Local Langlands for GL_2

Yiannis Fam

March 6, 2024

1 Hecke Algebras

In this section, we define the Hecke algebra $\mathcal{H}(G)$ associated to a locally profinite (unimodular) group G and explain how to switch between smooth representations of G and smooth modules of $\mathcal{H}(G)$. Under certain conditions on G we consider a particular subalgebra of $\mathcal{H}(G)$; the unramified Hecke algebra $\mathcal{H}(G, K)$, which turns out to be commutative by the Satake isomorphism. We use as reference Chapter 4 of [1] and Chapter 5 of [2].

If G is a finite group, representations of G are the same as $\mathbb{C}[G]$-modules. We want to extend this notion to smooth representations of locally profinite groups, where we need to correctly interpret the group algebra.

Let G be a locally profinite unimodular group and K an open compact subgroup of G. Let $C_c^\infty(G)$ be the space of locally constant compactly supported functions $G \to \mathbb{C}$ and $C_c^\infty(G//K)$ the K bi-invariant subspace.

These are naturally $\mathbb{C}$-vector spaces and we endow them with an associative (not necessarily unital) ring structure coming from convolution

$$f * h(g) := \int_G f(x)h(x^{-1}g)dx$$

where we fix a Haar measure $\mu = dx$ on G.

When G is discrete this is the usual product on $\mathbb{C}[G]$.

Definition 1.1. Let $\mathcal{H}(G)$ and $\mathcal{H}(G, K)$ denote $C_c^\infty(G)$ and $C_c^\infty(G//K)$ with the algebra structure specified above. We call $\mathcal{H}(G)$ the Hecke algebra of G.

We study these algebras in more detail:

The element $e_K = \mu(K)^{-1}\mathbb{1}_K \in \mathcal{H}(G)$ is idempotent and we have the property that

$$e_K * f = f \Leftrightarrow f \text{ is } K \text{ left invariant .}$$

Thus $\mathcal{H}(G, K) = e_K * \mathcal{H}(G) * e_K$, and this subalgebra now has a unit e_K. The compactness of K ensures $e_K \in C_c^\infty(G)$.

By Lemma 5.2.1 of [2], $\mathcal{H}(G)$ is spanned by indicator functions of K'-double cosets, where K' ranges over all compact open subgroups of G. If we normalise these indicator functions by defining

$$[K\alpha K] = \mu(K)^{-1}\mathbb{1}_{K\alpha K},$$

then we have the formula

$$[K\alpha K] * [K\beta K] = \sum_{i,j} [K\alpha_i\beta_j K]$$

where $K\alpha K = \sqcup K\alpha_i$ and $K\beta K = \sqcup \beta_j K$. This determines multiplication in the Hecke algebra.

1.1 Smooth representations and $\mathcal{H}(G)$-modules

The concepts of smooth representations of G and smooth modules over $\mathcal{H}(G)$ are interchangeable.

Because $\mathcal{H}(G)$ does not have a unit, not every $\mathcal{H}(G)$-module M satisfies $\mathcal{H}(G)M = M$. If it does, we say that M is smooth or nondegenerate.

Definition 1.2. From a representation V of G we define the action of $\mathcal{H}(G)$ on V via

$$f \cdot v := \int_G f(g)g \cdot v dg.$$

This is like a weighted average of the action of G on v, where the weighting comes from $f \in C_c^\infty(G)$. This defines an element of V when $f \in C_c^\infty(G)$ as the integral reduces to a finite sum.

Recall that the $e_K \in \mathcal{H}(G)$ are idempotents, and they induce the projection $V \to V^K$ onto the K-invariants of V. This is because e_K annihilates the K-complement $V(K)$ of V^K in V, and e_K is trivial on V^K. So $e_K \cdot V = V^K$ and this is a $\mathcal{H}(G,K)$-module where e_K acts via the identity.

Proposition 1.3. *V is a smooth representation of G if and only if it is a smooth $\mathcal{H}(G)$-module.*

Proof. If V is a smooth representation then any $v \in V$ is of the form $e_K \cdot v$ for K a compact open such that $v \in V^K$, so that V is a smooth $\mathcal{H}(G)$-module. Conversely, $\mathcal{H}(G)$ is the union of the $e_K * \mathcal{H}(G) * e_K = \mathcal{H}(G,K)$ over all compact open K, and so if V is a smooth $\mathcal{H}(G)$-module then any $v \in V$ is of the form $e_K * f * e_K \cdot v'$ for some K, f, v', from which we deduce $e_K \cdot v = v$ and so $v \in V^K$. $\square$

In the other direction, given M a smooth $\mathcal{H}(G)$-module, one can show that

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M = M$$

by smoothness. Then we can view M as a smooth G representation where G acts on the first factor by left translation. Concretely, if $m \in M$ there exists K such that $e_K \cdot m = m$. Then define

$$g \cdot m := \mu(K)^{-1} \mathbb{1}_{gK} \cdot m$$

where this is independent of K due to this normalisation of $\mu(K)^{-1}$.

1.2 Information in the K-invariants V^K

For a smooth representation V of G it is often easier to study the K-invariants V^K for compact open subgroups K of G.

Lemma 1.4. *If V is irreducible then V^K is either 0 or a simple $\mathcal{H}(G,K)$-module.*

Proof. If we had $0 \neq M \subset V^K$ a $\mathcal{H}(G,K)$-module, then $0 \neq \mathcal{H}(G)M \subset V$ as smooth $\mathcal{H}(G)$-modules. Since smooth $\mathcal{H}(G)$-modules are the same as smooth G-representations, and V is irreducible, we deduce $\mathcal{H}(G)M = V$. So then

$$V^K = e_K V = e_K * \mathcal{H}(G)M = e_K * \mathcal{H}(G) * e_K M = \mathcal{H}(G,K)M = M$$

which implies the result. $\square$

Remark 1.5. There is no parallel statement for G-representations because V^K is not a representation of G.

