Local Langlands for GL_2

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1 Locally Profinite Groups and Smooth Representations

The aim of this first section is to motivate the abstract notions of a *locally profinite group* and of a *smooth representation*, which will be our main object of study during the later subsections. To do so, we briefly recall some basic facts about non-Archimedean fields and we introduce a few important algebraic objects related to them. For the sake of brevity, we will omit proofs, and therefore we assume familiarity with the subject. For the unfamiliar reader, we encourage them to read **(insert here Gouvea reference, or others)**, where a detailed explanation is provided.

1.1 Local Fields and Locally Profinite Groups

We begin by recalling some basic objects from algebraic number theory. Given a field F, a **discrete valuation** on F is a surjective function $\nu : F \to \mathbb{Z} \cup \{\infty\}$ satisfying the three conditions

1. $\nu(xy) = \nu(x) + \nu(y)$ for any $x, y \in F$
2. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for any $x, y \in F$.
3. $\nu(x) = \infty$ if and only if $x = 0$.

Any discrete valuation ν induces an absolute value given by the formula

$$|x| = c^{\nu(x)}$$

for any $c \in (0, 1)$, and therefore it also induces a topology which is independent of the choice of c. One easily checks that this absolute value satisfies $|x + y| \leq \max\{|x|, |y|\}$ for any $x, y \in K$. Absolute values with this property are denoted as *non-Archimedean.*

A field F with an absolute value $|\cdot|$ induced by a discrete valuation ν is the fraction field of the **valuation ring**

$$R := \{x \in F : v(x) \geq 0\} = \{x \in K : |x| \leq 1\},$$

which contains a unique maximal ideal

$$\mathfrak{p} := \{x \in F : v(x) > 0\} = \{x \in K : |x| < 1\},$$

denoted as the **valuation ideal** or the **ring of integers of** F. The valuation idea is principal, and it is generated by any $\varpi \in K$ with $\nu(\varpi) = 1$. Such an element is called a uniformizer of F. Finally, the residue field κ of F is the quotient $R/\mathfrak{p}$. This motivates the following important definition.

Definition 1.1. A field F is a (non-Archimedean) *local field* if it is complete with respect to a topology induced by a discrete valuation and with finite residue field.

Remark 1.2. When the residue field is finite, it is conventional to write

$$|x| = q^{-\nu(x)},$$

where $q = |\kappa|$. From here onwards, we will follow this convention.

Remark 1.3. Local fields are ubiquitous in number theory. They arise as completions of number fields at non-archimedean places, if they have 0 characteristic, or as completions of finite extensions of $\mathbb{F}_p(t)$ at non-archimedean places, if the characteristic is positive.

As discussed above, the valuation ring R of a local field F is a local ring with unique principal ideal $\mathfrak{p}$. The ideals

$$\mathfrak{p}^n = \{x \in F : \nu(x) \geq n\} = \{x \in F : |x| \leq q^{-n}\} = \varpi^n R, \quad n \in \mathbb{N}$$

are a complete set of ideals of R and under the topology induced by the discrete valuation, they are also a fundamental system of neighbourhoods of the identity. Moreover, the field F (and therefore also R) is totally disconnected, and we also have a topological isomorphism

$$R \longrightarrow \varprojlim_{n\geq 1} R/\mathfrak{p}^n \quad x \mapsto (x \pmod{\mathfrak{p}^n})_{n\geq 1}$$

where the maps implicit in the right hand side are the obvious ones. In particular, since the residue field is finite and $\mathfrak{p}^n/\mathfrak{p}^{n+1} \cong \kappa$, all rings $R/\mathfrak{p}^n$ are finite and induced with the discrete topology. This shows that R, and also any $\mathfrak{p}^n$, is a profinite group, and in particular it is compact subring of F. We have therefore shown that F has the important property that any open subset of F contains an open compact subgroup (namely $\mathfrak{p}^n$ for a sufficiently large n).

We also remark that F satisfies the rather special property of being the union of its open compact subgroups, even though F itself is not compact. This fact has relevant consequences that will be discuss later.

We are now ready to give the main definition of this section.

Definition 1.4. A topological group G (which we always assume to be Hausdorff) is a *locally profinite group* if every open neighbourhood of the identity contains a compact open subgroup.

In this document we will be interested in studying the representation theory of many important groups and rings related to the local field F. The notion of a locally profinite group is an abstract one, but it has the great advantage of accomodating many important groups and rings associated to non-Archimedean local fields and their representation theory.

Examples 1.5. 1. In the preceding discussion, we have shown that F is a locally profinite group, where $\mathfrak{p}^n$ for $n \geq 1$ is a fundamental system of open compact subgrups.

2. The multiplicative group $F^\times$ is also a locally profinite group, where the congruence unit groups $U_F^n = 1+\mathfrak{p}^n$ for $n \geq 1$ is a fundamental system of open compact subgroups. We remark that unlike F, the group $F^\times$ is not the union of its open compact subgroups.

3. Given $m \geq 1$ an integer, the additive group $F^m = F \times \cdots \times F$ is also a locally profinite group endowed with the product topology. A fundamental system of open compact subgroups is given by $\mathfrak{p}^n \times \cdots \times \mathfrak{p}^n$ for $n \geq 1$. More generally, any product of locally profinite groups is locally profinite.

4. The matrix ring $M_m(F)$ is also locally profinite since it is isomorphic to F^{m^2} as additive groups. The open compact subgroups $\mathfrak{p}^n M_m(R)$ are a fundamental system of neighbourhood of the identity.

5. The group $\mathrm{GL}_m(F)$ of invertible matrices is an open subset of $M_m(F)$ since $\det : M_m(F) \to F$ is continuous and $F^\times$ is an open subset of F. Furthermore, mutiplication by a matrix $A \in M_m(F)$ and inversion of matrices are continous maps in $M_m(F)$ and therefore $\mathrm{GL}_m(F)$ is also a topological group and the subgroups

$$K = \mathrm{GL}_m(R), \quad K_n = 1 + \mathfrak{p}^n M_m(R), \quad n \geq 1,$$

are compact open, and a fundamental neighbourhood of the identity.

We give some further insight into the terminology used. If G is a locally profinite group, any open subgroup K of G is also a locally profinite group under the subspace topology and if H is a closed normal subgroup of G, then G/H is also locally profinite under the quotient topology.

Moreover, it is an easy exercise to prove that a profinite group is a compact locally profinite group. Rather strickingly, using a topological argument one can also show that the converse also holds. That is, if K is a compact locally profinite group, then

$$K \longrightarrow \varprojlim K/N$$

is a topological isomorphism where N ranges over the normal open subgroups, and the implicit maps are the obvious ones. Since K is compact and N is open, K/N must be finite and discrete, showing that K is profinite.

1.2 Characters of Local Fields

Now we turn our attention to the representation theory of locally profinite groups. We begin by recalling the notion of a representation of a group.

Definition 1.6. A representation of a group G over a field k is a pair (π, V) where V is a k-vector space and $\pi : G \to \mathrm{GL}_n(V)$ is a group homomorphism. We say that the $\dim V$ is the dimension of the representation.

Equivalently, a representation of G is a k-vector space V equipped with a k-linear G action. For $g \in G, v \in V$, we will often write $g \cdot v$ for $\pi(g)v$. Throughout this document, we will only be interested in complex representations, so for now on we will assume that $k = \mathbb{C}$, unless otherwise stated.

Naturally, we also define the notion of a homomorphism between representations.

