Hecke algebras for p-adic groups

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Fact (supercuspidal representations as building blocks)

For any irreducible object (π, V) in $\mathrm{Rep}(G)$, there is some parabolic subgroup $P \subseteq G$ with Levi subgroup M and supercuspidal representation σ of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$.

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There is a direct product decomposition

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Upshot: We study the irreducible objects $\mathrm{Irr}(G)_{[M,\sigma]}$ of each block individually, and the extensions between them.

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Example: If $\rho = \mathbf{1}$ is the trivial character, then

$$\mathcal{H}(G, K, \rho) = C_c(K \backslash G / K)$$

is the space of locally constant, compactly supported and K-invariant complex functions on G, equipped with the convolution product.

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Under mild conditions, one can associate to any $[M, \sigma] \in \mathfrak{J}(G)$ a pair (K, ρ) (called a $[M, \sigma]$-type) such that there is an equivalence of categories

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$$\begin{aligned} \mathrm{Rep}(G)_{[T,\mathbf{1}]} &\cong \text{ right } C_c(I \backslash G / I) - \text{modules} \\ &\cong \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\} \end{aligned}$$

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Main idea: Hecke algebras reduce infinite dimensional problems to finite-dimensional ones.

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so π is the only irreducible element of $\mathrm{Rep}(G)_{[G,\pi]}$, and it has no nontrivial extensions!

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4. $\mathcal{H}(W_{\mathrm{aff}}, S_{\mathrm{aff}}, q)$ has basis $\{T_w : w \in W_{\mathrm{aff}}\}$ and relations
   - $T_{w_1} T_{w_2} = T_{w_1 w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$,
   - $T_s^2 = (q-1)T_s + qT_1$ if $s \in S_{\mathrm{aff}}$.

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Open problem: Determine the parameter function $q : S(\rho)_{\mathrm{aff}} \to \mathbb{Q}_{>1}$ in the modular representation setting.

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- Conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ are in bijection with unramified Langlands parameters $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ (trivial on $\mathcal{I}_F \times \{1\}$) up to $\mathrm{GL}_n(\mathbb{C})$-conjugacy.

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This is precisely the statement of the Unramified LLC for $\mathrm{GL}_n(F)$!

Thank you for listening!