)v$ for $v \in V, k \in K$,*

$$V(K) = \bigoplus_{\substack{\rho \in \hat{K} \\ \rho \neq 1}} V^\rho \text{ and } V = V^K \oplus V(K)$$

and $V(K)$ is the unique K-complement of V^K in V.

Proof. [BH06, Proposition 2.3 and Corollary 1.2] □

As promised in §1.2, we now discuss the dual, restriction and induction functors in the context of smooth representations of locally profinite groups. From our previous discussion, two major problems arise in this context. Firstly, given a locally profinite group G and a subgroup H, there is no guarantee that H is locally profinite, and thus $\mathrm{Smo}(H)$ may not be well-defined. Secondly, when we perform some construction on a smooth representation (e.g., constructing its dual, inducing to a bigger group) there is no gurarantee that the resulting representation is smooth. Thankfully, both of these problems can be resolved in a straightforward way.

To ensure that H is locally profinite, we must add a condition on the topology of H. Based on Example 1.5(7), we just need to assume that H is a closed subgroup of G. In some cases, we will need to assume that H is also open, which is a more restrictive condition.

Definition 1.30. Let G be a locally profinite group. Define the *smoothness functor*

$$(-)^\infty : \mathrm{Rep}(G) \longrightarrow \mathrm{Smo}(G),$$
$$(\pi, V) \longmapsto (\pi^\infty, V^\infty)$$

by defining

$$V^\infty := \bigcup_K V^K \text{ and } \pi^\infty(g) := \pi(g)|_{V^\infty} \text{ for each } g \in G,$$

and K ranges over the compact open subgroups of G.

Remark 1.31. One should check that the smoothness functor is well-defined. In other words, we should check that the space V^∞ is preserved under the G-action, making it a G-representation. Let $v \in V^\infty$ and choose some open compact subgroup K such that $v \in V^K$. For any $g \in G$, we have that $\pi^\infty(g)v = \pi(g)v \in V^{gKg^{-1}} \subseteq V^\infty$ since $\pi(gkg^{-1})\pi(g)v = \pi(gk)v = \pi(g)v$ for any $k \in K$.

Furthermore, the functor $(-)^\infty$ is left-exact and it satisfies that

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty) \text{ for all } V \in \mathrm{Smo}(G), W \in \mathrm{Rep}(G).$$

Using these constructions, we can define the smooth dual, restriction and induction functors. Recall that a subgroup $H \leq G$ is also locally profinite if H is closed in G. In that case, the restriction functor (Definition

1.11) sends smooth representations of G to smooth representations of H. This is because the intersection of an open compact subgroup of G with H is still open compact in the subspace topology of H. The analogous statement does not hold for the dual and induction functors, so we must compose with the smoothness functor.

Definition 1.32. If G is a locally profinite group, define the *smooth dual functor*

$$\begin{aligned} \check{(-)} : \mathrm{Smo}(G) &\longrightarrow \mathrm{Smo}(G), \\ (\pi, V) &\longmapsto (\check{\pi}, \check{V}) \end{aligned}$$

by $(\check{\pi}, \check{V}) = (\pi^*, V^*)^\infty$.

The smooth dual satisfies an important property: if V is a smooth representation of G and $v \in V, v \neq 0$, then there is some $\check{v} \in \check{V}$ such that $\langle \check{v}, v \rangle \neq 0$. Consequently, the map $\delta : V \to \check{\check{V}}$ is injective, and the following proposition gives a criterion for surjectivity.

Proposition 1.33. *If G is a locally profinite group, and V is a smooth representation of G, the canonical map $\delta : V \longrightarrow \check{\check{V}}$ is an isomorphism if and only if (π, V) is admissible.*

Proof. [BH06, Proposition 2.9] □

We also define the smooth induction functor as the composition of the induction and smoothness functor.

Definition 1.34. Let G be a locally profinite group and $H \leq G$ a closed subgroup. Define the *smooth induction functor*

$$\begin{aligned} (\mathrm{Ind}_H^G(-))^\infty : \mathrm{Smo}(H) &\longrightarrow \mathrm{Smo}(G), \\ (\sigma, W) &\longmapsto (\Sigma, X)^\infty \end{aligned}$$

where we recall that X is the space of functions $f : G \to W$ satisfying $f(hg) = \sigma(h)f(g)$ for all $h \in H, g \in G$ and the action of Σ on X is given by right translation $\Sigma(g)f : x \mapsto f(xg)$.

Remark 1.35. Throghout this document, we will only be interested in studying the smooth induction of smooth representations. The idea is that smooth induction is the 'right' construction in this setting, which coincides with the abstract induction from Definition 1.11 when the group is finite with the discrete topology. Therefore, as it is common in the literature, we will use a slight abuse of notation and denote the smooth induction functor as Ind_H^G. We will write

$$\begin{aligned} \mathrm{Ind}_H^G : \mathrm{Smo}(H) &\longrightarrow \mathrm{Smo}(G), \\ (\sigma, W) &\longmapsto (\Sigma, X) \end{aligned}$$

where Σ is now the space of functions $f : G \to W$ satisfying:

1. For all $h \in H, g \in G$, we have $f(hg) = \sigma(h)f(g)$.

2. There is some open compact subgroup K of G such that $f(xg) = g(x)$ for all $x \in G$ and $g \in K$,

and Σ is the action on X by right translation.

The second condition is precisely the smoothness condition that appears after composing the abstract induction with the smoothness functor.

Since the action Σ on X is given by $\Sigma(g)f : x \mapsto f(xg)$, condition 2. is precisely the smoothness condition that $f \in X^K$ for some open compact subgroup K. As above, we will denote this representation of G by $\mathrm{Ind}_H^G\sigma$. Under these conditions, the first half of Frobenius reciprocity holds:

Theorem 1.36 (Frobenius reciprocity)**.** *Let (π, V) be a smooth representation of G, and (σ, W) a smooth representation of a closed subgroup H. Then the map*

$$\Psi : \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\sigma) \longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G\pi, \sigma),$$
$$\varphi \longmapsto a_\sigma \circ \varphi,$$

is a bijection that is functorial in both variables π, σ. Here $a_\sigma : \mathrm{Ind}_H^G\sigma \to W$ is the canonical map $a_\sigma(f) = f(1)$. In categorical terms,

$$\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G.$$

Proof. [BH06, 2.4 Frobenius Reciprocity] □

However, in this context, it is not the case that Ind_H^G is left adjoint to Res_H^G. With a small modification we can recover left exactness. Firstly, we note that to ensure that a_σ^c (to be defined shortly) is a H-homomorphism, we need the stronger assumption that H is open in G. Secondly, we observe that given representations (π, V) and (σ, W), of G and H respectively, $a_\sigma^c(w)$ is supported only in H for any $w \in W$. Hence, one should not consider the entire representation $\mathrm{Ind}_H^G\sigma$, but rather a subrepresentation of it. Here is the precise construction.

Definition 1.37. Let G be a locally profinite group, H a closed subgroup, and (σ, W) a smooth representation of H. Define the *compact induction functor*

$$c\text{-}\mathrm{Ind}_H^G : \mathrm{Smo}(H) \longrightarrow \mathrm{Smo}(G),$$
$$(\sigma, W) \longmapsto (\Sigma_c, X_c)$$

where, if $\mathrm{Ind}_H^G(\sigma, W) = (\Sigma, X)$, then

$$X_c := \{f \in X : \mathrm{supp} f \text{ in } H\backslash G \text{ is compact}\},$$

and Σ_c acts on X_c by right translation. We say that functions satisfying the later condition are *compactly supported modulo H*, and this condition is equivalent to $\mathrm{supp} f \subseteq HC$ for some compact subset C of G. The space X_c is closed under the action by Σ, so the functor is well-defined.

This construction is mainly of interest in the case when H is open in G, in which case a_σ^c is a H-homomorphism. This construction satisfies the second half of Frobenius reciprocity.

Theorem 1.38. *Let (π, V) be a smooth representation of G, and (σ, W) a smooth representation of an open subgroup H. Then the map*

$$\Psi^c : \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \sigma, \pi) \longrightarrow \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi)$$
$$\varphi \longmapsto \varphi \circ a_\sigma^c$$

is a bijection that is functorial in both variables π, σ. Here $a_\sigma^c : W \to c - \mathrm{Ind}_H^G \sigma$ is the map $w \mapsto f_w$, where f_w is supported in H and defined by $f_w(h) = hw$.

Proof. [BH06, 2.5 Theorem] □

In categorical terms, under the assumptions of this theorem we have

$$c\text{-}\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G.$$

1.5 Schur's Lemma

We end this section by discussing a version of Schur's Lemma for smooth representations of locally profinite groups. Throughout, G will denote a locally profinite group. We recall Schur's Lemma for finite groups.

Theorem 1.39. *Let $\mathbf{G}$ be a finite group and let (π, V) be a complex irreducible representation of $\mathbf{G}$. Then for any $\phi \in \mathrm{End}_{\mathbf{G}}(V)$, there is some $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$ for all $v \in V$. In other words, $\mathrm{End}_{\mathbf{G}}(V) \cong \mathbb{C}$.*

Schur's Lemma does not hold for complex smooth irreducible representations of a locally profinite group G. However, it is true under a mild hypothesis.

Hypothesis. For any compact open subgroup K of G, the set $K\backslash G$ is countable.

A short topological argument shows that if this hypothesis holds for one compact open subgroup K, then it holds for all of them.

Example 1.40. This hypothesis is satisfied by all locally profinite groups in Examples 1.5, which are the groups of interest for us. For example, if F is a local field of 0 characteristic, then $F = K_{\mathfrak{P}}$ is the completion of a number field at some prime $\mathfrak{P}$. Then the composite map $K \hookrightarrow K_{\mathfrak{P}} = F \twoheadrightarrow F/R$ is surjective. Since K is a number field, it is countable, which shows that F/R is countable too. The other cases are proven using similar (yet tedious) reasonings.

For the remainder of this section we assume the hypothesis.

Lemma 1.41. *Let (π, V) be an irreducible smooth representation of G. Then the dimension $\dim_{\mathbb{C}} V$ is countable.*

Proof. Let $v \in V$, $v \neq 0$ and let $K \leq G$ be an open compact subgroup such that $v \in V^K$. The set $\{\pi(g)v : g \in G\} = \{\pi(g)v : g \in K\backslash G\}$ spans V, by irreducibility of V, and it is countable. □

We are now ready to state and prove Schur's Lemma in our context.

Theorem 1.42 (Schur's Lemma)**.** *If (π, V) is a smooth irreducible representation of G, then* $\mathrm{End}_{\mathbb{C}} V \cong \mathbb{C}$.

Proof. [BH06, 2.6 Schur's Lemma] $\square$

This results has two important corollaries worth recalling. For the first one, we note that given a locally profinite group G, its centre Z is a closed subgroup of G and therefore a locally profinite group too.

Corollary 1.43. *Let (π, V) be an irreducible smooth representation of G. The centre Z of G acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$. In other words, $\pi(z)v = \omega_\pi(z)v$ for all $v \in V$ and $z \in Z$.*

Proof. For any $z \in Z$, the automorphism $\pi(z) : V \to V$ lies in $\mathrm{End}_G(V) \cong \mathbb{C}$. Hence, the desired map $\omega_\pi : Z \to \mathbb{C}^\times$ does indeed exist, and it is a group homomorphism. To prove smoothness, we note that if K is an open compact subgroup such that $V^K \neq 0$, then ω_π is trivial on the open compact subgroup $K \cap Z$ of Z. So ω_π is indeed a character of Z. $\square$

The character ω_π is called the *central character* of (π, V).

Corollary 1.44. *If G is abelian, any irreducible smooth represenation of G is one dimensional.*

This justifies the notation $\hat{K}$ for the set of equivalence classes of irreducible smooth representations of a locally profinite group K, since this notation can now be seen to coincide with the set of characters $\hat{F}$ of F.

2 Measure and the Duality Theorem

So far, we have introduced the central objects that we will study throughout: locally profinite groups and smooth representations. In addition, we have given a complete classification of the equivalence classes of irreducible smooth representations of a local field F. These are all 1-dimensional by Schur's Lemma, and we have Additive Duality, $\hat{F} \cong F$, by Theorem 1.21.

Classifying irreducible smooth representations of other locally profinite groups is considerably harder. Even the structure of the group of characters of $F^\times$ is more subtle. To describe the local Langlands correspondence for GL_2, we will need a classification theorem of irreducible smooth representations of $\mathrm{GL}_2(F)$. We will focus on a particular family of them, the so-called principal series representations. In order to study the group $\mathrm{GL}_2(F)$ and its subgroups, we study certain functions defined on them. To do so, we must first develop some measure theory on locally profinite groups. This is precisely the aim of this chapter, which follows a similar development to [BH06, Chapter 3].

We finish this section by studying the relationship between induction and duality, which is encapsulated by the Duality Theorem 2.11.

2.1 The Space $C_c^\infty(G)$ and the Haar Measure

Let G be a locally profinite group. Denote by $C_c^\infty(G)$ the space of functions $f : G \to \mathbb{C}$ that are locally constant and compactly supported.

Exercise. Show that a function $f : G \to \mathbb{C}$ lies in $C_c^\infty(G)$ if and only if it is a finite linear combination of characteristic functions of double cosets KgK for some open compact subgroup K of G.

The space $C_c^\infty(G)$ is a complex vector space and admits two natural actions of G by left and right translation:

$$\lambda_g f : x \longmapsto f(g^{-1}x), \quad \text{and} \quad \rho_g f : x \longmapsto f(xg),$$

for $x, g \in G$ and $f \in C_c^\infty(G)$. These actions endow $C_c^\infty(G)$ with the structure of a smooth representation of G, because characteristic functions of double cosets of K are invariant under translation by K.

Remark 2.1. The representation $(C_c^\infty(G), \rho) \in \mathrm{Smo}(G)$ is isomorphic to $c\text{-}\mathrm{Ind}_{\{1\}}^G \mathbb{1}$, where $\{1\}$ is the trivial subgroup of G.

We are now ready to define the notion of a Haar integral and Haar measure.

Definition 2.2. A *left Haar integral* on G is a non-zero linear functional

$$I : C_c^\infty(G) \longrightarrow \mathbb{C}$$

such that

(1) $I(\lambda_g f) = I(f),\ g \in G,\ f \in C_c^\infty(G)$, and

(2) $I(f) \geq 0$ for any $f \in C_c^\infty(G)$ such that $\mathrm{Im}(f) \subseteq \mathbb{R}_{\geq 0}$.

A *right Haar integral* is defined analogously by replacing λ_g with ρ_g.

The usefulness of the Haar integral relies on the fact that locally profinite groups possess essentially one unique left Haar integral.

