with B08902 - 065, 020, 142

Problem 3.

(a)

1° Suppose i^* is the top box of Hmax, after we remove i^* from Hmax, we obtain a lowest tower with height = $Hmax - h(i^*)$

2° Since OPT(I) is the heighest tower in the optimal solution, we know that every tower in optimal solution $\le$ OPT(I). Thus, by greedy-stacking we know, $Hmax - h(i^*) \le OPT(I)$.
($\because$ it's the lowest and if $Hmax - h(i^*) \ge OPT(I)$, there'll be some tower in optimal solution > OPT(I)).

3° Also, we know that $h(i^*) \le OPT(I)$ for sure. (or else OPT(I) isn't the highest).

4° So, from 3°, 4° we know that $Hmax - h(i^*) + h(i^*) = Hmax \le 2 \times OPT(I)$

$\Rightarrow Hmax \le 2 \cdot OPT(I) \Rightarrow \frac{Hmax}{OPT(I)} \le 2$, thus it is 2-approximation. QED.

(b)

1° Suppose i^* is the top box of Hmax, after we remove i^* from Hmax, we obtain a lowest tower with height = $Hmax - h(i^*)$. So consider two condition : $i^* \in B'$, $i^* \notin B'$

2° Suppose $i^* \in B'$, then since $B' \in B$, we know that $Hmax \le OPT(I)$ (or else we we could construct a higher tower in optimal solution)

3° Suppose $i^* \notin B'$ Since OPT(I) is the heighest tower in the optimal solution, we know every tower in optimal solution $\le$ OPT(I). Thus, we know, $Hmax - h(i^*) \le OPT(I)$.
Also, since $i^* \notin B'$, $h(i^*) \le \epsilon V \le \epsilon \cdot OPT(I)$. Thus $Hmax \le (1+\epsilon) \cdot OPT(I)$

4° So, from 4°, 5° we know $Hmax \le (1+\epsilon) \cdot OPT(I)$. Thus, it's $(1+\epsilon)$-approximation. QED

(c)

1° Consider two cases, the highest tower derived by f' and derived by greedy

2° If the highest tower is obtained by f', let $H'_{max} = \max\langle B', h', m\rangle$ and $H_{max} = \max\langle B', h, m\rangle$

$\because h(i) > \epsilon V \Rightarrow \frac{h(i)}{\mu} > \frac{\epsilon V}{\mu} = \frac{1}{\epsilon} \in Z^+ \Rightarrow \lfloor \frac{h(i)}{\mu} \rfloor \cdot \mu = h'(i) \geq \frac{\mu}{\epsilon} = \epsilon V$

Set there's at most N boxes $\Rightarrow N \cdot h'(i) = N \cdot \epsilon V \leq H'_{max} \leq V \Rightarrow N \leq \frac{1}{\epsilon}$.

$\therefore H_{max} - H'_{max} = N \cdot (h'(i) - h(i)) \leq N \cdot \mu \Rightarrow H_{max} \leq H'_{max} + N \cdot \mu$

$[N \cdot \mu = N \cdot \epsilon^2 V = \epsilon \cdot (N\epsilon V) \leq \epsilon \cdot V] \Rightarrow H_{max} \leq V + \epsilon V = (1+\epsilon) V$

3° Suppose $i^* \notin B'$ Since $OPT(I)$ is the heighest tower in the optimal solution, we know every tower in optimal solution $\leq OPT(I)$. Thus, we know, $H_{max} - h(i^*) \leq OPT(I)$.

Also, since $i^* \notin B'$, $h(i^*) \leq \epsilon V \leq \epsilon \cdot OPT(I)$. Thus $H_{max} \leq (1+\epsilon) \cdot OPT(I) \leq (1+\epsilon) \cdot V$

4° So, from 4°. 5° we know $H_{max} \leq (1+\epsilon) \cdot V$. QED

(d - 1)

1° let $h'(i) = N \cdot \mu$, $N \in Z^+$. Consider two cases: $h'(i) \leq V$, $h'(i) > V$

2° Suppose $h'(i) \leq V \Rightarrow N \cdot \mu \leq V = \frac{\mu}{\epsilon^2} \Rightarrow N \leq \frac{1}{\epsilon^2} \quad \therefore |\vec{n}| \leq \frac{1}{\epsilon^2}$

3° Suppose $h'(i) > V \Rightarrow H_{max} > V \quad \therefore$ return false.

4° Thus, from 2°. 3° we know, $|\vec{n}|$ is bounded by $\frac{1}{\epsilon^2}$. QED.

(d-2)

1°

$$F(\vec{n}) = \begin{cases} 1 + \min\{F(\vec{n}-\vec{u}),\ \forall \vec{u} \in U\}, & \text{if } \vec{n} \notin U \\ 1, & \text{if } \vec{n} \in U \end{cases}$$

$(U = \{\vec{u} \mid \Sigma_{i=1}^{|\vec{n}|} u_i \cdot (i \cdot u) \le V\},\ \forall u_i \Rightarrow u_i \le n_i)$

(d-3)

1° Since there is at most $(\frac{1}{\varepsilon})^2$ boxes in one tower, and there are $(\frac{1}{\varepsilon})^2$ kinds of boxes, constructing U take $O((\frac{1}{\varepsilon})^{2(\frac{1}{\varepsilon})^2}) = O(1)$

2° Also, each $F(\vec{n})$ will need to calculate all $\vec{n}-\vec{u}$ (memorized by top-down DP)

$\Rightarrow$ number of $\vec{n}-\vec{u} = \prod_{i=1}^{N} (n_i+1) \le (\Sigma_{i=1}^{n} n_i)^N = |B'|^N \le |B'|^{\frac{1}{\varepsilon^2}}$

3° Thus, the time complexity is $O(1) + O(|B'|^{\frac{1}{\varepsilon^2}}) = O(|B'|^{\frac{1}{\varepsilon^2}}) = O(|B|^{\frac{1}{\varepsilon^2}})$. QED

(e) 1° We know that $V \le 2^{\ell}$. So if we apply binary search on V and use PARTIAL-ROUNDED(I, V') to check on H_{max}, we can finish the algorithm in $O(\log V) \cdot O(|B'|^{\frac{1}{\varepsilon^2}}) = O(\ell \cdot |B|^{\frac{1}{\varepsilon^2}})$. It's polynomial time. QED.