

Hecke algebras for p -adic groups

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Introduction

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Fact (supercuspidal representations as building blocks)

For any irreducible object (π, V) in $\mathrm{Rep}(G)$, there is some parabolic subgroup $P \subseteq G$ with Levi subgroup M and supercuspidal representation σ of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$.

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Theorem (Bernstein)

There is a direct product decomposition

$$\mathrm{Rep}(G) \cong \prod_{[M,\sigma] \in \mathfrak{I}(G)} \mathrm{Rep}(G)_{[M,\sigma]}$$

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Upshot: We study the irreducible objects $\mathrm{Irr}(G)_{[M,\sigma]}$ of each block individually, and the extensions between them.

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Example: If $\rho = \mathbf{1}$ is the trivial character, then

$$\mathcal{H}(G, K, \rho) = C_c(K \backslash G / K)$$

is the space of locally constant, compactly supported and K -invariant complex functions on G , equipped with the convolution product.

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Under mild conditions, one can associate to any $[M, \sigma] \in \mathfrak{J}(G)$ a pair (K, ρ) (called a $[M, \sigma]$ -type) such that there is an equivalence of categories

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Example: The pair $(I, \mathbf{1})$ is a $[T, \mathbf{1}]$ -type. Thus,

$$\begin{aligned} \mathrm{Rep}(G)_{[T, \mathbf{1}]} &\cong \text{right } C_c(I \backslash G / I) - \text{modules} \\ &\cong \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\} \end{aligned}$$

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Main idea: Hecke algebras reduce infinite dimensional problems to finite-dimensional ones.

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so π is the only irreducible element of $\text{Rep}(G)_{[G, \pi]}$, and it has no nontrivial extensions!

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- 4 $\mathcal{H}(W_{\mathrm{aff}}, S_{\mathrm{aff}}, q)$ has basis $\{T_w : w \in W_{\mathrm{aff}}\}$ and relations
 - $T_{w_1} T_{w_2} = T_{w_1 w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$,
 - $T_s^2 = (q - 1)T_s + qT_1$ if $s \in S_{\mathrm{aff}}$.

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where $q : S(\rho)_{\text{aff}} \rightarrow \mathbb{Q}_{>1}$ is a parameter function.

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Open problem: Determine the parameter function $q : S(\rho)_{\text{aff}} \rightarrow \mathbb{Q}_{>1}$ in the modular representation setting.

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This is precisely the statement of the Unramified LLC for $\mathrm{GL}_n(F)$!

Thank you for listening!