Proposition 2.3. *There exists a left Haar integral $I : C_c^\infty(G) \to \mathbb{C}$. Moreover, a linear functional $I' : C_c^\infty(G) \to \mathbb{C}$ is a left Haar integral if and only if $I' = cI$ for some constant $c > 0$.*

Proof. [BH06, Proposition 3.1] □

Whenever we have a left Haar integral I, we can define the associated *left Haar measure* as follows. Let $S \subset G$ and let Γ_S be its characteristic function. Then $\Gamma_S \in C_c^\infty(G)$ if and only if S is open and compact. In that case, we define

$$\mu_G(S) = I(\Gamma_S)$$

to be the Haar measure of S. We note that $\mu_G(S) > 0$ when S is nonempty, and by left invariance, $\mu_G(gS) = \mu_G(S)$ for any $g \in G$. The relationship is commonly expressed by using the usual integral notation

$$I(f) = \int_G f(g) d\mu_G(g), \quad f \in C_c^\infty(G). \tag{†}$$

This choice of notation is motivated by the fact that one can also recover the left Haar integral from the left Haar measure. Indeed, since $f \in C_c^\infty(G)$ is locally constant and has constant support, we can express $f = \sum_{i=1}^r \alpha_i \mathbb{1}_{Kg_iK}$ for some open compact subgroup K, $g_i \in G, \alpha_i \in \mathbb{C}$ and $r \geq 1$. Then, Equation † represents the finite sum

$$I(f) = \sum_{i=1}^r \alpha_i \mu_G(Kg_iK)$$

from which we can recover the left Haar integral from the left Haar measure. Therefore, both notions carry essentially the same information. During our discussion, we will often fix a left Haar measure on G and then consider the Haar integral induced by the measure. This makes some arguments more natural to follow.

Example 2.4. The notion of a left Haar measure is only determined up to a constant. In practice, to uniquely determine the measure, we associate a particular open compact subset with a value. For example, if $G = F$ is a local field, one commonly chooses μ_F so that $\mu_F(R) = 1$, where R is the valuation ring. Under this choice, we calculate that $\mu_F(\mathfrak{p}^n) = q^{-n}$.

Left Haar measures behave predictably under usual group constructions. For example, if G_1, G_2 are profinite groups, then $G = G_1 \times G_2$ is also a profinite group, and we have an isomorphism

$$\begin{aligned} C_c^\infty(G_1) \otimes C_c^\infty(G_1) &\longrightarrow C_c^\infty(G) \\ \sum_{i=1}^r f_i^1 \otimes f_i^2 &\longmapsto \left((g_1, g_2) \mapsto \sum_{i=1}^r f_i^1(g_1) f_i^2(g_2) \right). \end{aligned}$$

If μ_i is a left Haar measure on G_i for $i = 1, 2$, then there is a unique left Haar measure μ_G on G such that

$$\int_G f_1 \otimes f_2(g) d\mu_G(g) = \int_{G_1} f_1(g_1) d\mu_1(g_1) \int_{G_2} f_2(g_2) d\mu_2(g_2),$$

usually dentoted $\mu_G = \mu_1 \otimes \mu_2$.

2.2 The Modular Character of a Group

Of course, the discussion from the previous subsection holds if we replace 'right' by 'left' throughout. At this point it is therefore natural to ask whether a left Haar integral I on G is also a right Haar integral. This important consideration motivates the following definition.

Definition 2.5. A locally profinite group G is *unimodular* if any left Haar integral (resp. measure) on G is also a right Haar integral (resp. measure).

As a first observation, we note that if the group G is abelian, then $\lambda_g f = \rho_{g^{-1}} f$, and therefore G is unimodular. However, for general locally profinite groups this is not always the case.

To investigate this, choose some left Haar measure μ_G on a locally profinite group G (not necessarily abelian), and consider the functional

$$\begin{aligned} I_g : C_c^\infty(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto \int_G f(xg) d\mu_G(x). \end{aligned}$$

In other words, if I is the associated left Haar integral of μ_G, then $I_g(f) = I(\rho_g f)$. Since the actions of G on $C_c^\infty(G)$ by left and right translation commute,

$$I_g(\lambda_h f) = I(\rho_g \lambda_h f) = I(\lambda_h \rho_g f) = I(\rho_g f) = I_g(f)$$

and so I_g is also a left Haar integral. Therefore, there is a unique $\delta_G(g) \in \mathbb{R}_+^\times$ such that $\delta_G(g) I_g(f) = I(f)$ for all $f \in C_c^\infty(G)$. In the integral notation, this means that

$$\delta_G(g) \int_G f(xg) d\mu_G(x) = \int_G f(x) d\mu_G(x)$$

for all $f \in C_c^\infty(G)$. Moreover, the map δ_G also interacts predictably with the left Haar measure. If S is an open compact subset of G and $f = \Gamma_S$ is its characteristic function then one obtains that

$$\delta_G(g) \mu_G(Sg) = \mu_G(S),$$

which also uniquely identifies $\delta_G(g)$.

Lemma 2.6. *The map $\delta_G : G \to \mathbb{R}_+^\times$ is a homomorphism independent of the choice of left Haar integral I and it is trivial on any open compact subgroup of G. In particular, δ_G is a character of G.*

Proof. By above, we have that

$$\delta_G(gh) I(\rho_{gh} f) = I(f) = \delta_G(g) I(\rho_g f) = \delta_G(g) \delta_G(h) I(\rho_h \rho_g f)$$

for any $g, h \in G$ and $f \in C_c^\infty(G)$. By uniqueness of δ_G and the fact that $\rho_{gh} = \rho_g \rho_h$, it follows that δ_G is a homomorphism. The fact that it is independent of the left Haar measure follows immediately from its definition and Proposition 2.3. If K is an open compact subgroup of G and $k \in K$, then by choosing $f = \Gamma_K$ to be the characteristic function of K, it follows that $\rho_k f = f$ and therefore $\delta_G(k) = 1$. $\square$

Definition 2.7. For G a locally profinite group, the character $\delta_G : G \to \mathbb{C}$ is called the *modular character* of G

Lemma 2.8. *Let G be a locally profinite group let and $\delta_G : G \to \mathbb{C}$ be its modular character. Then G is unimodular if and only if δ_G is trivial.*

Proof. Let I be a left Haar integral on G. Then G is unimodular if and only if I is a right Haar integral. This is equivalent to $I(f) = I(\rho_g f) = I_g(f) = \delta(g)^{-1} I(f)$ for every $g \in G$. But this is clearly equivalent to δ_G being trivial. $\square$

Finally, when the group G is not unimodular, the modular character δ_G gives a canonical relationship between left and right Haar integrals.

Lemma 2.9. *Let I be a left Haar integral on G with associated left Haar measure μ_G. If δ_G is the modular character of G, then the functional*

$$\begin{aligned} J : C_c^\infty(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x) \end{aligned}$$

is a right Haar integral for G.

Proof. The functional J can also be expressed as $J(f) = I(\delta_G^{-1} f)$. We note that $\delta_G^{-1} \rho_g(f) = \delta_G(g) \rho_g \delta_G^{-1} f$ as elements of $C_c^\infty(G)$ for all $g \in G$ and $f \in C_c^\infty(G)$. Hence,

$$J(\rho_g f) = I(\delta_G^{-1} \rho_g f) = \delta_G(g) I(\rho_g \delta_G^{-1} f) = I(\delta_G^{-1} f) = J(f)$$

for every $g \in G$ and $f \in C_c^\infty(G)$, as desired. $\square$

2.3 Positive Semi-invariant Measures and the Duality Theorem

To classify the principal series representations of $\mathrm{GL}_2(F)$ in the following section, one needs to understand the interaction between the induction and the duality functor for smooth representations of locally profinite groups and their closed subgroups. To this aim, we need to develop one last bit of machinery from measure theory called positive semi-invariant measures, which generalise the notion of Haar measures.

Let G be a locally profinite group and let H be a closed subgroup. Fix some character θ of H and consider the space of functions $f : G \to \mathbb{C}$ that are G-smooth under right translation, are compactly supported modulo H and satisfy

$$f(hg) = \theta(h) f(g), \quad h \in H, g \in G.$$

This space is the compact induction $c\text{-Ind}_H^G\theta$, but in analogy to $C_c^\infty(G) = c\text{-Ind}_{\{1\}}^G\mathbb{1}$ we denote it as $C_c^\infty(H\backslash G, \theta)$. At this point it is natural to ask if there exists some non-zero linear functional $I_\theta : C_c^\infty(H\backslash G, \theta) \to \mathbb{C}$ such that $I_\theta(\rho_g f) = I_\theta(f)$ for all $g \in G$. As it turns out, this is not always possible and there is a simple criterion to determine when it is.

Proposition 2.10. *Let $\theta : H \to \mathbb{C}^\times$ be a character of H. Then there exists a non-zero linear functional $I_\theta : C_c^\infty(H\backslash G, \theta) \to \mathbb{C}$ such that $I_\theta(\rho_g f) = I_\theta(f)$ for all $g \in G$ and $f \in C_c^\infty(H\backslash G, \theta)$ if and only if $\theta\delta_H = \delta_G|_H$.*

Furthermore, when this holds, the functional I_θ is uniquely determined up to a constant.

Proof. [BH06, Proposition 3.4] □

We remark that this is a generalisation of Proposition 2.3; indeed, by setting $H = \{1\}$ one recovers the usual right Haar integral on G. Similarly to the above case, when $\theta = \delta_H^{-1}\delta_G|_H$, one commonly expresses the functional I_θ with the integral notation

$$I_\theta(f) = \int_{H\backslash G} f(g)d\mu_{H\backslash G}(g), \quad f \in C_c^\infty(H\backslash G, \theta),$$

where $\mu_{H\backslash G}$ is called a *positive semi-invariant measure* on $H\backslash G$. Also, since such a θ for which Proposition 2.10 holds is uniquely defined, it is common to write $\delta_{H\backslash G}$ for $\delta_H^{-1}\delta_G|_H$. We now have all the required machinery to describe the Duality Theorem.

Theorem 2.11 (Duality Theorem)**.** *Let H be a closed subgroup of a locally profinite group G and let $\dot{\mu}$ be a positive semi-invariant measure on $H\backslash G$. Let (σ, W) be a smooth representation of H. Then there is a natural isomorphism*

$$\left(c\text{-Ind}_H^G\sigma\right)^{\check{}} \cong \text{Ind}_H^G(\delta_{H\backslash G} \otimes \check{\sigma}),$$

which only depends on the choice of $\dot{\mu}$.

Proof. We sketch a proof to motivate why one would expect $\delta_{H\backslash G}$ to appear. For a detailed proof, see [BH06, Theorem 3.5]. Throughout, we view the action of $\delta_{H\backslash G} \otimes \check{\sigma}$ naturally on $\check{W}$ (where $\check{\sigma}$ acts). For $\phi \in c\text{-Ind}_H^G\sigma$ and $\Phi \in \text{Ind}_H^G\delta_{H\backslash G}\otimes\check{\sigma}$, we have that $\phi(g) \in W$ and $\Phi(g) \in \check{W}$ for any $g \in G$. We can then consider the function

$$f : g \longmapsto \langle\Phi(g), \phi(g)\rangle, \quad g \in G$$

where $\langle\cdot,\cdot\rangle$ is the standard evaluation pairing on $\check{W} \times W$. This function satisfies

$$f(hg) = \langle\Phi(hg), \phi(hg)\rangle = \delta_{H\backslash G}(h)\langle\check{\sigma}(h)\Phi(g), \sigma(h)\phi(g)\rangle = \delta_{H\backslash G}(h)\langle\Phi(g), \phi(g)\rangle = \delta_{H\backslash G}(h)f(g) \quad h \in H, g \in G,$$

so $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$. Therefore, there is a well-defined pairing

$$\begin{aligned}\Psi : \text{Ind}_H^G(\delta_{H\backslash G} \otimes \check{\sigma}) \times c\text{-Ind}_H^G\sigma &\longrightarrow \mathbb{C},\\ (\Phi, \phi) &\longmapsto \int_{H\backslash G}\langle\Phi(x), \phi(x)\rangle d\dot{\mu}(x).\end{aligned}$$

Crucially, this pairing is G-invariant. Indeed,

$$\Psi(\rho_g\Phi, \rho_g\phi) = \int_{H\backslash G} \langle \Phi(xg), \phi(xg)\rangle d\dot{\mu}(x) = \int_{H\backslash G} \langle \Phi(x), \phi(x)\rangle d\dot{\mu}(x) = \Psi(\Phi, \phi)$$

by right translation invariance of the positive semi-invariant measure on H/G. This induces a G-homomorphism $\mathrm{Ind}_H^G(\delta_{H\backslash G} \otimes \check{\sigma}) \to \left(c\text{-}\mathrm{Ind}_H^G\sigma\right)\check{}$. The remainder of the proof consists of proving that this is an isomoprhism.

Lemma 2.12. *The above pairing identifies* $\left(\mathrm{Ind}_H^G(\delta_{H\backslash G} \otimes \check{\sigma})\right)^K$ *bijectively with the linear dual of* $\left(c\text{-}\mathrm{Ind}_H^G\sigma\right)^K$.

Proof. We omit the proof of this result. The advantage is that one can explicitly describe a canonical basis for each space, which are canonically identified by the pairing. For a complete description, check [BH06, Lemma 3.5.2]. □

This concludes the proof of the Duality Theorem. □

2.4 Measure Theory on $GL_2(F)$

We now focus on the group $G = \mathrm{GL}_2(F)$ over a non-Archimedean local field F. This group will be the main object of study of the next chapter, where we will study and classify a large family of irreducible representations of G. To that aim, we first need to develop some measure theory associated to the group, and we do this now.

We begin by introducing some notation. Let

$$B = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F^\times, b \in F \right\}, \quad T = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F^\times \right\} \cong F^\times \times F^\times \quad \text{and} \quad N = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in F\} \cong F$$

be the Borel subgroup B of upper triangular matrices, the maximal torus T and the subgroup of nilpotent elements N of B, respectively. A simple calculation shows that N is a normal subgroup of B while T is not. Furthermore, $B \cap N = \{1\}$ and $B = NT$ and so it follows that $B = N \rtimes T$.

Although G is unimodular ([BH06, Proposition 7.5]), the Borel subgroup B is not. The failure of B to be unimodular is a consequence of the subgroups T and N not commuting. As T and N are abelian, they are unimodular, and so we may pick Haar measures dt and dn on T and N respectively. Define a linear function I on $C_c^\infty(B) = C_c^\infty(T) \otimes C_c^\infty(N)$ by

$$I(\Phi) = \int_T \int_N \Phi(tn) dndt.$$

Proposition 2.13. *I is a left Haar integral on B.*

Proof. Let $b = sm \in TN$. By left invariance of dt we have

$$\int_T \int_N \Phi(smtn) dtdn = \int_T \int_N \Phi(mtn) dtdn = \int_T \int_N \Phi(tt^{-1}mtn) dtdn.$$

Since we integrate N first, we are integrating over fixed values of t so that $t^{-1}mt \in N$ is just constant, so left invariance of dn lets us pull out the $t^{-1}mt$ factor, and we recover $\int_T \int_N \Phi(tn) dndt$. □

Proposition 2.14. *The modular character δ_B of the group B is*

$$\delta_B : tn \mapsto |t_2/t_1|, \quad n \in N, t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$$

Proof. By a similar argument as above, we have

$$\int_T \int_N \Phi(tnsm)dtdn = \int_T \int_N \Phi(tss^{-1}nsm)dtdn = \int_T \int_N \Phi(ts^{-1}ns)dtdn.$$

Identifying $N \cong F$ this is

$$\int_T \int_N \Phi\left(t \cdot \left(\begin{smallmatrix} 1 & s_1^{-1}xs_2 \\ 0 & 1 \end{smallmatrix}\right)\right) d\mu_F(x) = |s_1/s_2| \int_T \int_N \Phi(tn)dtdn$$

so by definition of the modular character we have $\delta_B(sm) = |s_2/s_1|$. □

The family of irreducible representations that we will study in the next chapter arise as subrepresentations of $\mathrm{Ind}_B^G \sigma$ where σ is a smooth representation of T. We remark that the Borel subgroup B is a normal subgroup of G and that G/B is compact and therefore the functors Ind_B^G and c-Ind_B^G coincide.

