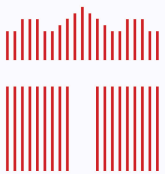
Slido: #ADA2020

CSIE 2136 Algorithm Design and Analysis, Fall 2020



National Taiwan University
國立臺灣大學

NP Completeness - II

Hsu-Chun Hsiao

3-week Agenda

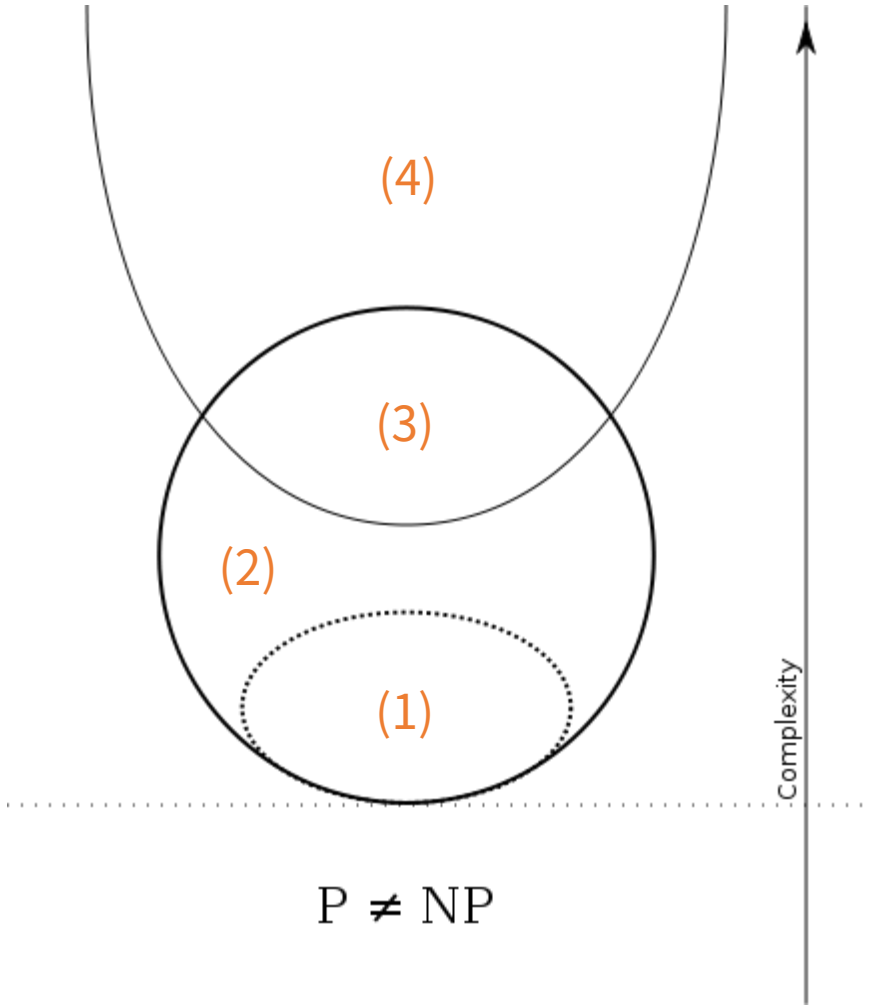
- NP-Completeness Overview
  - Warm up: four color problem
  - Decision vs. optimization
  - Complexity classes
  - Reduction
  - P-time solving vs. verification
- Proving NP-Completeness
  - Formula satisfiability problem
  - 3-CNF-SAT
  - The clique problem
  - The vertex-cover problem
  - The independent-set problem
  - Traveling salesman problem
  - Hamiltonian cycle
- Approximation algorithms
  - Vertex Cover
  - TSP
  - 3-CNF-SAT
- Randomized algorithms
  - Karger's min-cut algorithm
  - Probabilistic data structures

Closed-book final exam: 1/14

- Scope
  - Topics before midterm: roughly 1/3
  - Topics after the midterm: roughly 2/3
  - Easy: ~60%, medium: ~30%, hard: ~10%

- Exam questions may have different difficulty levels, move on if you get stuck!

Review: P, NP, NP-Hard, NP-Complete



Complexity classes

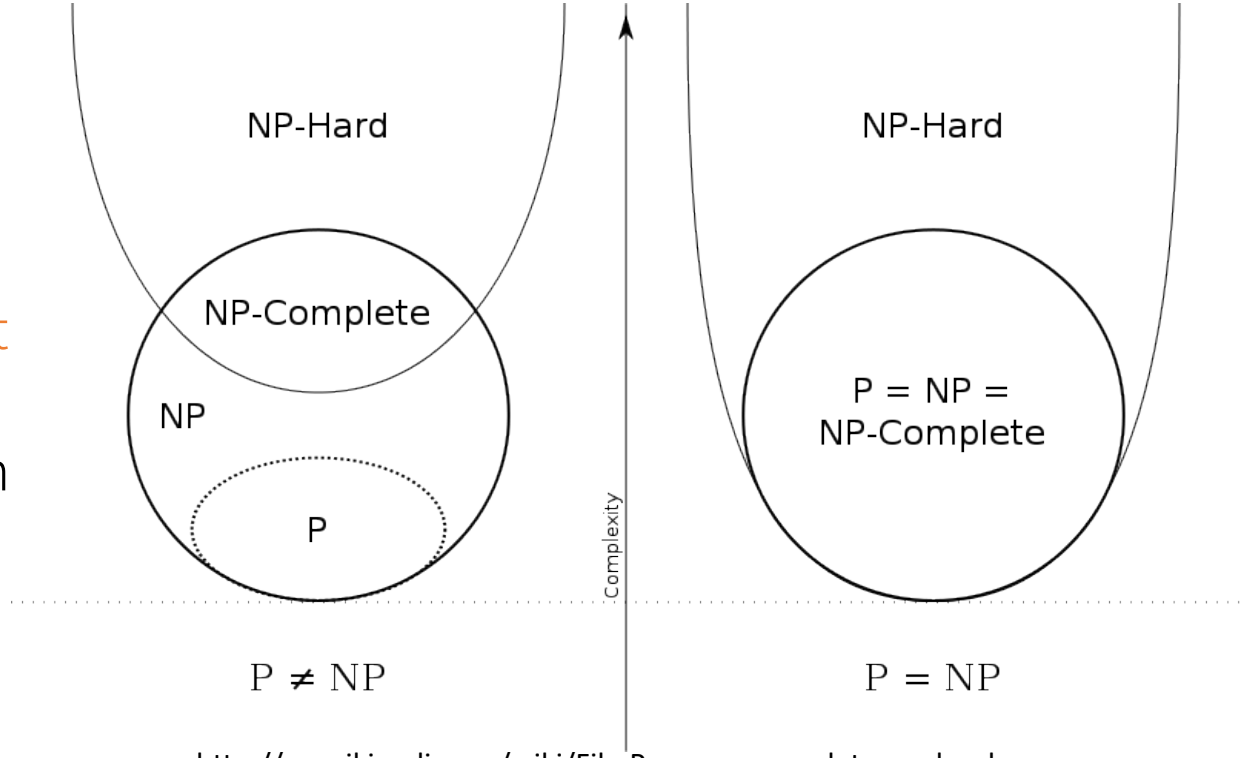
Class P: class of problems that can be solved in $O(n^k)$

Class NP: class of problems that can be verified in $O(n^k)$

Class NP-hard: a class of problems that are "at least as hard as the hardest problems" in NP

Class NP-complete (NPC): class of problems in both NP and NP-hard

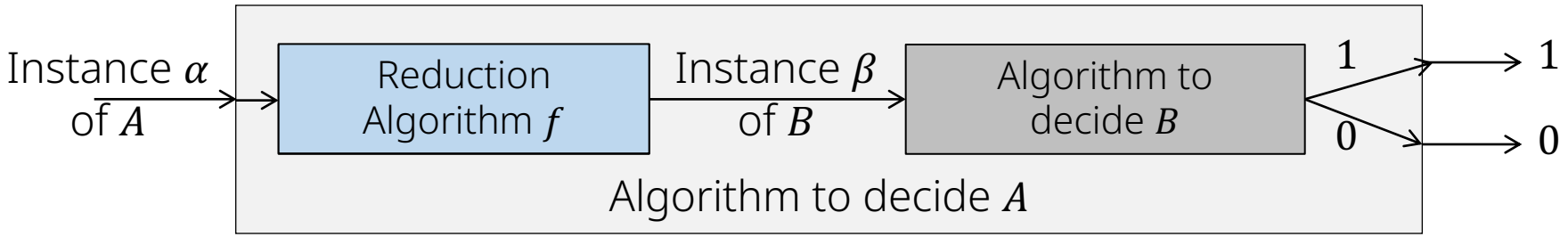
Hardness relationship can be determined via polynomial-time reduction



http://en.wikipedia.org/wiki/File:P_np_np-complete_np-hard.svg

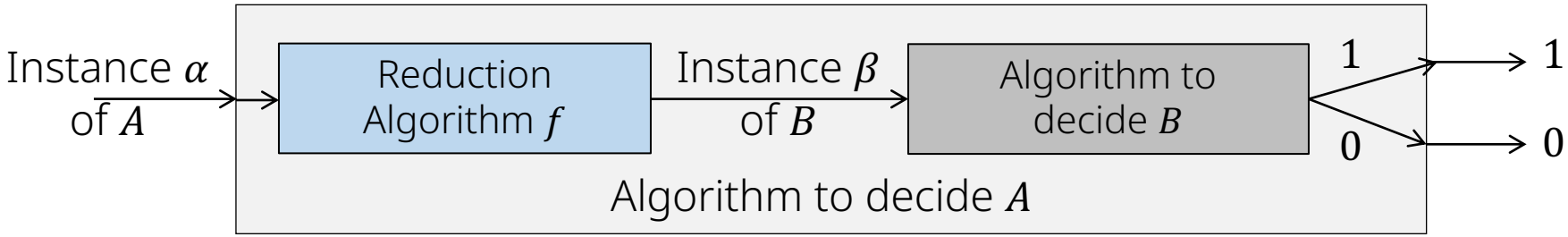
Reduction

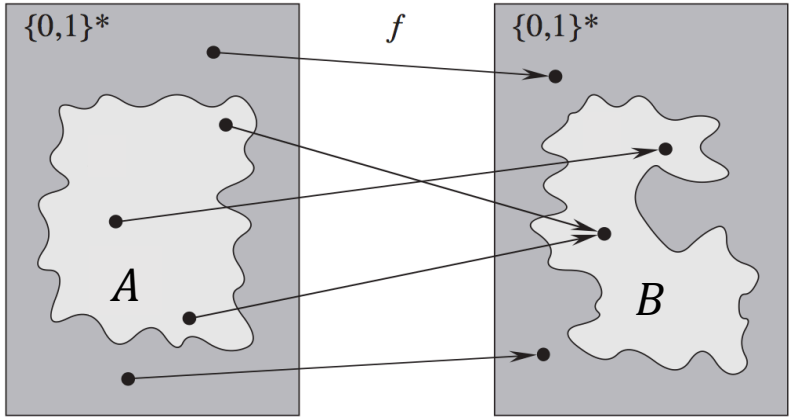
- A reduction is an algorithm for transforming every instance of a problem A into an instance of another problem B
  - We focus on decision problems here
- A polynomial-time reduction can help determine the hardness relationship between problems
  - We write $A \leq_p B$ if A can be reduced to B in p-time
  - $A \leq_p B$ implies A is no harder than B



Reduction

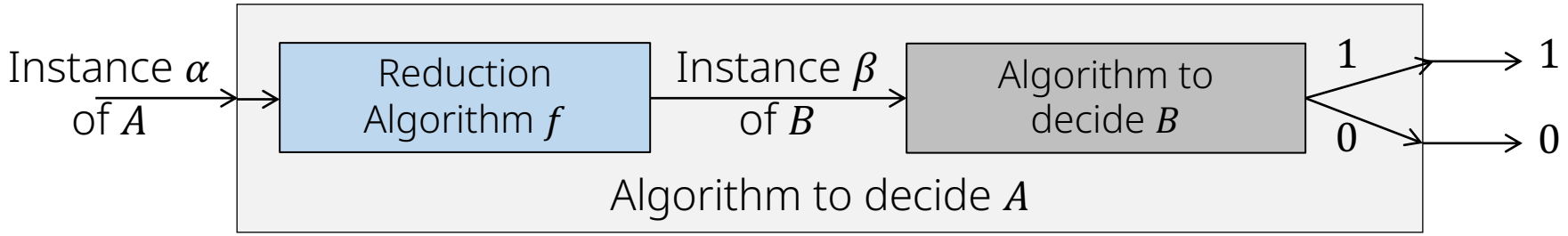
- To show the correctness of a p-time reduction $A \leq_p B$:
  - Show that B outputs 1 if and only if A outputs 1
  - That is, for all α, $AlgA(\alpha) = 1$ iff $AlgB(f(\alpha)) = 1$
  - Show that the reduction algorithm is in polynomial time