Definition 1.7. A morphism between two representations (π, V), (σ, W) of a group G is a linear map $\phi : V \to W$ compatible with the G action. That is, ϕ satisfies

$$\phi(\pi(g)v) = \sigma(g)\phi(v) \text{ for all } g \in G,\ v \in V.$$

In studying the representation theory of a locally profinite group G, it turns out that the entire set of representations of G is too big, so we need to restrict our attention to those representations satisfying a certain "smoothness" condition. To motivate this condition, we will first describe the simplest (yet very relevant!) case: one-dimensional representations of a local field F: that is, group homomorphisms $\phi : F \to \mathbb{C}^\times$. Later in this section we will also study the one-dimensional representations of $F^\times$.

As we have discussed in the previous section, locally profinite groups carry a certain topology, so a natural condition to impose is **continuity** with respect to the usual topologies in $\mathbb{C}^\times$ and G. A continuous homomorphism $\psi : G \to \mathbb{C}^\times$ will be denoted as a *character* of G.

Characters of a locally profinite group G are a group under multiplication, denoted by $\hat{G}$. It turns out that for one-dimensional representations, continuity coincides with the smoothness condition we require.

Firstly, we state this rather surprising result which we will use later.

Lemma 1.8. *Let G be a locally profinite group and $\psi : G \to \mathbb{C}^\times$ a homomorphism. Then ψ is continous if and only if $\ker \psi$ is open in G. Furthermore, if G is the union of its compact open subgroups, then*[1]

$$\psi(G) \subseteq \{z \in \mathbb{C}^\times : |z| = 1\} = S^1.$$

Proof. If $\ker \psi = \psi^{-1}(1)$ is open in G, then for any $z \in \operatorname{Im} \psi$, then $\psi^{-1}(z) = g \ker \psi$ is also open, where $\psi(g) = z$. So in fact, **for any** $U \subseteq \mathbb{C}^\times$,

$$\psi^{-1}(U) = \bigcup_{z \in U \cap \operatorname{Im} \psi} \psi^{-1}(z),$$

and so in particular it is continous. Conversely, if ψ is continous, then for any open neighbourhood $\mathcal{N}$ of 1, $\psi^{-1}(\mathcal{N})$ contains an open compact subgroup K of G. But $\mathcal{N}$ can be chosen sufficiently small so that it does not contain any non-trivial subgroup of $\mathbb{C}^\times$. Hence, $\psi(K) = 1$ so $K \subseteq \ker \psi$, and since K is open, so is $\ker \psi$. The last assertion is a direct consequence of the fact that the continuous image of a compact set is compact and S^1 is the unique maximal compact subgroup of $\mathbb{C}^\times$. $\square$

By the remark above Definition 1.4, all characters of F are unitary. However, this is not the case for $F^\times$. Indeed, the map $x \mapsto |x|$ is a character of $F^\times$, yet it is clearly not unitary.

Before stating the classification theorem for characters of F, we need one last definition.

Definition 1.9. Let ψ be a non-trivial character of F (resp. of $F^\times$). The **level** of ψ is defined as be the least $d \geq 0$ such that $\mathfrak{p}^d \subseteq \ker \psi$ (resp. $U_F^{d+1} \subseteq \ker \psi$).

We are now ready to give the classfication theorem for $\hat{F}$.

Theorem 1.10 (Additive duality)**.** *Let $\psi \in \hat{F}$ be a non-trivial character with level d.*

[1] Characters satisfying this property are called *unitary*

1. *Let $a \in F$. Then the map $a\psi : x \mapsto \psi(ax)$ is a character of F and if $a \neq 0$ then $a\psi$ has level $d - \nu_F(a)$.*

2. *The map $a \mapsto a\psi$ induces an isomorphism $F \cong \hat{F}$.*

Proof. For 1., it is clear that $a\psi$ is a character since if $x \in \mathfrak{p}^{d-\nu_F(a)}$, then $ax \in \mathfrak{p}^d$ so $a\psi(x) = 1$ so $\mathfrak{p}^{d-\nu_F(a)} \subseteq \ker(a\psi)$. Furthermore, there is some $y \in \mathfrak{p}^{d-1}$ such that $\psi(y) \neq 1$ so $a\psi(a^{-1}y) \neq 1$. Since $a^{-1}y \in \mathfrak{p}^{d-1-\nu_F(a)}$, this indeed shows that the level of $a\psi$ is $d - \nu_F(a)$. For 2. the map $a \mapsto a\psi$ is clearly a homomorphism. To prove injectivity, suppose that $a \neq b$ but $a\psi = b\psi$. Then it follows that $x(a-b) \in \ker \psi$ for all $x \in F$. But since $a - b \neq 0$, then $\ker \psi = F$, a contradiction.

The proof of surjectivity is more involved. Let $\theta \in \hat{F}$ non-trivial, and let l be the level of θ. The idea of the proof is to construct inductively a sequence some $u_0, u_1, \ldots \in F$ such that $u_n\psi$ agrees with θ on $\mathfrak{p}^{l-n}$ and $u_{n+1} \equiv u_n \pmod{\mathfrak{p}}$. More concretely, for any $u \in U_F$, the character $u\varpi^{d-l}\psi$ has level l, thus agreeing with θ at $\mathfrak{p}^l$, and any $u_0 = u\varpi^{d-l}$ will do. To construct u_1, we note that given $u, u' \in U_F$, then $u\varpi^{d-l}\psi$ and $u\varpi^{d-l}\psi$ agree on $\mathfrak{p}^{l-1}$ if and only if $u \equiv u' \pmod{\mathfrak{p}}$, i.e. when $u'u^{-1} \in U_F^1$. Since $\kappa^\times \cong (\mathfrak{p}^{l-1}/\mathfrak{p}^l)^\times \cong U_F/U_F^1$ are cyclic of order $q-1$, there are $q-1$ characters of $\mathfrak{p}^{l-1}$ trivial on $\mathfrak{p}^l$ and also $q-1$ characters of $\mathfrak{p}^{l-1}$ of the form $u\varpi^{d-l}\psi$ for $u \in U_F$. Then one of them, say $u_1\varpi^{d-l}\psi|_{\mathfrak{p}^{l-1}}$ equals $\theta|_{\mathfrak{p}^{l-1}}$.

This same idea can be iterated to find the sequence $\{u_n\}_{n\geq 1}$ described above. Since this sequence is Cauchy and F is complete, it converges to some $u \in F$ such that $u\varpi^{d-l}\psi = \theta$. □

1.3 Smooth representations of locally profinite groups

Let G be a locally profinite group. A representation of G is a pair (π, V) where V is a complex vector space (not necessarily finite-dimensional) and $\pi : G \to \mathrm{GL}(V)$ is a homomorphism.

Definition 1.11. A representation V of G is smooth if for $v \in V$ there exists a compact-open subgroup $K \subseteq G$ such that $kv = v$ for all $k \in K$ (i.e. $v \in V^K$). We say V is admissible if V^K is finite-dimensional for all compact-open K.

Let us define induced representations:

Definition 1.12. Let G be a locally profinite group, $H \subseteq G$ a closed subgroup, and W a smooth representation of H. We define the induced representation as the space of functions $f : G \to W$ satisfying

1. $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$

2. there is a compact open subgroup $K \subseteq G$ (depending on f) such that $f(gk) = f(g)$ for all $g \in G, k \in K$.

We let G act on this space via $(g \cdot f)(x) = f(xg)$. This is a smooth representation of G, denoted by $\mathrm{Ind}_H^G W$.

This construction satisfies the following important universal property:

Theorem 1.13 (Frobenius reciprocity)**.** *Let V be a smooth representation of G, and W a smooth representation* *of H. Then there's a natural bijection*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(V, W)$$

$$\varphi \mapsto \alpha_W \circ \varphi$$

where $\alpha_W : \mathrm{Ind}_H^G W \to W$ *is the canonical map* $\alpha_W(f) = f(1)$.