Hence, by the Duality Theorem, it follows that

$$(\mathrm{Ind}_B^G \sigma)\check{} \cong \mathrm{Ind}_B^G(\delta_B^{-1} \otimes \check{\sigma}),$$

for any smooth representation σ of T. This is slightly impractical, so one introduces a related functor that interacts well with duality.

Definition 2.15. Let G, B and T as above. Define the normalized induction functor

$$\begin{aligned} \iota_B^G : \mathrm{Smo}(T) &\longrightarrow \mathrm{Smo}(G), \\ \sigma &\longmapsto \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma). \end{aligned}$$

This functor is also additive and exact, and it gives the more natural formula

$$(\iota_B^G \sigma)\check{} \cong \iota_B^G \check{\sigma}.$$

3 Principal Series Representations of GL_2

Let F be a non-Archimedean local field, $G = \mathrm{GL}_2(F)$, and $B = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F^\times, b \in F\}$ the Borel subgroup of upper triangular matrices, so that $B = N \rtimes T$ for $T = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F^\times\} \cong F^\times \times F^\times$ and $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in F\} \cong F$. Between N and B we also have the mirabolic subgroup $M = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a \in F^\times, b \in F\}$ with $M/N \cong F^\times$.

In studying the local Langlands correspondence, we want to understand all the irreducible smooth representations of G. One method for producing representations of G is by induction from a subgroup of G. Typically one takes this subgroup to be 'parabolic'; in our case there is one nontrivial parabolic, namely B. From our decomposition $B = N \rtimes T$ (more generally we have a so-called Levi decomposition) we see that we can produce representations of B by inflating representations of the torus T. Since $T \cong F^\times \times F^\times$, the irreducible representations of T are products of characters of $F^\times$, which are relatively easy to get a handle on.

Definition 3.1. For $\chi : T \to \mathbb{C}^\times$ a character of the torus, we say that the representation $\mathrm{Ind}_B^G \chi$ is a *parabolically induced representation.* A *principal series representation* is an irreducible subrepresentation of a parabolically induced representation.

In this section, we will only concern ourselves with classifying the principal series representations of G. This means that we must understand how $\mathrm{Ind}_B^G \chi$ decomposes into irreducible representations of G, and also study the morphisms between them using Frobenius reciprocity.

To understand these decompositions, we want to study how they decompose into irreducibles over a less unwieldy subgroup of G, such as B. Note that restricting $\mathrm{Ind}_B^G \chi$ to B is analogous to applying Mackey theory in the finite group context. It turns out that the $\mathrm{Ind}_B^G \chi$ do not decompose any further over M than over B. On the other hand, the representation theory of M is very easy to classify - the combination of these two observations is what makes the mirabolic subgroup so 'miraculous'. To get representations of M we can induce from characters of N, or inflate from $M/N \cong F^\times$. There are many characters of $N \cong F$, in fact these are in bijection with F by Additive Duality 1.21. The key property of M is that conjugation by M acts transitively on these characters ψ, which greatly simplifies the representation theory of M coming via induction from N. The mirabolic M is also small enough that this induction, together with the characters of $F^\times$, give all irreducible representations of M.

In this section, we begin by studying the representations of N and introducing the Jacquet functor, before discussing representations of M. From there we determine that parabolically induced representations of G decompose over M with length at most 3. Theorem 3.21 gives the decomposition of $\mathrm{Ind}_B^G \chi$ into irreducible representations of G, and then Theorem 3.29 lists the isomorphism classes of principal series representations. The presentation follows sections 8 and 9 of [BH06].

3.1 Representations of N

We first study the representation theory of $N \cong F$. This is an abelian group so, by Schur's lemma, all irreducible representations are characters (Corollary 2.6.2 [BH06]). For finite abelian groups, any representation

V decomposes into a direct sum of characters. This is no longer true when $N \cong F$ is infinite, but it is still true that any vector in V is nonzero in some quotient on which N acts via a character. To formalise this, we define

Notation 3.2. Let V be a smooth representation of N and θ a character of N. Let $V(\theta) \leq V$ be the subspace spanned by $\{n \cdot v - \theta(n)v \mid n \in N, v \in V\}$. Set $V_\theta = V/V(\theta)$ so that N acts on V_θ by θ. When θ is trivial we write $V(N)$ and V_N respectively.

The following is a useful equivalent definition of $V(\theta)$:

Lemma 3.3. *The vector $v \in V$ lies in $V(\theta)$ if and only if*

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0$$

for some compact open subgroup N_0 of N.

In the lemma we restrict to compact opens for the integral to be well defined.

Proof. [BH06, Lemma 8.1]. □

Corollary 3.4. *The functor $V \mapsto V_\theta$ from smooth representations of N to complex vector spaces is exact.*

Proof. One checks formally that the functor is right exact. For left exactness we need to show that if $f : V \hookrightarrow V'$ is injective then $V_\theta \hookrightarrow V'_\theta$ is injective. If $v \in V$ with $f(v) \in V'(\theta)$, then

$$\int_{N_0} \theta(n)^{-1} n \cdot f(v) dn = 0$$

for some N_0 by the above lemma. Since f is compatible with the action of N, we can pull f out of the integral so that the injectivity of f implies

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0.$$

We deduce that $v \in V(\theta)$ by the above lemma. □

Proposition 3.5. *Let V be a smooth representation of N. For any $v \neq 0$ in V, there exists a character θ of N such that $v \notin V(\theta)$.*

Proof. [BH06, Proposition 8.1]. □

Corollary 3.6. *If V is a smooth representation of N such that $V_\theta = 0$ for all θ then $V = 0$.*

3.2 Representations of M

Now we consider V an irreducible smooth representation of M.

Lemma 3.7. *The subspace $V(N) \leq V$ is a representation of M, and so V_N is as well. Moreover, $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a \in F^\times\}$ permutes the subspaces $V(\theta)$ with $\theta \neq 1$ transitively, and hence the V_θ are isomorphic as vector spaces.*

Proof. The first claim comes from the computation

$$mn \cdot v - m \cdot v = n'm \cdot v - m \cdot v$$

for some $n' \in N$, using the fact that $N \lhd M$. For the second claim we have the computation

$$s(nv - \theta(n)v) = sns^{-1} \cdot sv - \theta(s^{-1}(sns^{-1})s)sv = n' \cdot sv - \theta(s^{-1}n's)sv$$

where $n' = sns^{-1} \in N$. Hence $sV(\theta) = V(\theta')$ where $\theta'(n) := \theta(s^{-1}ns)$. Now the computation

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

together with Additive Duality 1.21 implies the claim. $\square$

Theorem 3.8. *Let (π, V) be an irreducible smooth representation of M. Either*

- $\dim V = 1$ *and π is the inflation of a character of $M/N \cong F^\times$, or*
- $\dim V = \infty$ *and $\pi \cong c\text{--}\mathrm{Ind}_N^M \theta$, for any nontrivial character θ of N.*

This itself follows from the following theorems. To compare V and $c\text{--}\mathrm{Ind}_N^M \theta$, it is more natural to compare V and $\mathrm{Ind}_N^M V_\theta$. By Frobenius reciprocity,

$$\mathrm{Hom}_N(V, V_\theta) \cong \mathrm{Hom}_M(V, \mathrm{Ind}_N^M V_\theta).$$

Let $q_* : V \to \mathrm{Ind}_N^M(V_\theta)$ be the image of the quotient map $q : V \to V_\theta$.

Theorem 3.9. *The M-homomorphism $q_* : V \to \mathrm{Ind}_N^M V_\theta$ induces an isomorphism $V(N) \cong c\text{--}\mathrm{Ind}_N^M V_\theta$.*

Proof. [BH06, Theorem 8.3]. $\square$

Theorem 3.10. *For any nontrivial character θ of N, the smooth representation $c\text{--}\mathrm{Ind}_N^M \theta$ of M is irreducible.*

Proof. [BH06, Corollary 8.2] $\square$

Proof of Theorem 3.8. If V is an irreducible smooth representation of M, then either $V(N) = 0$ or $V(N) = V$. In the former case N acts trivially on V, so the action of M factors through $M/N \cong F^\times$. Schur's lemma implies that V is a character of M factoring through M/N.

In the latter case, $V_N = 0$, so we must have $V_\theta \neq 0$ for all nontrivial characters of N by Lemma 3.7 and Corollary 3.6. Thus the M-representation V must have infinite dimension, since there are infinitely many characters θ. Theorem 3.9 implies that $V = V(N)$ is isomorphic to $c\text{--}\mathrm{Ind}_N^M V_\theta$, which is a direct sum of copies of $c\text{--}\mathrm{Ind}_N^M \theta$. Since $c\text{--}\mathrm{Ind}_N^M \theta$ is irreducible by Theorem 3.10, we must have $V \cong c\text{--}\mathrm{Ind}_N^M \theta$. $\square$

3.3 Irreducible Principal Series Representations

Let V be a smooth representation of G. In the preceding subsections, we defined the quotient $V_N = V/V(N)$, called the N-coinvariants of V. As in Lemma 3.7, this is a representation of B (as $N \lhd B$). As N acts trivially on V_N, V_N inherits the structure of a representation of $T = B/N$.

Definition 3.11. Let V be a smooth representation of G (or B). The *Jacquet module* of V at N is the space of N-coinvariants V_N viewed as a representation of T. The *Jacquet functor* is the functor sending the G-representation (π, V) to the T-representation (π_N, V_N).

By Corollary 3.4, the Jacquet functor is exact.

If V is a representation of G, and χ is a character of T, then we have by Frobenius Reciprocity that

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_B(V, \chi)$$

But since χ as a character B has trivial N-action, maps $V \to \chi$ factor through V_N, and we obtain a version of Frobenius reciprocity for the Jacquet module:

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_T(V_N, \chi)$$

i.e. the Jacquet module is left adjoint to parabolic induction.

In the classical setting of representations of $\mathbf{G} = \mathrm{GL}_2(k)$ for a finite field k, we have the following dichotomy (where $\mathbf{B}, \mathbf{T}, \mathbf{N}$ are the appropriate subgroups of $\mathbf{G}$):

Lemma 3.12. *Let (π, V) be an irreducible representation of $\mathbf{G}$. The following are equivalent:*

1. *π contains the trivial character of $\mathbf{N}$*
2. *π is isomorphic to a $\mathbf{G}$-subrepresentation of $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \chi$ for some character χ of $\mathbf{T}$ inflated to $\mathbf{B}$.*

Proof. [BH06, Lemma 6.3]. □

Returning to $G = \mathrm{GL}_2(F)$, if (π, V) is a smooth representation, the restriction to N is no longer necessarily semisimple because F is of infinite order. We instead replace the condition that $\pi|_N$ contains the trivial character of N with the condition that N acts trivially on some nonzero quotient of V (which is an equivalent condition in the finite field case). This is measured by the Jacquet module V_N. There is the analogous dichotomy which tells us that principal series representations can be identified as the irreducible smooth representations of G with nonzero Jacquet module:

Proposition 3.13. *Let (π, V) be an irreducible smooth representation of G. The following are equivalent:*

1. *$V_N \neq 0$*
2. *π is isomorphic to a G-subrepresentation of $\mathrm{Ind}_B^G \chi$ for some character χ of T inflated to B.*

Proof sketch. (2) implies (1) is a consequence of Frobenius reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\chi) = \mathrm{Hom}_T(\pi_N, \chi)$$

Given (1), one shows by a technical argument that V_N is finitely generated as a representation of T. An application of Zorn's lemma allows us to construct a maximal T-subspace U of V_N, so that V_N/U is a nonzero irreducible T-representation, and is thus a character χ by Schur's lemma. The above Frobenius reciprocity implies (2). $\square$

Remark 3.14. The same proof holds for the finite field case, where we bypass the technical details in showing (1) implies (2) because any representation of the finite group T admits an irreducible quotient.

Remark 3.15. We ask for a nonzero Jacquet module V_N rather than a trivial N-subrepresentation of V because of the following fact:

Lemma 3.16. *Let (π, V) be an irreducible smooth representation of G with a nonzero vector $v \in V$ fixed by N. Then $\pi = \phi \circ \det$, for some character ϕ of $F^\times$. In particular, π is one dimensional.*

Proof sketch. The vector v is fixed by N, but also by a compact open subgroup K of G by smoothness. As we are working with F a nonarchimedean local field (as opposed to a finite field), this implies K contains a unipotent lower triangular matrix, and one shows that v is fixed by $\mathrm{SL}_2(F)$. Thus π factors through det. $\square$

Once again, let χ be a character of T and let (Σ, X) denote $\mathrm{Ind}_B^G\chi$. We want to study how X decomposes into irreducible G-representations. As mentioned earlier, we will begin by studying their decompositions over B or even M.

To begin with, X will never be irreducible over B because we always have the canonical B-homomorphism $\Sigma \to \chi$, given by sending $f \mapsto f(1) \in \mathbb{C}$. So we have an exact sequence of B-representations

$$0 \longrightarrow V \longrightarrow X \longrightarrow \mathbb{C} \longrightarrow 0,$$

where $V = \{f \in X \mid f(1) = 0\}$, and B acts on $\mathbb{C}$ via χ. Now we want to understand how V decomposes over B. We have another exact sequence of B-representations,

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0,$$

so we reduce to studying $V(N)$ and V_N. We will show that $V(N)$ is irreducible over B (and even over M), while V_N will be determined by the Restriction-Induction lemma.

The following lemma makes the structure of V more apparent.