In fact we have a converse:

Lemma 1.6. *A smooth representation V of G is irreducible if and only if each V^K is either 0 or a simple $\mathcal{H}(G,K)$-module for all compact open $K \leq G$.*

Proof. One direction is proved above. If V is not irreducible, and $W \neq 0$ is a proper subrepresentation, pick $v \in V - W$. By smoothness there exists K such that $v \in V^K$, but then $v \notin W^K$ so that V^K is not 0 or simple. $\square$

Surprisingly for any smooth representation V of G, V is determined by V^K with its structure as a $\mathcal{H}(G,K)$-module, provided $V^K \neq 0$.

Proposition 1.7. *The map $V \mapsto V^K$ induces a bijection between*

- *equivalence classes of irreducible smooth representations V of G with $V^K \neq 0$;*
- *isomorphism classes of simple $\mathcal{H}(G,K)$-modules.*

Proof. *Bushnell-Henniart Section 4.3. Recovering V from V^K is not very enlightening - it's to do with $\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$. $\square$

1.3 Unramified representations of G

It is interesting to study the smooth representations V with $V^K \neq 0$ as above. For example in an automorphic representation, Flath's theorem allows us to decompose into local factors, and furthermore tells us that almost all such local representations are unramified in the following sense:

Definition 1.8. We say that an irreducible smooth representation V of (the F-points of, where F is a nonarchimedean local field) a reductive group G is K-unramified if G is unramified and $V^K \neq 0.G$ being unramified is a techinical condition and is equivalent (I think) to the existence of a hyperspecial subgroup $K \leq G(F)$. This means that G has a model over $\mathcal{O}_F$ (for which the generic fibre recovers G and the special fibre is reductive?) for which the $\mathcal{O}_F$ points are K.

Definition 1.9. If $K \leq G(F)$ is hyperspecial then $\mathcal{H}(G,K)$ is called the unramified Hecke algebra.

We often denote $G(F)$ simply by G.

A corollary of the Satake isomorphism tells us that in this unramified case, the unramified Hecke algebra $\mathcal{H}(G,K) = C_c^\infty(G//K)$ is commutative. It follows that if V is K-unramified (in particular irreducible) then V^K is 1-dimensional (since it is irreducible by the previous subsection over a commutative algebra).

Definition 1.10. Thus $\mathcal{H}(G,K)$ acts on V^K via scaling, called the Hecke character of V. This is the $\mathbb{C}$-linear map

$$\begin{aligned} \mathcal{H}(G,K) &\to \mathbb{C} \\ f &\mapsto \mathrm{tr}\pi(f) \end{aligned}$$

where $f \cdot v = \mathrm{tr}\pi(f)v$ for any $v \in V^K$.

We give an alternative proof of the proposition of the previous subsection.

Proposition 1.11. *Let $K \leq G$ be a compact open subgroup. If V_1, V_2 are irreducible smooth representations of G such that V_1^K and V_2^K are nonzero and isomorphic as $\mathcal{H}(G,K)$-modules, then $V_1 \cong V_2$. In particular, unramified representations are determined by their Hecke characters.*

Proof. Proposition 7.1.1 of Getz-Hahn. The idea is to extend an isomorphism

$$I : V_1^K \to V_2^K$$

to a G-intertwining map $V_1 \to V_2$ of $\mathcal{H}(G)$-modules. Take an element $\pi_1(f) \cdot \phi \in V_1$, then the obvious thing is to map this to $\pi_2(f) \cdot I(\phi)$. Provided this is well defined, irreducibility of V_1, V_2 tell us that this gives an isomorphism $V_1 \cong V_2$.

To check this is well defined, it suffices to show that if $\pi_1(f)\phi = 0$ then $\pi_2(f)I(\phi) = 0$. We exploit the $\mathcal{H}(G,K)$-intertwining of I (for the second implication below). For all $f_1 \in \mathcal{H}(G)$ we have:

$$\pi_1(f)\phi = 0 \Rightarrow \pi_1(e_K * f_1 * f * e_K)\phi = 0 \Rightarrow \pi_2(e_K * f_1 * f * e_K)I(\phi) = 0.$$

This tells us that $\pi_2(f)I(\phi) = 0$, otherwise $\pi_2(f_1)\pi_2(f)I(\phi)$ generates V_2 by irreducibility, and the image under $\pi_2(e_K)$ must be all of V^K which is nonzero. ☐

1.4 Example computation of Hecke operators for GL_2

Let $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O})$ for F a nonarchimedean local field with uniformiser ϖ. We have the Cartan decomposition

$$G = \bigsqcup_{a \geq b \in \mathbb{Z}} K \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} K.$$

Let $S = K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K$ and $T = K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K$, viewed as elements of $\mathcal{H}(G,K)$ via their indicator functions.

Lemma 1.12. *The unramified Hecke algebra is $\mathcal{H}(G,K) \cong \mathbb{C}[S, S^{-1}, T]$. In particular, this is commutative.*

Proof. This is some induction argument using the formula for convolutions of these indicator functions. ☐

Remark 1.13. This fits into a general phenomenon - if G is unramified and K is a hyperspecial subgroup then the Satake isomorphism implies that the unramified Hecke algebra $\mathcal{H}(G, K)$ is always commutative.

Later we will be interested in principal series representations, which are representations of G coming from parabolic induction. So let $\chi = \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ be a character of the torus T, and consider the normalised induced representation

$$I(\chi) = \mathrm{Ind}_B^G \left(\chi \otimes \delta_B^{-1/2}\right)$$

where we recall that this is the space of functions $G \to \mathbb{C}$ with $f(bg) = \chi(b)\delta_B^{-1/2}(b)f(g)$ for $b \in B$.

We briefly discuss the module character δ_B. Although G is unimodular (see Bushnell-Henniart Section 7.5), the Borel subgroup is not. We have $B = NT$ with $N \cong F$, $T \cong F^\times \times F^\times$ and N normal in B. The failure of B to be unimodular is a consequence of T and N not commuting. We can then define a linear function I on $C_c^\infty(B) = C_c^\infty(T) \otimes C_c^\infty(N)$ by

$$I(\Phi) = \int_T \int_N \Phi(tn) dt dn$$

using Haar measures on T and N.