Note that $\{f(\alpha) | AlgA(\alpha) = 1\}$ is a subset of $\{\beta | AlgB(\beta) = 1\}$

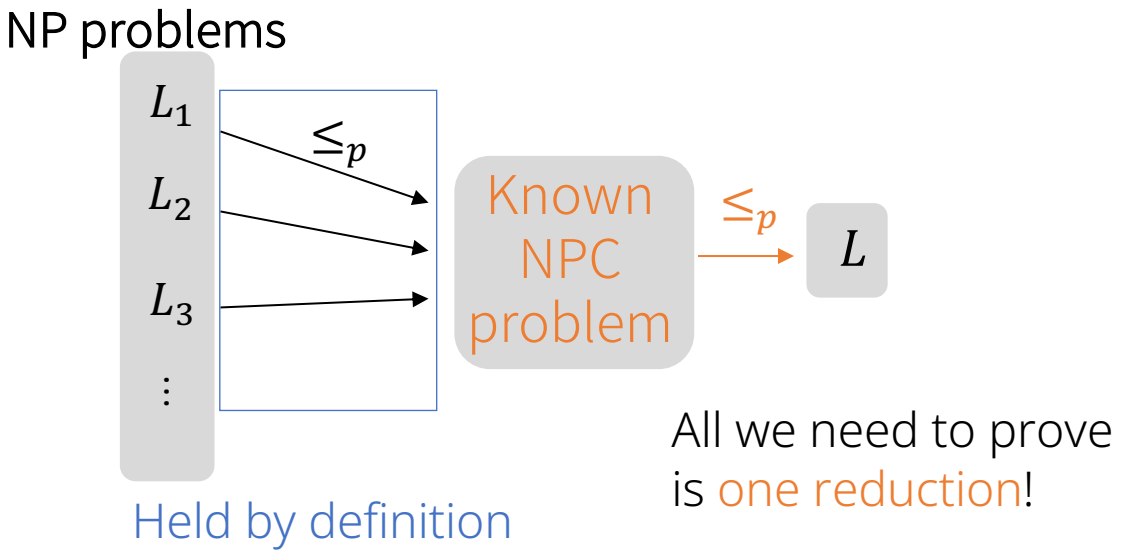
Applications of reduction: proving limits



- A common usage in NP-completeness proofs
  - Goal: prove that no known polynomial-time algorithm exists for $\boldsymbol{B}$
  - Known: no known polynomial-time algorithm exists for A
  - Approach: construct a polynomial-time reduction algorithm to convert α to β
  - Why correct? 用反證法說明：若 B 有p-time解，則根據 reduction，A 也能在 p-time 解決。矛盾，故得証 B 無 p-time 解。

Proving NP-Completeness

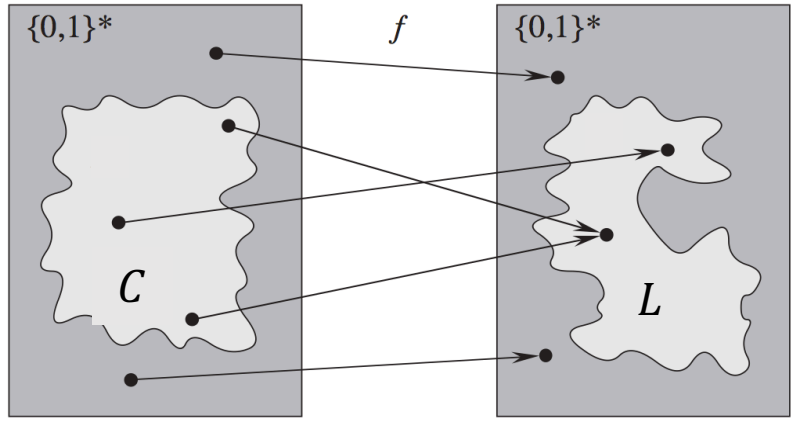
- Class NP-Complete (NPC): class of decision problems in both NP and NP-hard
- In other words, a decision problem L is NP-complete if
  1. $L \in$ NP
  2. $L \in$ NP-Hard (that is, $L' \leq_p L$ for every $L' \in$ NP)



Proving NP-Completeness

- $L \in$ NP-Complete $\Leftrightarrow L \in$ NP and $L \in$ NP-hard
- Step-by-step approach for proving L in NPC:

1. Prove $L \in$ NP
2. Prove $L \in$ NP-hard ($C \leq_p L$)
   - ① Select a known NPC problem C
   - ② Construct a reduction f transforming every instance of C to an instance of L
   - ③ Prove that x in C if and only if $f(x)$ in L for all x in $\{0,1\}^*$
   - ④ Prove that f is a polynomial time transformation



Slido Quizzes

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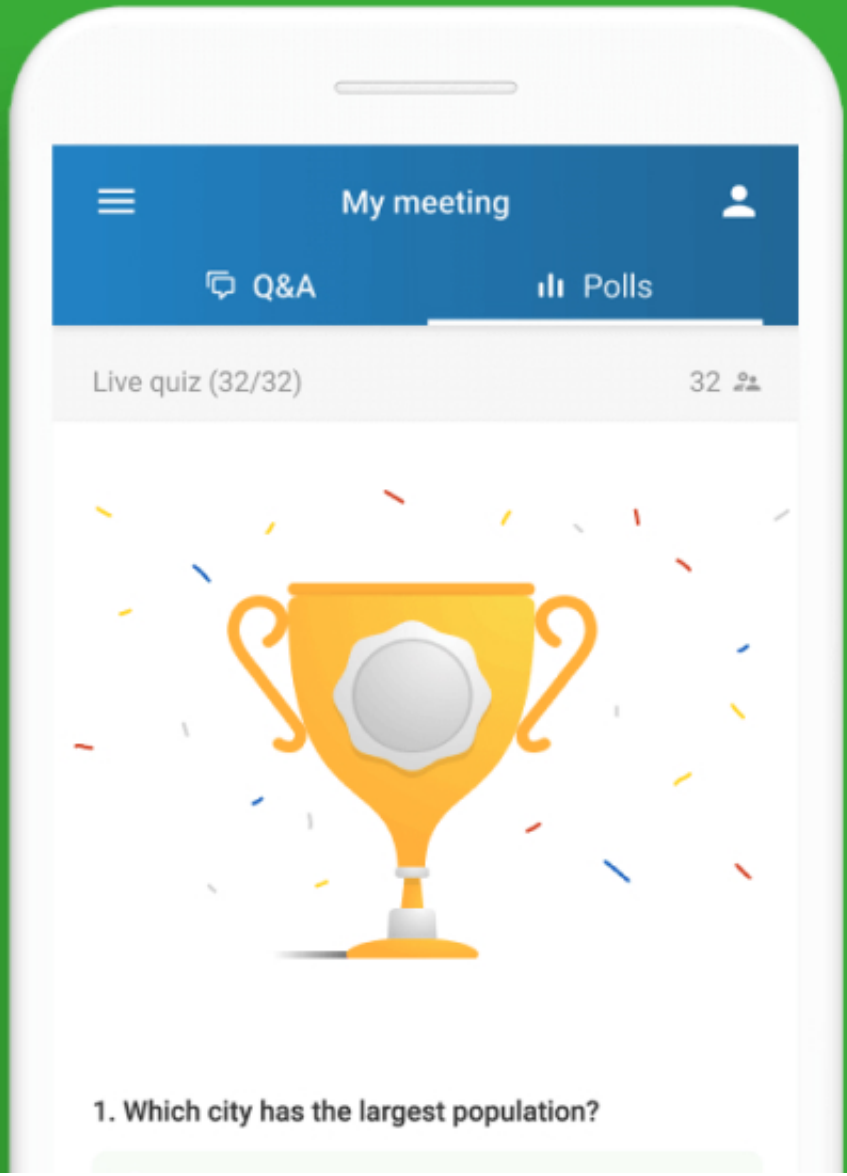


Polynomial-time Solving & Polynomial-time Verification

Chap. 34.1, 34.2

Abstract problems

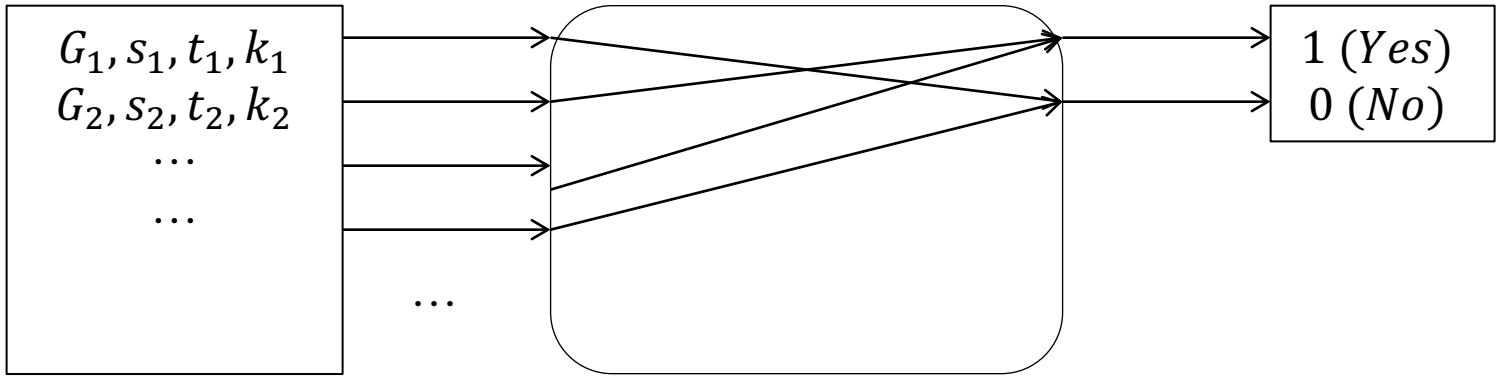
- I: the set of problem instances
- S: the set of problem solutions
- Q: an abstract problem, defined as a binary relation on I and S

Example of a decision problem, $\mathrm{SHORTEST-PATH}$:

$I: < G, src, dest, k >$
所有可能的圖、起訖點、長度上限

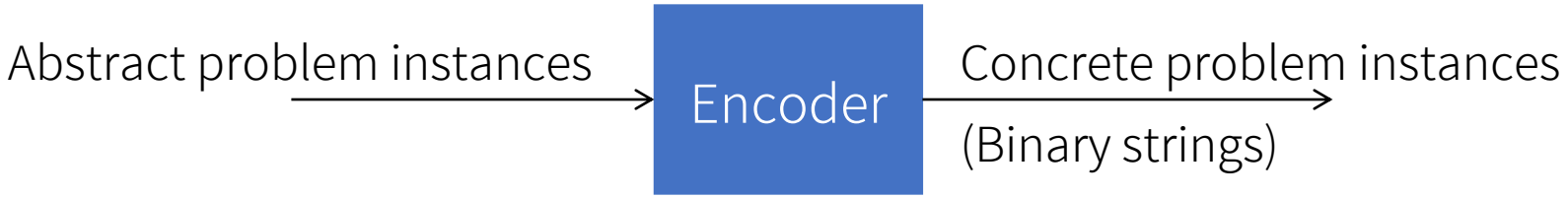
Q: $\mathrm{SHORTEST-PATH}$
是否找得到最短路徑長度 $\leq k$？

S: $\{0,1\}$



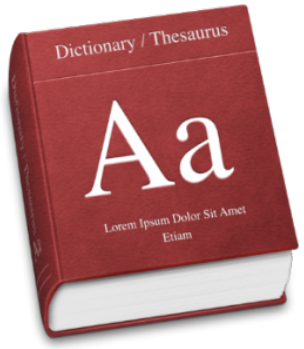
Encoding of problem instances

- Convert an abstract problem instance into a binary string that is fed to a computer program



- A concrete problem is polynomial-time solvable if there exists an algorithm that solves any concrete instance of length n in time $O(n^k)$ for some constant k
  - Solvable = can produce a solution

Formal-language theory



- An Alphabet Σ = a finite set of symbols
- A Language L over Σ = any set of strings made up from alphabet Σ
- Notations
  - ε: empty string
  - ϕ: empty language
  - Σ^*: the language of all strings over Σ

Q: In English, what is the alphabet Σ and the language L?

