Local Langlands for GL_2

Yiannis Fam, Albert Lopez Bruch, Jakab Schrettner

March 10, 2024

1 Locally Profinite Groups and Smooth Representations

1.1 Local Fields and Locally Profinite Groups

We begin by recalling some basic objects from algebraic number thoery. Given a field F, a **discrete valuation** on F is a surjective function $\nu : F \to \mathbb{Z} \cup \{\infty\}$ satisfying the three conditions

1. $\nu(xy) = \nu(x) + \nu(y)$ for any $x, y \in F$
2. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for any $x, y \in F$.
3. $\nu(x) = \infty$ if and only if $x = 0$.

Any discrete valuation ν induces an absolute value given by the formula

$$|x| = c^{\nu(x)}$$

for any $c \in (0, 1)$, and therefore it also induces a topology. We remark that this topology is independent of the choice of c. In addition, the absolute value satisfies $|x + y| \leq \max\{|x|, |y|\}$ for any $x, y \in K$. Absolute values with this property are denoted as *non-Archimedean*.

A field F with a an absolute value $|\cdot|$ induced by a discrete valuation ν is the fraction field of the valuation ring

$$R := \{x \in F : v(x) \geq 0\} = \{x \in K : |x| \leq 1\},$$

which contains a unique maximal ideal

$$\mathfrak{p} := \{x \in F : v(x) > 0\} = \{x \in K : |x| < 1\},$$

denoted as the valuation ideal. This ideal is principal, and it is generated by any $\varpi \in K$ with $\nu(\varpi) = 1$. Such an element is called a uniformizer of F. Finally, the residue field κ of F is the quotient $R/\mathfrak{p}$. This motivates the following definition.

Definition 1.1. A field F is a (non-Archimedean) *local field* if it is complete with respect to a topology induced by a discrete valuation and with finite residue field.

Remark 1.2. When the residue field is finite, it is conventional to write

$$|x| = q^{-\nu(x)},$$

where $q = |\kappa|$. From here onwards, we will follow this convention.

As discussed above, the valuation ring R of a local field F is a local ring with unique principal ideal $\mathfrak{p}$. Furthermore, the ideals

$$\mathfrak{p}^n = \{x \in F : \nu(x) \geq n\} = \{x \in F : |x| \leq q^{-n}\} = \varpi^n R, \quad n \in \mathbb{N}$$

are a complete set of ideals of R and also a fundamental system of neighbourhoods of the identity. Under the topology induced by the discrete valuation, the field F (and therefore also R) is totally disconnected, and furthermore we also have a topological isomorphism

$$R \longrightarrow \varprojlim_{n \geq 1} R/\mathfrak{p}^n,$$

where the maps implicit in the right hand side are the obvious ones. In particular, since the residue field is finite, all rings $R/\mathfrak{p}^n$ are finite and induced with the discrete topology. Hence the inverse limit, being a closed subset of the product of compact sets, is a compact set. This shows that R, and therefore also any $\mathfrak{p}^n$ for any $n \in \mathbb{Z}$ is a compact open subring of F. We have therefore shown that F has the important property that any open subset of F contains an open compact subgroup (namely $\mathfrak{p}^n$ for a sufficiently large n).

We also remark that F satisfies the rather special property of being the union of its open compact subgroups, even though F itself is clearly not. This fact has relevant consequences as we may discuss later.

We are now ready to give the main definition of this section.

Definition 1.3. A topological group G (which we always assume to be Hausdorff) is a *locally profinite group* if every open neighbourhood of the identity contains a compact open subgroup.

In this document we will be interested in studying the representation theory of many important groups and rings related to the local field F. The notion of a locally profinite group is an abstract one, but it has the great advantage of accomodating many important groups and rings associated to non-Archimedean local fields and their representation theory.

Examples 1.4.

1. In the preceding discussion, we have shown that F is a locally profinite group, where $\mathfrak{p}^n$ for $n \geq 1$ is a fundamental system of open compact subgrups.

2. The multiplicative group $F^\times$ is also a locally profinite group, where the congruence unit groups $U_F^n = 1+\mathfrak{p}^n$ for $n \geq 1$ is a fundamental system of open compact subgroups. We remark that unlike F, the group $F^\times$ is not the union of its open compact subgroups.

3. Given $n \geq 1$ an integer, the additive group $F^n = F \times \cdots \times F$ is also a locally profinite group endowed with the product topology. More generally, the product of locally profinite groups is locally profinite.

We give some further insight into the terminology used. If G is a locally profinite group, any open subgroup K of G is also a locally profinite group under the subspace topology. Also, if H is a closed normal subgroup of G, then G/H is also locally profinite. Recall that any profinite group is a locally profinite group and it is compact. Using a topological argument, one can also show that the converse also holds. That is, if K is a compact locally profinite group, then

$$K \longrightarrow \varprojlim K/N$$

is a topological isomorphism where N ranges over the normal open subgroups, and the implicit maps are the obvious ones.

1.2 Continuous Characters of Local Fields

2 Hecke Algebras

In this section, we define the Hecke algebra $\mathcal{H}(G)$ associated to a locally profinite (unimodular) group G and explain how to switch between smooth representations of G and smooth modules of $\mathcal{H}(G)$. Under certain conditions on G we consider a particular subalgebra of $\mathcal{H}(G)$; the unramified Hecke algebra $\mathcal{H}(G, K)$, which turns out to be commutative by the Satake isomorphism. We use as reference Chapter 4 of [BH06] and Chapter 5 of [GH24].

If G is a finite group, representations of G are the same as $\mathbb{C}[G]$-modules. We want to extend this notion to smooth representations of locally profinite groups, where we need to correctly interpret the group algebra.