Proposition 1.14. *I is a left Haar integral on B.*

Proof. Let $b = sm \in TN$. By left invariance of dt we have

$$\int_T \int_N \Phi(smtn) dt dn = \int_T \int_N \Phi(mtn) dt dn = \int_T \int_N \Phi(tt^{-1}mtn) dt dn.$$

Since we integrate N first, we are integrating over fixed values of t so that $t^{-1}mt \in N$ is just constant, so left invariance of dn let's us pull out the $t^{-1}mt$ factor. $\square$

Proposition 1.15. *The module δ_B of the group B is*

$$\delta_B : tn \mapsto |t_2/t_1|, \quad n \in N, t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$$

Proof. By a similar argument as above, we have

$$\int_T \int_N \Phi(tnsm) dt dn = \int_T \int_N \Phi(tss^{-1}nsm) dt dn = \int_T \int_N \Phi(ts^{-1}ns) dt dn.$$

Identifying $N \cong F$ this is

$$\int_T \int_N \Phi(t \cdot \left(\begin{smallmatrix} 1 & s_1^{-1}xs_2 \\ 0 & 1 \end{smallmatrix}\right)) d\mu_F(x) = |s_1/s_2| \int_T \int_N \Phi(tn) dt dn$$

so by definition of the module character we have $\delta_B(sm) = |s_2/s_1|$. $\square$

Going back to our principal series representation, the following proposition computes the action of the unramified Hecke algebra on the K-invariant subspace:

Proposition 1.16. *Let $\chi : T \to \mathbb{C}^\times$ be an unramified character of the torus (meaning trivial on $\begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix}$) and consider the normalised parabolic induction*

$$I(\chi) = \mathrm{Ind}_B^G(\chi \otimes \delta_B^{-1/2}).$$

For $K = \mathrm{GL}_2(\mathcal{O})$ as usual, the space $I(\chi)^K$ is 1-dimensional. As a $\mathcal{H}(G,K)$-module this is determined by the actions of S and T. Since χ is unramified we know $\chi_1(z) = \alpha^{v_F(z)}$ and $\chi_2(z) = \beta^{v_F(z)}$ for some $\alpha, \beta \in \mathbb{C}^\times$. Then S acts on $I(\chi)^K$ by scaling by $\alpha\beta$ and T acts by scaling by $q^{1/2}(\alpha+\beta)$.

Proof. We have the Iwasawa decomposition $G = BK$ so that the functions $f \in I(\chi)^K$ satisfy

$$f(bk) = f(b) = \chi(b)\delta_B^{-1/2}(b) \cdot f(1)$$

with $f(1) \in \mathbb{C}$, so the space is 1-dimensional spanned by $\hat{f}(bk) = \chi(b)\delta_B^{-1/2}(b)$.

The action of S is given by:

$$\begin{aligned} S \cdot f &= \mu(K)^{-1} \int_G \mathbb{1}_{K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K}(g) g \cdot f dg \\ &= \mu(K)^{-1} \int_K \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right) k \cdot f dk \\ &= \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right) \cdot f \\ &= \chi\left(\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)\right) \delta_B^{-1/2}\left(\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)\right) f \\ &= \alpha\beta f \end{aligned}$$

because $K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K = \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K$.

And for T we pick coset representatives for $K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K/K$ given by $\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)$, where a ranges over representatives of $\mathcal{O}/\varpi$. Writing down the integral for the action of T we decompose this into a sum over these left cosets and we deduce that T acts by

$$\chi_2(\varpi)|\varpi|^{-1/2} f + \sum_{a \in \mathcal{O}/\varpi} \chi_1(\varpi)|\varpi|^{1/2} f = q^{1/2}(\alpha+\beta)$$

since, for example, $\chi\left(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)\right) = \chi_1(\varpi) = \alpha$ and $\delta_B^{-1/2}\left(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)\right) = |\varpi|^{1/2}$. □

Remark 1.17. If we know the action of S, T on $I(\chi)^K$ for some unramified character χ of the torus T, then we can recover $\alpha, \beta \in \mathbb{C}^\times$ from the roots of the Satake polynomial $X^2 - q^{-1/2}TX + S \in \mathcal{H}(G,K)[X]$.

2 Modular forms as automorphic forms

This will be based on notes by Jeremy Booher, the LTCC notes, Getz-Hahn and Bump.

We will view modular forms as automorphic forms for GL_2. We first recall the classical definition.

Definition 2.1. For $\Gamma \leq \mathrm{GL}_2^+(\mathbb{Q})$ a subgroup commensurable with $\mathrm{SL}_2(\mathbb{Z})$ (meaning the intersection has finite index with each), we define a modular form of level Γ and weight (k,t) for $t \in \mathbb{R}$ to be a function $f : \mathcal{H} \to \mathbb{C}$ such that

- f is holomorphic;
- $f|_{(k,t)}\gamma = f$ for all $\gamma \in \Gamma$, where

$$f|_{(k,t)}\gamma(\tau) = (\det\gamma)^t (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

- $f_{(k,t)}\gamma(\tau)$ is bounded as $\tau \to i\infty$ for all $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$.

