Parameters of Hecke algebras for p-adic groups

Albert Lopez Bruch

5 September, 2025

Introduction

Setup: Let F be a non-archimedean local field with residue field $\mathbb{F}_q$ (think of F as a finite extension of $\mathbb{Q}_p$), and let G be a connected reductive group over F.

For example, think of G as $\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{Sp}_{2n}(F) \ldots$, but also as an exceptional group such as $G_2(F)$.

We will denote by $\mathrm{Rep}(G)$ the category of smooth admissible complex representations of G.

Fact (supercuspidal representations as building blocks)

For any irreducible object (π, V) in $\mathrm{Rep}(G)$, there is some parabolic subgroup $P \subseteq G$ with Levi subgroup M and supercuspidal representation σ of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$.

Bernstein Decomposition

Theorem (Bernstein)

There is a direct product decomposition

$$\mathrm{Rep}(G) \cong \prod_{[M,\sigma]\in\mathfrak{J}(G)} \mathrm{Rep}(G)_{[M,\sigma]}$$

into full indecomposable categories $\mathrm{Rep}(G)_{[M,\sigma]}$ *known as Bernstein blocks. The product ranges over conjugacy classes of pairs* (M,σ)*, denoted by* $[M,\sigma]\in\mathfrak{J}(G)$*.*

Example: $\mathrm{Rep}(G)_{[T,\mathbf{1}]}$ is the *principal block*.

Upshot: We study the irreducible objects $\mathrm{Irr}(G)_{[M,\sigma]}$ of each block individually, and the extensions between them.

Theory of types

Bushnell and Kutzko introduced a rich theory to study each block individually.

Definition (Types)

Let $[M, \sigma] \in \mathfrak{J}(G)$. A pair (K, ρ) is a $[M, \sigma]$-type if for any $(\pi, V) \in \mathrm{Irr}(G)$,

$$(\pi, V) \in \mathrm{Rep}(G)_{[M,\sigma]} \iff \pi|_K \text{ contains } \rho.$$

Example: The pair $(I, \mathbf{1})$ is a $[T, \mathbf{1}]$-type. Thus,

$$\mathrm{Rep}(G)_{[T,\mathbf{1}]} = \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}$$

Hecke Algebras

For a pair (K, ρ), we construct the associated Hecke algebra

$$\mathcal{H}(G, K, \rho) := \mathrm{End}_G(\mathrm{ind}_K^G \rho)$$

Theorem

Suppose that (K, ρ) is an $[M, \sigma]$-type. Then there is an equivalence of categories

$$\begin{aligned} \mathrm{Rep}(G)_{[M,\sigma]} &\longrightarrow \text{ right } \mathcal{H}(G, K, \rho) - \text{modules} \\ (\pi, V) &\longmapsto Hom_K(V, W) = Hom_G(\mathrm{ind}_K^G V, W) \end{aligned}$$

Main idea: Hecke algebras reduce infinite dimensional problems to finite-dimensional ones.

Examples

Example: $\mathcal{H}(G,K,\mathbf{1}) \cong C_c(K\backslash G/K)$ with the convolution product. There is a bijection

$$\{(\pi,V)\in \mathrm{Irr}(G) : V^K \neq 0\} \longleftrightarrow \text{ irreducible } C_c(K\backslash G/K)\text{-mod.}$$

Advanced example: Suppose that G is semisimple and that $\pi \in \mathrm{Irr}(G)$ is supercuspidal of depth-zero. Then $\pi = \mathrm{ind}_K^G \rho$ for some pair (K,ρ), and this is a $[G,\pi]$-type! Thus,

$$\mathcal{H}(G,K,\rho) = \mathrm{End}_G(\pi) = \mathbb{C},$$

so π is the only irreducible element of $\mathrm{Rep}(G)_{[G,\pi]}$, and it has no nontrivial extensions!

Questions

All these results are useful if we

1. know that types exist for the Bernstein blocks,
2. understand structure of Hecke algebras,
3. describe their irreducible modules.

The work of Kim and Yu, and Fintzen, Kaletha and Spice show that under mild conditions types can be constructed for all blocks.

Thank you for listening!