Lemma 3.17. *Let $V = \{f \in X : f(1) = 0\}$. The map*

$$V \to C_c^\infty(N)$$
$$f(-) \mapsto f(w-)$$

is an N-isomorphism (with N acting by right translation on either side), where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof. We have the Bruhat decomposition $G = B \sqcup BwN$. Since $f(1) = 0$, and f is induced from B, we must have that f is supported on BwN. G-smoothness of f implies that f is also zero on some compact open $K \leq G$. This will contain $\left(\begin{smallmatrix} 1 & 0 \\ \varpi^n\mathcal{O} & 1 \end{smallmatrix}\right)$ for some n, so that f vanishes on

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in Bw \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

for all $x \in \varpi^n\mathcal{O}$. Thus $f(w-)$ is supported on $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in N$ with $v(y) > -n$ and so is compactly supported. G-smoothness of f also implies that $f(w-)$ is N-smooth. Since f is induced from B and is supported on BwN, the map is injective. Conversely, any $g \in C_c^\infty(N)$ determines $f \in \mathrm{Ind}_B^G\chi$ such that $f(w-) = g$ and $f(B) = 0$. □

Proposition 3.18. *For V as above, $V(N)$ is irreducible over M (and hence over B). Moreover, $V(N)$ is infinite dimensional.*

Proof. The idea will be to use Theorem 3.9, which tells us $V(N) \cong c\text{–}\mathrm{Ind}_N^M V_\theta$. This is irreducible over M (and infinite dimensional) if we can show that V_θ is one dimensional, by Theorem 3.10.

By the above lemma we can identify $V \cong C_c^\infty(N)$ as N-representations. But M also acts via right translation on V (since $BwB = BwN = BwM$), which gives the structure of a M-representation on $C_c^\infty(N)$. We can calculate it explicitly (but we won't need it), where

$$f\left(bw\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = f\left(b\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w \begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}\right)$$

tells us that the corresponding $M = F^\times N$ action on $C_c^\infty(N)$ is the composite of right translation by N with the action

$$a \cdot \phi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \chi_2(a)\phi\begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}$$

of $a \in F^\times$.

So now we may consider $V = C_c^\infty(N)$. The benefit is that for this representation, the spaces of coinvariants of characters θ of N are very simple. In particular, the map $f \mapsto \theta f$ is a linear automorphism of $C_c^\infty(N)$ taking $V(N)$ to $V(\theta)$, since

$$n \cdot f - f \mapsto \theta(n \cdot f) - \theta f = \theta(n)^{-1} n \cdot (\theta f) - \theta f \in V(\theta).$$

Hence all the V_θ have the same dimension as $V_N = V/V(N)$, which has dimension 1 (we can see this from the characterisation of $V(N)$ as the zeros of some integral (Lemma 3.3), or from the Restriction-Induction lemma to follow). The result follows from Theorem 3.9 and Theorem 3.10. □

We turn our attention to the Jacquet module V_N. Recall V fits in the exact sequence

$$0 \longrightarrow V \longrightarrow X = \mathrm{Ind}_B^G\chi \xrightarrow{f \mapsto f(1)} \mathbb{C} \longrightarrow 0$$

of smooth representations of B, where B acts via χ on $\mathbb{C}$. Since the Jacquet functor is exact, we get the exact sequence

$$0 \longrightarrow V_N \longrightarrow X_N \longrightarrow \mathbb{C} \longrightarrow 0$$

of T-representations. The following lemma determines the structure of V_N as a T-representation. This can be stated in more generality:

Lemma 3.19 (Restriction-Induction lemma)**.** *Let (σ, U) be a smooth representation of T and $(\Sigma, X) = \mathrm{Ind}_B^G \sigma$. Then there is an exact sequence of smooth T representations:*

$$0 \longrightarrow \sigma^w \otimes \delta_B^{-1} \longrightarrow \Sigma_N \longrightarrow \sigma \longrightarrow 0.$$

Here, $\sigma^w(t) := \sigma(wtw)$ for $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so that if σ is the character $\chi_1 \otimes \chi_2$ of T, then $\sigma^w = \chi_2 \otimes \chi_1$.

Proof. The proof of Lemma 3.17 generalises to show that the vector space $V = \{f \in X \mid f(1) = 0\}$ is isomorphic, as N-representations, to the space $\mathcal{S}$ of smooth compactly supported functions $N \to U$, by identifying f with $f(w-)$.

We can define a map $\mathcal{S} \to U$ by

$$g = f(w-) \mapsto \int_N f(wn)dn,$$

where this integral is finite since g is compactly supported. By Lemma 3.3, this induces an isomorphism $\mathcal{S}_N \cong U$.

Now V also carries the structure of a B-representation as well, since $BwB = BwN$. We can repeat the same calculation as in the previous proposition, replacing $F^\times$ with $T \cong F^\times \times F^\times$, to compute the action of $B = TN$ on $\mathcal{S}$. As usual, N acts via right translation. If $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$, then for $\phi \in \mathcal{S}$,

$$t \cdot \phi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \sigma^w(t)\phi \begin{pmatrix} 1 & \frac{t_2}{t_1}x \\ 0 & 1 \end{pmatrix}.$$

Thus the T-representation structure on $U \cong \mathcal{S}_N \cong V_N$ is given by

$$t \cdot \int_N f(wn)dn = \sigma^w(t) \left|\frac{t_1}{t_2}\right| \int_N f(wn)dn,$$

which is $\sigma^w \otimes \delta_B^{-1}$. ∎

Corollary 3.20. *As a representation of B or M, $\mathrm{Ind}_B^G \chi$ has composition length 3. Two of the factors have dimension 1, and the other is infinite dimensional.*

Proof. This follows from the exact sequences

$$0 \longrightarrow V \longrightarrow \mathrm{Ind}_B^G \mathbb{C} \longrightarrow \chi \longrightarrow 0$$

and

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

where we saw that $V(N)$ is irreducible and infinite dimensional, and $V_N \cong \chi^w \otimes \delta_B^{-1}$. ∎

So we understand how $\mathrm{Ind}_B^G\chi$ decomposes into irreducible B-representations, and we want to understand its decomposition into G-representations. Our goal is to prove the following:

Theorem 3.21 (Irreducibility Criterion)**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let $X = \mathrm{Ind}_B^G\chi$.*

1. *The representation X of G is irreducible if and only if $\chi_1\chi_2^{-1}$ is either the trivial character of $F^\times$, or the character $x \mapsto |x|^2$ of $F^\times$.*

2. *Suppose X is reducible. Then*

    - *the G-composition length of X is 2*
    - *one factor has dimension 1, the other is infinite dimensional*
    - *X has a 1-dimensional G-subspace exactly when $\chi_1\chi_2^{-1} = 1$*
    - *X has a 1-dimensional G-quotient exactly when $\chi_1\chi_2^{-1}(x) = |x|^2$.*

We make some comments in preparation for the proof. Suppose X is a reducible representation of G, and X_0 a nonzero proper subrepresentation. If X_0 is finite-dimensional, then its composition factors over B can only consist of the 1-dimensional composition factors of X over B described in Corollary 3.20. If X_0 is infinite dimensional, then it contains the infinite-dimensional B-composition factor of Corollary 3.20, and so the quotient X/X_0 can only consist of the 1-dimensional factors. In all, if X is reducible then it has a finite dimensional (dimension 1 or 2) G-subspace or G-quotient. By taking duals we can assume we are in the first case. In the Irreducibility Criterion, we want to show that this implies $\chi_1 = \chi_2$ and that X has a 1-dimensional G-subspace.

Definition 3.22. Let π be a smooth representation of G and ϕ a character of $F^\times$. The twist of π by ϕ is the representation $\phi\pi$ of G defined by

$$\phi\pi(g) = \phi(\det g)\pi(g).$$

In this way, for a character $\chi = \chi_1 \otimes \chi_2$ of T, we have $\phi\chi = \phi\chi_1 \otimes \phi\chi_2$.

Lemma 3.23. *For χ a character of T and ϕ a character of $F^\times$, we have $\mathrm{Ind}_B^G(\phi\chi) = \phi\mathrm{Ind}_B^G\chi$.*

Proof. Since $\phi\chi(b) = \phi \circ \det(b)\chi(b)$ for any $b \in B$, where χ is inflated from T, we see that

$$(\phi \circ \det)(bg)f(bg) = \phi\chi(b)(\phi \circ \det)(g)f(g)$$

for any $f \in \mathrm{Ind}_B^G\chi$. Thus the map $f \mapsto (\phi \circ \det)f$ from $\mathrm{Ind}_B^G\chi \to \mathrm{Ind}_B^G(\phi\chi)$ is well defined on the underlying vector spaces. This induces an isomorphism of representations of G, $\phi\mathrm{Ind}_B^G\chi \cong \mathrm{Ind}_B^G(\phi\chi)$. □

Proposition 3.24. *The following are equivalent:*

1. $\chi_1 = \chi_2$
2. *X has a 1-dimensional N-subspace.*

If this holds then this subspace is also a G-subspace of X not contained in V.

Proof. (1) implies (2): since induction commutes with twisting we may assume $\chi_1 = \chi_2 = 1$. Then any nonzero constant function spans a 1-dimensional G-subspace (not just N-subspace) of $X = \mathrm{Ind}_B^G 1$.

(2) implies (1): suppose this subspace is spanned by f. The group N acts as a character on this subspace via right translation. We cannot have $f \in V$ (meaning $f(1) = 0$) because we saw earlier that f would then have support in some BwN_0 for $N_0 \leq N$ open compact, and this is not closed under multiplication by N.

So $f \notin V$ and therefore its image spans $X/V \cong \mathbb{C}$, on which B acts via χ. On this quotient, N acts trivially because χ was inflated from $B/N = T$. Thus f is in fact fixed by N under right translation. But f is also fixed under right translation by some compact open of G, so for sufficiently large $|x|$ we have

$$\begin{aligned} f(w) = f\left(w\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\right) \\ &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\right) \\ &= \chi_1(-1)\left(\chi_1^{-1}\chi_2(x)\right) f(1). \end{aligned}$$

The first equality comes from f being fixed by N. The third equality comes from f being fixed by a compact open subgroup of G.

This tells us that $\chi_1^{-1}\chi_2(x)$ is constant for $|x|$ sufficiently large. In particular, for large $|x|$ we have $\chi_1^{-1}\chi_2(x) = \chi_1^{-1}\chi_2(x^2) = (\chi_1^{-1}\chi_2(x))^2$. We deduce that $\chi_1(x) = \chi_2(x)$ for $|x|$ sufficiently large. Now for any $y \in F^\times$, we can pick $|x|$ large enough so that $\chi_1(x) = \chi_2(x)$ and $\chi_1(xy) = \chi_2(xy)$, from which we deduce that $\chi_1(y) = \chi_2(y)$.

$\square$

Proof of Irreducibility Criterion. Assume that X is reducible and we are in the case that X has a finite dimensional G-subspace. It has a 1-dimensional N-subspace L because N is abelian. Then L is also a G-subspace by the above proposition. Since G must act via a character on L, it factors as $\phi \circ \det$, where $\chi_1 = \phi = \chi_2$.

Let Y be the G-representation X/L. Since L spans the vector space X/V, the B-homomorphism $V \hookrightarrow X \to X/L$ is surjective. It is injective since $L \cap V = 0$. Thus $Y \cong V$ as B-representations.

We need to show that X has G-length 2. By the Corollary 3.20 it has length at most 3. We know that V has B-length 2 with a 1-dimensional quotient V_N. If Y had G-length 2, then the B-factors of V are also G-factors, so that G must act on V_N, necessarily by a character $\phi' \circ \det$. But this is impossible because $B \leq G$ acts on V_N by $\phi\delta_B^{-1}$ by Restriction-Induction, and this does not factor through det on B. So we must have that Y is irreducible over G and so X has G-length 2.

In the other case we have a finite dimensional G-quotient. The smooth dual $X^\vee$ is then in the first case, where the Duality Theorem 2.11 tells us that $X^\vee \cong \mathrm{Ind}_B^G \delta_B^{-1}\chi^\vee$. If we write $\delta_B^{-1}\chi^\vee = \psi_1 \otimes \psi_2$ then we must have $\psi_1 = \psi_2$. Computing $\psi_1(x) = |x|^{-1}\chi_1(x)$ and $\psi_2(x) = |x|\chi_2(x)$ gives $\chi_1\chi_2^{-1} = |\cdot|^2$.

The converse direction to (1) follows from the previous proposition. $\square$

3.4 Classification of Principal Series Representations

Now that we've seen how parabolically induced representations decompose into irreducibles, we want to classify the isomorphism classes.

Proposition 3.25. *Let χ, ξ be characters of T. The space* $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi)$ *is 1-dimensional if* $\xi = \chi$ *or* $\chi^w\delta_B^{-1}$ *and 0 otherwise.*

Proof. Frobenius reciprocity tells us

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi) \cong \mathrm{Hom}_T((\mathrm{Ind}\chi)_N, \xi).$$

From the Restriction-Induction lemma we have the exact sequence of T-modules

$$0 \longrightarrow \chi^w\delta_B^{-1} \longrightarrow (\mathrm{Ind}\chi)_N \longrightarrow \chi \longrightarrow 0.$$

By taking duals of these finite dimensional T-modules, we see that both χ and $\chi^w\delta_B^{-1}$ are subrepresentations of $(\mathrm{Ind}\chi)_N$. In the case $\chi \neq \chi^w\delta_B^{-1}$ we must have $(\mathrm{Ind}\chi)_N = \chi \oplus \chi^w\delta_B^{-1}$ and the result follows. If $\chi = \chi^w\delta_B^{-1}$ then $\chi_1\chi_2^{-1}(x) = |x|$ so $\mathrm{Ind}\chi$ is irreducible and the result still follows from Schur's lemma. □

Remark 3.26. In the case that $\mathrm{Ind}\chi$ is irreducible, we deduce that $\mathrm{Ind}\chi \cong \mathrm{Ind}\chi^w\delta_B^{-1}$. And in the case $\mathrm{Ind}\chi$ is reducible, it is not semisimple, else $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\chi)$ would have dimension strictly greater than 1.

We can be more explicit in the reducible case. One can check that the conditions for reducibility in the Irreducibility Criterion are equivalent to χ being of the form $\chi = \phi 1_T$ or $\chi = \phi\delta_B^{-1}$ for ϕ a character of $F^\times$. Untwisting, we may as well assume $\phi = 1$ in what follows.

Definition 3.27. The *Steinberg representation* of G is defined by the exact sequence

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow \mathrm{St}_G \longrightarrow 0,$$

and is an infinite-dimensional irreducible smooth representation. By Restriction-Induction, the Jacquet module is $(\mathrm{St}_G)_N \cong \delta_B^{-1}$. The representations $\phi\mathrm{St}_G$ are called 'twists of Steinberg' or 'special representations'.

The case $\chi = \delta_B^{-1}$ can be dealt with by taking smooth duals (which is exact by [BH06, Lemma 2.10]) to get

$$0 \longrightarrow \mathrm{St}_G^\vee \longrightarrow \mathrm{Ind}_B^G\delta_B^{-1} \longrightarrow 1_G \longrightarrow 0,$$

where we use the Duality Theorem 2.11. The Irreducibility Criterion implies that $\mathrm{St}_G^\vee$ is also irreducible, and in fact the previous proposition applied to $\chi = 1, \xi = \delta_B^{-1}$ implies that

$$\mathrm{St}_G \cong \mathrm{St}_G^\vee.$$

Notation 3.28. Define normalised induction by

$$\iota_B^G\sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma).$$

This has the benefit that $(\iota_B^G\sigma)^\vee \cong \iota_B^G\sigma^\vee$ (Theorem 2.11).