2.1 Adelic formulation of the modular curve

The usual action of $\mathrm{GL}_2^+(\mathbb{R})$ on the upper half plane $\mathcal{H}$ is transitive, and the stabiliser of i is $\mathbb{R}^+ \cdot \mathrm{SO}_2(\mathbb{R})$. Hence $\Gamma\backslash\mathcal{H}$ is in bijection with $\mathbb{R}^+\Gamma\backslash\mathrm{GL}_2^+(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

We want to make this adelic. If $K \leq \mathrm{GL}_2(\mathbb{A}_f)$ is a compact open subgroup, we have the quotient

$$Y(K) := \mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{SO}_2(\mathbb{R}) \cdot K = \mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f) \times \mathcal{H}/K$$

where $\mathrm{GL}_2^+(\mathbb{Q})$ acts diagonally and K acts by right translation on $\mathrm{GL}_2(\mathbb{A}_f)$ only. Given Γ as above we will associate an adelic modular curve $Y(K)$ for K the closure of Γ in the diagonal embedding of $\mathrm{GL}_2^+(\mathbb{Q})$ in $\mathrm{GL}_2(\mathbb{A}_f)$. For example, the closure of $\Gamma_0(N)$ is the group $K_0(N)$ of matrices in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ which are upper triangular mod N.

Theorem 2.2. *Strong approximation holds for* SL_2*, in the sense that* $\mathrm{SL}_2(\mathbb{Q})$ *is dense in* $\mathrm{SL}_2(\mathbb{A}_f)$*.*

Proof. Since $\mathrm{SL}_2(\mathbb{Z})$ surjects onto $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for all N, we see that the closure of $\mathrm{SL}_2(\mathbb{Q})$ contains $\mathrm{SL}_2(\hat{\mathbb{Z}})$. The Cartan decomposition tells us that

$$\mathrm{SL}_2(\mathbb{A}_f) = \bigsqcup_{m\geq 1} \mathrm{SL}_2(\hat{\mathbb{Z}})\begin{pmatrix} m & \\ & m^{-1}\end{pmatrix}\mathrm{SL}_2(\hat{\mathbb{Z}})$$

so that the closure of $\mathrm{SL}_2(\mathbb{Q})$ contains everything. $\square$

Proposition 2.3. *There is a bijection between* $\mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/K$ *for K compact open in* $\mathrm{GL}_2(\mathbb{A}_f)$*, and* $\mathbb{Q}^+\backslash\mathbb{A}_f^\times/\det(K)$*. In particular, if* $\det(K) = \hat{\mathbb{Z}}$ *(for example when* $K = K_0(N)$*), both sides are in bijection with the class group of* $\mathbb{Q}$*, which is trivial.*

Proof. The determinant map gives the exact sequence

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{A}_f) \longrightarrow \mathrm{GL}_2(\mathbb{A}_f) \longrightarrow \mathbb{A}_f^\times \longrightarrow 1$$

from which we get

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_f)/K\cap\mathrm{SL}_2(\mathbb{A}_f) \longrightarrow \mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/K \longrightarrow \mathbb{Q}^+\backslash\mathbb{A}_f^\times/\det(K) \longrightarrow 1.$$

Strong approximation for SL_2 tells us the first term is trivial. $\square$

Theorem 2.4. *For* $K \leq \mathrm{GL}_2(\mathbb{A}_f)$ *compact open,* $Y(K)$ *is a manifold with finitely many connected components, each (non-canonically) isomorphic to a quotient of* $\mathcal{H}$*. More precisely, if* $g_1, \ldots, g_n \in \mathrm{GL}_2(\mathbb{A}_f)$ *are representatives of* $\mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/K$ *(by the above proposition this is equivalent to the determinants being representatives of* $\mathbb{A}_f^\times/\mathbb{Q}^+\det(K)$*), then defining*

$$\Gamma_i := \mathrm{GL}_2^+(\mathbb{Q}) \cap g_i K g_i^{-1},$$

these Γ_i *are commensurable with* $\mathrm{SL}_2(\mathbb{Z})$ *and*

$$\begin{aligned} \bigsqcup \Gamma_i\backslash\mathcal{H} &\to Y(K) \\ \tau &\mapsto (g_i, \tau) \end{aligned}$$

is the isomorphism.

Proof. Let $\gamma_i \in \Gamma_i$. To show the map is well defined we need that $(g_i, \gamma_i\tau) \sim (g_i, \tau)$. Certainly $(g_i, \tau) \sim (\gamma_i g_i, \gamma_i\tau)$ and then this is equivalent to $(g_i, \gamma_i\tau)$ since $\gamma_i \in g_i K g_i^{-1}$. The map is then well defined and is injective and surjective by construction (from the definition of the g_i). The Γ_i are commensurable with $\mathrm{SL}_2(\mathbb{Z})$ because K is compact open and we can check this commensurability locally. $\square$

Note the Γ_i are left quotients but K is a right quotient.

Example 2.5. When $K = K_0(N)$ (or $K_1(N) = \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$) using $g_1 = 1$ we recover $\Gamma_0(N)\backslash\mathcal{H} \cong Y(K_0(N))$ via $\tau \mapsto (1, \tau)$.

Definition 2.6. An adelic modular form of weight (k, t) is a function

$$F : \mathrm{GL}_2(\mathbb{A}_f) \times \mathcal{H} \to \mathbb{C}$$

such that

- $F(g, \tau)$ is holomorphic in τ for every g.
- There exists open compact $K \leq \mathrm{GL}_2(\mathbb{A}_f)$ such that F is invariant under right translation by K in the first factor.
- $F(\gamma g, -) = F(g, -)|_{(k,t)}\gamma^{-1}$ for all $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ (the inverse is to go between left and right actions).
- For all $g \in \mathrm{GL}_2(\mathbb{A}_f)$, $F(g, \tau)$ is bounded as $\tau \to i\infty$.

Remark 2.7. For $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ we compute

$$F(\gamma g, \gamma\tau) = F(g, \gamma\tau)|_{(k,t)}\gamma^{-1} = F(g, \tau) \cdot C$$

where C is some constant only depending on γ (from the j-factor and the determinant). Under some appropriate renormalisation this should give a function

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathrm{SO}_2(\mathbb{R})K \to \mathbb{C}$$

which resembles the usual definition of an automorphic form (the -1 determinant is absorbed in replacing $\mathrm{GL}_2^+(\mathbb{R})$ with $\mathrm{GL}_2(\mathbb{R})$). This possibly (probably) trades off K-invariance with K-finiteness.