Let G be a locally profinite unimodular group and K an open compact subgroup of G. Let $C_c^\infty(G)$ be the space of locally constant compactly supported functions $G \to \mathbb{C}$ and $C_c^\infty(G//K)$ the K bi-invariant subspace.

These are naturally $\mathbb{C}$-vector spaces and we endow them with an associative (not necessarily unital) ring structure coming from convolution

$$f * h(g) := \int_G f(x)h(x^{-1}g)dx$$

where we fix a Haar measure $\mu = dx$ on G.

When G is discrete this is the usual product on $\mathbb{C}[G]$.

Definition 2.1. Let $\mathcal{H}(G)$ and $\mathcal{H}(G, K)$ denote $C_c^\infty(G)$ and $C_c^\infty(G//K)$ with the algebra structure specified above. We call $\mathcal{H}(G)$ the Hecke algebra of G.

We study these algebras in more detail:

The element $e_K = \mu(K)^{-1}\mathbb{1}_K \in \mathcal{H}(G)$ is idempotent and we have the property that

$$e_K * f = f \Leftrightarrow f \text{ is } K \text{ left invariant .}$$

Thus $\mathcal{H}(G, K) = e_K * \mathcal{H}(G) * e_K$, and this subalgebra now has a unit e_K. The compactness of K ensures $e_K \in C_c^\infty(G)$.

By Lemma 5.2.1 of [GH24], $\mathcal{H}(G)$ is spanned by indicator functions of K'-double cosets, where K' ranges over all compact open subgroups of G. If we normalise these indicator functions by defining

$$[K\alpha K] = \mu(K)^{-1}\mathbb{1}_{K\alpha K},$$

then we have the formula

$$[K\alpha K] * [K\beta K] = \sum_{i,j}[K\alpha_i\beta_j K]$$

where $K\alpha K = \sqcup K\alpha_i$ and $K\beta K = \sqcup \beta_j K$. This determines multiplication in the Hecke algebra.

2.1 Smooth representations and $\mathcal{H}(G)$-modules

We now explain how the concepts of smooth representations of G and smooth modules over $\mathcal{H}(G)$ are interchangeable. To define these smooth modules, we note that the Hecke algebra $\mathcal{H}(G)$ does not in general have a unit. Consequently, not every $\mathcal{H}(G)$-module M satisfies $\mathcal{H}(G)M = M$.

Definition 2.2. We say that a $\mathcal{H}(G)$-module M is smooth if $\mathcal{H}(G)M = M$.

Definition 2.3. From a representation V of G we define the action of $\mathcal{H}(G)$ on V via

$$f \cdot v := \int_G f(g)g \cdot v dg.$$

This can be viewed as a weighted average of the action of G on v, where the weighting is described by $f \in C_c^\infty(G)$. The integral defines an element of V when $f \in C_c^\infty(G)$ as the integral reduces to a finite sum.

Lemma 2.4. *Under this action, $e_K \in \mathcal{H}(G)$, for $K \leq G$ a compact open, is the projection $V \to V^K$ onto the K-invariants of V. In particular, e_K is an idempotent element of $\mathcal{H}(G)$, it is the identity element of $\mathcal{H}(G, K)$, and V^K is a $\mathcal{H}(G, K)$-module.*

Proof. Let $V(K) \leq V$ be the subspace spanned by vectors of the form $k \cdot v - v$. Since e_K is invariant under K-translation, it is zero on $V(K)$. The normalisation of e_K is such that e_K is the identity on V^K, and this implies the result. □

Proposition 2.5. *A representation V of G is smooth if and only if it is a smooth $\mathcal{H}(G)$-module.*

Proof. If V is a smooth representation, then any $v \in V$ is K-invariant for some compact open K, and so $v = e_K \cdot v$. This implies that V is a smooth $\mathcal{H}(G)$-module. Conversely, $\mathcal{H}(G)$ is the union of $e_K * \mathcal{H}(G) * e_K = \mathcal{H}(G, K)$ over all compact open K, and so if V is a smooth $\mathcal{H}(G)$-module then any $v \in V$ is of the form $e_K * f * e_K \cdot v'$ for some K, f, v'. Then $e_K \cdot v = v$ and so $v \in V^K$. □

So we can view smooth representations of G as smooth $\mathcal{H}(G)$-module. In the other direction, given M a smooth $\mathcal{H}(G)$-module, we have

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M = M$$

by smoothness. We can then view M as a smooth G representation by letting G act on the first factor by left translation. Concretely, if $m \in M$ there exists K such that $e_K \cdot m = m$. Then define

$$g \cdot m := \mu(K)^{-1} \mathbb{1}_{gK} \cdot m,$$

where this is independent of K due to the normalisation factor $\mu(K)^{-1}$.

2.2 Information in the K-invariants V^K

For a smooth representation V of G it is often easier to study the K-invariants V^K for compact open subgroups K of G.

Lemma 2.6. *A smooth representation V of G is irreducible if and only if each V^K is either 0 or a simple $\mathcal{H}(G,K)$-module for all compact open $K \leq G$.*

Proof. Suppose V is irreducible. If we had $0 \neq M \subset V^K$ a $\mathcal{H}(G,K)$-module, then $0 \neq \mathcal{H}(G)M \subset V$ as smooth $\mathcal{H}(G)$-modules. Since smooth $\mathcal{H}(G)$-modules are the same as smooth G-representations, and V is irreducible, we deduce $\mathcal{H}(G)M = V$. So then

$$V^K = e_K V = e_K * \mathcal{H}(G)M = e_K * \mathcal{H}(G) * e_K M = \mathcal{H}(G,K)M = M$$

which implies the result.

If V is not irreducible, and $W \neq 0$ is a proper subrepresentation, pick $v \in V - W$. By smoothness, there exists K such that $v \in V^K$, but then $v \notin W^K$ so that V^K is not 0 or simple. □

The next result tells us that for any K, any smooth representation V of G is determined by V^K with its structure as a $\mathcal{H}(G,K)$-module, provided $V^K \neq 0$.

