Bias-variance tradeoff

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1 Introduction

Previously, we talked about the bias-variance tradeoff and briefly mentioned the James-Stein estimator. The main take-away is that a biased estimator may have a better performance than an unbiased estimator, in terms of achieving a lower MSE. In statistics, the most notable example is the James-Stein estimator.

We use R to demonstrate the superiority of James-Stein estimator for a particular model, which is also a good exercise for R programming and statistical computing.

2 Set-up

The hiring problem. Alice is a hiring manager. She looks at N different aspects of each candidate's characteristics (say communication skills, analytical skills, etc). A candidate's true characteristic in aspect i is μ_i. Alice cannot observe μ_i, but she can have an imperfect measure z_i of each μ_i. Alice wishes to have a good estimate of a candidate's characteristics $\mu = (\mu_1, \dots, \mu_N)$ based on the measures $z = (z_1, \dots, z_n)$.

We build a simple model for the hiring problem:

- Data are generated from

$$z_i | \mu_i \sim N(\mu_i, 1), \quad i = 1, \dots, N.$$

- In plain words, the first observation z_1 is drawn from the normal distribution with mean μ_1 and variance 1, the second observation z_2 is drawn from the normal distribution with mean μ_2 and variance 1, and so on.
- Note that for each candidate μ, we have only **one measurement** z, which is composed of N numbers, $(z_1, \dots, z_N)$.

Question: How to obtain a good estimate of $\mu = (\mu_1, \ldots, \mu_N)$ based on the one observation $z = (\mu_1, \ldots, \mu_N)$?

1. The **maximum-likelihood estimator** (MLE) is $\hat{\mu}^{(MLE)} = z$. We can verify that this estimator is unbiased:
$$\mathbb{E}\hat{\mu}^{(MLE)} = \mu.$$

2. The James-Stein Estimator is
$$\hat{\mu}^{(JS)} = (1 - \frac{N-2}{||z||^2})z.$$

Clearly, the James-Stein Estimator is biased. Indeed, it simply "shrinks" the unbiased MLE by $1 - \frac{N-2}{||z||^2}$. When $N \geq 3$, we have $1 - \frac{N-2}{||z||^2} < 1$.

> James-Stein Theorem (1961)
>
> For $N \geq 3$, the James-Stein Estimator dominates the MLE in terms of achieving a lower MSE:
> $$\mathbb{E}||\hat{\mu}^{(JS)} - \mu||^2 < \mathbb{E}||\hat{\mu}^{(MLE)} - \mu||^2,$$
> where $||\hat{\mu} - \mu||^2 = \sum_{i=1}^{N}(\hat{\mu}_i - \mu_i)^2$.

Proving this theorem may be thorny if you have limited exposure to mathematical statistics before. However, we can use R to run some experiments/simulations, and data will tell us whether this theorem holds in reality.

3 Experiments

The function `run_experiment()` performs the following operations:

1. Take an input `N` and generate some vector μ with length `N`
2. Repeat the following steps `nrep(=100)` times:
   - generate a measurement `z` based on
   - compute the `_MLE` and `_JSE` based on `z`
   - compute the `err_MLE` and `err_JSE` based on `z` and
3. Return a data frame containing all the `err_MLE` and `err_JSE`

```
run_experiment = function(N) {
    nrep = 100
    err_MLE = rep(0,nrep)
    err_JSE = rep(0,nrep)
      = rnorm(N) # generated from standard normal dist

    for (i in 1:nrep) {
        e = rnorm(N,0,1)
        z =   + e
         _MLE = z
         _JSE = (1-(N-2)/sum(z^2))*z
        err_MLE[i] = sum(( _MLE- )^2)/N
        err_JSE[i] = sum(( _JSE- )^2)/N
    }
    err_both = as.data.frame(cbind(err_MLE,err_JSE))
    names(err_both) = c("err_MLE","err_JSE")
    return(err_both)
}
```