Notation 2.8. Let $M_{k,t}$ and $S_{k,t}$ be the space of such (cuspidal) adelic modular forms.

The spaces $M_{k,t}$ and $S_{k,t}$ are representations of $\mathrm{GL}_2(\mathbb{A}_f)$ under right translation on the first factor, and by definition (invariance under some compact open K) this representation is smooth.

Proposition 2.9. *Evaluation at the $g_1, \ldots, g_n$ in Theorem 2.4 gives an isomorphism*

$$(S_{k,t})^K = \bigoplus_{i=1}^{n} S_{k,t}(\Gamma_i)$$

where the right hand side consists of cusp forms in the classical sense. A similar result holds for $M_{k,t}$. In particular, these gives admissible representations of $\mathrm{GL}_2(\mathbb{A}_f)$.

Proof. Invert the isomorphism of Theorem **??** and check that the axioms defining a modular form match up. $\square$

Remark 2.10. It is convenient to work with the space of all (cuspidal) adelic modular forms without having to specify the level, instead incorporating the level through the fixed points.

Example 2.11. If $K = K_1(N) = \{\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right)\}$ we only have $g_1 = 1$ and we recover $S_{k,t}(\Gamma_1(N))$.

3 Principal series representations of GL_2

Let F be a nonarchimedean local field, $G = \mathrm{GL}_2(F)$, $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$ the Borel subgroup of upper triangular matrices, so that $B = NT$ for $T = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\} \cong F^\times \times F^\times$ and $N = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \cong F$ with $N \lhd B$. Between N and B we also have the mirabolic subgroup $M = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}$ with $M/N \cong F^\times$.

Ultimately we want to understand irreducible representations of G (for example, modular forms give rise to automorphic forms, the space of which is a smooth representation of $\mathrm{GL}_2(\mathbb{A})$, and this decomposes under Flath's theorem to give smooth representations of G). Initially this is too difficult so we restrict to simpler subgroups like B, and then further to N and T, although it is more natural to view T as a quotient of B. Then we get representations of B inflating from characters of T, and inducting to G is called parabolic induction, giving so-called principal series representations. We want to understand how these decompose into irreducibles, and from there we can classify all (irreducible) principal series representations using Frobenius reciprocity.

To understand the decomposition of parabolically induced representations into irreducibles as G-representations, we want to see how they decompose into irreducibles over a less unwieldy subgroup of G, such as B. It turns out that these do not decompose any further over M than over B. On the other hand, the representation theory of M is very easy to classify - this is what makes the mirabolic subgroup so 'miraculous'. To get representations of M we can induct from characters of N, or inflate from $M/N \cong F^\times$. There are many characters of $N \cong F$, in fact these are in bijection with F under $\psi(x) \mapsto \psi(ax)$ for $a \in F$ and any nontrivial character ψ. The key property of M is that conjugation by M acts transitively on these characters ψ, which greatly simplifies the representation theory of M coming from induction from N. M is also small enough that this induction, together with the characters of $F^\times$, give all irreducible representations of M.

3.1 Representations of N

For an abelian group, all the irreducible representations are characters (Schur's lemma?), and when the group is finite, any representation decomposes as a direct sum of characters. $N \cong F$ is not finite, so we lose this decomposition, but it is still true that any vector is nonzero in some quotient on which N acts via a character. To formalise this, we define

Notation 3.1. Let V be a smooth representation of N and θ a character of N. Let $V(\theta) \leq V$ be the subspace spanned by $n \cdot v - \theta(n)v$ for $n \in N$, and set $V_\theta = V/V(\theta)$ so that N acts on V_θ by θ. When θ is trivial we write $V(N)$ and V_N respectively.

The following is a useful equivalent definition of $V(\theta)$:

Lemma 3.2. *The vector $v \in V$ lies in $V(\theta)$ if and only if*

$$\int_{N_0} \theta(n)^{-1} \cdot n \cdot v dn = 0$$

for some compact open subgroup N_0 of N (we restrict to compact opens for the integral to be well defined).

Proof. [1] Lemma 8.1. □

Corollary 3.3. *The functo $V \mapsto V_\theta$ is exact from representations of N to complex vector spaces.*

Proof. Taking quotients in this way is right exact. So we need to show that if $f : V \hookrightarrow V'$ then $V_\theta \hookrightarrow V'_\theta$. If $v \in V$ with $f(v) \in V'(\theta)$ then

$$\int_{N_0} \theta(n)^{-1} n \cdot f(v) dn = 0$$

for some N_0. But this integral is a finite sum and f is compatible with the action of N so that we can pull f out of the integral. Injectivity of f implies

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0$$

from which we deduce that $v \in V(\theta)$ by the above lemma. □

Proposition 3.4. *For any $v \neq 0$ in V, there exists a character θ of N such that $v \notin V(\theta)$.*

Proof. Bushnell-Henniart Proposition 8.1. □

Corollary 3.5. *If V is a smooth representation of N with $V_\theta = 0$ for all θ then $V = 0$.*

3.2 Representations of M

Now we consider V an (irreducible) representation of M. Note that $V(N)$ is still a representation of M because N is normal in M ($mn \cdot v - m \cdot v = n'm \cdot v - m \cdot v$ for some $n' \in N$), and so V_N is also a representation of M (but V_θ is not). Since V is irreducible, either $V(N) = 0$, so that N acts trivially on V and so we just get a character of $F^\times$, or $V(N) = V$. In the latter case $V_N = 0$ so we must have $V_\theta \neq 0$ for all nontrivial characters of N (since the V_θ are conjugate under M), so that the M-representation V must have infinite dimension. In fact there is only one such V, and we can prove more specifically:

Theorem 3.6. *Let (π, V) be an irreducible smooth representation of M. Either*

- *$\dim V = 1$ and π is the inflation of a character of $M/N \cong F^\times$, or*
- *$\dim V = \infty$ and $\pi \cong c - \mathrm{Ind}_N^M \theta$ for any nontrivial character θ of N.*

This itself follows from the following theorem. To compare V and $c - \mathrm{Ind}_N^M \theta$ it is more natural to compare V and $\mathrm{Ind}_N^M V_\theta$. By Frobenius reciprocity,

$$\mathrm{Hom}_N(V, V_\theta) \cong \mathrm{Hom}_M(V, \mathrm{Ind}_N^M V_\theta).$$

Let $q_* : V \to \mathrm{Ind}_N^M(V_\theta)$ be the image of the quotient map $V \to V_\theta$.