There is also a different variant of induction, called compact induction:

Definition 1.14. Let G be a locally profinite group, H a closed subgroup, and W a smooth representation of H. Consider the space of functions $f : G \to W$ that satisfy (1) and (2) in Definition 1.12, and are also compactly supported mod H, i.e. the support $\mathrm{supp} f \subseteq H\backslash G$ is compact. The group G also acts on this space by right translations, so we get a representation of G, denoted by $c - \mathrm{Ind}_H^G W$.

This construction is mainly of interest in the case when H is open in G. In this case, it satisfies a version of Frobenius reciprocity:

Theorem 1.15. *Let $H \subseteq G$ be open, W a smooth representation of H and V a smooth representation of G. We have a natural bijection*

$$\mathrm{Hom}_G(c - \mathrm{Ind}_H^G W, V) \cong \mathrm{Hom}_H(W, V)$$

$$\varphi \mapsto \varphi \circ \alpha_W^c$$

Where $\alpha_W^c : W \to c - \mathrm{Ind}_H^G W$ *is the map* $w \mapsto f_w$ *where* f_w *is supported in* H *and satisfies* $f_w(h) = hw$.

2 Hecke Algebras

[Possibly put this whole section at the end with modular forms. I think we only need the computation of δ_B from here.]

In this section, we define the Hecke algebra $\mathcal{H}(G)$ associated to a locally profinite (unimodular) group G and explain how to switch between smooth representations of G and smooth modules of $\mathcal{H}(G)$. Under certain conditions on G we consider a particular subalgebra of $\mathcal{H}(G)$; the unramified Hecke algebra $\mathcal{H}(G, K)$, which turns out to be commutative by the Satake isomorphism. We use as reference Chapter 4 of [BH06] and Chapter 5 of [GH24].

If G is a finite group, representations of G are the same as $\mathbb{C}[G]$-modules. We want to extend this notion to smooth representations of locally profinite groups, where we need to correctly interpret the group algebra.

Let G be a locally profinite unimodular group and K an open compact subgroup of G. Let $C_c^\infty(G)$ be the space of locally constant compactly supported functions $G \to \mathbb{C}$ and $C_c^\infty(G//K)$ the K bi-invariant subspace.

These are naturally $\mathbb{C}$-vector spaces and we endow them with an associative (not necessarily unital) ring structure coming from convolution

$$f * h(g) := \int_G f(x)h(x^{-1}g)dx$$

where we fix a Haar measure $\mu = dx$ on G.

When G is discrete this is the usual product on $\mathbb{C}[G]$.

Definition 2.1. Let $\mathcal{H}(G)$ and $\mathcal{H}(G,K)$ denote $C_c^\infty(G)$ and $C_c^\infty(G//K)$ with the algebra structure specified above. We call $\mathcal{H}(G)$ the Hecke algebra of G.

We study these algebras in more detail:

The element $e_K = \mu(K)^{-1}\mathbb{1}_K \in \mathcal{H}(G)$ is idempotent and we have the property that

$$e_K * f = f \Leftrightarrow f \text{ is } K \text{ left invariant .}$$

Thus $\mathcal{H}(G,K) = e_K * \mathcal{H}(G) * e_K$, and this subalgebra now has a unit e_K. The compactness of K ensures $e_K \in C_c^\infty(G)$.

By Lemma 5.2.1 of [GH24], $\mathcal{H}(G)$ is spanned by indicator functions of K'-double cosets, where K' ranges over all compact open subgroups of G. If we normalise these indicator functions by defining

$$[K\alpha K] = \mu(K)^{-1}\mathbb{1}_{K\alpha K},$$

then we have the formula

$$[K\alpha K] * [K\beta K] = \sum_{i,j}[K\alpha_i\beta_j K]$$

where $K\alpha K = \sqcup K\alpha_i$ and $K\beta K = \sqcup \beta_j K$. This determines multiplication in the Hecke algebra.

2.1 Smooth representations and $\mathcal{H}(G)$-modules

We now explain how the concepts of smooth representations of G and smooth modules over $\mathcal{H}(G)$ are interchangeable. To define these smooth modules, we note that the Hecke algebra $\mathcal{H}(G)$ does not in general have a unit. Consequently, not every $\mathcal{H}(G)$-module M satisfies $\mathcal{H}(G)M = M$.

Definition 2.2. We say that a $\mathcal{H}(G)$-module M is smooth if $\mathcal{H}(G)M = M$.

Definition 2.3. From a representation V of G we define the action of $\mathcal{H}(G)$ on V via

$$f \cdot v := \int_G f(g)g \cdot v dg.$$

This can be viewed as a weighted average of the action of G on v, where the weighting is described by $f \in C_c^\infty(G)$. The integral defines an element of V when $f \in C_c^\infty(G)$ as the integral reduces to a finite sum.

Lemma 2.4. *Under this action, $e_K \in \mathcal{H}(G)$, for $K \leq G$ a compact open, is the projection $V \to V^K$ onto the K-invariants of V. In particular, e_K is an idempotent element of $\mathcal{H}(G)$, it is the identity element of $\mathcal{H}(G,K)$, and V^K is a $\mathcal{H}(G,K)$-module.*

Proof. Let $V(K) \leq V$ be the subspace spanned by vectors of the form $k \cdot v - v$. Since e_K is invariant under K-translation, it is zero on $V(K)$. The normalisation of e_K is such that e_K is the identity on V^K, and this implies the result. □

Proposition 2.5. *A representation V of G is smooth if and only if it is a smooth $\mathcal{H}(G)$-module.*

Proof. If V is a smooth representation, then any $v \in V$ is K-invariant for some compact open K, and so $v = e_K \cdot v$. This implies that V is a smooth $\mathcal{H}(G)$-module. Conversely, $\mathcal{H}(G)$ is the union of $e_K * \mathcal{H}(G) * e_K = \mathcal{H}(G, K)$ over all compact open K, and so if V is a smooth $\mathcal{H}(G)$-module then any $v \in V$ is of the form $e_K * f * e_K \cdot v'$ for some K, f, v'. Then $e_K \cdot v = v$ and so $v \in V^K$. $\square$

So we can view smooth representations of G as smooth $\mathcal{H}(G)$-module. In the other direction, given M a smooth $\mathcal{H}(G)$-module, we have

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M = M$$

by smoothness. We can then view M as a smooth G representation by letting G act on the first factor by left translation. Concretely, if $m \in M$ there exists K such that $e_K \cdot m = m$. Then define

$$g \cdot m := \mu(K)^{-1} \mathbb{1}_{gK} \cdot m,$$

where this is independent of K due to the normalisation factor $\mu(K)^{-1}$.

2.2 Information in the K-invariants V^K

For a smooth representation V of G it is often easier to study the K-invariants V^K for compact open subgroups K of G.

Lemma 2.6. *A smooth representation V of G is irreducible if and only if each V^K is either 0 or a simple $\mathcal{H}(G, K)$-module for all compact open $K \leq G$.*

Proof. Suppose V is irreducible. If we had $0 \neq M \subset V^K$ a $\mathcal{H}(G, K)$-module, then $0 \neq \mathcal{H}(G)M \subset V$ as smooth $\mathcal{H}(G)$-modules. Since smooth $\mathcal{H}(G)$-modules are the same as smooth G-representations, and V is irreducible, we deduce $\mathcal{H}(G)M = V$. So then

$$V^K = e_K V = e_K * \mathcal{H}(G)M = e_K * \mathcal{H}(G) * e_K M = \mathcal{H}(G, K)M = M$$

which implies the result.

If V is not irreducible, and $W \neq 0$ is a proper subrepresentation, pick $v \in V - W$. By smoothness, there exists K such that $v \in V^K$, but then $v \notin W^K$ so that V^K is not 0 or simple. $\square$

The next result tells us that for any K, any smooth representation V of G is determined by V^K with its structure as a $\mathcal{H}(G, K)$-module, provided $V^K \neq 0$.