Proposition 2.7. *The map $V \mapsto V^K$ induces a bijection between*

- *equivalence classes of irreducible smooth representations V of G with $V^K \neq 0$;*
- *isomorphism classes of simple (by definition nonzero) $\mathcal{H}(G,K)$-modules.*

Proof. Proposition 4.3 of [BH06]. □

2.3 Unramified representations of G

It is interesting to study the smooth representations V with $V^K \neq 0$ as above. For example, in an automorphic representation, Flath's theorem ([GH24] Section 5.7) allows us to decompose into local factors, and furthermore tells us that almost all such local representations are unramified in the following sense:

Definition 2.8. We consider the case $G = \mathrm{GL}_2(F)$. We say that an irreducible smooth representation V of G is unramified if $V^K \neq 0$ for $K = \mathrm{GL}_2(\mathcal{O}_F)$. See Section 5.5 of [GH24] for a more general definition for reductive groups.

For the remainder of this subsection we work in the context of $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O}_F)$ for simplicity. The results generalise to reductive groups G as in Sections 5.5 and 7.1 of [GH24].

Definition 2.9. For K as above, $\mathcal{H}(G, K)$ is called the unramified Hecke algebra of G.

An application of the Satake isomorphism ([GH24] Theorem 5.5.1) tells us that in this unramified case, the unramified Hecke algebra $\mathcal{H}(G, K)$ is commutative. It follows that if V is K-unramified (in particular irreducible) then V^K is 1-dimensional by Lemma 2.6. Thus $\mathcal{H}(G, K)$ acts on V^K via scaling, called the Hecke character of V.

Definition 2.10. The Hecke character (with respect to K) of a smooth representation (π, V) of G is the $\mathbb{C}$-linear map

$$\mathcal{H}(G, K) \to \mathbb{C}$$
$$f \mapsto \mathrm{tr}\pi(f)$$

defined by $f \cdot v =: \mathrm{tr}\pi(f)v$ for any $v \in V^K$.

We give an alternative proof of Proposition 2.7.

Proposition 2.11. *Let $K \leq G$ be a compact open subgroup. If V_1, V_2 are irreducible smooth representations of G such that V_1^K and V_2^K are nonzero and isomorphic as $\mathcal{H}(G, K)$-modules, then $V_1 \cong V_2$. In particular, unramified representations are determined by their Hecke characters.*

Proof. This is Proposition 7.1.1 of [GH24]. The idea is to extend an isomorphism

$$I : V_1^K \to V_2^K$$

to a G-intertwining map $V_1 \to V_2$ of $\mathcal{H}(G)$-modules. By irreducibility, $V_i = \mathcal{H}(G)V_i^K$. Take an element $\pi_1(f) \cdot \phi \in V_1$, with $f \in \mathcal{H}(G), \phi \in V_1^K$, then the obvious choice is to map this to $\pi_2(f) \cdot I(\phi)$. Provided this is well defined, this is a nonzero homomorphism of $\mathcal{H}(G)$-modules, so irreducibility of V_1, V_2 implies this is an isomorphism $V_1 \cong V_2$.

To check this is well defined, it suffices to show that if $\pi_1(f)\phi = 0$ then $\pi_2(f)I(\phi) = 0$. We exploit the $\mathcal{H}(G, K)$-intertwining of I (for the second implication below). For all $f_1 \in \mathcal{H}(G)$ we have:

$$\pi_1(f)\phi = 0 \Rightarrow \pi_1(e_K * f_1 * f * e_K)\phi = 0 \Rightarrow \pi_2(e_K * f_1 * f * e_K)I(\phi) = \pi_2(e_K * f_1 * f)I(\phi) = 0.$$

By Lemma 2.4, e_K acts on V_2 by projection to V_2^K. If $\pi_2(f)I(\phi) \neq 0$, then $\pi_2(f_1)\pi_2(f)I(\phi)$, over all $f_1 \in \mathcal{H}(G)$, generates V_2 by irreducibility. The image under $\pi_2(e_K)$ is the exactly V^K, which is nonzero, contradicting the implication above. $\square$

2.4 Example computation of Hecke operators for $\mathrm{GL}_2(F)$

[I haven't checked this subsection. Some parts might be more suitable for a section on modular forms. The computation of the modular character of B will be needed in the main text. And the last proposition naturally goes with the unramified representations above.]

Let $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O})$ for F a nonarchimedean local field with uniformiser ϖ. We have the Cartan decomposition

$$G = \bigsqcup_{a \geq b \in \mathbb{Z}} K \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} K.$$

Let $S = K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K$ and $T = K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K$, viewed as elements of $\mathcal{H}(G, K)$ via their indicator functions.

Lemma 2.12. *The unramified Hecke algebra is $\mathcal{H}(G, K) \cong \mathbb{C}[S, S^{-1}, T]$. In particular, this is commutative.*

Proof. This is some induction argument using the formula for convolutions of these indicator functions. $\square$

Remark 2.13. This fits into a general phenomenon - if G is unramified and K is a hyperspecial subgroup then the Satake isomorphism implies that the unramified Hecke algebra $\mathcal{H}(G, K)$ is always commutative.

Later we will be interested in principal series representations, which are representations of G coming from parabolic induction. So let $\chi = \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ be a character of the torus T, and consider the normalised induced representation

$$I(\chi) = \mathrm{Ind}_B^G \left(\chi \otimes \delta_B^{-1/2}\right)$$

where we recall that this is the space of functions $G \to \mathbb{C}$ with $f(bg) = \chi(b)\delta_B^{-1/2}(b)f(g)$ for $b \in B$.