Theorem 3.29 (Classification Theorem)**.** *The following are all the isomorphism classes of principal series representations of G:*

- *the irreducible induced representations* $\iota_B^G\chi$ *when* $\chi \neq \phi\delta_B^{\pm 1/2}$ *for a character* ϕ *of* $F^\times$.

- *the one-dimensional representations* $\phi \circ \det$ *for* ϕ *a character of* $F^\times$.

- *the twists of Steinberg (special representations)* $\phi\mathrm{St}_G$ *for* ϕ *a character of* $F^\times$.

These are all distinct isomorphism classes except in the first case where $\iota_B^G\chi \cong \iota_B^G\chi^w$.

4 Functional Equation for GL_1

In the previous section, we classified the prinicipal series representations of $G = \mathrm{GL}_2(F)$ over a non-Archimedean local field F. For characters χ of $\mathrm{GL}_1(F)$, Tate's thesis [Tat67] associates a space $\mathcal{Z}(\chi)$ of zeta functions in a complex variable s. This space will, in a sense to be made precise, be generated by a single element, the L-function $L(\chi, s)$. The zeta functions will also satisfy a functional equation depending on the 'local constant' $\epsilon(\chi, s, \psi)$. Here $\psi : F \to \mathbb{C}^\times$ is a character whose purpose is to fix a form of Fourier transform on F. These definitions and results in Tate's thesis are intended to mimic the classical theory of L-functions due largely to Hecke, which encompass the Riemann zeta function. The L-function and local constant of a character $\chi : F^\times \to \mathbb{C}^\times$ will turn out to carry the essential information of χ. In the classical setting see, for example, the converse theorem of Weil reproduced in [Bum97, Theorem 1.5.1].

In the setting of irreducible smooth representations π of G, in particular the principal series representations π, we want to again associate a space $\mathcal{Z}(\pi)$ of zeta functions, an L-function $L(\pi, s)$ and a local constant $\epsilon(\pi, s, \psi)$ determining a functional equation.

We begin this section with a brief review of harmonic and Fourier analysis and the role it plays in representation theory. For more details, see [Bum97, Chapter 3.1]. Following the presentation in [GH24], we define the L-functions and local constants of characters of $F^\times$. We explain how this theory generalises to irreducible smooth representations π of G, culminating in the Theorems 5.14 and 5.24, which determine the functional equations satisfied by the zeta functions associated to π. Propositions 5.9 and 5.21 prove these in the case where $\pi = \iota_B^G \chi$ is a principal series representation. The case where $\iota_B^G \chi$ is reducible, so that π is only a subquotient, requires slightly more work. The results are summarised in Table 1. Finally, we prove a converse theorem for principal series representations of G.

4.1 Review of Harmonic Analysis

Take as motivation the representation theory of a finite group H. Every irreducible representation of H appears as a direct summand of the regular representation $\mathbb{C}[H]$, with some multiplicity. For a locally compact topological group $\mathbb{G}$ with Haar measure dg, the correct generalisation of $\mathbb{C}[H]$ is the space $L^2(\mathbb{G})$ of measurable functions $f : \mathbb{G} \to \mathbb{C}$ for which

$$\int_{\mathbb{G}} |f(g)|^2 dg < \infty.$$

The action of $\mathbb{G}$ is by right translation. If $\mathbb{G}$ is additionally abelian, the group $\hat{\mathbb{G}}$ of (unitary) characters of $\mathbb{G}$ is also a locally compact abelian group, the Pontryagin dual of $\mathbb{G}$.

Example 4.1. The Pontryagin duals of $\mathbb{G} = \mathbb{R}, \mathbb{Z}, \mathbb{R}/\mathbb{Z}$ are $\mathbb{R}, \mathbb{R}/\mathbb{Z}, \mathbb{Z}$ respectively. The characters of $\mathbb{R}$ are of the form $x \mapsto e^{-2\pi i x y}$ for $y \in \mathbb{R}$. The characters of $\mathbb{Z}$ are of the form $n \mapsto e^{-2\pi i n x}$ for $x \in \mathbb{R}/\mathbb{Z} \cong S^1$. The characters of $\mathbb{R}/\mathbb{Z}$ are of the form $x \mapsto e^{-2\pi i n x}$ for $n \in \mathbb{Z}$. In particular, $\mathbb{R}$ is self-dual.

On a suitable dense subset of $L^2(\mathbb{G})$ (the Schwartz space), one can define the Fourier transform $\hat{f} \in L^2(\hat{\mathbb{G}})$

of f by

$$\hat{f}(\chi) = \int_{\mathbb{G}} f(g)\chi(g)dg.$$

The Fourier transform uniquely extends to a map $L^2(\mathbb{G}) \to L^2(\hat{\mathbb{G}})$. For suitable choices of Haar measures there is then a Fourier inversion formula

$$\hat{\hat{f}}(g) = f(-g),$$

so that the above map is a bijection.

Example 4.2. For $\mathbb{G} = \mathbb{R}$, the Fourier transform of f is

$$\hat{f}(x) = \int_{\mathbb{R}} f(y)e^{-2\pi ixy}dy$$

which is the classical Fourier transform. Identifying $\hat{\mathbb{R}} = \mathbb{R}$, the Fourier transform gives an invertible map $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, so that any element of $L^2(\mathbb{R})$ can be expressed as an integral of elements of $\hat{\mathbb{R}}$.

Inside the representation $L^2(\mathbb{R})$ of $\mathbb{R}$ we therefore see this 'continuous spectrum' of the irreducible unitary representations (characters) of $\mathbb{R}$, parametrised by $\mathbb{R}$. Note, however, that each such character can not be realised as a subrepresentation of $L^2(\mathbb{R})$; for $y \in \mathbb{R}$ the character $x \mapsto e^{-2\pi ixy}$ is realised as the Fourier transform of a function on $\mathbb{R}$ supported only at y, but such a function is not in $L^2(\mathbb{R})$.

Example 4.3. For $\mathbb{G} = \mathbb{Z}$, the Fourier transform of f is

$$\hat{f}(x) = \sum_{\mathbb{Z}} f(n)e^{-2\pi inx}.$$

So any element of $L^2(\mathbb{R}/\mathbb{Z})$ can be expressed as a sum of unitary characters of $\mathbb{Z}$; we have a 'discrete spectrum'.

Remark 4.4. The terminology of discrete and continuous spectra comes from the analogy with the spectral theory of the Laplacian. Over $\mathbb{R}$, the Laplacian is $\Delta = \frac{\partial^2}{\partial x^2}$, and the characters $x \mapsto e^{-2\pi ixy}$ are eigenfunctions.

The dichotomy in the above examples is reflected in the compactness of S^1 and non compactness of $\mathbb{R}$. More generally,

Theorem 4.5 (Peter-Weyl)**.** *Let K be a compact Hausdorff topological group. Any unitary representation of K decomposes into a completed Hilbert space direct sum of irreducible unitary subrepresentations. There is a unitary equivalence*

$$L^2(K) \cong \widehat{\bigoplus}_{\pi \in \hat{K}} \mathrm{End}(V_\pi)$$

of representations of $K \times K$, where (π, V_π) ranges over the set $\hat{K}$ of equivalence classes of irreducible representations of K, and $\hat{\oplus}$ denotes the completed Hilbert space direct sum.

Proof. [DE09, Theorem 7.3.2] and [DE09, Theorem 7.2.3]. □

Even more generally, for so-called Type I groups one can decompose unitary representations through a combination of integrals and Hilbert space direct sums. See [GH24, Section 3.10] for further details.

Returning to $G = \mathrm{GL}_2(F)$, as this is not compact we would expect the regular representation $L^2(G)$ to decompose according to both a continuous spectra and a discrete spectra. This continous spectra is provided by the parabolically induced representations $\iota_B^G \chi$, where χ ranges over the characters of $T \cong F^\times \times F^\times$.

In order to compare representations of G and Galois representations through the local Langlands correspondence, we would like to classify them according to some common language. This is provided by the zeta functions, L-functions and functional equations discussed in this section.

The prototypical example of an L-function is the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$.

Proposition 4.6. *The function $\zeta(s) = \sum_{n \geq 1} n^{-s}$ satisfies the following properties:*

- *(Analytic continuation) The Riemann zeta functions converges absolutely to a holomorphic function on* $\mathrm{Re}(s) > 1$. *It has a unique analytic continuation to the complex plane, except the point $s = 1$ where $\zeta(s)$ has a simple pole.*

- *(Euler product) We have the identity*

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

  convergent for $\mathrm{Re}(s) > 1$.

- *(Functional equation) There is an explicit function $\gamma(s)$ such that $\zeta(1-s) = \gamma(s)\zeta(s)$.*

The approach of Tate in his thesis was to view the Riemann (And Dedekind) zeta functions from an adelic perspective. There the Euler product formulation is immediate and we only need to study the zeta functions locally. Attached to any character $\chi : F^\times \to \mathbb{C}^\times$ there is an associated space $\mathcal{Z}(\chi)$ of zeta functions $\zeta(\Phi, \chi, s)$, where $\Phi \in C_c^\infty(F)$. The factor at the prime p of the Riemann zeta function corresponds to the trivial character of $\mathbb{Q}_p^\times$ and the function $\mathbb{1}_{\mathbb{Z}_p} \in C_c^\infty(\mathbb{Q}_p)$. A key ingredient in the proof of the functional equation of the Riemann zeta function is the Fourier transform over $\mathbb{C}$. In general, the functional equation associated to χ relates zeta functions $\zeta(\hat{\Phi}, \chi^{-1}, 1-s)$ and $\zeta(\Phi, \chi, s)$, where $\hat{\Phi}$ is the Fourier transform of Φ in $C_c^\infty(F)$.

4.2 The L-function of a Character of $F^\times$

Let F be a non-Archimedean local field, ϖ be a uniformiser and q be the size of the residue field. We will later define L-functions attached to an irreducible smooth representation of $\mathrm{GL}_2(F)$ and determine a functional equation they satisfy. The ideas involved in the proof that such objects exists are similar (and highly dependent) on the ideas associated to the development of the theory discussed in the previous section for the local field F.

Therefore, we explain these results first in the context of irreducible smooth representations χ of $\mathrm{GL}_1(F)$, necessarily a character $\chi : F^\times \to \mathbb{C}^\times$. To mimic the development we aim to follow for the case of irreducible representations of $\mathrm{GL}_2(F)$, we will first define the space $\mathcal{Z}(\chi)$ from which the definition of the L-function $L(\chi, s)$ arises naturally. Afterwards, we will introduce the analogue of the Fourier transform over F. This will lead us to the proof of the functional equation of GL_1 and the construction of the local constant $\varepsilon(\chi, s, \psi)$.

Let $\chi : F^\times \to \mathbb{C}^\times$ be a character of $F^\times$. We want to attach to this character an L-function $L(\chi, s)$ in the formal variable s. This is defined to be $(1 - \chi(\varpi)q^{-s})^{-1}$ when χ is unramified, and 1 otherwise. In order to generalise to GL_2 it would be preferable to have a more intrinsic definition.

Definition 4.7. For $\Phi \in C_c^\infty(F)$ and $\chi : F^\times \to \mathbb{C}^\times$, define the zeta function $\zeta(\Phi, \chi, s)$ to be

$$\zeta(\Phi, \chi, s) := \int_{F^\times} \Phi(x)\chi(x)|x|^s d^*x,$$

in the formal variable s, where $d\mu^*(x) = d^*x$ denotes any choice of Haar measure on $F^\times$.

We remark that we can equivalently rewrite the zeta function as

$$\zeta(\Phi, \chi, s) = \sum_{m \in \mathbb{Z}} z_m q^{-ms}$$

where

$$z_m = \int\limits_{\varpi^m \mathcal{O}_F^\times} \Phi(x)\chi(x)d^*x.$$

Note that $z_m = z_m(\Phi, \chi)$ vanishes for $m \ll 0$ because Φ is compactly supported on F, so $\mathrm{supp}\Phi \subseteq \mathfrak{p}^N$ for some sufficiently small N. In this way it is clear that $\zeta(\Phi, \chi, s) \in \mathbb{C}((q^{-s}))$. The zeta function $\zeta(\Phi, \chi, s)$ only depends on d^*x up to scaling. To remove this dependence we define the following space.

Definition 4.8. Let χ be a character of $F^\times$. Then we define the space of ζ-functions associated to χ as

$$\mathcal{Z}(\chi) = \{\zeta(\Phi, \chi, s) \mid \Phi \in C_c^\infty(F)\}.$$

Notation 4.9. For $a \in F^\times$ and $\Phi \in C_c^\infty(F)$, denote by $a\Phi$ the function $x \mapsto \Phi(a^{-1}x)$.

Lemma 4.10. *The space $\mathcal{Z}(\chi)$ is a $\mathbb{C}[q^{-s}, q^s]$-module, containing $\mathbb{C}[q^{-s}, q^s]$.*

Proof. Let $a \in F^\times$ of valuation $v_F(a)$. Then

$$\zeta(a\Phi, \chi, s) = \chi(a)q^{-v_f(a)s}\zeta(\Phi, \chi, s),$$

giving the desired module structure. To establish the containment we show that $\mathcal{Z}(\chi)$ contains a nonzero constant. Let d be such that $\chi|_{U_F^d} = 1$. Taking $\Phi = \mathbb{1}_{U_F^d}$, we see that

$$Z(\Phi, \chi, s) = \mu^*(U_F^d) \neq 0.$$

□

Proposition 4.11. *Let $\chi : F^\times \to \mathbb{C}^\times$ be a character. There exists a unique polynomial $P_\chi \in \mathbb{C}[X]$ with $P_\chi(0) = 1$ such that*

$$\mathcal{Z}(\chi) = P_\chi(q^{-s})^{-1} \cdot \mathbb{C}[q^{-s}, q^s].$$

Moreover, we have

$$P_\chi(X) = \begin{cases} 1 - \chi(\varpi)X & \textit{if } \chi \textit{ is unramified,} \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Suppose $\Phi(0) = 0$. Then $\Phi|_{F^\times} \in C_c^\infty(F^\times)$, and so Φ is identically zero on $\varpi^m \mathcal{O}_F^\times$ for $|m| >> 0$. Thus only finitely many of the coefficients z_m are nonzero, so that $\Phi \in \mathbb{C}[q^{-s}, q^s]$.