Proposition 2.7. *The map $V \mapsto V^K$ induces a bijection between*

- *equivalence classes of irreducible smooth representations V of G with $V^K \neq 0$;*
- *isomorphism classes of simple (by definition nonzero) $\mathcal{H}(G, K)$-modules.*

Proof. Proposition 4.3 of [BH06]. $\square$

2.3 Unramified representations of G

It is interesting to study the smooth representations V with $V^K \neq 0$ as above. For example, in an automorphic representation, Flath's theorem ([GH24] Section 5.7) allows us to decompose into local factors, and furthermore tells us that almost all such local representations are unramified in the following sense:

Definition 2.8. We consider the case $G = \mathrm{GL}_2(F)$. We say that an irreducible smooth representation V of G is unramified if $V^K \neq 0$ for $K = \mathrm{GL}_2(\mathcal{O}_F)$. See Section 5.5 of [GH24] for a more general definition for reductive groups.

For the remainder of this subsection we work in the context of $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O}_F)$ for simplicity. The results generalise to reductive groups G as in Sections 5.5 and 7.1 of [GH24].

Definition 2.9. For K as above, $\mathcal{H}(G, K)$ is called the unramified Hecke algebra of G.

An application of the Satake isomorphism ([GH24] Theorem 5.5.1) tells us that in this unramified case, the unramified Hecke algebra $\mathcal{H}(G, K)$ is commutative. It follows that if V is K-unramified (in particular irreducible) then V^K is 1-dimensional by Lemma 2.6. Thus $\mathcal{H}(G, K)$ acts on V^K via scaling, called the Hecke character of V.

Definition 2.10. The Hecke character (with respect to K) of a smooth representation (π, V) of G is the $\mathbb{C}$-linear map

$$\mathcal{H}(G, K) \to \mathbb{C}$$
$$f \mapsto \mathrm{tr}\pi(f)$$

defined by $f \cdot v =: \mathrm{tr}\pi(f)v$ for any $v \in V^K$.

We give an alternative proof of Proposition 2.7.

Proposition 2.11. *Let $K \leq G$ be a compact open subgroup. If V_1, V_2 are irreducible smooth representations of G such that V_1^K and V_2^K are nonzero and isomorphic as $\mathcal{H}(G, K)$-modules, then $V_1 \cong V_2$. In particular, unramified representations are determined by their Hecke characters.*

Proof. This is Proposition 7.1.1 of [GH24]. The idea is to extend an isomorphism

$$I : V_1^K \to V_2^K$$

to a G-intertwining map $V_1 \to V_2$ of $\mathcal{H}(G)$-modules. By irreducibility, $V_i = \mathcal{H}(G)V_i^K$. Take an element $\pi_1(f) \cdot \phi \in V_1$, with $f \in \mathcal{H}(G), \phi \in V_1^K$, then the obvious choice is to map this to $\pi_2(f) \cdot I(\phi)$. Provided this is well defined, this is a nonzero homomorphism of $\mathcal{H}(G)$-modules, so irreducibility of V_1, V_2 implies this is an isomorphism $V_1 \cong V_2$.

To check this is well defined, it suffices to show that if $\pi_1(f)\phi = 0$ then $\pi_2(f)I(\phi) = 0$. We exploit the $\mathcal{H}(G, K)$-intertwining of I (for the second implication below). For all $f_1 \in \mathcal{H}(G)$ we have:

$$\pi_1(f)\phi = 0 \Rightarrow \pi_1(e_K * f_1 * f * e_K)\phi = 0 \Rightarrow \pi_2(e_K * f_1 * f * e_K)I(\phi) = \pi_2(e_K * f_1 * f)I(\phi) = 0.$$

By Lemma 2.4, e_K acts on V_2 by projection to V_2^K. If $\pi_2(f)I(\phi) \neq 0$, then $\pi_2(f_1)\pi_2(f)I(\phi)$, over all $f_1 \in \mathcal{H}(G)$, generates V_2 by irreducibility. The image under $\pi_2(e_K)$ is the exactly V^K, which is nonzero, contradicting the implication above. $\square$

2.4 Example computation of Hecke operators for $\mathrm{GL}_2(F)$

[I haven't checked this subsection. Some parts might be more suitable for a section on modular forms. The computation of the modular character of B will be needed in the main text. And the last proposition naturally goes with the unramified representations above.]

Let $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O})$ for F a nonarchimedean local field with uniformiser ϖ. We have the Cartan decomposition

$$G = \bigsqcup_{a \geq b \in \mathbb{Z}} K \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} K.$$

Let $S = K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K$ and $T = K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K$, viewed as elements of $\mathcal{H}(G,K)$ via their indicator functions.

Lemma 2.12. *The unramified Hecke algebra is $\mathcal{H}(G,K) \cong \mathbb{C}[S, S^{-1}, T]$. In particular, this is commutative.*

Proof. This is some induction argument using the formula for convolutions of these indicator functions. $\square$

Remark 2.13. This fits into a general phenomenon - if G is unramified and K is a hyperspecial subgroup then the Satake isomorphism implies that the unramified Hecke algebra $\mathcal{H}(G,K)$ is always commutative.

Later we will be interested in principal series representations, which are representations of G coming from parabolic induction. So let $\chi = \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ be a character of the torus T, and consider the normalised induced representation

$$I(\chi) = \mathrm{Ind}_B^G \left(\chi \otimes \delta_B^{-1/2}\right)$$

where we recall that this is the space of functions $G \to \mathbb{C}$ with $f(bg) = \chi(b)\delta_B^{-1/2}(b)f(g)$ for $b \in B$.

We briefly discuss the module character δ_B. Although G is unimodular (see Bushnell-Henniart Section 7.5), the Borel subgroup is not. We have $B = NT$ with $N \cong F$, $T \cong F^\times \times F^\times$ and N normal in B. The failure of B to be unimodular is a consequence of T and N not commuting. We can then define a linear function I on $C_c^\infty(B) = C_c^\infty(T) \otimes C_c^\infty(N)$ by

$$I(\Phi) = \int_T \int_N \Phi(tn)dtdn$$

using Haar measures on T and N.

Proposition 2.14. *I is a left Haar integral on B.*

Proof. Let $b = sm \in TN$. By left invariance of dt we have

$$\int_T \int_N \Phi(smtn)dtdn = \int_T \int_N \Phi(mtn)dtdn = \int_T \int_N \Phi(tt^{-1}mtn)dtdn.$$

Since we integrate N first, we are integrating over fixed values of t so that $t^{-1}mt \in N$ is just constant, so left invariance of dn let's us pull out the $t^{-1}mt$ factor. $\square$

Proposition 2.15. *The module δ_B of the group B is*

$$\delta_B : tn \mapsto |t_2/t_1|, \quad n \in N, t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$$

Proof. By a similar argument as above, we have

$$\int_T \int_N \Phi(tnsm)dtdn = \int_T \int_N \Phi(tss^{-1}nsm)dtdn = \int_T \int_N \Phi(ts^{-1}ns)dtdn.$$

Identifying $N \cong F$ this is

$$\int_T \int_N \Phi(t \cdot \left(\begin{smallmatrix} 1 & s_1^{-1}xs_2 \\ 0 & 1 \end{smallmatrix}\right))d\mu_F(x) = |s_1/s_2| \int_T \int_N \Phi(tn)dtdn$$

so by definition of the module character we have $\delta_B(sm) = |s_2/s_1|$. $\square$

Going back to our principal series representation, the following proposition computes the action of the unramified Hecke algebra on the K-invariant subspace:

Proposition 2.16. *Let $\chi : T \to \mathbb{C}^\times$ be an unramified character of the torus (meaning trivial on $\left(\begin{smallmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{smallmatrix}\right)$) and consider the normalised parabolic induction*

$$I(\chi) = \mathrm{Ind}_B^G(\chi \otimes \delta_B^{-1/2}).$$

For $K = \mathrm{GL}_2(\mathcal{O})$ as usual, the space $I(\chi)^K$ is 1-dimensional. As a $\mathcal{H}(G,K)$-module this is determined by the actions of S and T. Since χ is unramified we know $\chi_1(z) = \alpha^{v_F(z)}$ and $\chi_2(z) = \beta^{v_F(z)}$ for some $\alpha, \beta \in \mathbb{C}^\times$. Then S acts on $I(\chi)^K$ by scaling by $\alpha\beta$ and T acts by scaling by $q^{1/2}(\alpha+\beta)$.