- $\Sigma = \{a, b, c, \dots, z\}$
- L = a set of English words

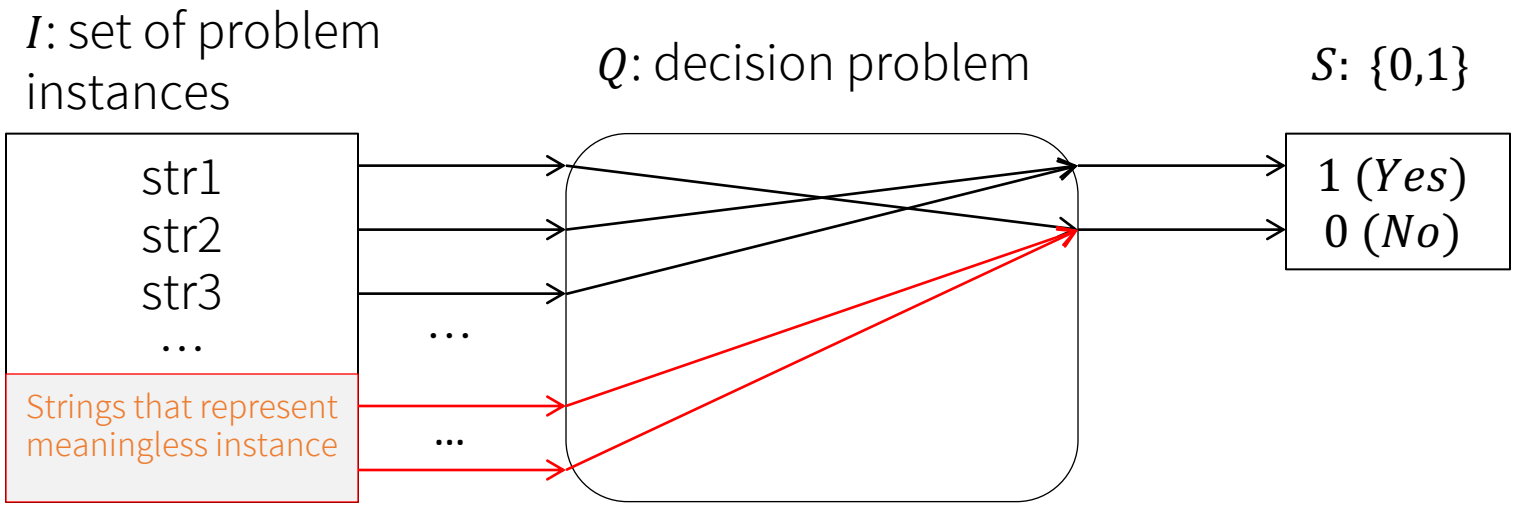
Q: In the language of binary representation of prime numbers, what is the alphabet Σ and the language L?

- $\Sigma = \{0,1\}$
- $L = \{10, 11, 101, 111, \dots\}$

Q: If $\Sigma = \{0,1\}$, what is Σ^*?

- $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$

Representation of a decision problem

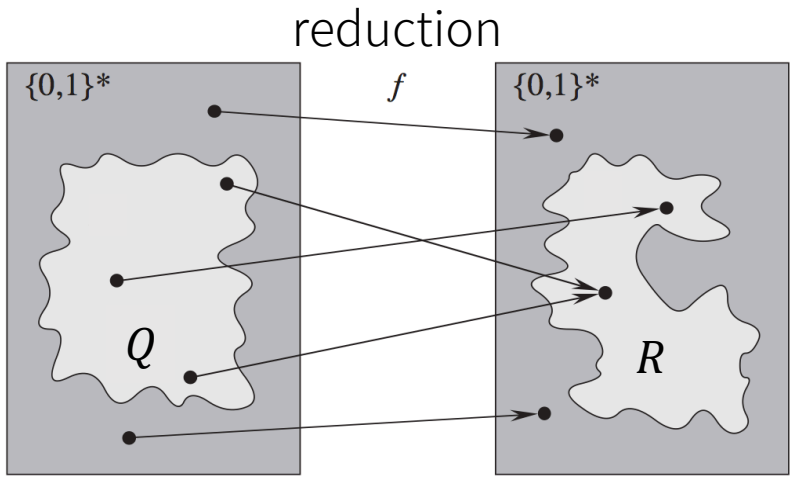


- I = the set of problem instances = Σ^*, where $\Sigma = \{0, 1\}$
- Q = a decision problem = a language L over $\Sigma = \{0, 1\}$ such that $L = \{x \in \Sigma^*: Q(x) = 1\}$
  - 以答案為 1 的 instances 定義 decision problem Q!
  - $L = \{str2, str3\}$ in this simple example

Decision problem

A decision problem Q can be defined as
a language L over $\Sigma = \{0, 1\}$ such that $L = \{x \in \Sigma^* : Q(x) = 1\}$

- 以答案為 1 的 instances 定義 decision problem Q
- Q can be reduced to R means $x \in Q \Leftrightarrow f(x) \in R$



Definition of the complexity class P in language-theoretic framework

- An algorithm A accepts a string $x \in \{0,1\}^*$ if $A(x) = 1$
- An algorithm A rejects a string $x \in \{0,1\}^*$ if $A(x) = 0$
- An algorithm A accepts a language L means…
  - A accepts every string $x \in L$
  - 對所有在 L 中的 string 都要能正確地輸出 YES
  - 對不在 L 中的 string 可能輸出 NO, 也可能無法判斷（e.g., loop forever)
- An algorithm A decides a language L if A accepts L and A rejects every string $x \notin L$
  - 對所有的string都要能正確地輸出YES/NO

Definition of the complexity class P in language-theoretic framework

- Class P: a class of decision problems solvable in p-time
  - That is, given an instance x of a decision problem Q, its solution $Q(x)$ (i.e., YES or NO) can be found in polynomial time
- Hence, an alternative definition of class P:

Definition of Class P

$P = \{L \subseteq \{0,1\}^* |$ there exists an algorithm that decides L in p-time$\}$

Theorem 34.2

P is also the class of language that can be accepted in polynomial time.
$P = \{L \subseteq \{0,1\}^* | L$ is accepted by a polynomial-time algorithm$\}$

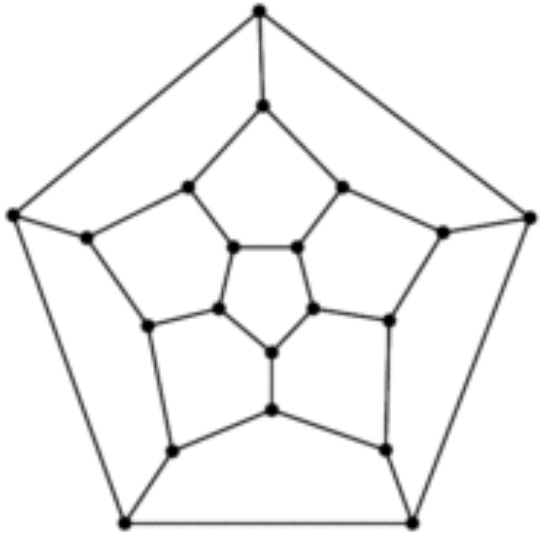
Hamiltonian-Cycle Problem

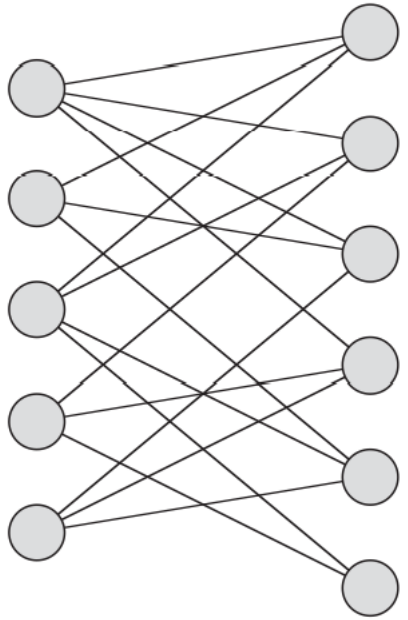
$\mathrm{HAM\text{-}CYCLE} = \{\langle G \rangle \mid G \text{ has a Hamiltonian cycle}\}$

Q: Can you find a cycle visiting each vertex exactly once in the examples?

Q: Is this language decidable?

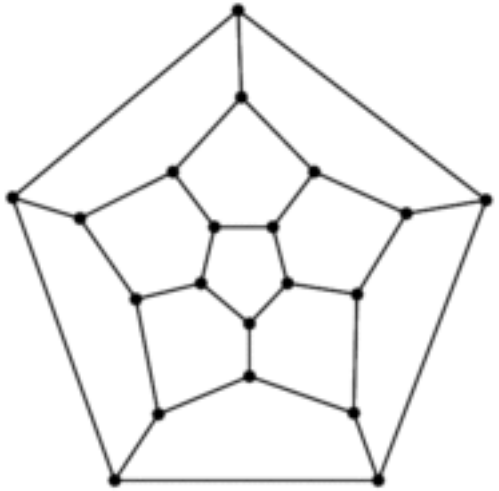
Q: Is this language decidable in polynomial time?





Polynomial-time Verification: Hamiltonian-cycle problem

- Suppose a nice TA gives you a certificate – a vertex sequence that forms a Hamiltonian cycle in a given graph G
- With this certificate, how much time does it take to verify that G indeed contains a Hamiltonian cycle?

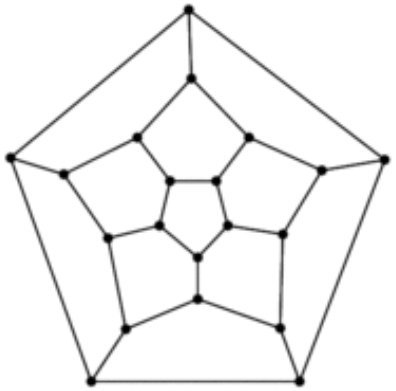


Verification algorithms

- Verification algorithms verify memberships in language

$$\text{HAM-CYCLE} = \{\langle G \rangle \mid G \text{ has a Hamiltonian cycle}\}$$

Input x



Certificate y

Verification algorithm: Is y a Hamiltonian cycle in the graph (encoded in x)?

YES

(x is in HAM-CYCLE)

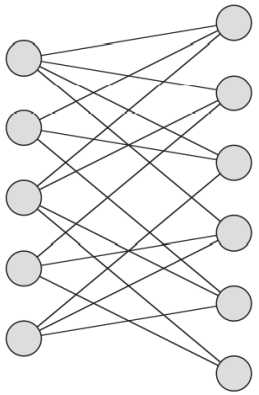
There exists a certificate for each YES instance

Verification algorithms

- Verification algorithms verify memberships in language

$$\text{HAM}-\text{CYCLE} = \{\langle G \rangle \mid G \text{ has a Hamiltonian cycle}\}$$

Input x



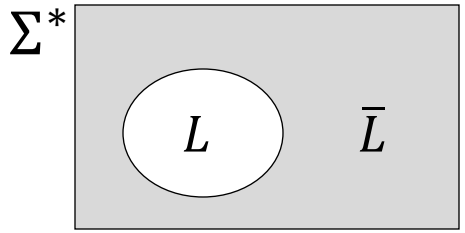
Certificate y ??

Verification algorithm: Is y a Hamiltonian cycle in the graph (encoded in x)?

NO

No conclusion

Co-NP

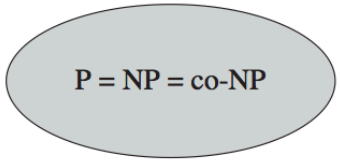


- Class co-NP: class of decision problems whose complement are in NP
  - Complement of $L = \bar{L} = \Sigma^* - L$
- $\mathbf{HAM{-}CYCLE}$: Given a graph G, is there a simple cycle that visits each vertex on G exactly once?
- $\overline{\mathbf{HAM{-}CYCLE}}$: Given a graph G, does G contain no Hamiltonian cycle?
- It's still an open problem whether $\mathbf{HAM{-}CYCLE} \in$ co-NP
  - Equivalently, whether $\overline{\mathbf{HAM{-}CYCLE}} \in$ NP
  - 是否能提供 certificate 證明 graph G 沒有 Hamiltonian cycle?

