Hecke algebras for p-adic groups

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Introduction

Setup: Let F be a non-archimedean local field with residue field $\mathbb{F}_q$ (think of F as a finite extension of $\mathbb{Q}_p$), and let G be a connected reductive group over F.

For example, think of G as $\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{Sp}_{2n}(F) \ldots$, but also as an exceptional group such as $G_2(F)$.

We will denote by $\mathrm{Rep}(G)$ the category of smooth admissible complex representations of G.

Fact (supercuspidal representations as building blocks)

For any irreducible object (π, V) in $\mathrm{Rep}(G)$, there is some parabolic subgroup $P \subseteq G$ with Levi subgroup M and supercuspidal representation σ of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G \sigma$.

Bernstein Decomposition

Theorem (Bernstein)

There is a direct product decomposition

$$\mathrm{Rep}(G) \cong \prod_{[M,\sigma]\in\mathfrak{J}(G)} \mathrm{Rep}(G)_{[M,\sigma]}$$

into full indecomposable categories $\mathrm{Rep}(G)_{[M,\sigma]}$ *known as Bernstein blocks. The product ranges over conjugacy classes of pairs* (M,σ)*, denoted by* $[M,\sigma]\in\mathfrak{J}(G)$*.*

Example: $\mathrm{Rep}(G)_{[T,\mathbf{1}]}$ is the *principal block*.

Upshot: We study the irreducible objects $\mathrm{Irr}(G)_{[M,\sigma]}$ of each block individually, and the extensions between them.

Hecke Algebras

Consider a pair (K, ρ) where

- K is a compact open subgroup of G.
- (ρ, W) is a smooth irreducible representation of K.

For such a pair, we construct the associated Hecke algebra

$$\mathcal{H}(G, K, \rho) := \mathrm{End}_G(\mathrm{ind}_K^G \rho)$$

Example: If $\rho = \mathbf{1}$ is the trivial character, then

$$\mathcal{H}(G, K, \rho) = C_c(K \backslash G / K)$$

is the space of locally constant, compactly supported and K-invariant complex functions on G, equipped with the convolution product.

Theory of types

To study each block using Hecke algebras, we use the theory of types introduced by Bushnell-Kutzko.

Theorem (Kim, Yu, Fintzen, Kaletha, Spice)

Under mild conditions, one can associate to any $[M, \sigma] \in \mathfrak{J}(G)$ a pair (K, ρ) (called a $[M, \sigma]$-type) such that there is an equivalence of categories

$$\mathrm{Rep}(G)_{[M,\sigma]} \cong \textit{ right } \mathcal{H}(G, K, \rho) - \textit{modules.}$$

Example: The pair $(\mathrm{I}, \mathbf{1})$ is a $[T, \mathbf{1}]$-type. Thus,

$$\begin{aligned}\mathrm{Rep}(G)_{[T,\mathbf{1}]} &\cong \text{ right } C_c(I \backslash G / I) - \text{modules} \\ &\cong \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}\end{aligned}$$

Main idea: Hecke algebras reduce infinite dimensional problems to finite-dimensional ones.

Questions

All these results are useful if we

1. understand structure of Hecke algebras,
2. describe their irreducible modules.

Example: Suppose that G is semisimple and that $\pi \in \mathrm{Irr}(G)$ is supercuspidal of depth-zero. Then $\pi = \mathrm{ind}_K^G \rho$ for some pair (K, ρ), and this is a $[G, \pi]$-type! Thus,

$$\mathcal{H}(G, K, \rho) = \mathrm{End}_G(\pi) = \mathbb{C},$$

so π is the only irreducible element of $\mathrm{Rep}(G)_{[G,\pi]}$, and it has no nontrivial extensions!

The Iwahori-sperical Hecke algebra

Let G be a semisimple split adjoint group (e.g. $G = \mathrm{PGL}_n(F)$) with maximal torus T. Associated to G, there is the extended affine Weyl group $\widetilde{W} = N_G(T)(F)/T(\mathcal{O}_F)$ with the properties:

1. There is a canonical $\mathbb{C}$-basis $\{T_w : w \in \widetilde{W}\}$ of $\mathcal{H}(G, I, \mathbf{1})$.
2. There is a semidirect product $\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$ with Ω finite group and $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ an affine Coxeter group.
3. This decomposition induces an isomorphism of $\mathbb{C}$-algebras
$$\mathcal{H}(G, I, \mathbf{1}) = \mathcal{H}(W_{\mathrm{aff}}, S_{\mathrm{aff}}, q) \,\tilde{\otimes}\, \mathbb{C}[\Omega].$$
4. $\mathcal{H}(W_{\mathrm{aff}}, S_{\mathrm{aff}}, q)$ has basis $\{T_w : w \in W_{\mathrm{aff}}\}$ and relations
    - $T_{w_1} T_{w_2} = T_{w_1 w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$,
    - $T_s^2 = (q-1)T_s + qT_1$ if $s \in S_{\mathrm{aff}}$.

Hecke algebras in general

Similar results hold for general types (K, ρ).

1. There is a group $W(\rho)$ and a canonical $\mathbb{C}$-basis $\{T_w : w \in W(\rho)\}$ of $\mathcal{H}(G, K, \rho)$.
2. There is a semidirect product $W(\rho) = W(\rho)_{\mathrm{aff}} \rtimes \Omega(\rho)$ with $(W(\rho)_{\mathrm{aff}}, S(\rho)_{\mathrm{aff}})$ an affine Coxeter group.
3. This decomposition induces an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}(G, K, \rho) = \mathcal{H}(W_{\mathrm{aff}}(\rho), S_{\mathrm{aff}}(\rho), q) \tilde{\otimes} \mathbb{C}[\Omega],$$

where $q : S(\rho)_{\mathrm{aff}} \to \mathbb{Q}_{>1}$ is a parameter function.

Open problem: Determine the parameter function $q : S(\rho)_{\mathrm{aff}} \to \mathbb{Q}_{>1}$ in the modular representation setting.

Unramified Langlands correspondence for $\mathrm{GL}_n(F)$

For simplicity, we assume now that $G = \mathrm{GL}_n(F)$. The work of Kazhdan–Lusztig gives a complete classification of the irreducible modules of $\mathcal{H}_n = \mathcal{H}(\mathrm{GL}_n(F), I, \mathbf{1}) = C_c(I \backslash \mathrm{GL}_n(F)/I)$:

- For each $x \in \mathrm{GL}_n(\mathbb{C}) = G^\vee(\mathbb{C})$, we construct a canonical $\mathcal{H}_n$-representation V_x using Borel–Moore homology.
- This representation has one unique irreducible subquotient, say π_x, depending only on the conjugacy class of $x \in \mathrm{GL}_n(\mathbb{C})$.
- All irreducible $\mathcal{H}_n$-modules arise this way.
- Conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ are in bijection with unramified Langlands parameters $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ (trivial on $\mathcal{I}_F \times \{1\}$) up to $\mathrm{GL}_n(\mathbb{C})$-conjugacy.

This is precisely the statement of the Unramified LLC for $\mathrm{GL}_n(F)$!

Thank you for listening!