Theorem 3.7. *The M-homomorphism $q_* : V \to \mathrm{Ind}_N^M V_\theta$ induces an isomorphism $V(N) \cong c - \mathrm{Ind}_N^M V_\theta$. Moreover, this compact induction is an irreducible representation of M.*

Proof. Bushnell-Henniart Theorem 8.3. □

3.3 Irreducible principal series representations

Let V be a smooth representation of G. By restriction this gives a representation of B, and so does the space of N-coinvariants $V_N = V/V(N)$, again because N is normal in B. Then V_N inherits a representation π_N of $T = B/N$, and we call this the Jacquet module of V at N. As shown before, the Jacquet functor $V \mapsto V_N$ is exact.

Parallel to the classical finite field setting, we want to study when V arises from parabolic induction. We have the analogous result:

Proposition 3.8. *The following are equivalent:*

- $V_N \neq 0$
- *π is isomorphic to a G-subrepresentation of $\mathrm{Ind}_B^G \chi$ for some character χ of T inflated to B.*

Proof. (2) implies (1) comes from Frobenius reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\chi) = \mathrm{Hom}_B(\pi, \chi) = \mathrm{Hom}_T(\pi_N, \chi)$$

where the second equality is due to any B-homomorphism $\pi \to \chi$ factoring through π_N (because χ is trivial on N).

Given (1), one shows by a technical argument that V_N is finitely generated as a representation of T. An application of Zorn's lemma allows us to construct a maximal T-subspace U of V_N so that V_N/U is an irreducible T-representation and is thus a character (Schur's lemma) χ. Frobenius reciprocity implies the result. □

Remark 3.9. The same proof holds for the finite field case (noting the notion of having a subrepresentation where N acts trivially is the same as having a nonzero quotient where N acts trivially). The proof that (1) implies (2) bypasses the technical details because V_N as a representation of T obviously admits an irreducible quotient as V_N is finite dimensional.

Remark 3.10. In the general case we ask for a nonzero quotient of V on which N acts trivially as opposed to having a subrepresentation, because one can show in this latter case that all we get are characters $\pi = \phi \circ \det$ for some character ϕ of $F^\times$. In fact any finite dimensional smooth representation is of this form. The difference with the finite field case is that smoothness tells us that if $v \in V$ is fixed by N, it is also fixed by an open compact subgroup of G. Over a finite field, N is open, but in general it is not and we fix v by too much (all of SL_2).

We restrict our attention to principal series representations and want to understand how $\mathrm{Ind}_B^G\chi$ decomposes into irreducible G-representations. As mentioned earlier, we will first study how they decompose as representations of B or even M.

These induced representations will never be irreducible over B because we always have the canonical B-homomorphism $X = \mathrm{Ind}_B^G\chi \to \chi$ given by sending $f \mapsto f(1) \in \mathbb{C}$. So we have an exact sequence of B-representations

$$0 \longrightarrow V \longrightarrow X \longrightarrow \mathbb{C} \longrightarrow 0$$

where $V = \{f \in X\chi : f(1) = 0\}$, with B acting on $\mathbb{C}$ via χ. Now we want to understand how V decomposes. We have another exact sequence of B-representations,

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

so we reduce to studying $V(N)$ and V_N. We will show that $V(N)$ is irreducible over B (and even over M), while V_N will be determined by the Restriction-Induction lemma (which generally treats the exact sequence obtained by applying the Jacquet functor to the first exact sequence, where we may replace χ by any smooth representation σ of T).

Firstly we want to understand $V = \{f \in X : f(1) = 0\}$ better.

Lemma 3.11. *For V as above, the map*

$$V \to C_c^\infty(N)$$
$$f(-) \mapsto f(w-)$$

is an N-isomorphism, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

Proof. We have the decomposition $G = B \sqcup BwN$. Since $f(1) = 0$ and f is induced from B we must have that f is supported on BwN. G-smoothness of f implies that $f(1) = 0$ is fixed by right translation by some compact open subgroup $K \leq G$. This will contain $\left(\begin{smallmatrix} 1 & 0 \\ \pi^n\mathcal{O} & 0 \end{smallmatrix}\right)$ for some n, so that f vanishes on

$$\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \in Bw\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)$$

for all $x \in \pi^n\mathcal{O}$. Thus $f(w-)$ is supported on $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in N$ with $v(y) > -n$ (so $y \in \pi^{-n}\mathcal{O}$). G-smoothness of f also implies that $f(w-)$ is N-smooth, and that the above map is an N-homomorphism. The decomposition $G = B \sqcup BwB$ implies that it is in fact an isomorphism. $\square$

Proposition 3.12. *For V as above, $V(N)$ is irreducible over M (and hence over B).*

Proof. By the above lemma we can identify $V \cong C_c^\infty(N)$ with M acting via right translation on V. This gives the structure of a M-representation on $C_c^\infty(N)$. We can calculate it explicitly (but we won't need it) where

$$f(bw\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)) = f(b\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)w\left(\begin{smallmatrix} 1 & a^{-1}x \\ 0 & 1 \end{smallmatrix}\right))$$

tells us that the corresponding $M = F^\times N$ action on $C_c^\infty(N)$ is the composite of right translation by N with the action

$$a\phi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \chi_2(a)\phi\left(\begin{smallmatrix} 1 & a^{-1}x \\ 0 & 1 \end{smallmatrix}\right)$$

So now we may consider $V = C_c^\infty(N)$. The benefit is that for this representation, the spaces of coinvariants of characters of N are very simple. In particular, the map $f \mapsto \theta f$ is a linear automorphism of $C_c^\infty(N)$ taking $V(N)$ to $V(\theta)$ since

$$n \cdot f - f \mapsto \theta(n \cdot f) - \theta f = \theta(n)^{-1} n \cdot (\theta f) - \theta f \in V(\theta).$$

Hence all the V_θ have the same dimension as $V_N = V/V(N)$, which has dimension 1 (we can see this from the characterisation of $V(N)$ as the zeros of some integral, or from the Restriction-Induction lemma to follow).