The space $C_c^\infty(F)$ is spanned by $C_c^\infty(F^\times)$ and $\mathbb{1}_{\mathcal{O}_F}$. We compute

$$\zeta(\mathbb{1}_{\mathcal{O}_F}, \chi, s) = \sum_{m \geq 0} \chi(\varpi^m) q^{-ms} \int_{\mathcal{O}_F^\times} \chi(x) d^*x.$$

If χ is unramified (trivial on $\mathcal{O}_F^\times$), this gives us

$$\sum_{m \geq 0} \chi(\varpi)^m q^{-ms} \mu^*(\mathcal{O}_F^\times) = (1 - \chi(\varpi) q^{-s})^{-1} \mu^*(\mathcal{O}_F^\times).$$

When χ is ramified the integral is zero. Indeed, translation invariance of d^*x implies

$$\int_{\mathcal{O}_F^\times} \chi(x) d^*x = \int_{\mathcal{O}_F^\times} \chi(xy) d^*x = \chi(y) \int_{\mathcal{O}_F^\times} \chi(x) d^*x$$

for any $y \in \mathcal{O}_F^\times$, so that this is zero if there is some y with $\chi(y) \neq 1$. This computation, together with the previous lemma, establish the result. $\square$

Remark 4.12. The computation in the proof above shows, in the case $\chi = 1$, that $\zeta(\mathbb{1}_{\mathcal{O}_F}, 1, s) = (1 - q^{-s})^{-1}$, provided that $\mu^*(\mathcal{O}_F^\times) = 1$. If $F = K_v$ is the completion of a number field K at a non-Archimedean place v, we recover the Euler factor of the Dedekind zeta function $\zeta_K(s)$ at the place v. This explains the naming of our zeta functions.

Remark 4.13. The computations of Proposition 4.11 show that each $\zeta(\Phi, \chi, s)$ converges absolutely and uniformly in vertical strips in some right half plane, and admit analytic continuation to a rational function in q^{-s}.

Definition 4.14. The L-function attached to a character χ of $F^\times$ is defined to be

$$L(\chi, s) = P_\chi(q^{-s})^{-1} = \begin{cases} (1 - \chi(\varpi) q^{-s})^{-1} & \text{if } \chi \text{ is unramified} \\ 1 & \text{otherwise,} \end{cases}$$

which indeed coincides with the classical language.

4.3 The Functional Equation

Next, taking from the classical study of the Riemann zeta function and its functional equation, we want to introduce an analogue of the Fourier transform over F. We replace the additive character $x \mapsto e^{2\pi i x}, x \in \mathbb{R}$ with any choice of additive character $\psi : F \to \mathbb{C}^\times$ with $\psi \neq 1$. In this way, by Additive Duality, all characters of F are of the form $y \mapsto \psi(ay), y \in F$ for some $a \in F$. The functions we will apply the Fourier transform to will be the algebra $C_c^\infty(F)$ of locally constant compactly supported functions $F \to \mathbb{C}$. For any choice of Haar measure μ on F, we now define the Fourier transform.

Definition 4.15. Let $\Phi \in C_c^\infty(F)$, $\psi : F \to \mathbb{C}^\times$ be an additive character of F, and μ be a Haar measure on F. The Fourier transform of Φ (with respect to ψ and μ) is

$$\hat{\Phi}(x) := \int_F \Phi(y)\psi(xy)d\mu(y).$$

To match the classical definition over $\mathbb{R}$, we would like the Fourier transform to preserve $C_c^\infty(F)$, and to have a Fourier inversion formula. Indeed:

Proposition 4.16. *The Fourier transform on $C_c^\infty(F)$ satisfies the following:*

- *For any $\Phi \in C_c^\infty(F)$, we have $\hat{\Phi} \in C_c^\infty(F)$.*
- *For any $\psi : F \to \mathbb{C}^\times$ with $\psi \neq 1$, there is a unique Haar measure μ_ψ on F such that for the associated Fourier transform we have*
$$\hat{\hat{\Phi}}(x) = \Phi(-x)$$
for any $\Phi \in C_c^\infty(F)$ and $x \in F$. This measure satisfies that $\mu_\psi(\mathfrak{o}) = q^{l/2}$, where l is the level of ψ.

Proof. [BH06, Proposition 23.1] $\square$

Notation 4.17. For the remainder of this subsection, $\psi \neq 1$ will be an additive character of F, and $\mu = \mu_\psi$ will denote the associated self-dual Haar measure on F.

As with the Riemann zeta function, we have functional equations for the zeta functions.

Theorem 4.18. *Let $\chi : F^\times \to \mathbb{C}^\times$. There is a unique $\gamma(\chi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \check{\chi}, 1-s) = \gamma(\chi, s, \psi)\zeta(\Phi, \chi, s)$$

for all $\Phi \in C_c^\infty(F)$, where $\check{\chi} = 1/\chi : F^\times \to \mathbb{C}^\times$.

Proof. [BH06, Theorem 23.3]. $\square$

Since $\mathcal{Z}(\chi) = L(\chi, s) \cdot \mathbb{C}[q^{-s}, q^s]$, it is natural to consider the terms $\frac{\zeta(\Phi,\chi,s)}{L(\chi,s)} \in \mathbb{C}[q^{-s}, q^s]$. This allows us to treat the case of χ ramified and unramified evenly.

Definition 4.19. Given characters χ and ψ of $F^\times$ and F respectively, we define

$$\varepsilon(\chi, s, \psi) := \gamma(\chi, s, \psi)\frac{L(\chi, s)}{L(\check{\chi}, 1-s)}.$$

Then $\varepsilon(\chi, s, \psi) \in \mathbb{C}(q^{-s})$ is known as *Tate's local constant.*

The functional equation for ζ can be rewritten as

$$\frac{\zeta(\hat{\Phi}, \check{\chi}, 1-s)}{L(\check{\chi}, 1-s)} = \varepsilon(\chi, s, \psi)\frac{\zeta(\Phi, \chi, s)}{L(\chi, s)}.$$

Corollary 4.20. *The local constant satisfies the functional equation*

$$\varepsilon(\chi, s, \psi)\varepsilon(\check{\chi}, 1-s, \psi) = \chi(-1).$$

The local constant is of the form

$$\varepsilon(\chi, s, \psi) = aq^{bs}$$

for some $a \in \mathbb{C}^\times$, $b \in \mathbb{Z}$.

Proof. The first statement comes from the Fourier inversion formula, where the $\chi(-1)$ term comes from the minus sign in $\hat{\hat{\Phi}}(x) = \Phi(-x)$. The functional equation implies that $\varepsilon(\chi, s, \psi)$ is a unit in $\mathbb{C}[q^{-s}, q^s]$, and the units are precisely the elements of the form aq^{bs} for $a \in \mathbb{C}^\times$ and $b \in \mathbb{Z}$. $\square$

5 Functional Equation for GL_2

We turn now to smooth representations π of $G = \mathrm{GL}_2(F)$ and define the L-functions and local constants in an analogous manner to the characters $\chi : F^\times \to \mathbb{C}^\times$. We proceed in a similar manner to the previous section; first we will construct the space of ζ-functions $\mathcal{Z}(\pi)$ that will motivate the definition of the L-function $L(\pi, s)$. Afterwards, we will define the analogous notion of the Fourier transform in this context, and this will lead us to the statement and proof of the functional equation for GL_2.

5.1 The L-function of a Principal Series Representation

Recall that for a character $\chi : F^\times \to \mathbb{C}^\times$ we defined, for any $\Phi \in C_c^\infty(F)$, a zeta function

$$\zeta(\Phi, \chi, s) = \int_{F^\times} \Phi(x)\chi(x)|x|^s d^*x.$$

To replicate this for smooth representations $\pi : G \to \mathrm{GL}(V)$ we need to extract scalar values from $\pi(g) \in \mathrm{GL}(V)$. These will come from matrix coefficients.

Definition 5.1. Let (π, V) be a smooth representation of G with smooth dual $\check{V}$. For vectors $v \in V, \check{v} \in \check{V}$, define the smooth function $\gamma_{v\otimes\check{v}} : G \to \mathbb{C}$ by

$$\gamma_{\check{v}\otimes v} : g \mapsto \langle \check{v}, \pi(g)v \rangle,$$

where $\langle , \rangle$ denotes the natural evaluation pairing $\check{V} \otimes V \to \mathbb{C}$. Let $\mathcal{C}(\pi)$ be the vector space spanned by the functions $\gamma_{\check{v}\otimes v}$. Elements of $\mathcal{C}(\pi)$ are called the *matrix coefficients* of π.

Remark 5.2. If $\pi = \chi : F^\times \to \mathbb{C}^\times$ is a character, any matrix coefficient (defined in the analogous way for $F^\times$) of χ is some scalar multiple of χ.

If V is the tautological representation of G with basis e_1, e_2, then $\gamma_{\check{e_i}\otimes e_j}(g)$ is precisely the (i, j)-th entry of g as a matrix with respect to the basis e_1, e_2.

An important aspect of matrix coefficients is that the interact well with the action of the centre Z of G by left translation.

Lemma 5.3. *Let (π, V) be an irreducible smooth representation of G, and let Z be its centre. For any $f \in \mathcal{C}(\pi), z \in Z$ and $g \in G$ we have $f(zg) = \omega_\pi(z)f(g)$, where $\omega_\pi : Z \to \mathbb{C}^\times$ is the central character defined in Corollary 1.43.*

Fix a smooth representation π of G. We may now define zeta functions for any $f \in \mathcal{C}(\pi)$.

Definition 5.4. For $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$, define the *zeta function* $\zeta(\Phi, f, s)$ to be

$$\zeta(\Phi, f, s) := \int_G \Phi(x)f(x)|\det x|^s d^*x,$$

in the formal variable s, where $d\mu^*(x) = d^*x$ denotes any choice of Haar measure on G.

Lemma 5.5. *For any $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$ we have $\zeta(\Phi, f, s) \in \mathbb{C}((q^{-s}))$ in the formal variable s.*

Proof. This follows from [BH06, Lemma 24.4.1]. $\square$

Definition 5.6. Let (π, V) be a smooth representation of G. We define the space of ζ-functions associated to π as

$$\mathcal{Z}(\pi) = \left\{ \zeta\left(\Phi, f, s + \frac{1}{2}\right) \mid \Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi) \right\}.$$

Remark 5.7. The addition of 1/2 will be explained in the case of principal series representations by the appearance of the modular character δ_B.

Lemma 5.8. *The space $\mathcal{Z}(\pi)$ is a $\mathbb{C}[q^{-s}, q^s]$-module, containing $\mathbb{C}[q^{-s}, q^s]$.*

Proof. [BH06, Lemma 24.4.2]. $\square$

Consider now the situation where $\pi = \iota_B^G \chi$ is a parabolically induced representation, where $\chi = \chi_1 \otimes \chi_2$ is a character of T. We want to study the space $\mathcal{Z}(\pi)$ and prove an analogous result to Proposition 4.11. The following fundamental result provides a complete answer to this question.

Proposition 5.9. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let $(\pi, V) = \iota_B^G \chi$. Then, formally, we have*

$$\mathcal{Z}(\pi) = \mathcal{Z}(\chi_1)\mathcal{Z}(\chi_2) \subset \mathbb{C}((q^{-s})).$$

In particular, there is a unique polynomial $P_\pi \in \mathbb{C}[X]$ with $P_\pi(0) = 1$ such that

$$\mathcal{Z}(\pi) = P_\pi(q^{-s})^{-1} \cdot \mathbb{C}[q^{-s}, q^s].$$

Moreover, $P_\pi(X) = P_{\chi_1}(X) P_{\chi_2}(X)$.

We make some comments in preparation for the proof. The proposition concerns the zeta integrals

$$\zeta\left(\Phi, f, s + \frac{1}{2}\right) = \int_G \Phi(x) f(x) |\det x|^{s+\frac{1}{2}} d^* x.$$

The matrix coefficients $\mathcal{C}(\pi)$ are spanned by

$$\gamma_{\tau \otimes \theta} : g \mapsto \langle \tau, \pi(g)\theta \rangle$$

over $\theta \in V, \tau \in \check{V}$. Here $\theta \in \iota_B^G \chi$ is viewed as a smooth function $\theta : G \to \mathbb{C}$ satisfying

$$\theta(ntg) = \delta_B^{-1/2}(t) \chi(t) \theta(g)$$

for any $t \in T, n \in N, g \in G$. The Duality Theorem 2.11 identifies $\check{V} \cong \iota_B^G \check{\chi}$. In this way we view τ as a smooth function $\tau : G \to \mathbb{C}$ satisfying

$$\tau(ntg) = \delta_B^{-1/2}(t) \chi(t)^{-1} \tau(g)$$

for any $t \in T, n \in N, g \in G$. The proof of the Duality Theorem 2.11 shows that the pairing between V and $\check{V}$ gives

$$\gamma_{\tau \otimes \theta}(g) = \langle \tau, \pi(g)\theta \rangle = \int_{B \backslash G} \tau(x) \theta(xg) d\dot{x}$$

for a positive semi-invariant measure $d\dot{x}$ on $B\backslash G$. Let $K = \mathrm{GL}_2(\mathcal{O}_F)$. Since we have a bijection $B\backslash G \leftrightarrow K\cap B\backslash K$ and $\delta_B(tn) = \delta_B(t) = |t_2/t_1|$ (Proposition 2.14) is trivial on $K \cap B$, we can rewrite this as

$$\gamma_{\tau\otimes\theta}(g) = \int_K \tau(k)\theta(kg)dk$$

for some Haar measure dk on K ([BH06, Corollary 7.6]). Moreover, [BH06, Equation 7.6.2] tells us that there is a left Haar measure db on B such that

$$\int_G \phi(g)dg = \int_K \int_B \phi(bk)dbdk$$

for all $\phi \in C_c^\infty(G)$. Using this, our zeta integrals reduce to integrals over B and K. Integration over K is easier to handle using the smoothness of our representations. We can write $db = dndt$ to view integration over B as integration over T and N. In order to relate $\zeta(\Phi, f, s + \frac{1}{2})$ to zeta functions coming from $\chi : T \to \mathbb{C}^\times$, we want to express the integrals over B solely in terms of integrals over T. To do so we use the following lemma.

Lemma 5.10. *Let D be the algebra of diagonal matrices in A so that $D^\times = T$. Let $\Phi \in C_c^\infty(A)$. There is a unique function $\Phi_T \in C_c^\infty(D)$ whose restriction to T is given by*

$$\Phi_T(t) = |t_1| \int_N \Phi(tn)dn, \qquad t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right).$$

The map $\Phi \mapsto \Phi_T$ is a linear surjection $C_c^\infty(A) \to C_c^\infty(D)$.

Proof. The space $C_c^\infty(A)$ is spanned by functions of the form

$$\Phi = (\phi_{uv}) : (a_{uv}) \mapsto \prod_{u,v} \phi_{uv}(a_{uv})$$

for $\phi_{uv} \in C_c^\infty(F)$ and $1 \leq u, v \leq 2$. For such Φ we compute (identifying $N \cong F$)

$$\begin{aligned}
\Phi_T(t) &= |t_1| \int_F \phi_{11}(t_1)\phi_{12}(t_1 n)\phi_{21}(0)\phi_{22}(t_2)dn \\
&= \phi_{11}(t_1)\phi_{22}(t_2)\phi_{21}(0)|t_1| \int_F \phi_{12}(t_1 n)dn \\
&= \phi_{11}(t_1)\phi_{22}(t_2)\phi_{21}(0) \int_F \phi_{12}(n)dn
\end{aligned}$$

which uniquely extends to a function in $C_c^\infty(D)$. Surjectivity is now clear. □

Remark 5.11. The content of the lemma is that the function Φ_T is compactly supported, for which the introduction of the factor of $|t_1|$ is necessary.