P, NP, and co-NP

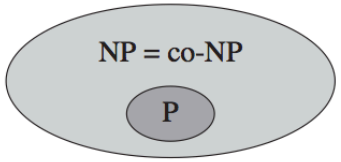
- Class P: class of decision problems solvable in $O(n^k)$
  - $O(n^k)$ means polynomial in the input size (k is a constant)
- Class NP: class of decision problems verifiable in $O(n^k)$
- Class co-NP: class of decision problems whose complement are in NP

If P = NP = co-NP



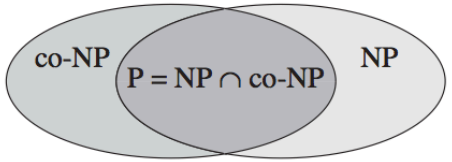
(a)

If P ≠ NP and NP = co-NP



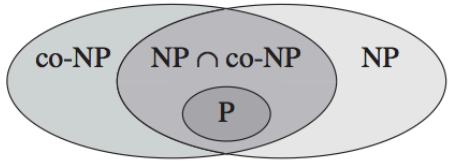
(b)

If P = NP ∩ co-NP



(c)

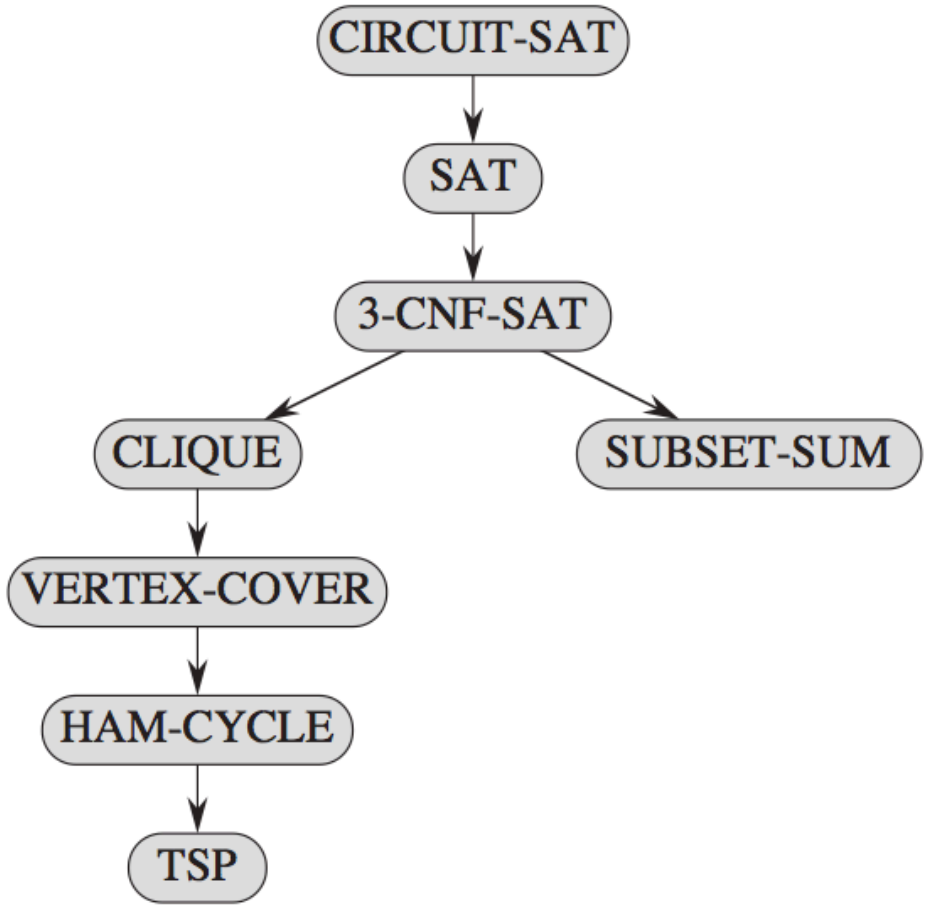
Most likely case!



(d)

Classic NP-Complete Problems

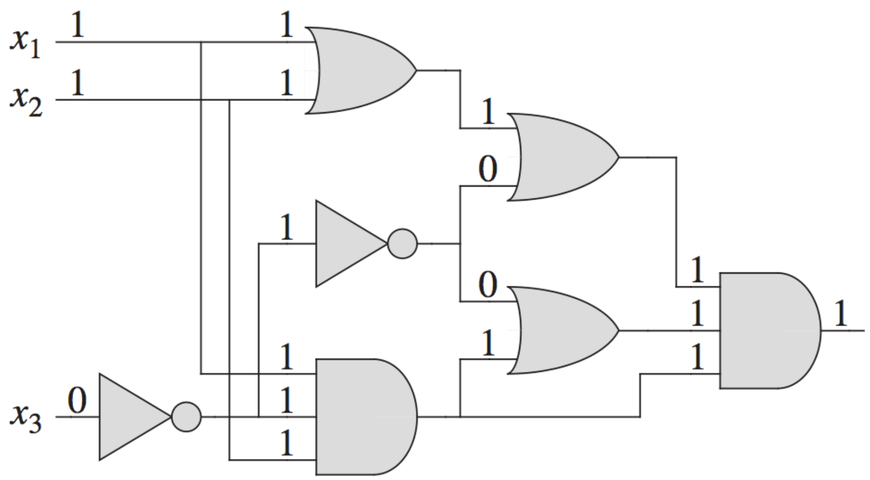
Roadmap



$A \to B : A \leq_p B$

Circuit-Satisfiability Problem

CIRCUIT-SAT $= \{\langle C \rangle : C$ is a satisfiable Boolean combinational circuit$\}$



Cook's Theorem

The Circuit-Satisfiability Problem (CIRCUIT-SAT) is NP-complete

Formula satisfiability problem (SAT)

$\mathrm{SAT} = \{\Phi \mid \Phi$ is a Boolean Formula with a satisfying assignment $\}$

- Given a Boolean formula ϕ, is there a variable assignment satisfying ϕ?
  - $\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT), $\rightarrow$ (implication), $\leftrightarrow$ (if and only if)
  - Satisfiable = Φ is evaluated to 1
- Example: $\phi = \Big((x_1 \rightarrow x_2) \vee \neg\big((\neg x_1 \leftrightarrow x_3) \vee x_4\big)\Big) \wedge \neg x_2$
  - <0, 0, 1, 1> is one solution

Formula satisfiability problem (SAT)

$\mathbf{SAT} = \{\Phi \mid \Phi$ is a Boolean Formula with a satisfying assignment $\}$

- Is $\mathbf{SAT} \in$ NP-Complete?
- To prove that $\mathbf{SAT}$ is NP-Complete, we show that

1. $\mathbf{SAT} \in$ NP
2. $\mathbf{SAT} \in$ NP-hard ($\mathbf{CIRCUIT{-}SAT} \leq_p \mathbf{SAT}$)
   ① $\mathbf{CIRCUIT{-}SAT}$ is a known NPC problem
   ② Construct a reduction f transforming every instance of $\mathbf{CIRCUIT{-}SAT}$ to an instance of $\mathbf{SAT}$
   ③ Prove that $x \in \mathbf{CIRCUIT{-}SAT} \Leftrightarrow f(x) \in \mathbf{SAT}$
   ④ Prove that f is a polynomial time transformation

SAT ∈ NP

- Polynomial-time verification: replaces each variable in the formula with the corresponding value in the certificate and then evaluates the expression
- Example: $\phi = \Big((x_1 \rightarrow x_2) \vee \neg\big((\neg x_1 \leftrightarrow x_3) \vee x_4\big)\Big) \wedge \neg x_2$

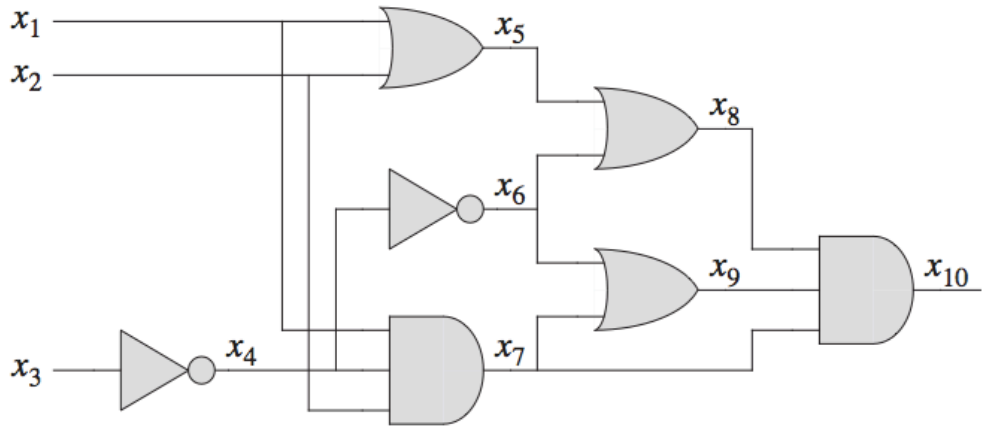
$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 1$$
$$x_4 = 1$$

SAT ∈ NP-hard

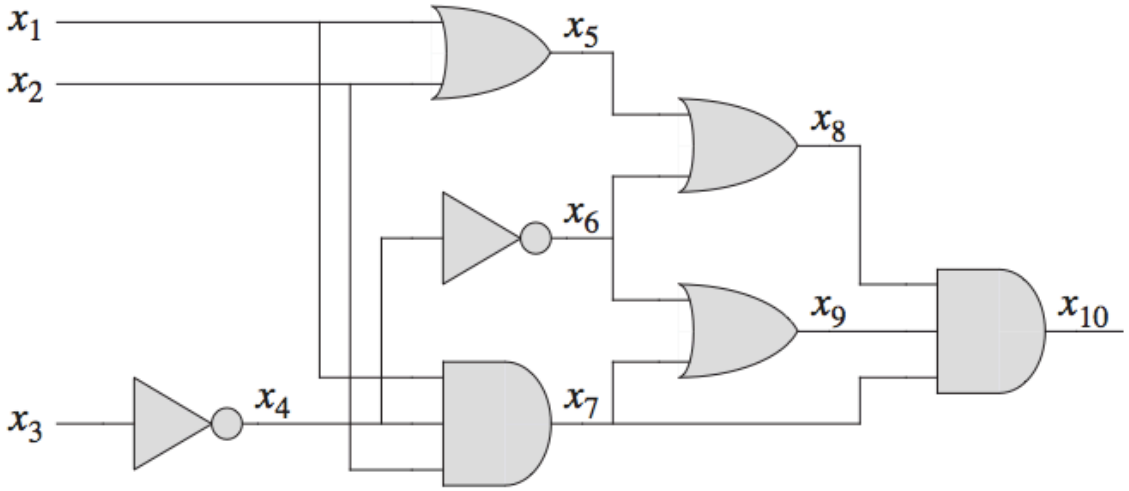
① $\mathbf{CIRCUIT{-}SAT}$ is a known NPC problem

② Construct a reduction f transforming every instance of $\mathbf{CIRCUIT{-}SAT}$ to an instance of $\mathbf{SAT}$

- Assign a variable to each wire in circuit C
- Represent the operation of each gate using a formula, e.g., $x_{10} \leftrightarrow x_7 \wedge x_8 \wedge x_9$
- ϕ = AND the output variable and the operations of all gates



SAT ∈ NP-hard



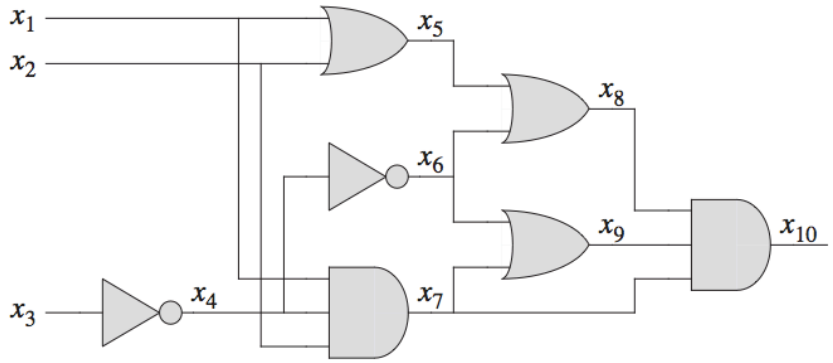
$$\begin{aligned}\phi = x_{10} &\wedge (x_4 \leftrightarrow \neg x_3) \\ &\wedge (x_5 \leftrightarrow (x_1 \vee x_2)) \\ &\wedge (x_6 \leftrightarrow \neg x_4) \\ &\wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\ &\wedge (x_8 \leftrightarrow (x_5 \vee x_6)) \\ &\wedge (x_9 \leftrightarrow (x_6 \vee x_7)) \\ &\wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9))\end{aligned}$$