But then Theorem 3.7 implies that for our M-representation V, we have $V(N) \cong c - \mathrm{Ind}_N^M V_\theta$ where $V_\theta \cong \theta$ as it is one dimensional. This is irreducible as a M-representation by the same Theorem. $\square$

We turn our attention to V_N where we recall V fits in the exact sequence

$$0 \longrightarrow V \longrightarrow X = \mathrm{Ind}_B^G \chi \xrightarrow{f \mapsto f(1)} \chi \longrightarrow 0$$

of smooth representations of B. Since the Jacquet functor is exact, we get the exact sequence

$$0 \longrightarrow V_N \longrightarrow X_N \longrightarrow \chi \longrightarrow 0$$

of T-representations. We can say in more generality,

Lemma 3.13 (Restriction-Induction Lemma)**.** *Let (σ, U) be a smooth representation of T and $(\Sigma, X) = \mathrm{Ind}_B^G \sigma$. Then there is an exact sequence of smooth T representations:*

$$0 \longrightarrow \sigma^w \otimes \delta_B^{-1} \longrightarrow \Sigma_N \longrightarrow \sigma \longrightarrow 0$$

Proof. The proof of Lemma 3.11 generalises to show that the vector space V is isomorphic to the space $\mathcal{S}$ of smooth compactly supported functions $N \to U$ by identifying f with $f(w-)$.

We can define a map $\mathcal{S} \to U$ by

$$g = f(w-) \mapsto \int_N g(n) = f(wn) dn$$

where this integral is finite since g is compactly supported. By Lemma 3.2, this induces an isomorphism $\mathcal{S}_N \cong U$.

The B-representation structure on $\mathcal{S}$ coming from V is by right translation, where $b = sm \in TN$ acts by

$$f(wnsm) = f(wss^{-1}nsm) = f(wswws^{-1}nsm) = \sigma(s^w) f(ws^{-1}nsm)$$

where $s^{-1}nsm \in N$. Under the isomorphism $\mathcal{S}_N \cong U$, this induces a T representation structure on U where $s \in T$ acts by

$$s \cdot \int_N f(wn) dn = \sigma(s^w) \int_N f(ws^{-1}ns) dn = \sigma(s^w) |\frac{s_1}{s_2}| \int_N f(wn) dn$$

which is $\sigma^w \otimes \delta_B^{-1}$. $\square$

Corollary 3.14. *As a representation of B or M, $\mathrm{Ind}_B^G \chi$ has composition length 3. Two of the factors have dimension 1, and the other is infinite dimensional.*

Proof. This follows from the exact sequences

$$0 \longrightarrow V \longrightarrow \mathrm{Ind}_B^G \chi \longrightarrow \chi \longrightarrow 0$$

and

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

where we saw that $V(N)$ is irreducible (and infinite dimensional by Theorem 3.6), and $V_N \cong \chi^w \otimes \delta_B^{-1}$. □

So we understand how $\mathrm{Ind}_B^G \chi$ decomposes into irreducible B representations, and we want to understand its decomposition into G representations. Our goal is to prove the following

Theorem 3.15 (Irreducibility Criterion)**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let $X = \mathrm{Ind}_B^G \chi$.*

1. *X is irreducible if and only if $\chi_1 \chi_2^{-1}$ is either the trivial character of $F^\times$, or the character $x \mapsto |x|^2$ of $F^\times$.*

2. *Suppose X is reducible, then*

   - *the G-composition length of X is 2*
   - *one factor has dimension 1, the other is infinite dimensional*
   - *X has a 1-dimensional G-subspace exactly when $\chi_1 \chi_2^{-1} = 1$*
   - *X has a 1-dimensional G-quotient exactly when $\chi_1 \chi_2^{-1}(x) = |x|^2$.*

We make some comments in preparation for the proof. By the above Corollary, if X is reducible then it has a finite dimensional (dimension 1 or 2) G-subspace or G-quotient. By taking duals we can assume we are in the first case. In the Irreducibility Criterion, we want to show that this implies $\chi_1 = \chi_2$ and that X has a 1-dimensional G-subspace.

Definition 3.16. Let π be a smooth representation of G and ϕ a character of $F^\times$. The twist of π by ϕ is the representation $\phi\pi$ of G defined by

$$\phi\pi(g) = \phi(\det g)\pi(g).$$

In this way, for a character $\chi = \chi_1 \otimes \chi_2$ of T, we have $\phi\chi = \phi\chi_1 \otimes \phi\chi_2$. Then

$$\mathrm{Ind}_B^G(\phi\chi) = \phi \mathrm{Ind}_B^G \chi.$$

Proposition 3.17. *The following are equivalent:*

1. *$\chi_1 = \chi_2$*
2. *X has a 1-dimensional N-subspace.*

If this holds then this subspace is unique, and is also a G-subspace of X not contained in V.

Proof. (1) implies (2): since induction commutes with twisting we may assume $\chi_1 = \chi_2 = 1$, then the nonzero constant function spans a 1-dimensional G-subspace (not just N-subspace) of $X = \mathrm{Ind}_B^G 1$.

(2) implies (1): suppose this subspace is spanned by f. N acts by right translation as a character. We cannot have $f \in V$ ($f(1) = 0$) else we earlier saw that f would then have support in some BwN_0 for $N_0 \le N$ open compact, and this is not closed under multiplication by N.