Proof. We have the Iwasawa decomposition $G = BK$ so that the functions $f \in I(\chi)^K$ satisfy

$$f(bk) = f(b) = \chi(b)\delta_B^{-1/2}(b) \cdot f(1)$$

with $f(1) \in \mathbb{C}$, so the space is 1-dimensional spanned by $\hat{f}(bk) = \chi(b)\delta_B^{-1/2}(b)$.

The action of S is given by:

$$\begin{aligned} S \cdot f &= \mu(K)^{-1} \int_G \mathbb{1}_{K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K}(g) g \cdot f dg \\ &= \mu(K)^{-1} \int_K \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right) k \cdot f dk \\ &= \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right) \cdot f \\ &= \chi\left(\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)\right) \delta_B^{-1/2}\left(\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)\right) f \\ &= \alpha\beta f \end{aligned}$$

because $K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K = \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K$.

And for T we pick coset representatives for $K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K/K$ given by $\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)$, where a ranges over representatives of $\mathcal{O}/\varpi$. Writing down the integral for the action of T we decompose this into a sum over these left cosets and we deduce that T acts by

$$\chi_2(\varpi)|\varpi|^{-1/2} f + \sum_{a \in \mathcal{O}/\varpi} \chi_1(\varpi)|\varpi|^{1/2} f = q^{1/2}(\alpha+\beta)$$

since, for example, $\chi\left(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)\right) = \chi_1(\varpi) = \alpha$ and $\delta_B^{-1/2}\left(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)\right) = |\varpi|^{1/2}$. $\square$

Remark 2.17. If we know the action of S, T on $I(\chi)^K$ for some unramified character χ of the torus T, then we can recover $\alpha, \beta \in \mathbb{C}^\times$ from the roots of the Satake polynomial $X^2 - q^{-1/2}TX + S \in \mathcal{H}(G, K)[X]$.

3 Principal series representations of GL_2

Let F be a nonarchimedean local field, $G = \mathrm{GL}_2(F)$, and $B = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F^\times, b \in F\}$ the Borel subgroup of upper triangular matrices, so that $B = N \rtimes T$ for $T = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F^\times\} \cong F^\times \times F^\times$ and $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in F\} \cong F$. Between N and B we also have the mirabolic subgroup $M = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a \in F^\times, b \in F\}$ with $M/N \cong F^\times$.

In studying the local Langlands correspondence, we want to understand all the irreducible smooth representations of G. One method for producing representations of G is by induction from a subgroup of G. Typically one takes this subgroup to be 'parabolic'; in our case there is one nontrivial parabolic, namely B. From our decomposition $B = N \rtimes T$ (more generally we have a so-called Levi decomposition) we see that we can produce representations of B by inflating representations of the torus T. Since $T \cong F^\times \times F^\times$, the irreducible representations of T are products of characters of $F^\times$, which are relatively easy to get a handle on.

Definition 3.1. For $\chi : T \to \mathbb{C}^\times$ a character of the torus, we say that the representation $\mathrm{Ind}_B^G \chi$ is a parabolically induced representation. A principal series representation is an irreducible subrepresentation of a parabolically induced representation.

In this section, we will only concern ourselves with classifying the principal series representations of G. This means that we must understand how $\mathrm{Ind}_B^G \chi$ decomposes into irreducible representations of G, and also study the morphisms between them using Frobenius reciprocity.

To understand these decompositions, we want to study how they decompose into irreducibles over a less unwieldy subgroup of G, such as B. Note that restricting $\mathrm{Ind}_B^G \chi$ to B is analogous to applying Mackey theory in the finite group context. It turns out that the $\mathrm{Ind}_B^G \chi$ do not decompose any further over M than over B. On the other hand, the representation theory of M is very easy to classify - the combination of these two observations is what makes the mirabolic subgroup so 'miraculous'. To get representations of M we can induce from characters of N, or inflate from $M/N \cong F^\times$. There are many characters of $N \cong F$, in fact these are in bijection with F [REFER TO SECTION 1]. The key property of M is that conjugation by M acts transitively on these characters ψ, which greatly simplifies the representation theory of M coming via induction from N. The mirabolic M is also small enough that this induction, together with the characters of $F^\times$, give all irreducible representations of M.

In this section, we begin by studying the representations of N and introducing the Jacquet functor, before discussing representations of M. From there we determine that parabolically induced representations of G decompose over M with length at most 3. Theorem 3.20 gives the decomposition of $\mathrm{Ind}_B^G \chi$ into irreducible representations of G, and then Theorem 3.27 lists the isomorphism classes of principal series representations. The presentation follows sections 8 and 9 of [BH06].

3.1 Representations of N

We first study the representation theory of $N \cong F$. This is an abelian group so, by Schur's lemma, all irreducible representations are characters (Corollary 2.6.2 [BH06]). For finite abelian groups, any representation V decomposes into a direct sum of characters. This is no longer true when $N \cong F$ is infinite, but it is still true that any vector in V is nonzero in some quotient on which N acts via a character. To formalise this, we define

Notation 3.2. Let V be a smooth representation of N and θ a character of N. Let $V(\theta) \leq V$ be the subspace spanned by $\{n \cdot v - \theta(n)v \mid n \in N, v \in V\}$. Set $V_\theta = V/V(\theta)$ so that N acts on V_θ by θ. When θ is trivial we write $V(N)$ and V_N respectively.

The following is a useful equivalent definition of $V(\theta)$:

Lemma 3.3. *The vector $v \in V$ lies in $V(\theta)$ if and only if*

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0$$

for some compact open subgroup N_0 of N.

In the lemma we restrict to compact opens for the integral to be well defined.

Proof. [BH06] Lemma 8.1. □

Corollary 3.4. *The functor $V \mapsto V_\theta$ from smooth representations of N to complex vector spaces is exact.*

Proof. One checks formally that the functor is right exact. For left exactness we need to show that if $f : V \hookrightarrow V'$ is injective then $V_\theta \hookrightarrow V'_\theta$ is injective. If $v \in V$ with $f(v) \in V'(\theta)$, then

$$\int_{N_0} \theta(n)^{-1} n \cdot f(v) dn = 0$$

for some N_0 by the above lemma. Since f is compatible with the action of N, we can pull f out of the integral so that the injectivity of f implies

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0.$$

We deduce that $v \in V(\theta)$ by the above lemma. □

Proposition 3.5. *Let V be a smooth representation of N. For any $v \neq 0$ in V, there exists a character θ of N such that $v \notin V(\theta)$.*

Proof. [BH06] Proposition 8.1. □

Corollary 3.6. *If V is a smooth representation of N such that $V_\theta = 0$ for all θ then $V = 0$.*

3.2 Representations of M

Now we consider V an irreducible smooth representation of M.