We have the data for all the MSEs on 100 samples of James-Stein estimator and Maximum Likelihood estimator. We use the box plot to compare them.

```
get_box = function(N) {
    err = run_experiment(N)
    boxplot(err, ylab = "Error: ||hat{ }- ||/N")
    title(paste(" _i is generated from Normal(0,1), sample size N=",N,sep=""))
}
```

```
get_box(N=3)
```

```
Warning in (function (main = NULL, sub = NULL, xlab = NULL, ylab = NULL, :
conversion failure on 'Error: ||hat{ }- ||/N' in 'mbcsToSbcs': dot substituted
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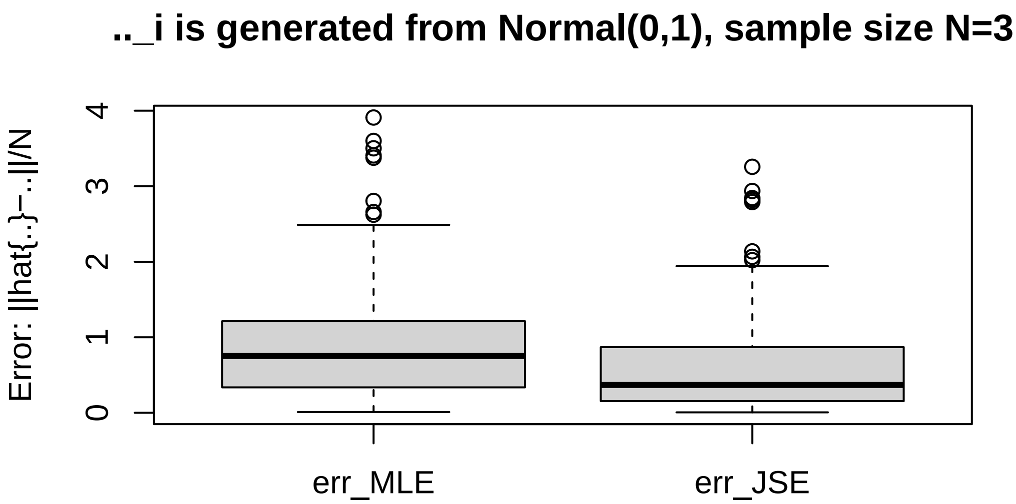
```




```
get_box(N=5)
```

```
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conversion failure on 'Error: ||hat{ }- ||/N' in 'mbcsToSbcs': dot substituted
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```

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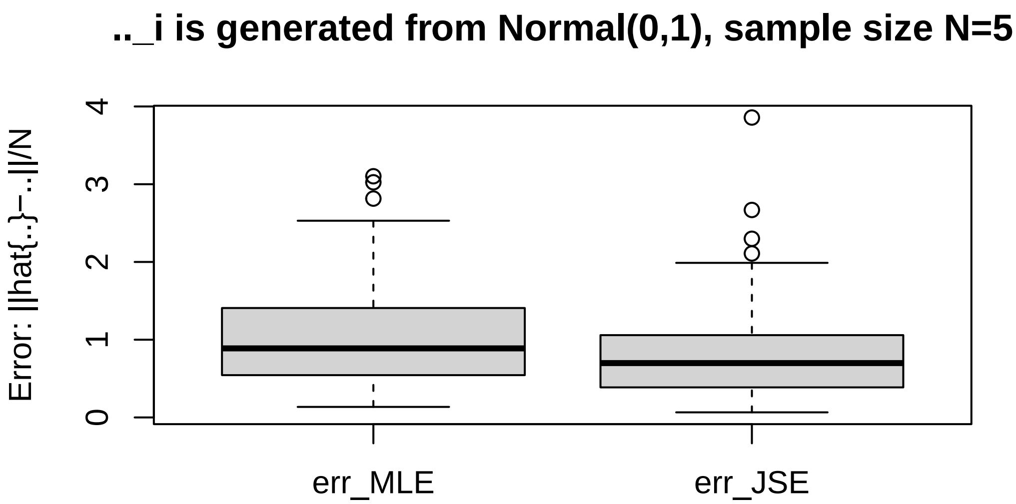
'mbcsToSbcs': dot substituted for <bc>
```




```
get_box(N=2)
```

```
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conversion failure on 'Error: ||hat{ }- ||/N' in 'mbcsToSbcs': dot substituted
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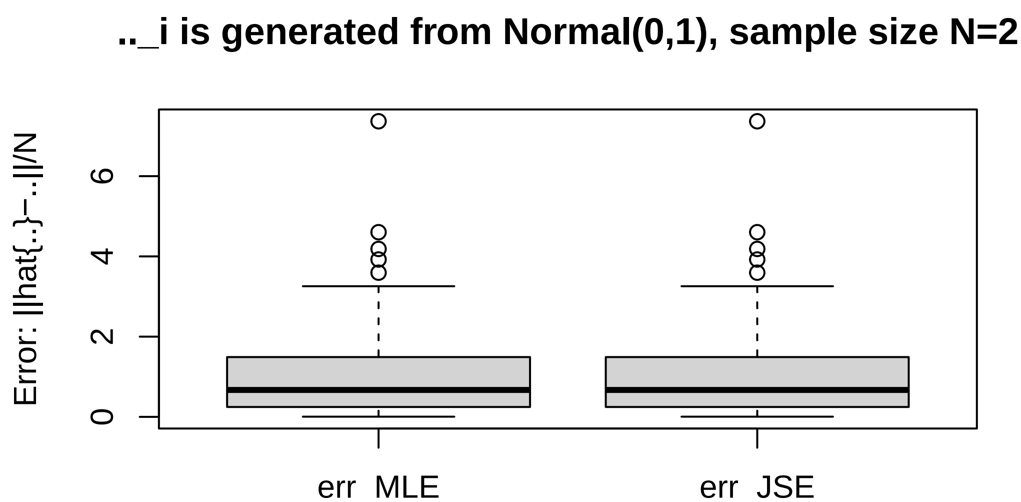
conversion failure on ' _i is generated from Normal(0,1), sample size N=2' in
'mbcsToSbcs': dot substituted for <bc>
```



The statistical experiments confirm the superiority of James-Stein estimator when $N \geq 3$.