SAT ∈ NP-hard

③ Prove that $x \in \mathbf{CIRCUIT{-}SAT} \Leftrightarrow f(x) \in \mathbf{SAT}$

- $x \in \mathbf{CIRCUIT{-}SAT} \Rightarrow f(x) \in \mathbf{SAT}$
- $f(x) \in \mathbf{SAT} \Rightarrow x \in \mathbf{CIRCUIT{-}SAT}$

④ f is a polynomial time transformation



$$\begin{aligned}\phi = x_{10} \wedge\ & (x_4 \leftrightarrow \neg x_3) \\ \wedge\ & (x_5 \leftrightarrow (x_1 \vee x_2)) \\ \wedge\ & (x_6 \leftrightarrow \neg x_4) \\ \wedge\ & (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\ \wedge\ & (x_8 \leftrightarrow (x_5 \vee x_6)) \\ \wedge\ & (x_9 \leftrightarrow (x_6 \vee x_7)) \\ \wedge\ & (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9))\end{aligned}$$

Satisfiability of Boolean formulas in 3-conjunctive normal form (3-CNF)

$\mathbf{3\text{-}CNF\text{-}SAT} = \{\phi \mid \phi$ is a Boolean Formula in 3-conjunctive normal form (3-CNF) with a satisfying assignment $\}$

- 3-conjunctive normal form (3-CNF)
  - 3-CNF = AND of clauses, each is the OR of exactly 3 distinct literals
  - A literal is an occurrence of a variable or its negation, e.g., x_1 or $\neg x_1$
- Example: $(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$
  - <0, 0, 1, 1> is one solution

Satisfiability of Boolean formulas in 3-conjunctive normal form (3-CNF)

$\mathbf{3-CNF-SAT} = \{\phi \mid \phi$ is a Boolean Formula in 3-conjunctive normal form (3-CNF) with a satisfying assignment $\}$

- Is $\mathbf{3-CNF-SAT} \in$ NP-Complete?
- To prove $\mathbf{3-CNF-SAT}$ is NP-Complete, we show that

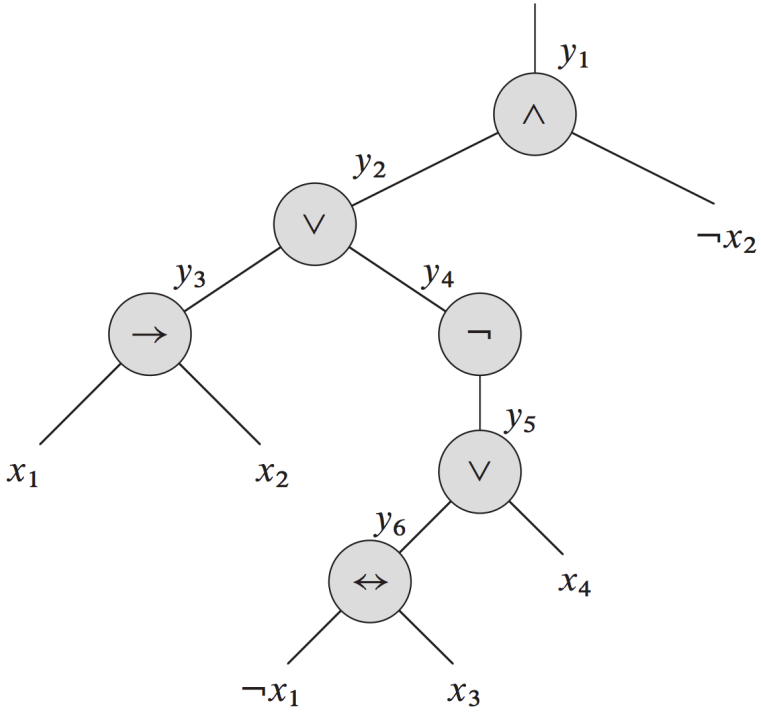
1. $\mathbf{3-CNF-SAT} \in$ NP
2. $\mathbf{3-CNF-SAT} \in$ NP-hard ($\mathbf{SAT} \leq_p \mathbf{3-CNF-SAT}$)
   ① $\mathbf{SAT}$ is a known NPC problem
   ② Construct a reduction f transforming every instance of $\mathbf{SAT}$ to an instance of $\mathbf{3-CNF-SAT}$
   ③ Prove that $x \in \mathbf{SAT} \Leftrightarrow f(x) \in \mathbf{3-CNF-SAT}$
   ④ Prove that f is a polynomial time transformation

* We will focus on the construction of reduction functions from now on; but a full proof requires showing the other conditions are true as well

$\mathrm{SAT} \leq_p 3\mathrm{-CNF-SAT}$

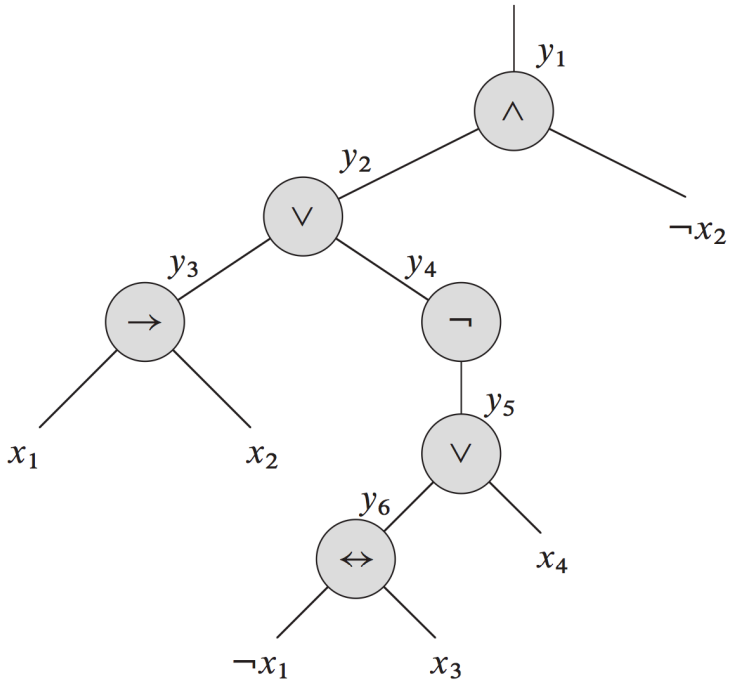
a) Construct a binary parser tree for input formula ϕ and introduce a variable y_i for the output of each internal node

$$\phi = \Big((x_1 \to x_2) \vee \neg\big((\neg x_1 \leftrightarrow x_3) \vee x_4\big)\Big) \wedge x_2$$



$\mathrm{SAT} \leq_p 3\mathrm{-CNF-SAT}$

b) Rewrite ϕ as the AND of the root variable and clauses describing the operation of each node



ϕ'_1

$$\begin{aligned}\phi' = {} & y_1 \\ & \wedge \boxed{(y_1 \leftrightarrow (y_2 \wedge \neg x_2))} \\ & \wedge (y_2 \leftrightarrow (y_3 \vee y_4)) \\ & \wedge (y_3 \leftrightarrow (x_1 \rightarrow x_2)) \\ & \wedge (y_4 \leftrightarrow \neg y_5) \\ & \wedge (y_5 \leftrightarrow (y_6 \vee x_4)) \\ & \wedge (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3))\end{aligned}$$

$\mathrm{SAT} \leq_p 3\mathrm{-CNF-SAT}$

c) Convert each clause ϕ_i' to CNF
- Construct a truth table for each clause ϕ_i'
- Construct the disjunctive normal form for $\neg\phi_i'$
- Apply DeMorgan's Law to get the CNF formula ϕ_i''

$$\boxed{\begin{aligned} \neg(a \wedge b) &= \neg a \vee \neg b \\ \neg(a \vee b) &= \neg a \wedge \neg b \end{aligned}}$$

			ϕ_1'	
y_1	y_2	x_2	$(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$	$\neg\phi_1'$
1	1	1	0	1
1	1	0	1	0
1	0	1	0	1
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1
0	0	1	1	0
0	0	0	1	0

$$\neg\phi_1' = (y_1 \wedge y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge \neg x_2) \vee (\neg y_1 \wedge y_2 \wedge \neg x_2)$$

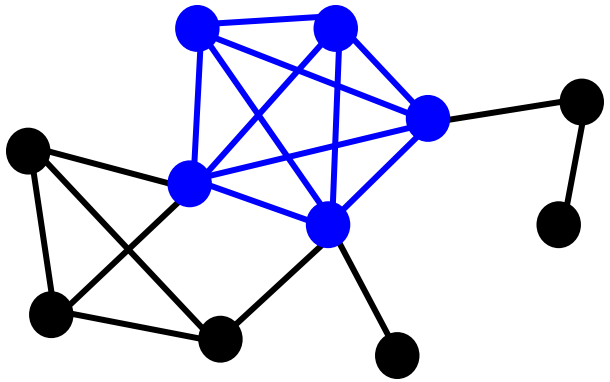
$$\begin{aligned} \phi_1'' = {} & (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2) \\ & \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2) \end{aligned}$$

$\mathrm{SAT} \leq_p 3\mathrm{-CNF-SAT}$

d) Construct ϕ''' s.t. each clause C_i has exactly 3 distinct literals
   - C_i has 3 distinct literals: do nothing
   - C_i has 2 distinct literals: $C_i = I_1 \vee I_2 = (I_1 \vee I_2 \vee p) \wedge (I_1 \vee I_2 \vee \neg p)$
   - C_i has 1 literal: $C_i = I = (I \vee p \vee q) \wedge (I \vee \neg p \vee q) \wedge (I \vee p \vee \neg q) \wedge (I \vee \neg p \vee \neg q)$
- Show that ϕ''' is satisfiable iff ϕ is satisfiable
- Show that all transformation can be done in polynomial time
- => $3\mathrm{-CNF-SAT}$ is NP-Complete

The clique problem

- A clique in $G = (V, E)$ is a complete subgraph of G
  - Each pair of vertices in a clique is connected by an edge in E
  - Size of a clique = # of vertices it contains
- Optimization problem: find a clique of maximum size in G
- Decision problem: find a clique of size k in G



Does G contain a clique of size 4? YES

Does G contain a clique of size 5? YES

Does G contain a clique of size 6? NO

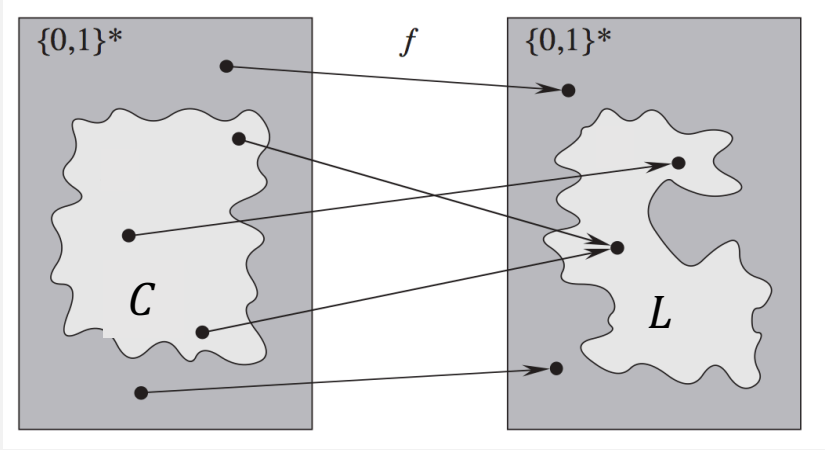
The Clique Problem

$\mathbf{CLIQUE} = \{\langle G, k\rangle : G \text{ is a graph containing a clique of size } k\}$

- Prove that $\mathbf{CLIQUE} \in$ NP-COMPLETE
- Key idea: $\mathbf{3-CNF-SAT} \leq_p \mathbf{CLIQUE}$

Step-by-step approach for proving L in NPC:

1. Prove $L \in$ NP
2. Prove $L \in$ NP-hard ($C \leq_p L$)
   - ① Select a known NPC problem C
   - ② Construct a reduction f transforming every instance of C to an instance of L
   - ③ Prove that x in C if and only if $f(x)$ in L for all x in $\{0,1\}^*$
   - ④ Prove that f is a polynomial time transformation



The Clique Problem

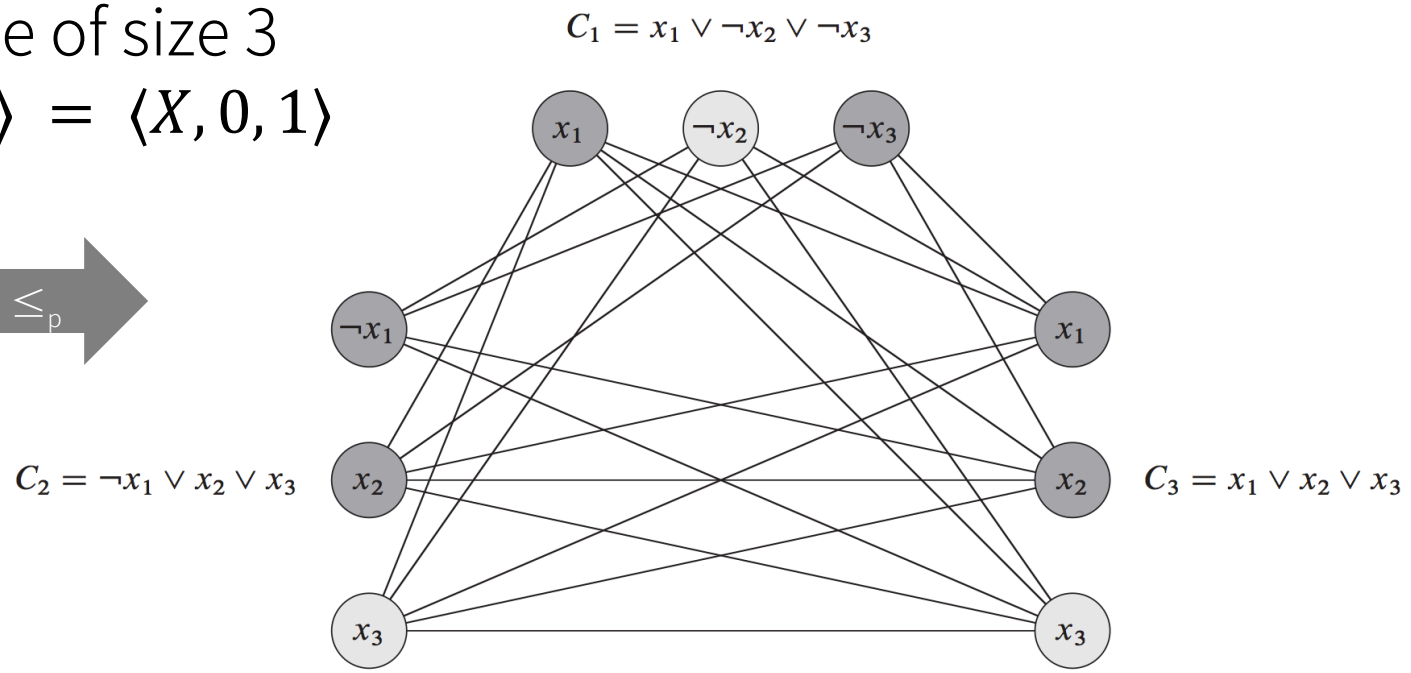
$\mathbf{CLIQUE} = \{\langle G, k\rangle\colon\ G$ is a graph containing a clique of size $k\}$

- Prove that $\mathbf{CLIQUE} \in$ NP-COMPLETE
- Key idea: $\mathbf{3-CNF-SAT} \leq_p \mathbf{CLIQUE}$
  - Construct a graph G s.t. ϕ with k clauses is satisfiable $\Leftrightarrow$ G has a clique of size k

Example: $\phi = 1 \Leftrightarrow G$ has a clique of size 3
Satisfying assignment $\langle x_1, x_2, x_3\rangle = \langle X, 0, 1\rangle$

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

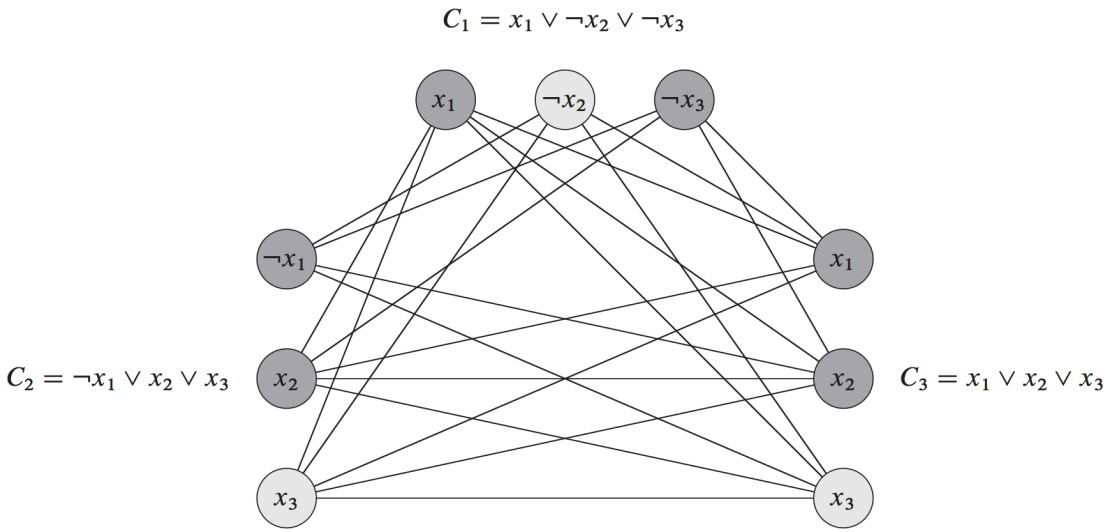
$\leq_p$



$3\text{−CNF−SAT} \leq_p \text{CLIQUE}$

Polynomial-time reduction:

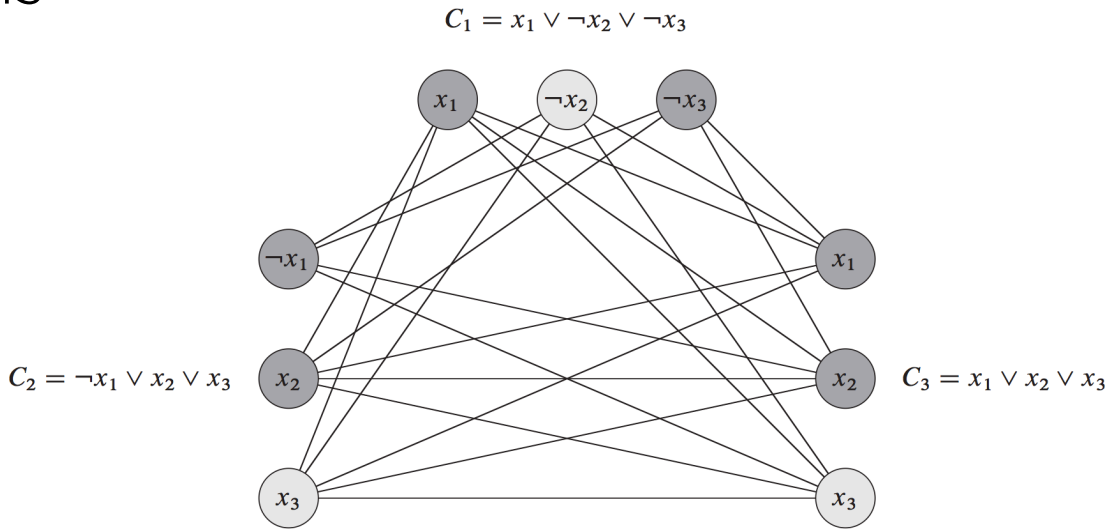
- Let $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_k$ be a Boolean formula in 3-CNF with k clauses, and each C_r has exactly 3 distinct literals l_1^r, l_2^r, l_3^r
- For each $C_r = (l_1^r \vee l_2^r \vee l_3^r)$ in ϕ, introduce a triple of vertices v_1^r, v_2^r, v_3^r in V
- Build an edge between v_i^r, v_j^s if both of the following hold:
  - v_i^r and v_j^s are in different triples, and
  - l_i^r is not the negation of l_j^s



3-CNF-SAT $\leq_p$ CLIQUE

Correctness proof: ϕ is satisfiable $\Rightarrow$ G has a clique of size k

If ϕ is satisfiable,
$\Rightarrow$ Each C_r contains at least one $l_i^r = 1$ and each such literal corresponds to a vertex v_i^r
$\Rightarrow$ Picking a $TRUE$ literal from each C_r forms a set of V' of k vertices
$\Rightarrow$ For any two vertices $v_i^r, v_j^s \in V'$, edge $(v_i^r, v_j^s) \in E$, because $r \neq s$ and $l_i^r = l_j^s =$ (
(l_i^r, l_j^s cannot be complements)

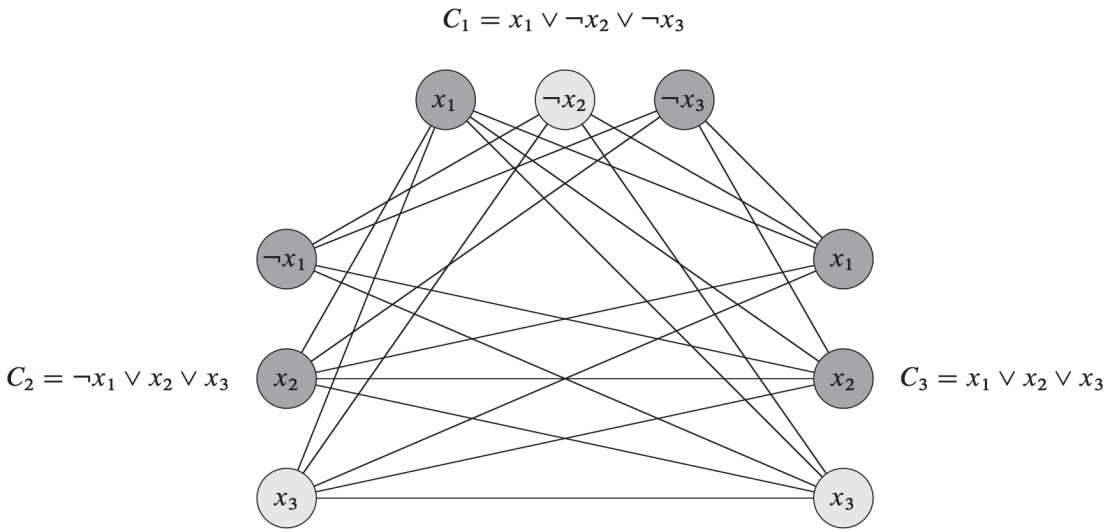


$3-\mathrm{CNF}-\mathrm{SAT} \leq_p \mathrm{CLIQUE}$

Correctness proof: G has a clique of size $k \Rightarrow \phi$ is satisfiable

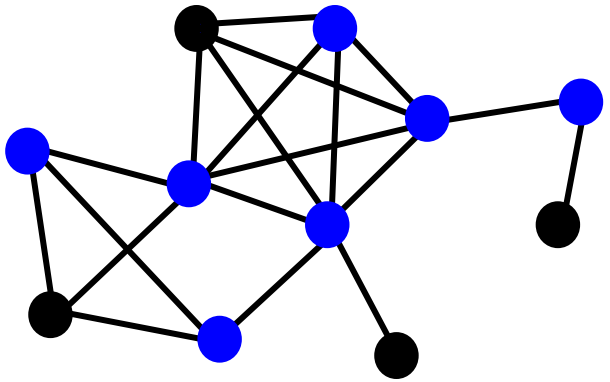
- If G has a clique V' of size k
- $\Rightarrow V'$ contains exactly one vertex per triple since no edges connect vertices in the same triple
- $\Rightarrow$ Assign $TRUE$ to each l_i^r where $v_i^r \in V'$
- $\Rightarrow$ each C_r is satisfied, and so is ϕ

Q: why is this a feasible assignment?