Proof of Proposition 5.9. We first establish the containment $\mathcal{Z}(\pi) \subset \mathcal{Z}(\chi_1)\mathcal{Z}(\chi_2)$. We must show that for any $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$ we have $\zeta(\Phi, f, s + \frac{1}{2}) \in \mathcal{Z}(\chi_1)\mathcal{Z}(\chi_2)$. Since $\mathcal{C}(\pi)$ is spanned by the coefficients $\gamma_{\tau\otimes\theta}$, for $\theta \in V, \tau \in \check{V}$, we assume f is of this form.

Formally expanding, for any $\Phi \in C_c^\infty(A)$

$$\begin{aligned}\zeta\left(\Phi, f, s+\frac{1}{2}\right) &= \int_G \Phi(g) f(g) |\det g|^{s+\frac{1}{2}} dg \\ &= \int_G \int_K \Phi(g)\tau(k)\theta(kg)|\det g|^{s+\frac{1}{2}} dk dg \\ &= \int_K \int_G \Phi(k^{-1}g)\tau(k)\theta(g)|\det g|^{s+\frac{1}{2}} dg dk \\ &= \int_K \int_K \int_B \Phi(k^{-1}bk')\tau(k)\theta(bk')|\det b|^{s+\frac{1}{2}} db dk' dk.\end{aligned}$$

Smoothness of Φ, θ and τ imply there is some open normal subgroup K_1 of K for which Φ is left and right translation invariant, and θ and τ are right translation invariant. Let $\{k_i\}$ be a finite set of coset representatives of K/K_1, and let $\Phi^{ij}(x) = \Phi(k_i^{-1}xk_j)$. Then $\zeta(\Phi, f, s+\frac{1}{2})$ can be expressed as a finite linear combination over $\mathbb{C}$ of terms of the form

$$\int_B \Phi^{ij}(b)\tau(k_i)\theta(bk_j)|\det b|^{s+\frac{1}{2}} db.$$

Using the formula $\theta(bk_j) = \delta_B^{-1/2}(t)\chi(t)\theta(k_j)$, we can express the above as

$$\theta(k_j)\tau(k_i)\int_T \int_N \Phi^{ij}(tn)\chi(t)\delta_B^{-1/2}(t)|\det b|^{s+\frac{1}{2}} dn dt.$$

We have $|\det b| = |\det t| = |t_1||t_2|$ and $\delta_B^{-1/2}(t) = |t_2/t_1|^{-1/2}$. Combining with the previous lemma, we deduce that $\zeta(\Phi, f, s+\frac{1}{2})$ can be expressed as a linear combination of terms of the form

$$\theta(k_j)\tau(k_i)\int_T \Phi_T^{ij}(t)\chi(t)|\det t|^s dt.$$

If Φ^{ij} is of the form (ϕ_{uv}) for $\phi_{uv} \in C_c^\infty(F)$, then the above term is a scalar multiple of $\zeta(\phi_{11}, \chi_1, s)\zeta(\phi_{22}, \chi_2, s)$ so that $\zeta(\Phi, f, s+\frac{1}{2}) \in \mathcal{Z}(\chi_1)\mathcal{Z}(\chi_2)$. In general, Φ^{ij} is a linear combination of terms of this form, so that we always have $\zeta(\Phi, f, s+\frac{1}{2}) \in \mathcal{Z}(\chi_1)\mathcal{Z}(\chi_2)$.

In the other direction, we wish to find $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$ such that $\zeta(\Phi, f, s+\frac{1}{2})$ is a constant multiple of $L(\chi_1, s)L(\chi_2, s)$. We will find f of the form $\gamma_{\tau\otimes\theta}$ and reverse the above calculation. Suppose we were in the situation where Φ is left and right invariant under K, and θ and τ are right invariant under K. Then the above computation shows that

$$\zeta\left(\Phi, f, s+\frac{1}{2}\right) = \mu(K)^2\theta(1)\tau(1)\int_T \Phi_T(t)\chi(t)|\det t|^s dt.$$

Therefore, if we could choose Φ left and right invariant under K with $\Phi_T = \phi_1 \otimes \phi_2$, where $\phi_i \in C_c^\infty(F)$ satisfy $\zeta(\phi_i, \chi_i, s) = L(\chi_i, s)$, and also choose $\theta \in \iota_B^G\chi$, $\tau \in \iota_B^G\check{\chi}$, with $\theta(1), \tau(1) \neq 0$, and θ, τ right invariant under K, then we would be done. Unfortunately, if this was the case then

$$\theta(bk) = \chi(b)\delta_B^{-1/2}(b)\theta(1)$$

for all $b \in B, k \in K$. But this is not well defined - we would require $1 = \chi(b)\delta_B^{-1/2}(b) = \chi(b)$ for all $b \in B \cap K$. This only occurs when χ_1 and χ_2 are both unramified.

Instead, let K_1 be any open normal subgroup of K such that χ is trivial on $B \cap K_1$, and let $\{k_i\}$ be a finite set of coset representatives of K/K_1. There are then unique $\theta \in \iota_B^G \chi$ and $\tau \in \iota_B^G \check{\chi}$, each supported on BK_1, invariant under right translation by K_1, and with $\theta(1) = 1 = \tau(1)$. Let $f = \gamma_{\tau \otimes \theta}$.

For $\Phi \in C_c^\infty(A)$ left and right invariant under K_1, our previous computation gives us

$$\zeta\left(\Phi, f, s + \frac{1}{2}\right) = \mu(K_1)^2 \sum_{i,j} \int_T \theta(k_j)\tau(k_i)\Phi_T^{ij}(t)\chi(t)|\det t|^s dt.$$

To control the terms over all i, j, we would like to choose Φ such that

$$\theta(k_j)\tau(k_i)\Phi_T^{ij}(t) = \Phi_T(t)$$

for all $t \in T$, and all i, j such that $k_i, k_j \in BK_1$. Then, by construction of θ and τ, each term $\theta(k_j)\tau(k_i)\Phi_T^{ij}(t)$ is either 0 or $\Phi_T(t)$, and at least one is $\Phi_T(t)$, so that

$$\zeta(\Phi, f, s + \frac{1}{2}) = c \int_T \Phi_T(t)\chi(t)|\det t|^s dt$$

for some $c > 0$. If $k_j = b_j k \in BK_1$, then $\theta(k_j) = \chi(b_j)\delta_B^{-1/2}(b_j)\theta(1) = \chi(b_j)$ because $\delta_B = 1$ on $B \cap K$. Similarly, if $k_i = b_i k \in BK_1$, then $\tau(k_i) = \chi(b_i)^{-1}$. The condition

$$\theta(k_j)\tau(k_i)\Phi_T^{ij}(t) = \Phi_T(t),$$

together with the K_1 invariance of Φ, reduces to the condition

$$\chi(b_j)\chi(b_i)^{-1} \int_N \Phi(b_i^{-1} t n b_j) dn = \int_N \Phi(tn) dn$$

for all $b_i, b_j \in B \cap K_1$, as functions of $t \in T$.

To summarise, we want to construct $\Phi \in C_c^\infty(A)$ with the following properties:

- The function Φ is invariant under left and right translation by K_1.
- For all $b_i, b_j \in B \cap K_1$ and $b \in B$ we have

$$\chi(b_j)\chi(b_i)^{-1}\Phi(b_i^{-1} b b_j) = \Phi(b).$$

- For our chosen $\phi_1, \phi_2 \in C_c^\infty(F)$ satisfying $\zeta(\phi_u, \chi_u, s) = L(\chi_u, s)$, we have $\Phi_T = c \cdot \phi_1 \otimes \phi_2 \in C_c^\infty(D)$ for some $c \neq 0$.

Since we may have chosen any open $K_1 \lhd K$, provided χ is trivial on $B \cap K_1$, we are free to shrink K_1 and adjust τ and θ accordingly. We can remove the dependence on K_1 by strengthening the second condition above, and now ask for $\Phi \in C_c^\infty(A)$ with the following properties:

- For all $x, y \in B \cap K$ and $b \in B$ we have

$$\chi(xy)\Phi(xby) = \Phi(b).$$

- For some $\phi_1, \phi_2 \in C_c^\infty(F)$ satisfying $\zeta(\phi_u, \chi_u, s) = L(\chi_u, s)$, we have $\Phi_T = c \cdot \phi_1 \otimes \phi_2 \in C_c^\infty(D)$ for some $c \neq 0$.

If we take Φ of the form $\Phi = (\phi_{uv})$, and set $\phi_{12} = \phi_{21} = \mathbb{1}_{\mathcal{O}_F}$, then the computation of Lemma 5.10 shows that for $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right)$,

$$\Phi_T(t) = \mu(\mathcal{O}_F)\phi_{11}(t_1)\phi_{22}(t_2).$$

Taking $\phi_{uu} = \phi_u$, it suffices to find for each $u = 1, 2$ some $\phi_u \in C_c^\infty(F)$ such that

- For all $x, y \in \mathcal{O}_F^\times$ and $a \in F^\times$ we have

$$\chi_u(xy)\phi_u(xay) = \phi_u(a).$$

- We have $\zeta(\phi_u, \chi_u, s) = c \cdot L(\chi_u, s)$ for some $c \neq 0$.

Here we divide into cases. If χ_u is unramified, then we may take $\phi_u = \mathbb{1}_{\mathcal{O}_F}$ by the proof of Proposition 4.11. If χ_u is ramified, and the restriction to U_F^n is trivial, then we take

$$\phi_u = \sum_{z \in \mathcal{O}_F^\times / U_F^n} \chi_u(z)^{-1}\mathbb{1}_{zU_F^n}.$$

One sees that this satisfies the first condition. For the second we have

$$\zeta(\phi_u, \chi_u, s) = \sum_z \int_{U_F^n} \chi_i(z)^{-1}\chi_i(zx)|x|^s d^*x = \mu(\mathcal{O}_F^\times)$$

which is a constant (and $L(\chi_u, s) = 1$ in the ramified case). We have proven $\mathcal{Z}(\chi_1)\mathcal{Z}(\chi_2) \subset \mathcal{Z}(\pi)$. □

Remark 5.12. The computations of Proposition 5.9 show that each $\zeta(\Phi, f, s)$ converges absolutely and uniformly in vertical strips in some right half plane, and admit analytic continuation to a rational function in q^{-s}.

Definition 5.13. Define the *L-function* attached to $\pi = \iota_B^G\chi$, where $\chi = \chi_1 \otimes \chi_2$ is a character of T, to be

$$L(\pi, s) = P_\pi(q^{-s})^{-1} = L(\chi_1, s)L(\chi_2, s),$$

where the $L(\chi_i, s)$ are the L-functions defined in 4.14.

For context, we state more general versions of these results that hold for any irreducible smooth representation π of G.

Theorem 5.14. *Let π be an irreducible smooth representation of G. There is a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$, satsifying $P_\pi(0) = 1$, and*

$$\mathcal{Z}(\pi) = P_\pi(q^{-s})^{-1}\mathbb{C}[q^{-s}, q^s].$$

Proof. [BH06, Theorem 24.2.1]. □

Notation 5.15. Set $L(\pi, s) = P_\pi(q^{-s})^{-1}$.

It remains to describe the L-functions of $\phi \circ \det$ and $\phi \mathrm{St}_G$. This could probably be done here.

5.2 The Functional Equation

Having calculated the L-functions associated to all principal series representations, we now turn our attention towards the functional equation. Just like the case for $F^\times$, we begin by defining the notion of the Fourier transform. In this context, we need an additive character of $A = M_2(F)$, which we will take to be $\psi_A = \psi \circ \mathrm{tr}_A$ for some non-trivial additive character $\psi : F \to \mathbb{C}^\times$ of F. We will apply the Fourier transform to the F-algebra $\Phi \in C_c^\infty(A)$ of locally constant compactly supported functions on $M_2(F)$. We remark that $M_2(F)$ is also the union of its open compact subgroups, and in particular it is unimodular, so any left Haar measure is also a left Haar measure.

Definition 5.16. With respect to a Haar measure μ in A, and $\psi_A = \psi \circ \mathrm{tr}_A$ an additive character of A, we define, for any $\Phi \in C_c^\infty(A)$

$$\hat{\Phi}(x) = \int_A \Phi(y)\psi_A(xy)d\mu(y).$$

Similarly to the previous case, this construction also satisfies the following desired properties analogous to the classical setting.

Proposition 5.17. *The following holds:*

- *For any $\Phi \in C_c^\infty(A)$, we have $\hat{\Phi} \in C_c^\infty(A)$.*
- *For any $\psi : F \to \mathbb{C}^\times$ with $\psi \neq 1$, there is a unique Haar measure μ_{ψ_A} on A such that for the associated Fourier transform we have*

$$\hat{\hat{\Phi}}(x) = \Phi(-x)$$

  for any $\Phi \in C_c^\infty(A)$ and $x \in A$.

Notation 5.18. For the remainder of this subsection, we fix an additive character $\psi \neq 1$ of F and $\psi_A = \psi \circ \mathrm{tr}_A$. Throughout, $\mu = \mu_{\psi_A}$ will denote the associated self-dual Haar measure on A.

We now turn to the functional equations satisfied by the zeta functions $\zeta(\Phi, f, s)$. This involves understanding these zeta functions when we replace Φ with its Fourier transform, $\hat{\Phi}$. From the computations of Proposition 5.9, this boils down to relating the map $\Phi \mapsto \Phi_T$ to the various Fourier transforms over A and D. The first step towards this aim is to undertand the interaction between the Fourier transform and the map $\Theta \mapsto \Theta_T$. The following result states that these two operators, in fact, commute.

Lemma 5.19. *For $\Phi \in C_c^\infty(A)$, we have $(\hat{\Phi})_T = \widehat{\Phi_T}$.*

Proof. [BH06, Lemma 26.3]. □

In addition, the Fouirer transform also commutes with another operator that naturally arised during the proof of Propostion 5.9.

Lemma 5.20. *For $k_i, k_j \in K$ let Φ^{ij} denote the function $x \mapsto \Phi(k_i^{-1}xk_j)$ for $\Phi \in C_c^\infty(A)$. Then $\hat{\Phi}^{ji} = \widehat{\Phi^{ij}}$.*

Proof. We calculate

$$\hat{\Phi}^{ji}(x) = \int_A \Phi(y)\psi_A(k_j^{-1}xk_iy)dy$$

and

$$\widehat{\Phi^{ij}}(x) = \int_A \Phi(k_i^{-1}yk_j)\psi_A(xy)dy = \int_A \Phi(y)\psi_A(xk_iyk_j^{-1})dy.$$

Since $\psi_A = \psi \circ \mathrm{tr}_A$ and tr_A is invariant under conjugation, we have $\psi_A(k_j^{-1}xk_iy) = \psi_A(xk_iyk_j^{-1})$. □

We require one last element of notation before we can state and prove the functional equation for G, the main result of this section. Recall that for the $F^\times$ case, the functional equation related $\zeta(\hat{\Phi}, \check{\chi}, 1-s)$ with $\zeta(\Phi, \chi, s)$, where $\check{\chi}(g) = \chi(g^{-1})$. Analogously, given a matrix coefficient $f \in \mathcal{C}(\pi)$, we write $\check{f} \in \mathcal{C}(\check{\pi})$ for the matrix coefficient $\check{f}(g) = f(g^{-1})$.