We briefly discuss the module character δ_B. Although G is unimodular (see Bushnell-Henniart Section 7.5), the Borel subgroup is not. We have $B = NT$ with $N \cong F$, $T \cong F^\times \times F^\times$ and N normal in B. The failure of B to be unimodular is a consequence of T and N not commuting. We can then define a linear function I on $C_c^\infty(B) = C_c^\infty(T) \otimes C_c^\infty(N)$ by

$$I(\Phi) = \int_T \int_N \Phi(tn) dt dn$$

using Haar measures on T and N.

Proposition 2.14. *I is a left Haar integral on B.*

Proof. Let $b = sm \in TN$. By left invariance of dt we have

$$\int_T \int_N \Phi(smtn) dt dn = \int_T \int_N \Phi(mtn) dt dn = \int_T \int_N \Phi(tt^{-1}mtn) dt dn.$$

Since we integrate N first, we are integrating over fixed values of t so that $t^{-1}mt \in N$ is just constant, so left invariance of dn let's us pull out the $t^{-1}mt$ factor. $\square$

Proposition 2.15. *The module δ_B of the group B is*

$$\delta_B : tn \mapsto |t_2/t_1|, \quad n \in N, t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$$

Proof. By a similar argument as above, we have

$$\int_T \int_N \Phi(tnsm) dt dn = \int_T \int_N \Phi(tss^{-1}nsm) dt dn = \int_T \int_N \Phi(ts^{-1}ns) dt dn.$$

Identifying $N \cong F$ this is

$$\int_T \int_N \Phi(t \cdot \left(\begin{smallmatrix} 1 & s_1^{-1}xs_2 \\ 0 & 1 \end{smallmatrix}\right)) d\mu_F(x) = |s_1/s_2| \int_T \int_N \Phi(tn) dt dn$$

so by definition of the module character we have $\delta_B(sm) = |s_2/s_1|$. $\square$

Going back to our principal series representation, the following proposition computes the action of the unramified Hecke algebra on the K-invariant subspace:

Proposition 2.16. *Let $\chi : T \to \mathbb{C}^\times$ be an unramified character of the torus (meaning trivial on $\begin{pmatrix} \mathcal{O}^\times & \\ & \mathcal{O}^\times \end{pmatrix}$) and consider the normalised parabolic induction*

$$I(\chi) = \mathrm{Ind}_B^G(\chi \otimes \delta_B^{-1/2}).$$

For $K = \mathrm{GL}_2(\mathcal{O})$ as usual, the space $I(\chi)^K$ is 1-dimensional. As a $\mathcal{H}(G,K)$-module this is determined by the actions of S and T. Since χ is unramified we know $\chi_1(z) = \alpha^{v_F(z)}$ and $\chi_2(z) = \beta^{v_F(z)}$ for some $\alpha, \beta \in \mathbb{C}^\times$. Then S acts on $I(\chi)^K$ by scaling by $\alpha\beta$ and T acts by scaling by $q^{1/2}(\alpha+\beta)$.

Proof. We have the Iwasawa decomposition $G = BK$ so that the functions $f \in I(\chi)^K$ satisfy

$$f(bk) = f(b) = \chi(b)\delta_B^{-1/2}(b) \cdot f(1)$$

with $f(1) \in \mathbb{C}$, so the space is 1-dimensional spanned by $\hat{f}(bk) = \chi(b)\delta_B^{-1/2}(b)$.

The action of S is given by:

$$\begin{aligned} S \cdot f &= \mu(K)^{-1} \int_G \mathbb{1}_{K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K}(g) g \cdot f dg \\ &= \mu(K)^{-1} \int_K \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right) k \cdot f dk \\ &= \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right) \cdot f \\ &= \chi\left(\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)\right) \delta_B^{-1/2}\left(\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)\right) f \\ &= \alpha\beta f \end{aligned}$$

because $K\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K = \left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)K$.

And for T we pick coset representatives for $K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K/K$ given by $\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)$, where a ranges over representatives of $\mathcal{O}/\varpi$. Writing down the integral for the action of T we decompose this into a sum over these left cosets and we deduce that T acts by

$$\chi_2(\varpi)|\varpi|^{-1/2} f + \sum_{a \in \mathcal{O}/\varpi} \chi_1(\varpi)|\varpi|^{1/2} f = q^{1/2}(\alpha + \beta)$$

since, for example, $\chi\left(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)\right) = \chi_1(\varpi) = \alpha$ and $\delta_B^{-1/2}\left(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)\right) = |\varpi|^{1/2}$. □

Remark 2.17. If we know the action of S, T on $I(\chi)^K$ for some unramified character χ of the torus T, then we can recover $\alpha, \beta \in \mathbb{C}^\times$ from the roots of the Satake polynomial $X^2 - q^{-1/2}TX + S \in \mathcal{H}(G,K)[X]$.

3 Principal series representations of GL_2

Let F be a nonarchimedean local field, $G = \mathrm{GL}_2(F)$, and $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$ the Borel subgroup of upper triangular matrices, so that $B = N \rtimes T$ for $T = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\} \cong F^\times \times F^\times$ and $N = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \cong F$. Between N and B we also have the mirabolic subgroup $M = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}$ with $M/N \cong F^\times$.

In studying the local Langlands correspondence, we want to understand all the irreducible smooth representations of G. One method for producing representations of G is by induction from a subgroup of G. Typically one takes this subgroup to be 'parabolic'; in our case there is one nontrivial parabolic, namely B. From our decomposition $B = N \rtimes T$ (more generally we have a so-called Levi decomposition) we see that we can produce representations of B by inflating representations of the torus T. Since $T \cong F^\times \times F^\times$, the irreducible representations of T are products of characters of $F^\times$, which are relatively easy to get a handle on.