Lemma 3.7. *The subspace $V(N) \leq V$ is a representation of M, and so V_N is as well. Moreover, $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a \in F^\times\}$ permutes the subspaces $V(\theta)$ with $\theta \neq 1$ transitively, and hence the V_θ are isomorphic as vector spaces.*

Proof. The first claim comes from the computation

$$mn \cdot v - m \cdot v = n'm \cdot v - m \cdot v$$

for some $n' \in N$, using the fact that $N \lhd M$. For the second claim we have the computation

$$s(nv - \theta(n)v) = sns^{-1} \cdot sv - \theta(s^{-1}(sns^{-1})s)sv = n' \cdot sv - \theta(s^{-1}n's)sv$$

where $n' = sns^{-1} \in N$. Hence $sV(\theta) = V(\theta')$ where $\theta'(n) := \theta(s^{-1}ns)$. Now the computation

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

together with [ADDITIVE DUALITY] implies the claim. □

If V is an irreducible smooth representation of M, then either $V(N) = 0$ or $V(N) = V$. In the former case N acts trivially on V, so the action of M factors through $M/N \cong F^\times$. Irreducibility implies that V is a character of M factoring through M/N. and we just get a character of $F^\times$, or $V(N) = V$. In the latter case, $V_N = 0$, so we must have $V_\theta \neq 0$ for all nontrivial characters of N by the above Lemma and Corollary 3.6. Thus the M-representation V must have infinite dimension. In fact there is only one such V, and we have more specifically:

Theorem 3.8. *Let (π, V) be an irreducible smooth representation of M. Either*

- $\dim V = 1$ *and π is the inflation of a character of $M/N \cong F^\times$, or*
- $\dim V = \infty$ *and $\pi \cong c\text{--}\mathrm{Ind}_N^M \theta$, for any nontrivial character θ of N.*

Proof. If $V(N) = 0$ then N acts trivially on V, and V is a representation of $M/N \cong F^\times$. Hence by Schur's lemma, it's one-dimensional and we're in the first case.

If $V(N) \neq 0$ then $V(N) = V$ and $V_N = 0$. Hence $V_\theta \neq 0$ for all nontrivial characters θ, and $\dim V = \infty$. The following theorem implies that $V = V(N)$ is isomorphic to $c\text{--}\mathrm{Ind}_N^M V_\theta$, which is a direct sum of copies of $c\text{--}\mathrm{Ind}_N^M \theta$. Since it is irreducible, we must have $V \cong c\text{--}\mathrm{Ind}_N^M \theta$. □

This itself follows from the following theorem. To compare V and $c\text{--}\mathrm{Ind}_N^M \theta$, it is more natural to compare V and $\mathrm{Ind}_N^M V_\theta$. By Frobenius reciprocity,

$$\mathrm{Hom}_N(V, V_\theta) \cong \mathrm{Hom}_M(V, \mathrm{Ind}_N^M V_\theta).$$

Let $q_* : V \to \mathrm{Ind}_N^M(V_\theta)$ be the image of the quotient map $q : V \to V_\theta$.

Theorem 3.9. *The M-homomorphism $q_* : V \to \mathrm{Ind}_N^M V_\theta$ induces an isomorphism $V(N) \cong c\text{–}\mathrm{Ind}_N^M V_\theta$. The smooth representation $c\text{–}\mathrm{Ind}_N^M \theta$ of M is irreducible.*

Proof. Theorem 8.3 and Corollary 8.2 of [BH06]. □

3.3 Irreducible principal series representations

Let V be a smooth representation of G. In the preceding subsections, we defined the quotient $V_N = V/V(N)$, called the N-coinvariants of V. As in Lemma 3.7, this is a representation of B (as $N \lhd B$). As N acts trivially on V_N, V_N inherits the structure of a representation of $T = B/N$.

Definition 3.10. Let V be a smooth representation of G (or B). The Jacquet module of V at N is the space of N-coinvariants V_N viewed as a representation of T. The Jacquet functor is the functor sending the G-representation (π, V) to the T-representation (π_N, V_N).

By Corollary 3.4, the Jacquet functor is exact.

If V is a representation of G, and χ is a character of T, then we have by Frobenius Reciprocity that

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_B(V, \chi)$$

But since χ as a character B has trivial N-action, maps $V \to \chi$ factor through V_N, and we obtain a version of Frobenius reciprocity for the Jacquet module:

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_T(V_N, \chi)$$

i.e. the Jacquet module is left adjoint to parabolic induction.

In the classical setting of representations of $\mathbf{G} = \mathrm{GL}_2(k)$ for a finite field k, we have the following dichotomy (where $\mathbf{B}, \mathbf{T}, \mathbf{N}$ are the appropriate subgroups of $\mathbf{G}$):

Lemma 3.11. *Let (π, V) be an irreducible representation of $\mathbf{G}$. The following are equivalent:*

1. *π contains the trivial character of $\mathbf{N}$*
2. *π is isomorphic to a $\mathbf{G}$-subrepresentation of $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \chi$ for some character χ of $\mathbf{T}$ inflated to $\mathbf{B}$.*

Proof. Lemma 6.3 of [BH06]. □

Returning to $G = \mathrm{GL}_2(F)$, if (π, V) is a smooth representation, the restriction to N is no longer necessarily semisimple because F is of infinite order. We instead replace the condition that $\pi|_N$ contains the trivial character of N with the condition that N acts trivially on some nonzero quotient of V (which is an equivalent condition in the finite field case). This is measured by the Jacquet module V_N. There is the analogous dichotomy which tells us that principal series representations can be identified as the irreducible smooth representations of G with nonzero Jacquet module:

Proposition 3.12. *Let (π, V) be an irreducible smooth representation of G. The following are equivalent:*

1. $V_N \neq 0$

2. π *is isomorphic to a* G*-subrepresentation of* $\mathrm{Ind}_B^G \chi$ *for some character* χ *of* T *inflated to* B*.*

Proof sketch. (2) implies (1) is a consequence of Frobenius reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\chi) = \mathrm{Hom}_T(\pi_N, \chi)$$

Given (1), one shows by a technical argument that V_N is finitely generated as a representation of T. An application of Zorn's lemma allows us to construct a maximal T-subspace U of V_N, so that V_N/U is a nonzero irreducible T-representation, and is thus a character χ by Schur's lemma. The above Frobenius reciprocity implies (2). □

Remark 3.13. The same proof holds for the finite field case, where we bypass the technical details in showing (1) implies (2) because any representation of the finite group T admits an irreducible quotient.

Remark 3.14. We ask for a nonzero Jacquet module V_N rather than a trivial N-subrepresentation of V because of the following fact:

Lemma 3.15. *Let* (π, V) *be an irreducible smooth representation of* G *with a nonzero vector* $v \in V$ *fixed by* N*. Then* $\pi = \phi \circ \det$*, for some character* ϕ *of* $F^\times$*. In particular,* π *is one dimensional.*

Proof sketch. The vector v is fixed by N, but also by a compact open subgroup K of G by smoothness. As we are working with F a nonarchimedean local field (as opposed to a finite field), this implies K contains a unipotent lower triangular matrix, and one shows that v is fixed by $\mathrm{SL}_2(F)$. Thus π factors through det. □

Once again, let χ be a character of T and let (Σ, X) denote $\mathrm{Ind}_B^G \chi$. We want to study how X decomposes into irreducible G-representations. As mentioned earlier, we will begin by studying their decompositions over B or even M.

To begin with, X will never be irreducible over B because we always have the canonical B-homomorphism $\Sigma \to \chi$, given by sending $f \mapsto f(1) \in \mathbb{C}$. So we have an exact sequence of B-representations

$$0 \longrightarrow V \longrightarrow X \longrightarrow \mathbb{C} \longrightarrow 0,$$

where $V = \{f \in X \mid f(1) = 0\}$, and B acts on $\mathbb{C}$ via χ. Now we want to understand how V decomposes over B. We have another exact sequence of B-representations,

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0,$$

so we reduce to studying $V(N)$ and V_N. We will show that $V(N)$ is irreducible over B (and even over M), while V_N will be determined by the Restriction-Induction lemma.