Proposition 5.21. *Let $\pi = \iota_B^G\chi$ where $\chi = \chi_1 \otimes \chi_2$ is a character of T. There is a unique $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$, depending on the additive character $\psi \neq 1$ of F defining the Fourier transform, such that*

$$\zeta\left(\hat{\Phi}, \check{f}, (1-s) + \frac{1}{2}\right) = \gamma(\pi, s, \psi)\zeta\left(\Phi, f, s + \frac{1}{2}\right)$$

for all $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$. Moreover,

$$\gamma(\pi, s, \psi) = \gamma(\chi_1, s, \psi)\gamma(\chi_2, s, \psi).$$

Proof. Since the zeta function is linear in the matrix coefficients, as is the operation $f \mapsto \check{f}$, it suffices to prove such γ exists for all $\Phi \in C_c^\infty(A)$ and f of the form $\gamma_{\tau\otimes\theta}$ as in the proof of Proposition 5.9. We calculated that

$$f(g) = \int_{B\backslash G} \tau(x)\theta(xg)d\dot{x} = \int_K \tau(k)\theta(kg)dk,$$

for some Haar measure dk on K, so that by right invariance of $d\dot{x}$ we have

$$\check{f}(g) = \int_{B\backslash G} \tau(xg)\theta(x)d\dot{x} = \int_K \tau(kg)\theta(k)dk.$$

The same computation as the proof of Proposition 5.9 gives (for the same K_1 and coset representatives k_i of K/K_1)

$$\begin{aligned}\zeta\left(\hat{\Phi}, \check{f}, (1-s) + \frac{1}{2}\right) &= \mu(K_1)^2 \sum_{i,j} \theta(k_j)\tau(k_i) \int_T (\hat{\Phi}^{ji})_T(t)\chi(t)^{-1}|\det t|^{1-s}dt \\ &= \mu(K_1)^2 \sum_{i,j} \theta(k_j)\tau(k_i) \int_T \widehat{(\Phi_T^{ij})}(t)\chi(t)^{-1}|\det t|^{1-s}dt\end{aligned}$$

by Lemma 5.20. Therefore, it suffices to show that

$$\begin{aligned}&\int_{F^\times}\int_{F^\times} \widehat{(\Phi_T^{ij})}(t)\chi_1(t_1)^{-1}\chi_2(t_2)^{-1}|t_1t_2|^{1-s}dt_2dt_1 \\ &\qquad = \gamma(\chi_1, s, \psi)\gamma(\chi_2, s, \psi)\int_{F^\times}\int_{F^\times} \Phi_T^{ij}(t)\chi_1(t_1)\chi_2(t_2)|t_1t_2|^s dt_2dt_1,\end{aligned}$$

where $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$. By Theorem 4.18, this equality holds whenever we replace $\Phi_T^{ij} \in C_c^\infty(D)$ by a function of the form $\phi_{11}(t_1) \otimes \phi_{22}(t_2) \in C_c^\infty(D)$. But such functions span $C_c^\infty(D)$, so we are done by linearity of the integrals. □

Definition 5.22. Define the *Godement-Jacquet local constant* $\varepsilon(\pi, s, \psi)$ of $\pi = \iota_B^G \chi$ by

$$\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\check{\pi}, 1-s)}.$$

Corollary 5.23. *For $\pi = \iota_B^G \chi$ we have*

$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi).$$

Proof. This follows from Proposition 5.21 and Proposition 5.9. □

Similarly to case of the L-functions, one can also prove more general versions of the functional equation and the local constant that hold for any irreducible smooth representation π of G.

Theorem 5.24. *Let π be an irreducible smooth representation of G. There is a unique rational function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta\left(\hat{\Phi}, \check{f}, (1-s) + \frac{1}{2}\right) = \gamma(\pi, s, \psi)\zeta\left(\Phi, f, s + \frac{1}{2}\right)$$

for all $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$.

Proof. [BH06, Theorem 24.2.2]. □

The above theorem holds for any irreducible representation π of G, including cuspidal representations. With the work we have done so far, we can easily calculate $\gamma(\pi, s, \chi)$ whenever π is a principal series representation.

Lemma 5.25. *If (π, V) is a composition factor of $\Sigma := \iota_B^G \chi$ for some character $\chi = \chi_1 \otimes \chi_2$ of T, then*

$$\gamma(\pi, s, \psi) = \gamma(\Sigma, s, \psi) = \gamma(\chi_1, s, \psi)\gamma(\chi_2, s, \psi).$$

Proof. By the classification of principal series representations, we may assume that π is a subrepresentation of Σ. In this case, by definition, we have that $\mathcal{C}(\pi) \subseteq \mathcal{C}(\Sigma)$. Consequently, $\mathcal{Z}(\pi) \subseteq \mathcal{Z}(\Sigma)$ and, in particular, the convergence for the zeta functions also hold for π. Therefore, for any $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$ we have that $\check{f} \in \mathcal{C}(\check{\pi}) \subseteq \mathcal{C}(\check{\Sigma})$ and therefore

$$\zeta\left(\hat{\Phi}, \check{f}, \frac{3}{2} - s\right) = \gamma(\Sigma, s, \psi)\zeta\left(\Phi, f, s + \frac{1}{2}\right),$$

whence $\gamma(\pi, s, \psi) = \gamma(\Sigma, s, \psi)$ as desired. □

Definition 5.26. Define the *Godement-Jacquet local constant* $\varepsilon(\pi, s, \psi)$ of an irreducible smooth representation π of G by

$$\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\check{\pi}, 1-s)}.$$

Corollary 5.27. *The local constant satisfies the functional equation*

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1-s, \psi) = \omega_\pi(-1).$$

The local constant is of the form

$$\varepsilon(\pi, s, \psi) = aq^{bs}$$

for some $a \in \mathbb{C}^\times$, $b \in \mathbb{Z}$.

Proof. The first statement comes from the Fourier inversion formula and Theorem 5.24. The $\omega_\pi(-1)$ term comes from the minus sign in $\hat{\hat{\Phi}}(x) = \Phi(-x)$ and the observation that for a matrix coefficient $f \in \mathcal{C}(\pi)$ we have $f(-g) = \omega_\pi(-1)f(g)$. The functional equation and Theorem 5.14 implies that ε is a unit in $\mathbb{C}[q^{-s}, q^s]$, and the units are precisely the elements of the form aq^{bs} for $b \in \mathbb{Z}$. $\square$

The Propositions 5.9 and 5.21 prove the Theorems 5.14 and 5.24 in the case that $\pi = \iota_B^G\chi$ and π is irreducible. As in Theorem 3.29, the representations $\pi = \iota_B^G\chi$ are typically irreducible - they are only reducible when $\chi = \phi\delta_B^{\pm 1/2}$ for some character ϕ of $F^\times$. In this case the composition factors are characters $\phi \circ \det$, and twists of Steinberg $\phi\mathrm{St}_G$. We state without proof the L-functions and local constants in the case that π is one of these composition factors. For more detail see Sections 26.5 - 26.8 of [BH06]. The results for all principal series representations are summarised in the following table:

Principal series representation π	$L(\pi, s)$	$\varepsilon(\pi, s, \psi)$
$\iota_B^G\chi$, $\chi = \chi_1 \otimes \chi_2$, $\chi \neq \phi\delta_B^{\pm 1/2}$	$L(\chi_1, s)L(\chi_2, s)$	$\varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi)$
$\phi \circ \det$, $\phi : F^\times \to \mathbb{C}^\times$ ramified	1	$\varepsilon(\phi, s - \frac{1}{2}, \psi)\varepsilon(\phi, s + \frac{1}{2}, \psi)$
$\phi\mathrm{St}_G$, $\phi : F^\times \to \mathbb{C}^\times$ ramified	1	$\varepsilon(\phi, s - \frac{1}{2}, \psi)\varepsilon(\phi, s + \frac{1}{2}, \psi)$
$\phi \circ \det$, $\phi : F^\times \to \mathbb{C}^\times$ unramified	$L(\phi, s - \frac{1}{2})L(\phi, s + \frac{1}{2})$	$\varepsilon(\phi, s - \frac{1}{2}, \psi)\varepsilon(\phi, s + \frac{1}{2}, \psi)$
$\phi\mathrm{St}_G$, $\phi : F^\times \to \mathbb{C}^\times$ unramified	$L(\phi, s + \frac{1}{2})$	$-\varepsilon(\phi, s, \psi)$

Figure 1: L-functions and local constants of principal series representations of G

In particular, if π is a composition factor of $\iota_B^G\chi$ then $L(\pi, s) = L(\chi_1, s)L(\chi_2, s)$, unless $\pi = \phi\mathrm{St}_G$ for some unramified character $\phi : F^\times \to \mathbb{C}^\times$.

5.3 Converse Theorem

Attached to any principal series representation π of G we have an associated L-function $L(\pi, s)$ and local constant $\varepsilon(\pi, s, \psi)$. In some sense this is enough information to distinguish them as irreducible smooth representations of G. More precisely, one can also define L-functions and local constants for the cuspidal representations of G, and the following holds.

Theorem 5.28 (Converse Theorem)**.** *Let $\psi : F \to \mathbb{C}^\times$ be an additive character with $\psi \neq 1$. Let π_1, π_2 be irreducible smooth representations of $G = \mathrm{GL}_2(F)$. Suppose that*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s) \text{ and } \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi),$$

for all characters $\chi : F^\times \to \mathbb{C}^\times$. Then $\pi_1 \cong \pi_2$.

Recall that the twist $\chi\pi$ denotes the representation $g \mapsto \chi(\det(g))\pi(g)$.

We take as fact the following result for cuspidal representations.

Proposition 5.29. *Let π be an irreducible cuspidal representation of G. Then $L(\pi, s) = 1$.*

Proof. [BH06, Corollary 24.5]. $\square$

Then we can distinguish between cuspidal and principal series representations as follows.

Proposition 5.30. *An irreducible smooth representation π of G is cuspidal if and only if $L(\phi\pi, s) = 1$ for all characters ϕ of $F^\times$.*

Proof. Since twisting preserves principal series representations, it preserves cuspidal representations. Proposition 5.29 implies that if π is cuspidal then $L(\phi\pi, s) = 1$ for all ϕ. In the other direction, suppose that π is a composition factor of $\iota_B^G\chi$ for $\chi = \chi_1 \otimes \chi_2$ a character of T. Taking $\phi = \chi_2^{-1}$, $\phi\pi$ is a composition factor of $\iota_B^G\phi\chi$ with $\phi\chi = \chi_1\chi_2^{-1} \otimes 1$. Now, except for the case $\phi\pi$ is a twist of Steinberg by an unramified character, we have $L(\phi\pi, s) = L(\chi_1\chi_2^{-1}, s)L(1, s)$, and then $L(1, s) = (1 - q^{-s})^{-1}$ is nontrivial. In the case it is a twist of Steinberg by an unramified character, the L-function is still nontrivial as seen in Table 1. $\square$

Proof of Theorem 5.28 for principal series representations. Twisting π, we may assume that $L(\pi, s) \neq 1$ as in the proof of Proposition 5.30. Then $L(\pi, s)$ has degree 2 (as a rational function of q^{-s}).

Suppose $L(\pi, s)$ has degree 2. From Table 1, π is either $\iota_B^G\chi$ for some $\chi = \chi_1 \otimes \chi_2$, with $\chi_1\chi_2^{-1} \neq |-|^{\pm 1}$ and χ_i unramified, or $\pi = \phi \circ \det$ for some unramified character $\phi : F^\times \to \mathbb{C}^\times$. In either case, we have $L(\pi, s) = L(\chi_1, s)L(\chi_2, s)$ for unramified characters χ_i of $F^\times$, where $\pi = \phi \circ \det$ corresponds to $\chi_i = \phi|-|^{\pm 1}$. But since an unramified character χ is determined by $\chi(\varpi)$, it is determined by $L(\chi, s)$. Since $\iota_B^G(\chi_1 \otimes \chi_2) \cong \iota_B^G(\chi_2 \otimes \chi_1)$, it follows that $L(\pi, s)$ is enough to distinguish all principal series representations π for which $L(\pi, s)$ has degree 2.

Suppose $L(\pi, s)$ has degree 1, and is $L(\theta, s)$ for some unramified character θ of $F^\times$. As above, we can recover θ from $L(\theta, s)$. From Table 1, π is either $\iota_B^G(\theta' \otimes \theta)$ for some ramified character θ', or $\pi = \theta' \mathrm{St}_G$ for $\theta' = \theta|-|^{-1/2}$. In the latter case, θ' is unramified and so for any ramified character ϕ we have $L(\phi\pi, s) = 1$. This distinguishes it from the former case where if we take $\phi = (\theta')^{-1}$, a ramified character, we have $\phi\pi = \iota_B^G(1 \otimes \phi\theta)$ so that $L(\phi\pi, s) \neq 1$. To recover θ' in this case, we can choose some ramified character ϕ such that $L(\phi\pi, s) \neq 1$, say $L(\phi\pi, s) = L(\theta'', s)$ fo a unique unramified character θ'' of $F^\times$. Since $\phi\pi = \iota_B^G(\phi\theta' \otimes \phi\theta, s)$, and $\phi\theta$ is ramified, we have $L(\phi\pi, s) = L(\phi\theta', s)$. Therefore $\theta' = \phi^{-1}\theta''$. $\square$

Remark 5.31. The proof of Theorem 5.28 for principal series representations shows that the isomorphism class of π is determined solely by the L-functions $L(\phi\pi, s)$ as we range over all characters $\phi : F^\times \to \mathbb{C}^\times$. For cuspidal representations, all L-functions are 1 and they are instead distinguished solely by the local constants.

References

[BH06] C. J. Bushnell and G. Henniart, *The Local Langlands Conjecture for GL(2)*, Grundlehren der mathematischen Wissenschaften, Springer Berlin, 2006.

[Bum97] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1997.

[DE09] A. Deitmar and S. Echterhoff, *Principles of Harmonic Analysis*, Universitext, Springer, New York, 2009.

[GH24] J. R. Getz and H. Hahn, *An Introduction to Automorphic Representations*, Graduate Texts in Mathematics, Springer Cham, 2024.

[Gou20] F. Q. Gouvêa, *p-adic Numbers*, Universitext, Springer Cham, 2020.

[Tat67] J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, Algebraic Number Theory, edited by J.W.S. Cassels and A. Frölich, London Mathematical Society, 1967.