Definition 3.1. For $\chi : T \to \mathbb{C}^\times$ a character of the torus, we call the process of inflating to B followed by inducing to G parabolic induction. The representation $\mathrm{Ind}_B^G \chi$ is a parabolically induced representation. A principal series representation is an irreducible subrepresentation of a parabolically induced representation.

In this section, we will only concern ourselves with classifying the principal series representations of G. This means that we must understand how $\mathrm{Ind}_B^G \chi$ decomposes into irreducible representations of G, and also study the morphisms between them using Frobenius reciprocity.

To understand these decompositions, we want to study how they decompose into irreducibles over a less unwieldy subgroup of G, such as B. Note that restricting $\mathrm{Ind}_B^G \chi$ to B is analogous to applying Mackey theory in the finite group context. It turns out that the $\mathrm{Ind}_B^G \chi$ do not decompose any further over M than over B. On the other hand, the representation theory of M is very easy to classify - the combination of these two observations is what makes the mirabolic subgroup so 'miraculous'. To get representations of M we can induct from characters of N, or inflate from $M/N \cong F^\times$. There are many characters of $N \cong F$, in fact these are in bijection with F under $a \mapsto \psi(a-)$, for $a \in F$ and any nontrivial character ψ of F. The key property of M is that conjugation by M acts transitively on these characters ψ, which greatly simplifies the representation theory of M coming via induction from N. The mirabolic M is also small enough that this induction, together with the characters of $F^\times$, give all irreducible representations of M.

In this section, we begin by studying the representations of N and introducing the Jacquet functor, before discussing representations of M. From there we determine that parabolically induced representations of G decompose over M with length at most 3. Theorem 3.17 gives the decomposition of $\mathrm{Ind}_B^G \chi$ into irreducible representations of G, and then Theorem 3.24 lists the isomorphism classes of principal series representations. The presentation follows sections 8 and 9 of [BH06].

3.1 Representations of N

We first study the representation theory of $N \cong F$. This is an abelian group so, by Schur's lemma, all irreducible representations are characters (Corollary 2.6.2 [BH06]). For finite abelian groups, any representation V decomposes into a direct sum of characters. This is no longer true when $N \cong F$ is infinite, but it is still true that any vector in V is nonzero in some quotient on which N acts via a character. To formalise this, we define

Notation 3.2. Let V be a smooth representation of N and θ a character of N. Let $V(\theta) \leq V$ be the subspace spanned by $\{n \cdot v - \theta(n)v \mid n \in N, v \in V\}$. Set $V_\theta = V/V(\theta)$ so that N acts on V_θ by θ. When θ is trivial we write $V(N)$ and V_N respectively.

The following is a useful equivalent definition of $V(\theta)$:

Lemma 3.3. *The vector $v \in V$ lies in $V(\theta)$ if and only if*

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0$$

for some compact open subgroup N_0 of N (we restrict to compact opens for the integral to be well defined).

Proof. [BH06] Lemma 8.1. □

Corollary 3.4. *The functor $V \mapsto V_\theta$ from representations of N to complex vector spaces is exact.*

Proof. One checks formally that the functor is right exact. For left exactness we need to show that if $f : V \hookrightarrow V'$ is injective then $V_\theta \hookrightarrow V'_\theta$ is injective. If $v \in V$ with $f(v) \in V'(\theta)$, then

$$\int_{N_0} \theta(n)^{-1} n \cdot f(v) dn = 0$$

for some N_0 by the above lemma. Since f is compatible with the action of N, we can pull f out of the integral so that the injectivity of f implies

$$\int_{N_0} \theta(n)^{-1} n \cdot v dn = 0.$$

We deduce that $v \in V(\theta)$ by the above lemma. □

Proposition 3.5. *For any $v \neq 0$ in V, there exists a character θ of N such that $v \notin V(\theta)$.*

Proof. [BH06] Proposition 8.1. □

Corollary 3.6. *If V is a smooth representation of N such that $V_\theta = 0$ for all θ then $V = 0$.*

3.2 Representations of M

Now we consider V an irreducible smooth representation of M.

Lemma 3.7. *The subspace $V(N) \leq V$ is a representation of M, and so V_N is as well. The V_θ are conjugate under M (so in particular are not representations of M).*

Proof. The first line comes from the computation

$$mn \cdot v - m \cdot v = n'm \cdot v - m \cdot v$$

for some $n' \in N$, using the fact that $N \lhd M$. The second line is because conjugation by M acts transitively on the nontrivial elements of N, and hence on the nontrivial characters of $N \cong F$, where we use the bijection $F \leftrightarrow \hat{F}$. □

So if V is irreducible, either $V(N) = 0$, so that N acts trivially on V and we just get a character of $F^\times$, or $V(N) = V$. In the latter case, $V_N = 0$, so we must have $V_\theta \neq 0$ for all nontrivial characters of N by the above Lemma and Corollary 3.6. Thus the M-representation V must have infinite dimension. In fact there is only one such V, and we have more specifically:

Theorem 3.8. *Let (π, V) be an irreducible smooth representation of M. Either*

- $\dim V = 1$ *and π is the inflation of a character of $M/N \cong F^\times$, or*
- $\dim V = \infty$ *and $\pi \cong c\text{-}\mathrm{Ind}_N^M \theta$, for any nontrivial character θ of N.*

Proof. Corollary 8.3 [BH06]. □

This itself follows from the following theorem. To compare V and $c\text{-}\mathrm{Ind}_N^M \theta$, it is more natural to compare V and $\mathrm{Ind}_N^M V_\theta$. By Frobenius reciprocity,

$$\mathrm{Hom}_N(V, V_\theta) \cong \mathrm{Hom}_M(V, \mathrm{Ind}_N^M V_\theta).$$

Let $q_* : V \to \mathrm{Ind}_N^M(V_\theta)$ be the image of the quotient map $q : V \to V_\theta$.