The following lemma makes the structure of V more apparent.

Lemma 3.16. *Let $V = \{f \in X : f(1) = 0\}$. The map*

$$V \to C_c^\infty(N)$$
$$f(-) \mapsto f(w-)$$

is an N-isomorphism (with N acting by right translation on either side), where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof. We have the Bruhat decomposition $G = B \sqcup BwN$. Since $f(1) = 0$, and f is induced from B, we must have that f is supported on BwN. G-smoothness of f implies that f is also zero on some compact open $K \leq G$. This will contain $\left(\begin{smallmatrix} 1 & 0 \\ \varpi^n \mathcal{O} & 1 \end{smallmatrix}\right)$ for some n, so that f vanishes on

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in Bw \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

for all $x \in \varpi^n \mathcal{O}$. Thus $f(w-)$ is supported on $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in N$ with $v(y) > -n$ and so is compactly supported. G-smoothness of f also implies that $f(w-)$ is N-smooth. Since f is induced from B and is supported on BwN, the map is injective. Conversely, any $g \in C_c^\infty(N)$ determines $f \in \mathrm{Ind}_B^G \chi$ such that $f(w-) = g$ and $f(B) = 0$. $\square$

Proposition 3.17. *For V as above, $V(N)$ is irreducible over M (and hence over B). Moreover, $V(N)$ is infinite dimensional.*

Proof. The idea will be to use Theorem 3.9, which tells us $V(N) \cong c\text{--}\mathrm{Ind}_N^M V_\theta$. This is irreducible over M (and infinite dimensional) if we can show that V_θ is one dimensional, by the same theorem.

By the above lemma we can identify $V \cong C_c^\infty(N)$ as N-representations. But M also acts via right translation on V (since $BwB = BwN = BwM$), which gives the structure of a M-representation on $C_c^\infty(N)$. We can calculate it explicitly (but we won't need it), where

$$f\left(bw\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = f\left(b\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w \begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}\right)$$

tells us that the corresponding $M = F^\times N$ action on $C_c^\infty(N)$ is the composite of right translation by N with the action

$$a \cdot \phi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \chi_2(a)\phi \begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}$$

of $a \in F^\times$.

So now we may consider $V = C_c^\infty(N)$. The benefit is that for this representation, the spaces of coinvariants of characters θ of N are very simple. In particular, the map $f \mapsto \theta f$ is a linear automorphism of $C_c^\infty(N)$ taking $V(N)$ to $V(\theta)$, since

$$n \cdot f - f \mapsto \theta(n \cdot f) - \theta f = \theta(n)^{-1} n \cdot (\theta f) - \theta f \in V(\theta).$$

Hence all the V_θ have the same dimension as $V_N = V/V(N)$, which has dimension 1 (we can see this from the characterisation of $V(N)$ as the zeros of some integral (Lemma 3.3), or from the Restriction-Induction lemma to follow). The result follows from Theorem 3.9. $\square$

We turn our attention to the Jacquet module V_N. Recall V fits in the exact sequence

$$0 \longrightarrow V \longrightarrow X = \mathrm{Ind}_B^G \chi \xrightarrow{f \mapsto f(1)} \mathbb{C} \longrightarrow 0$$

of smooth representations of B, where B acts via χ on $\mathbb{C}$. Since the Jacquet functor is exact, we get the exact sequence

$$0 \longrightarrow V_N \longrightarrow X_N \longrightarrow \mathbb{C} \longrightarrow 0$$

of T-representations. The following lemma determines the structure of V_N as a T-representation. This can be stated in more generality:

Lemma 3.18 (Restriction-Induction lemma)**.** *Let (σ, U) be a smooth representation of T and $(\Sigma, X) = \mathrm{Ind}_B^G \sigma$. Then there is an exact sequence of smooth T representations:*

$$0 \longrightarrow \sigma^w \otimes \delta_B^{-1} \longrightarrow \Sigma_N \longrightarrow \sigma \longrightarrow 0.$$

Here, $\sigma^w(t) := \sigma(wtw)$ for $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so that if σ is the character $\chi_1 \otimes \chi_2$ of T, then $\sigma^w = \chi_2 \otimes \chi_1$.

Proof. The proof of Lemma 3.16 generalises to show that the vector space $V = \{f \in X \mid f(1) = 0\}$ is isomorphic, as N-representations, to the space $\mathcal{S}$ of smooth compactly supported functions $N \to U$, by identifying f with $f(w-)$.

We can define a map $\mathcal{S} \to U$ by

$$g = f(w-) \mapsto \int_N f(wn)dn,$$

where this integral is finite since g is compactly supported. By Lemma 3.3, this induces an isomorphism $\mathcal{S}_N \cong U$.

Now V also carries the structure of a B-representation as well, since $BwB = BwN$. We can repeat the same calculation as in the previous proposition, replacing $F^\times$ with $T \cong F^\times \times F^\times$, to compute the action of $B = TN$ on $\mathcal{S}$. As usual, N acts via right translation. If $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$, then for $\phi \in \mathcal{S}$,

$$t \cdot \phi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \sigma^w(t) \phi \begin{pmatrix} 1 & \frac{t_2}{t_1} x \\ 0 & 1 \end{pmatrix}.$$

Thus the T-representation structure on $U \cong \mathcal{S}_N \cong V_N$ is given by

$$t \cdot \int_N f(wn)dn = \sigma^w(t) \left| \frac{t_1}{t_2} \right| \int_N f(wn)dn,$$

which is $\sigma^w \otimes \delta_B^{-1}$. ∎

Corollary 3.19. *As a representation of B or M, $\mathrm{Ind}_B^G \chi$ has composition length 3. Two of the factors have dimension 1, and the other is infinite dimensional.*

Proof. This follows from the exact sequences

$$0 \longrightarrow V \longrightarrow \mathrm{Ind}_B^G \mathbb{C} \longrightarrow \chi \longrightarrow 0$$

and

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

where we saw that $V(N)$ is irreducible and infinite dimensional, and $V_N \cong \chi^w \otimes \delta_B^{-1}$. $\square$

So we understand how $\mathrm{Ind}_B^G \chi$ decomposes into irreducible B-representations, and we want to understand its decomposition into G-representations. Our goal is to prove the following:

Theorem 3.20 (Irreducibility Criterion)**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let $X = \mathrm{Ind}_B^G \chi$.*

1. *The representation X of G is irreducible if and only if $\chi_1 \chi_2^{-1}$ is either the trivial character of $F^\times$, or the character $x \mapsto |x|^2$ of $F^\times$.*

2. *Suppose X is reducible, then*

   - *the G-composition length of X is 2*
   - *one factor has dimension 1, the other is infinite dimensional*
   - *X has a 1-dimensional G-subspace exactly when $\chi_1 \chi_2^{-1} = 1$*
   - *X has a 1-dimensional G-quotient exactly when $\chi_1 \chi_2^{-1}(x) = |x|^2$.*

We make some comments in preparation for the proof. By the above Corollary, if X is reducible then it has a finite dimensional (dimension 1 or 2) G-subspace or G-quotient. By taking duals we can assume we are in the first case. In the Irreducibility Criterion, we want to show that this implies $\chi_1 = \chi_2$ and that X has a 1-dimensional G-subspace.