The vertex-cover problem

- A vertex cover of $G = (V, E)$ is a subset $V' \subseteq V$ such that if $(w, v) \in E$, then $w \in V'$ or $v \in V'$ or both
  - A vertex cover "covers" every edge in G
- Optimization problem: find a vertex cover of minimum size in G
- Decision problem: find a vertex cover of size k in G



Does G have a vertex cover of size 11?

Does G have a vertex cover of size 7?

Does G have a vertex cover of size 6?

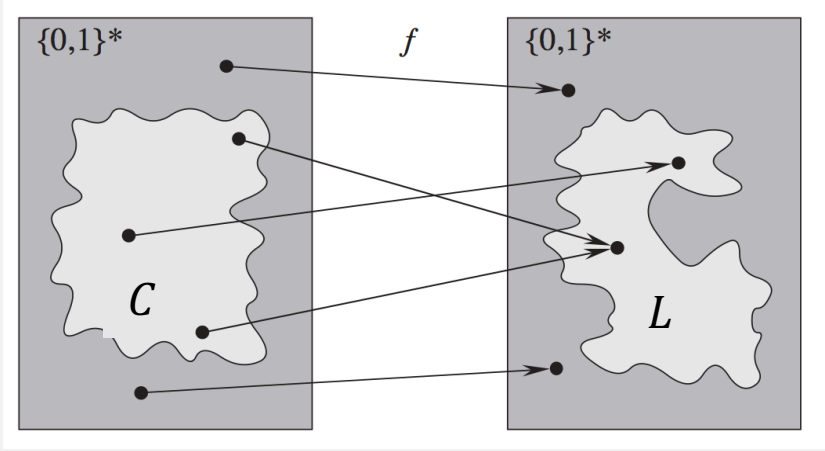
The Vertex-Cover Problem

$\mathbf{VERTEX\text{-}COVER} = \{\langle G, k\rangle$: graph G has a vertex cover of size $k\}$

- Prove that $\mathbf{VERTEX\text{-}COVER} \in$ NP-Complete
- Polynomial-time reduction: $\mathbf{CLIQUE} \leq_p \mathbf{VERTEX\text{-}COVER}$

Step-by-step approach for proving L in NPC:

1. Prove $L \in$ NP
2. Prove $L \in$ NP-hard ($C \leq_p L$)
   ① Select a known NPC problem C
   ② Construct a reduction f transforming every instance of C to an instance of L
   ③ Prove that x in C if and only if $f(x)$ in L for all x in $\{0,1\}^*$
   ④ Prove that f is a polynomial time transformation

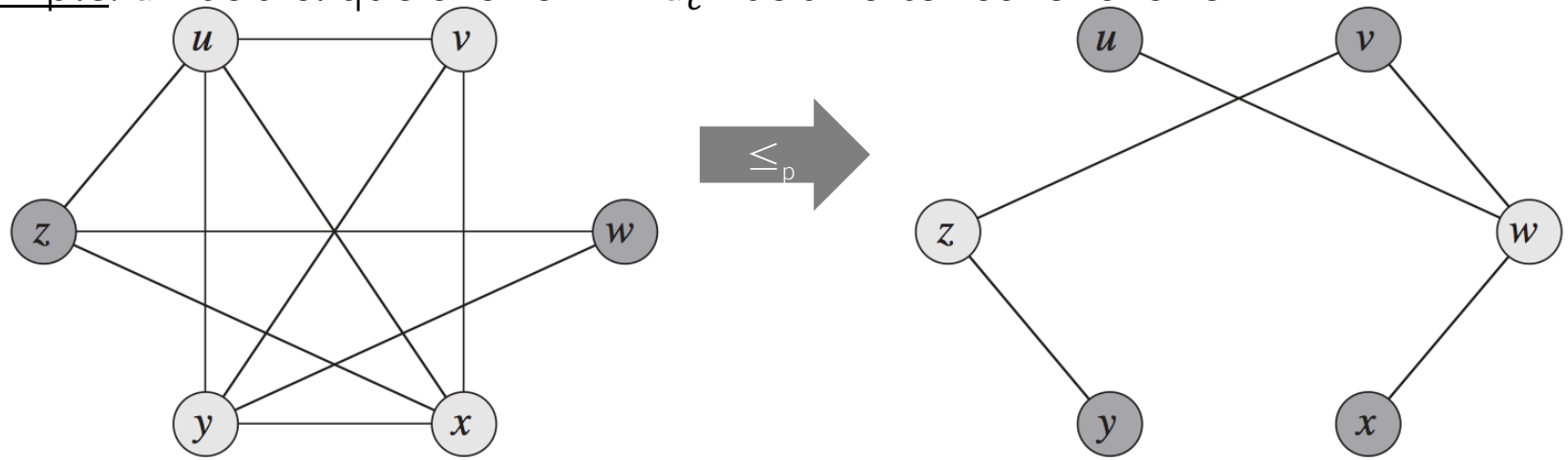


The Vertex-Cover Problem

$\mathbf{VERTEX\text{-}COVER} = \{\langle G, k\rangle$: graph G has a vertex cover of size $k\}$

- Prove that $\mathbf{VERTEX\text{-}COVER} \in$ NP-Complete
- Polynomial-time reduction: $\mathbf{CLIQUE} \leq_p \mathbf{VERTEX\text{-}COVER}$
  - Compute the complement of G
  - Given $G = \langle V, E\rangle$, G_c is defined as $\langle V, E_c\rangle$ s.t. $E_c = \{(u, v) | (u, v) \notin E\}$
  - We want to prove that G has a clique of size $k \Leftrightarrow G's$ complement (G_c) has a vertex cover of size $|V| - k$

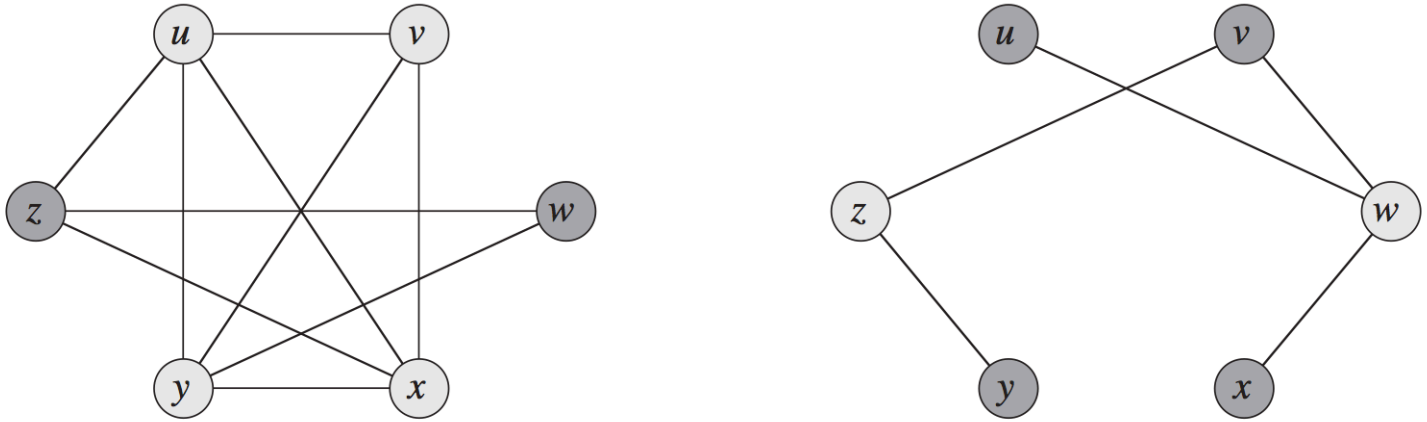
Example: G has a clique of size 4 $\Leftrightarrow$ G_c has a vertex cover of size 2



CLIQUE $\leq_p$ VERTEX-COVER

Correctness proof: G has a clique of size $k \Rightarrow G_c$ has as a vertex cover of size $|V| - k$

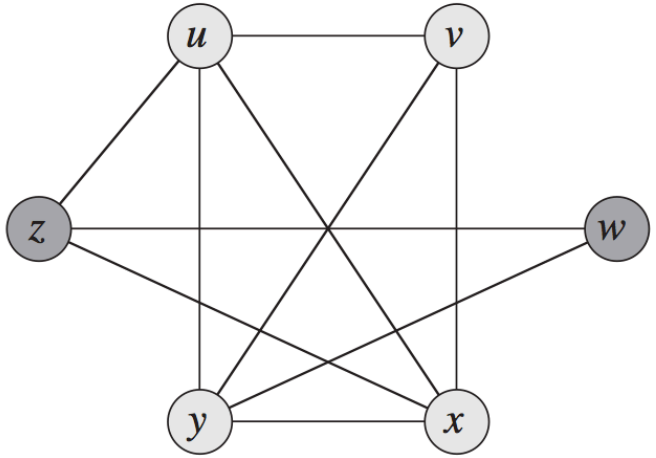
- Suppose that G has a clique $V' \subseteq V$ with $|V'| = k$
- $\Rightarrow$ for all $(w, v) \in E_c$, at least one of w or $v \notin V'$
- $\Rightarrow w \in V - V'$ or $v \in V - V'$ (or both)
- $\Rightarrow$ edge (w, v) is covered by $V - V'$
- $\Rightarrow V - V'$ forms a vertex cover of G_c, and $|V - V'| = |V| - k$

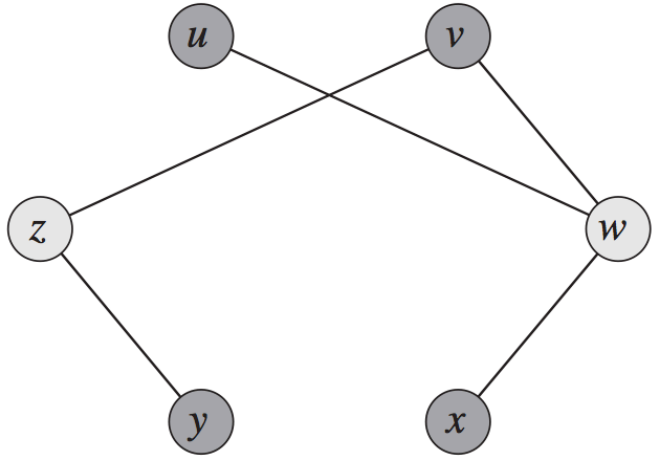


$\mathrm{CLIQUE} \leq_p \mathrm{VERTEX-COVER}$

Correctness proof: G_c has as a vertex cover of size $|V| - k \Rightarrow G$ has a clique of size k

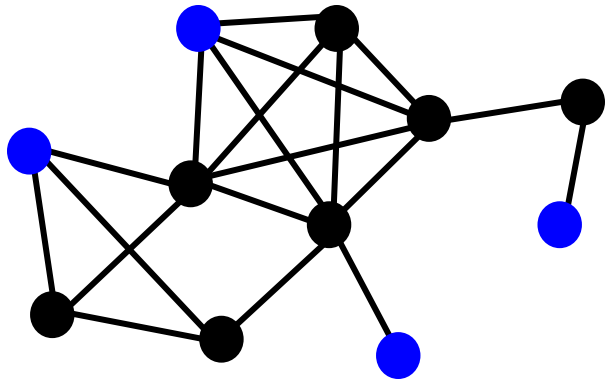
- Suppose G_c has a vertex cover $V' \subseteq V$ with $|V'| = |V| - k$
- $\Rightarrow \forall a, b \in V$, if $(a, b) \in E_c$, then $a \in V'$ or $b \in V'$ or both
- $\Rightarrow \forall a, b \in V$, if $a \notin V'$ and $b \notin V'$, then $(a, b) \notin E_c$; that is, $(a, b) \in E$
- $\Rightarrow V - V'$ is a clique, and $|V - V'| = k$





The independent-set problem

- An independent set of $G = (V, E)$ is a subset $V' \subseteq V$ such that G has no edge between any pair of vertices in V'
- Optimization problem: find an independent set of maximum size in G
- Decision problem: find an independent set of size k in G



Does G have an independent set of size 1?

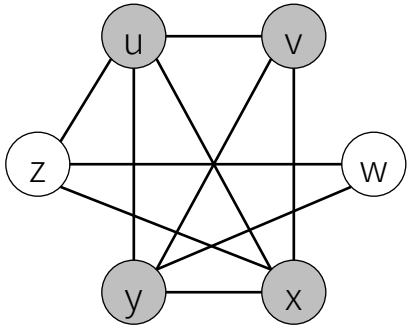
Does G have an independent set of size 4?