Theorem 3.9. *The M-homomorphism $q_* : V \to \mathrm{Ind}_N^M V_\theta$ induces an isomorphism $V(N) \cong c\text{-}\mathrm{Ind}_N^M V_\theta$. Moreover, this compact induction is an irreducible representation of M.*

Proof. Theorem 8.3 and Corollary 8.2 of [BH06]. □

3.3 Irreducible principal series representations

Let V be a smooth representation of G. By restriction this gives a representation of B, and so does the space of N-coinvariants $V_N = V/V(N)$, again because N is normal in B. Then V_N inherits a representation π_N of $T = B/N$, and we call this the Jacquet module of V at N. As shown before, the Jacquet functor $V \mapsto V_N$ is exact.

Parallel to the classical finite field setting, we want to study when V arises from parabolic induction. We have the analogous result:

Proposition 3.10. *The following are equivalent:*

- $V_N \neq 0$
- *π is isomorphic to a G-subrepresentation of $\mathrm{Ind}_B^G \chi$ for some character χ of T inflated to B.*

Proof. (2) implies (1) comes from Frobenius reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\chi) = \mathrm{Hom}_B(\pi, \chi) = \mathrm{Hom}_T(\pi_N, \chi)$$

where the second equality is due to any B-homomorphism $\pi \to \chi$ factoring through π_N (because χ is trivial on N).

Given (1), one shows by a technical argument that V_N is finitely generated as a representation of T. An application of Zorn's lemma allows us to construct a maximal T-subspace U of V_N so that V_N/U is an irreducible T-representation and is thus a character (Schur's lemma) χ. Frobenius reciprocity implies the result. □

Remark 3.11. The same proof holds for the finite field case (noting the notion of having a subrepresentation where N acts trivially is the same as having a nonzero quotient where N acts trivially). The proof that (1) implies (2) bypasses the technical details because V_N as a representation of T obviously admits an irreducible quotient as V_N is finite dimensional.

Remark 3.12. In the general case we ask for a nonzero quotient of V on which N acts trivially as opposed to having a subrepresentation, because one can show in this latter case that all we get are characters $\pi = \phi \circ \det$ for some character ϕ of $F^\times$. In fact any finite dimensional smooth representation is of this form. The difference with the finite field case is that smoothness tells us that if $v \in V$ is fixed by N, it is also fixed by an open compact subgroup of G. Over a finite field, N is open, but in general it is not and we fix v by too much (all of SL_2).

We restrict our attention to principal series representations and want to understand how $\mathrm{Ind}_B^G \chi$ decomposes into irreducible G-representations. As mentioned earlier, we will first study how they decompose as representations of B or even M.

These induced representations will never be irreducible over B because we always have the canonical B-homomorphism $X = \mathrm{Ind}_B^G \chi \to \chi$ given by sending $f \mapsto f(1) \in \mathbb{C}$. So we have an exact sequence of B-representations

$$0 \longrightarrow V \longrightarrow X \longrightarrow \mathbb{C} \longrightarrow 0$$

where $V = \{f \in X\chi : f(1) = 0\}$, with B acting on $\mathbb{C}$ via χ. Now we want to understand how V decomposes. We have another exact sequence of B-representations,

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

so we reduce to studying $V(N)$ and V_N. We will show that $V(N)$ is irreducible over B (and even over M), while V_N will be determined by the Restriction-Induction lemma (which generally treats the exact sequence obtained by applying the Jacquet functor to the first exact sequence, where we may replace χ by any smooth representation σ of T).

Firstly we want to understand $V = \{f \in X : f(1) = 0\}$ better.

Lemma 3.13. *For V as above, the map*

$$V \to C_c^\infty(N)$$
$$f(-) \mapsto f(w-)$$

is an N-isomorphism, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

Proof. We have the decomposition $G = B \sqcup BwN$. Since $f(1) = 0$ and f is induced from B we must have that f is supported on BwN. G-smoothness of f implies that $f(1) = 0$ is fixed by right translation by some compact open subgroup $K \leq G$. This will contain $\left(\begin{smallmatrix} 1 & 0 \\ \pi^n \mathcal{O} & 0 \end{smallmatrix}\right)$ for some n, so that f vanishes on

$$\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \in Bw\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)$$

for all $x \in \pi^n\mathcal{O}$. Thus $f(w-)$ is supported on $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in N$ with $v(y) > -n$ (so $y \in \pi^{-n}\mathcal{O}$). G-smoothness of f also implies that $f(w-)$ is N-smooth, and that the above map is an N-homomorphism. The decomposition $G = B \sqcup BwB$ implies that it is in fact an isomorphism. $\square$

Proposition 3.14. *For V as above, $V(N)$ is irreducible over M (and hence over B).*

Proof. By the above lemma we can identify $V \cong C_c^\infty(N)$ with M acting via right translation on V. This gives the structure of a M-representation on $C_c^\infty(N)$. We can calculate it explicitly (but we won't need it) where

$$f(bw(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix})) = f(b(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix})w\left(\begin{smallmatrix} 1 & a^{-1}x \\ 0 & 1 \end{smallmatrix}\right))$$

tells us that the corresponding $M = F^\times N$ action on $C_c^\infty(N)$ is the composite of right translation by N with the action

$$a\phi(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) = \chi_2(a)\phi\left(\begin{smallmatrix} 1 & a^{-1}x \\ 0 & 1 \end{smallmatrix}\right)$$

So now we may consider $V = C_c^\infty(N)$. The benefit is that for this representation, the spaces of coinvariants of characters of N are very simple. In particular, the map $f \mapsto \theta f$ is a linear automorphism of $C_c^\infty(N)$ taking $V(N)$ to $V(\theta)$ since

$$n \cdot f - f \mapsto \theta(n \cdot f) - \theta f = \theta(n)^{-1} n \cdot (\theta f) - \theta f \in V(\theta).$$

Hence all the V_θ have the same dimension as $V_N = V/V(N)$, which has dimension 1 (we can see this from the characterisation of $V(N)$ as the zeros of some integral, or from the Restriction-Induction lemma to follow).