So $f \notin V$ ($f(1) \neq 0$) and so its image spans $X/V \cong \mathbb{C}$ on which N acts trivially (since we inflate χ to be trivial on N). Thus N fixes f under right translation. f is also fixed under right translation by some compact open of G, so for sufficiently large $|x|$ we have

$$\begin{aligned} f(w) = f\left(w\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\right) \\ &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\right) \\ &= \chi_1(-1)\left(\chi_1^{-1}\chi_2(x)\right) f(1) \end{aligned}$$

For this to hold for all $|x|$ sufficiently large, it follows that we must have $\chi_1 = \chi_2$ (if $\chi_1(y) \neq \chi_2(y)$ then $\chi_1(xy) \neq \chi_2(xy)$ for all sufficiently large x, but then xy is also large). The uniqueness of the 1-dimensional subspace comes from the fact that it must span $X/V \cong \mathbb{C}$. □

Proof of Irreducibility Criterion. Assume that X is reducible and we are in the case that X has a finite dimensional G-subspace. Then it has a 1-dimensional N-subspace L, which is also a G subspace by the above Proposition with G acting via $\phi \circ \det$, where $\phi = chi_1 = \chi_2$. Since $L \cap V = 0$, we see that $Y = X/L \cong V$ as B-representations. We need to show X has G-length 2. By the previous corollary it has length at most 3. We know that V has B-length 2 with a 1-dimensional quotient V_N. Thus if Y had G-length 2, then the B-factors of V are also G-factors, so that G must act on V_N, necessarily by a character $\phi' \circ \det$ (see 9.2 Exercise 2). But this is impossible because $B \le G$ acts by $\phi\delta_B^{-1}$ by restriction-induction, and this does not factor through det on B. So we must have that X has G-length 2.

In the other case we have a finite dimensional G-quotient. The smooth dual $X^\vee$ is then in the first case, where the Duality Theorem tells us $X^\vee \cong \mathrm{Ind}_B^G \delta_B^{-1}\chi^\vee$. If we write $\delta_B^{-1}\chi^\vee = \psi_1 \otimes \psi_2$ then we must have $\psi_1 = \psi_2$. The result follows from computing $\psi_1(x) = |x|^{-1}\chi_1(x)$ and $\psi_2(x) = |x|\chi_2(x)$.

The converse direction to (1) follows from the previous Proposition. □

3.4 Classification of principal series representations

Now that we've seen how parabolically induced representations decompose into irreducibles, we want to classify the isomorphism classes.

Proposition 3.18. *Let χ, ξ be characters of T. The space $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi)$ is 1-dimensional if $\xi = \chi$ or $\chi^w\delta_B^{-1}$ and 0 otherwise.*

Proof. Frobenius reciprocity tells us

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi) \cong \mathrm{Hom}_T((\mathrm{Ind}\chi)_N, \xi).$$

From restriction-induction we have

$$0 \longrightarrow \chi^w \delta_B^{-1} \longrightarrow (\mathrm{Ind}\chi)_N \longrightarrow \chi \longrightarrow 0.$$

In the case $\chi \neq \chi^w \delta_B^{-1}$ the sequence splits and the result follows. If $\chi = \chi^w \delta_B^{-1}$ then $\chi_1 \chi_2^{-1}(x) = |x|$ so $\mathrm{Ind}\chi$ is irreducible and the result still follows. $\square$

Remark 3.19. Hence, in the case $\mathrm{Ind}\chi$ is irreducible, we have $\mathrm{Ind}\chi \cong \mathrm{Ind}\chi^w \delta_B^{-1}$.

And in the case $\mathrm{Ind}\chi$ is reducible, it is not semisimple, else the Hom space would be 2-dimensional.

We can be more explicit in the reducible case. One can check that the conditions in the Irreducibility Criterion of reducibility are equivalent to χ being of the form $\chi = \phi 1_T$ or $\chi = \phi\delta_B^{-1}$. Untwisting, we may as well assume $\phi = 1$.

Definition 3.20. The Steinberg representation is defined by the exact sequence

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow \mathrm{St}_G \longrightarrow 0$$

which is an infinite dimensional irreducible representation with Jacquet module $(\mathrm{St}_G)_N \cong \delta_B^{-1}$ by restriction-induction. If $\chi = \phi 1_T$ we would instead get a twist of Steinberg, $\phi\mathrm{St}_G$.

The case $\chi = \delta_B^{-1}$ can be dealt with by taking smooth duals (which is exact (Lemma 2.10 of Bushnell-Henniart) and preserves irreducibles (by checking on V^K)) to get

$$0 \longrightarrow \mathrm{St}_G^\vee \longrightarrow \mathrm{Ind}_B^G \delta_B^{-1} \longrightarrow 1_G \longrightarrow 0$$

The Proposition applied to $\chi = 1$ then implies

$$\mathrm{St}_G \cong \mathrm{St}_G^\vee.$$

Notation 3.21. Define normalised induction by

$$\iota_B^G \sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma).$$

This has the benefit that $(\iota_B^G\sigma)^\vee \cong \iota_B^G \sigma^\vee$.

Theorem 3.22 (Classification Theorem)**.** *The following are all the isomorphism classes of principal series representations of G:*

- *the irreducible induced representations $\iota_B^G\chi$ when $\chi \neq \phi\delta_B^{\pm 1/2}$ for a character ϕ of $F^\times$.*
- *the one-dimensional representations $\phi \circ \det$ for ϕ a character of $F^\times$.*
- *the twists of Steinberg (special representations) $\phi\mathrm{St}_G$ for ϕ a character of $F^\times$.*

These are all distinct isomorphism classes except in the first case where $\iota_B^G\chi \cong \iota_B^G\chi^w$.

References

[1] C. J. Bushnell and G. Henniart. *The Local Langlands Conjecture for GL(2)*. Grundlehren der mathematischen Wissenschaften. Springer Berlin, 2006.

[2] J. R. Getz and H. Hahn. *An Introduction to Automorphic Representations*. Graduate Texts in Mathematics. Springer Cham, 2024.