Definition 3.21. Let π be a smooth representation of G and ϕ a character of $F^\times$. The twist of π by ϕ is the representation $\phi\pi$ of G defined by

$$\phi\pi(g) = \phi(\det g)\pi(g).$$

In this way, for a character $\chi = \chi_1 \otimes \chi_2$ of T, we have $\phi\chi = \phi\chi_1 \otimes \phi\chi_2$. Then

$$\mathrm{Ind}_B^G(\phi\chi) = \phi \mathrm{Ind}_B^G \chi.$$

Proposition 3.22. *The following are equivalent:*

1. *$\chi_1 = \chi_2$*

2. *X has a 1-dimensional N-subspace.*

If this holds then this subspace is unique, and is also a G-subspace of X not contained in V.

Proof. (1) implies (2): since induction commutes with twisting we may assume $\chi_1 = \chi_2 = 1$, then the nonzero constant function spans a 1-dimensional G-subspace (not just N-subspace) of $X = \mathrm{Ind}_B^G 1$.

(2) implies (1): suppose this subspace is spanned by f. The group N acts as a character on this subspace via right translation. We cannot have $f \in V$ (meaning $f(1) = 0$) because we saw earlier that f would then have support in some BwN_0 for $N_0 \leq N$ open compact, and this is not closed under multiplication by N.

So $f \notin V$ and therefore its image spans $X/V \cong \mathbb{C}$. On this quotient, N acts trivially because χ was inflated from $B/N = T$. Thus f is in fact fixed by N under right translation. But f is also fixed under right translation by some compact open of G, so for sufficiently large $|x|$ we have

$$\begin{aligned} f(w) = f\left(w\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\right) \\ &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\right) \\ &= \chi_1(-1)\left(\chi_1^{-1}\chi_2(x)\right) f(1) \end{aligned}$$

This tells us that $\chi_1^{-1}\chi_2(x)$ is constant for x sufficiently large. Since this is also true for x^2, we see that $\chi_1(x) = \chi_2(x)$ for x sufficiently large. Now for any $y \in F^\times$, we can pick x large enough so that $\chi_1(x) = \chi_2(y)$ and $\chi_1(xy) = \chi_2(xy)$, from which we deduce that $\chi_1 = \chi_2$.

The uniqueness of the 1-dimensional subspace comes from the fact that it must span $X/V \cong \mathbb{C}$. □

Proof of Irreducibility Criterion. Assume that X is reducible and we are in the case that X has a finite dimensional G-subspace. Then it has a 1-dimensional N-subspace L because N is abelian, which is also a G-subspace by the above proposition. Since G must act via a character on L, it factors as $\phi \circ \det$, where $\chi_1 = \phi = \chi_2$.

Let Y be the G-representation X/L. Since L spans the vector space X/V, the B-homomorphism $V \hookrightarrow X \to X/L$ is surjective. It is injective since $L \cap V = 0$. Thus $Y \cong V$ as B-representations.

We need to show that X has G-length 2. By the Corollary 3.19 it has length at most 3. We know that V has B-length 2 with a 1-dimensional quotient V_N. If Y had G-length 2, then the B-factors of V are also G-factors, so that G must act on V_N, necessarily by a character $\phi' \circ \det$. But this is impossible because $B \leq G$ acts on V_N by $\phi\delta_B^{-1}$ by Restriction-Induction, and this does not factor through det on B. So we must have that Y is irreducible over G and so X has G-length 2.

In the other case we have a finite dimensional G-quotient. The smooth dual $X^\vee$ is then in the first case, where the Duality Theorem (Theorem 3.5 [BH06]) tells us that $X^\vee \cong \mathrm{Ind}_B^G \delta_B^{-1}\chi^\vee$. If we write $\delta_B^{-1}\chi^\vee = \psi_1 \otimes \psi_2$ then we must have $\psi_1 = \psi_2$. Computing $\psi_1(x) = |x|^{-1}\chi_1(x)$ and $\psi_2(x) = |x|\chi_2(x)$ gives $\chi_1\chi_2^{-1} = |\cdot|^2$.

The converse direction to (1) follows from the previous proposition. □

3.4 Classification of principal series representations

Now that we've seen how parabolically induced representations decompose into irreducibles, we want to classify the isomorphism classes.

Proposition 3.23. *Let χ, ξ be characters of T. The space $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi)$ is 1-dimensional if $\xi = \chi$ or $\chi^w\delta_B^{-1}$ and 0 otherwise.*

Proof. Frobenius reciprocity tells us

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi) \cong \mathrm{Hom}_T((\mathrm{Ind}\chi)_N, \xi).$$

From the Restriction-Induction lemma we have

$$0 \longrightarrow \chi^w\delta_B^{-1} \longrightarrow (\mathrm{Ind}\chi)_N \longrightarrow \chi \longrightarrow 0.$$

In the case $\chi \neq \chi^w \delta_B^{-1}$ the sequence splits and the result follows. If $\chi = \chi^w \delta_B^{-1}$ then $\chi_1 \chi_2^{-1}(x) = |x|$ so $\mathrm{Ind}\chi$ is irreducible and the result still follows. □

Remark 3.24. In the case that $\mathrm{Ind}\chi$ is irreducible, we deduce that $\mathrm{Ind}\chi \cong \mathrm{Ind}\chi^w \delta_B^{-1}$. And in the case $\mathrm{Ind}\chi$ is reducible, it is not semisimple, else $\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \chi)$ would have dimension strictly greater than 1.

We can be more explicit in the reducible case. One can check that the conditions for reducibility in the Irreducibility Criterion are equivalent to χ being of the form $\chi = \phi 1_T$ or $\chi = \phi \delta_B^{-1}$. Untwisting, we may as well assume $\phi = 1$ in what follows.

Definition 3.25. The Steinberg representation of G is defined by the exact sequence

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow \mathrm{St}_G \longrightarrow 0,$$

and is an infinite dimensional irreducible smooth representation. By Restriction-Induction, the Jacquet module is $(\mathrm{St}_G)_N \cong \delta_B^{-1}$. The representations $\phi \mathrm{St}_G$ are called 'twists of Steinberg' or 'special representations'.

The case $\chi = \delta_B^{-1}$ can be dealt with by taking smooth duals (which is exact by [BH06] Lemma 2.10) to get

$$0 \longrightarrow \mathrm{St}_G^\vee \longrightarrow \mathrm{Ind}_B^G \delta_B^{-1} \longrightarrow 1_G \longrightarrow 0,$$

where we use the Duality Theorem, [BH06] Theorem 3.5. The Irreducibility Criterion implies that $\mathrm{St}_G^\vee$ is also irreducible, and in fact the previous proposition applied to $\chi = 1, \xi = \delta_B^{-1}$ implies that

$$\mathrm{St}_G \cong \mathrm{St}_G^\vee.$$

Notation 3.26. Define normalised induction by

$$\iota_B^G \sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma).$$

This has the benefit that $(\iota_B^G \sigma)^\vee \cong \iota_B^G \sigma^\vee$ ([BH06] Theorem 3.5).

Theorem 3.27 (Classification Theorem)**.** *The following are all the isomorphism classes of principal series representations of G:*

- *the irreducible induced representations $\iota_B^G \chi$ when $\chi \neq \phi \delta_B^{\pm 1/2}$ for a character ϕ of $F^\times$.*
- *the one-dimensional representations $\phi \circ \det$ for ϕ a character of $F^\times$.*
- *the twists of Steinberg (special representations) $\phi \mathrm{St}_G$ for ϕ a character of $F^\times$.*

These are all distinct isomorphism classes except in the first case where $\iota_B^G \chi \cong \iota_B^G \chi^w$.

4 Maths

References

[BH06] C. J. Bushnell and G. Henniart, *The Local Langlands Conjecture for GL(2)*, Grundlehren der mathematischen Wissenschaften, Springer Berlin, 2006.

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