But then Theorem 3.9 implies that for our M-representation V, we have $V(N) \cong c - \mathrm{Ind}_N^M V_\theta$ where $V_\theta \cong \theta$ as it is one dimensional. This is irreducible as a M-representation by the same Theorem. $\square$

We turn our attention to V_N where we recall V fits in the exact sequence

$$0 \longrightarrow V \longrightarrow X = \mathrm{Ind}_B^G \chi \xrightarrow{f \mapsto f(1)} \chi \longrightarrow 0$$

of smooth representations of B. Since the Jacquet functor is exact, we get the exact sequence

$$0 \longrightarrow V_N \longrightarrow X_N \longrightarrow \chi \longrightarrow 0$$

of T-representations. We can say in more generality,

Lemma 3.15 (Restriction-Induction Lemma)**.** *Let (σ, U) be a smooth representation of T and $(\Sigma, X) = \mathrm{Ind}_B^G \sigma$. Then there is an exact sequence of smooth T representations:*

$$0 \longrightarrow \sigma^w \otimes \delta_B^{-1} \longrightarrow \Sigma_N \longrightarrow \sigma \longrightarrow 0$$

Proof. The proof of Lemma 3.13 generalises to show that the vector space V is isomorphic to the space $\mathcal{S}$ of smooth compactly supported functions $N \to U$ by identifying f with $f(w-)$.

We can define a map $\mathcal{S} \to U$ by

$$g = f(w-) \mapsto \int_N g(n) = f(wn)dn$$

where this integral is finite since g is compactly supported. By Lemma 3.3, this induces an isomorphism $\mathcal{S}_N \cong U$.

The B-representation structure on $\mathcal{S}$ coming from V is by right translation, where $b = sm \in TN$ acts by

$$f(wnsm) = f(wss^{-1}nsm) = f(wswws^{-1}nsm) = \sigma(s^w)f(ws^{-1}nsm)$$

where $s^{-1}nsm \in N$. Under the isomorphism $\mathcal{S}_N \cong U$, this induces a T representation structure on U where $s \in T$ acts by

$$s \cdot \int_N f(wn)dn = \sigma(s^w) \int_N f(ws^{-1}ns)dn = \sigma(s^w)|\frac{s_1}{s_2}| \int_N f(wn)dn$$

which is $\sigma^w \otimes \delta_B^{-1}$. □

Corollary 3.16. *As a representation of B or M, $\mathrm{Ind}_B^G\chi$ has composition length 3. Two of the factors have dimension 1, and the other is infinite dimensional.*

Proof. This follows from the exact sequences

$$0 \longrightarrow V \longrightarrow \mathrm{Ind}_B^G\chi \longrightarrow \chi \longrightarrow 0$$

and

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

where we saw that $V(N)$ is irreducible (and infinite dimensional by Theorem 3.8), and $V_N \cong \chi^w \otimes \delta_B^{-1}$. □

So we understand how $\mathrm{Ind}_B^G\chi$ decomposes into irreducible B representations, and we want to understand its decomposition into G representations. Our goal is to prove the following

Theorem 3.17 (Irreducibility Criterion)**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let $X = \mathrm{Ind}_B^G\chi$.*

1. *X is irreducible if and only if $\chi_1\chi_2^{-1}$ is either the trivial character of $F^\times$, or the character $x \mapsto |x|^2$ of $F^\times$.*

2. *Suppose X is reducible, then*
   - *the G-composition length of X is 2*
   - *one factor has dimension 1, the other is infinite dimensional*
   - *X has a 1-dimensional G-subspace exactly when $\chi_1\chi_2^{-1} = 1$*
   - *X has a 1-dimensional G-quotient exactly when $\chi_1\chi_2^{-1}(x) = |x|^2$.*

We make some comments in preparation for the proof. By the above Corollary, if X is reducible then it has a finite dimensional (dimension 1 or 2) G-subspace or G-quotient. By taking duals we can assume we are in the first case. In the Irreducibility Criterion, we want to show that this implies $\chi_1 = \chi_2$ and that X has a 1-dimensional G-subspace.

Definition 3.18. Let π be a smooth representation of G and ϕ a character of $F^\times$. The twist of π by ϕ is the representation $\phi\pi$ of G defined by

$$\phi\pi(g) = \phi(\det g)\pi(g).$$

In this way, for a character $\chi = \chi_1 \otimes \chi_2$ of T, we have $\phi\chi = \phi\chi_1 \otimes \phi\chi_2$. Then

$$\mathrm{Ind}_B^G(\phi\chi) = \phi\mathrm{Ind}_B^G\chi.$$

Proposition 3.19. *The following are equivalent:*

1. $\chi_1 = \chi_2$
2. *X has a 1-dimensional N-subspace.*

If this holds then this subspace is unique, and is also a G-subspace of X not contained in V.

Proof. (1) implies (2): since induction commutes with twisting we may assume $\chi_1 = \chi_2 = 1$, then the nonzero constant function spans a 1-dimensional G-subspace (not just N-subspace) of $X = \mathrm{Ind}_B^G 1$.

(2) implies (1): suppose this subspace is spanned by f. N acts by right translation as a character. We cannot have $f \in V$ ($f(1) = 0$) else we earlier saw that f would then have support in some BwN_0 for $N_0 \leq N$ open compact, and this is not closed under multiplication by N.

So $f \notin V$ ($f(1) \neq 0$) and so its image spans $X/V \cong \mathbb{C}$ on which N acts trivially (since we inflate χ to be trivial on N). Thus N fixes f under right translation. f is also fixed under right translation by some compact open of G, so for sufficiently large $|x|$ we have

$$\begin{aligned} f(w) = f\left(w\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\right) \\ &= f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right)\right) \\ &= \chi_1(-1)\left(\chi_1^{-1}\chi_2(x)\right) f(1) \end{aligned}$$

For this to hold for all $|x|$ sufficiently large, it follows that we must have $\chi_1 = \chi_2$ (if $\chi_1(y) \neq \chi_2(y)$ then $\chi_1(xy) \neq \chi_2(xy)$ for all sufficiently large x, but then xy is also large). The uniqueness of the 1-dimensional subspace comes from the fact that it must span $X/V \cong \mathbb{C}$. □

Proof of Irreducibility Criterion. Assume that X is reducible and we are in the case that X has a finite dimensional G-subspace. Then it has a 1-dimensional N-subspace L, which is also a G subspace by the above Proposition with G acting via $\phi \circ \det$, where $\phi = chi_1 = \chi_2$. Since $L \cap V = 0$, we see that $Y = X/L \cong V$ as B-representations. We need to show X has G-length 2. By the previous corollary it has length at most 3. We know that V has B-length 2 with a 1-dimensional quotient V_N. Thus if Y had G-length 2, then the B-factors of V are also G-factors, so that G must act on V_N, necessarily by a character $\phi' \circ \det$ (see 9.2 Exercise 2). But this is impossible because $B \leq G$ acts by $\phi\delta_B^{-1}$ by restriction-induction, and this does not factor through det on B. So we must have that X has G-length 2.

In the other case we have a finite dimensional G-quotient. The smooth dual $X^\vee$ is then in the first case, where the Duality Theorem tells us $X^\vee \cong \mathrm{Ind}_B^G\delta_B^{-1}\chi^\vee$. If we write $\delta_B^{-1}\chi^\vee = \psi_1 \otimes \psi_2$ then we must have $\psi_1 = \psi_2$. The result follows from computing $\psi_1(x) = |x|^{-1}\chi_1(x)$ and $\psi_2(x) = |x|\chi_2(x)$.

The converse direction to (1) follows from the previous Proposition. □

3.4 Classification of principal series representations

Now that we've seen how parabolically induced representations decompose into irreducibles, we want to classify the isomorphism classes.

Proposition 3.20. *Let χ, ξ be characters of T. The space $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi)$ is 1-dimensional if $\xi = \chi$ or $\chi^w\delta_B^{-1}$ and 0 otherwise.*

Proof. Frobenius reciprocity tells us

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi) \cong \mathrm{Hom}_T((\mathrm{Ind}\chi)_N, \xi).$$

From restriction-induction we have

$$0 \longrightarrow \chi^w\delta_B^{-1} \longrightarrow (\mathrm{Ind}\chi)_N \longrightarrow \chi \longrightarrow 0.$$

In the case $\chi \neq \chi^w\delta_B^{-1}$ the sequence splits and the result follows. If $\chi = \chi^w\delta_B^{-1}$ then $\chi_1\chi_2^{-1}(x) = |x|$ so $\mathrm{Ind}\chi$ is irreducible and the result still follows. □

Remark 3.21. Hence, in the case $\mathrm{Ind}\chi$ is irreducible, we have $\mathrm{Ind}\chi \cong \mathrm{Ind}\chi^w\delta_B^{-1}$.

And in the case $\mathrm{Ind}\chi$ is reducible, it is not semisimple, else the Hom space would be 2-dimensional.

We can be more explicit in the reducible case. One can check that the conditions in the Irreducibility Criterion of reducibility are equivalent to χ being of the form $\chi = \phi 1_T$ or $\chi = \phi\delta_B^{-1}$. Untwisting, we may as well assume $\phi = 1$.

Definition 3.22. The Steinberg representation is defined by the exact sequence

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow \mathrm{St}_G \longrightarrow 0$$

which is an infinite dimensional irreducible representation with Jacquet module $(\mathrm{St}_G)_N \cong \delta_B^{-1}$ by restriction-induction. If $\chi = \phi 1_T$ we would instead get a twist of Steinberg, $\phi\mathrm{St}_G$.

The case $\chi = \delta_B^{-1}$ can be dealt with by taking smooth duals (which is exact (Lemma 2.10 of Bushnell-Henniart) and preserves irreducibles (by checking on V^K)) to get

$$0 \longrightarrow \mathrm{St}_G^\vee \longrightarrow \mathrm{Ind}_B^G\delta_B^{-1} \longrightarrow 1_G \longrightarrow 0$$

The Proposition applied to $\chi = 1$ then implies

$$\mathrm{St}_G \cong \mathrm{St}_G^\vee.$$

Notation 3.23. Define normalised induction by

$$\iota_B^G\sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma).$$

This has the benefit that $(\iota_B^G\sigma)^\vee \cong \iota_B^G\sigma^\vee$.

Theorem 3.24 (Classification Theorem). *The following are all the isomorphism classes of principal series representations of G:*

- *the irreducible induced representations $\iota_B^G \chi$ when $\chi \neq \phi \delta_B^{\pm 1/2}$ for a character ϕ of $F^\times$.*
- *the one-dimensional representations $\phi \circ \det$ for ϕ a character of $F^\times$.*
- *the twists of Steinberg (special representations) $\phi \mathrm{St}_G$ for ϕ a character of $F^\times$.*

These are all distinct isomorphism classes except in the first case where $\iota_B^G \chi \cong \iota_B^G \chi^w$.

References

[BH06] C. J. Bushnell and G. Henniart, *The Local Langlands Conjecture for GL(2)*, Grundlehren der mathematischen Wissenschaften, Springer Berlin, 2006.

[GH24] J. R. Getz and H. Hahn, *An Introduction to Automorphic Representations*, Graduate Texts in Mathematics, Springer Cham, 2024.