Does G have an independent set of size 5?

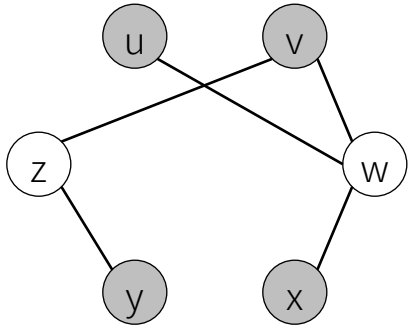
Practice: show that IND-SET is NP-Complete (Textbook Problem 34-1)

Clique, Independent-Set, Vertex-Cover

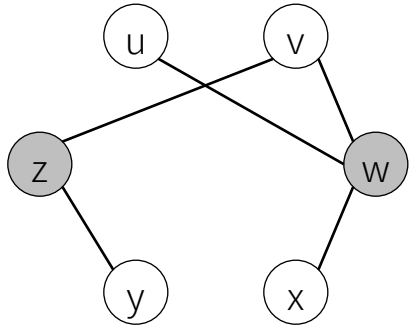
- The following are equivalent for $G = (V, E)$ and a subset V' of V:
  1. V' is a clique of G
  2. V' is an independent set of G_c
  3. $V - V'$ is a vertex cover of G_c



Clique
V' = {u, v, x, y} in G



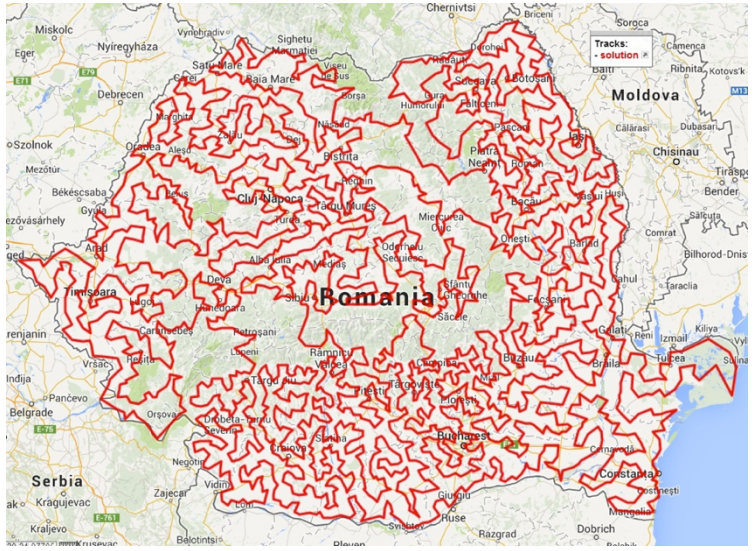
Independent set
V' = {u, v, x, y} in G_c



Vertex cover
V - V' = {z, w} in G_c

Traveling Salesman Problem (TSP)

- Optimization problem: Given a set of cities and their pairwise distances, find a tour of lowest cost that visits each city exactly once.
- Decision problem: Given a set of cities and their pairwise distances, find a tour of cost at most k that visits each city exactly once.



Romanian TSP: 2950 nodes
http://cadredidactice.ub.ro/ceraselacrisan/cercetare/

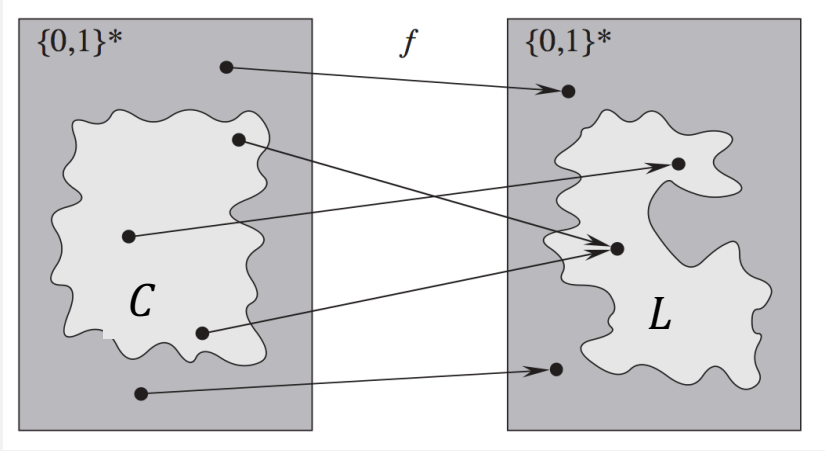
The TSP Problem

$TSP = \{\langle G, c, k\rangle : G = (V, E)$ is a complete graph, c is a non-negative cost function for edges, G has a traveling-salesman tour with cost at most $k\}$

- Prove that $\mathbf{TSP} \in$ NP-COMPLETE
- Polynomial-time reduction: $\mathbf{HAM{-}CYCLE} \leq_p \mathbf{TSP}$

Step-by-step approach for proving L in NPC:

1. Prove $L \in$ NP
2. Prove $L \in$ NP-hard ($C \leq_p L$)
   - ① Select a known NPC problem C
   - ② Construct a reduction f transforming every instance of C to an instance of L
   - ③ Prove that x in C if and only if $f(x)$ in L for all x in $\{0,1\}^*$
   - ④ Prove that f is a polynomial time transformation

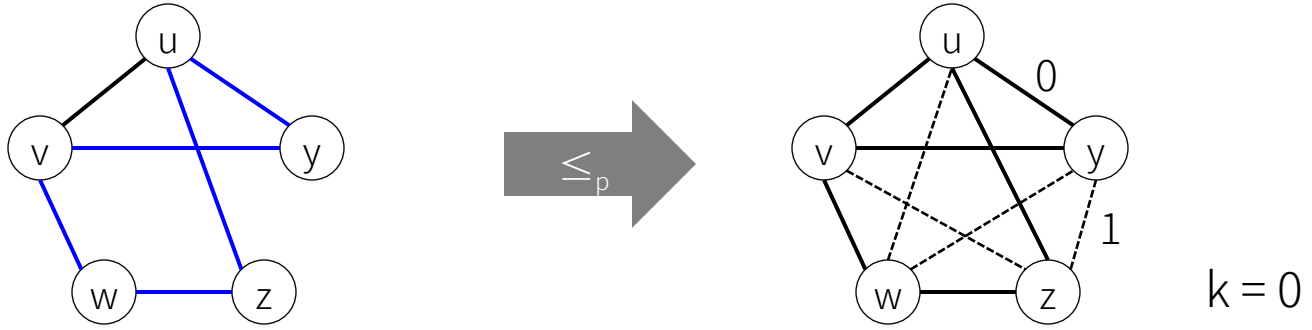


The TSP Problem

$TSP = \{\langle G, c, k\rangle : G = (V, E)$ is a complete graph, c is a non-negative cost function for edges, G has a traveling-salesman tour with cost at most $k\}$

- Prove that $\mathbf{TSP} \in$ NP-COMPLETE
- Polynomial-time reduction: $\mathbf{HAM\text{-}CYCLE} \leq_p \mathbf{TSP}$
  - Given a HAM-CYCLE instance, we construct a TSP instance in which G is a complete graph and

$$c(i, j) = \begin{cases} 0 & \text{if } (i, j) \in E\ , \\ 1 & \text{if } (i, j) \notin E\ . \end{cases}$$

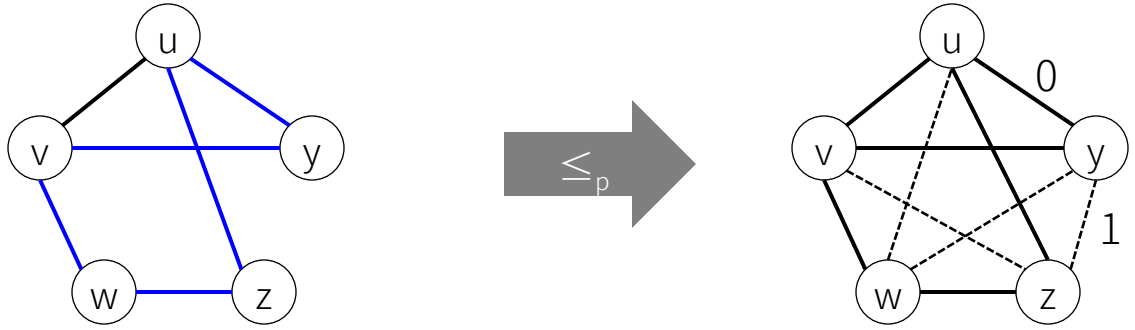


$\mathrm{HAM{-}CYCLE} \leq_p \mathrm{TSP}$

Correctness proof: $x \in \text{HAM-CYCLE} \Leftrightarrow f(x) \in \text{TSP}$

- More specifically, we want to prove that G contains a Hamiltonian cycle $h = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$ if and only if $\langle v_1, v_2, \ldots, v_n, v_1 \rangle$ is a traveling-salesman tour with cost 0

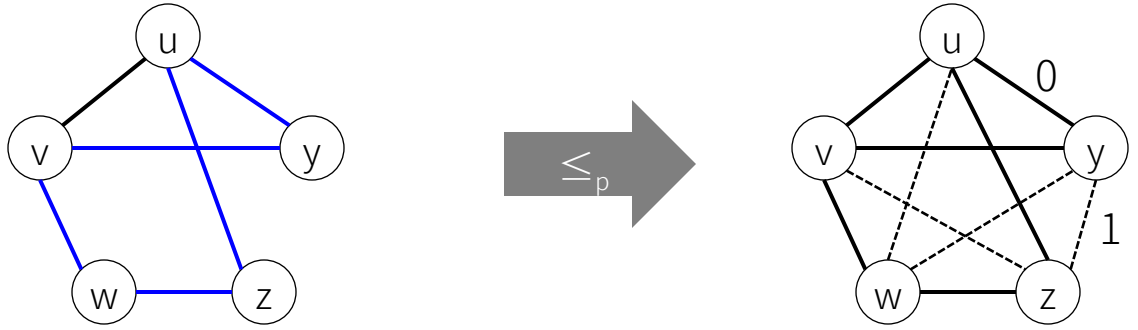
u, y, v, w, z, u is a Hamiltonian cycle ⇔ u, y, v, w, z, u is a traveling-salesman tour with cost 0



$\mathrm{HAM{-}CYCLE} \leq_p \mathrm{TSP}$

Correctness proof: $x \in \mathrm{HAM{-}CYCLE} \Rightarrow f(x) \in \mathrm{TSP}$

- Suppose the Hamiltonian cycle is $h = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$
- $\Rightarrow h$ is also a tour in the transformed TSP instance
- $\Rightarrow$ The distance of the tour h is 0 since there are n consecutive edges in E, and so has distance 0 in $f(x)$
- $\Rightarrow f(x) \in \mathrm{TSP}$ ($f(x)$ has a TSP tour with cost ≤ 0)



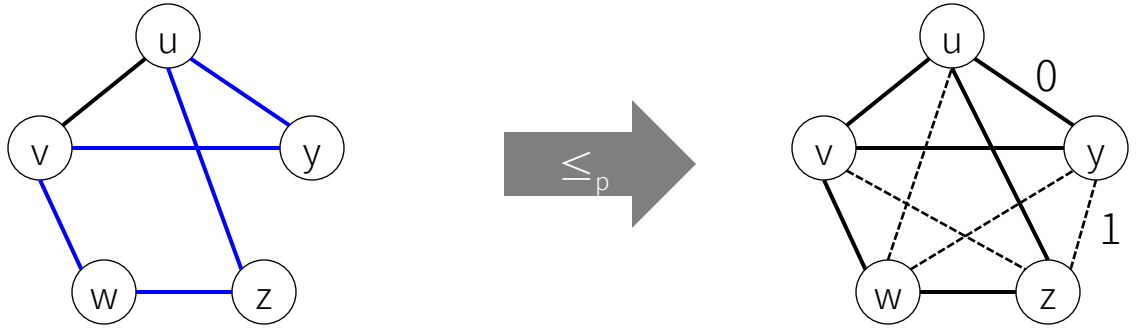
u, y, v, w, z, u is a Hamiltonian cycle

u, y, v, w, z, u is a traveling-salesman tour with cost 0

$\mathrm{HAM{-}CYCLE} \leq_p \mathrm{TSP}$

Correctness proof: $f(x) \in \mathrm{TSP} \Rightarrow x \in \mathrm{HAM{-}CYCLE}$

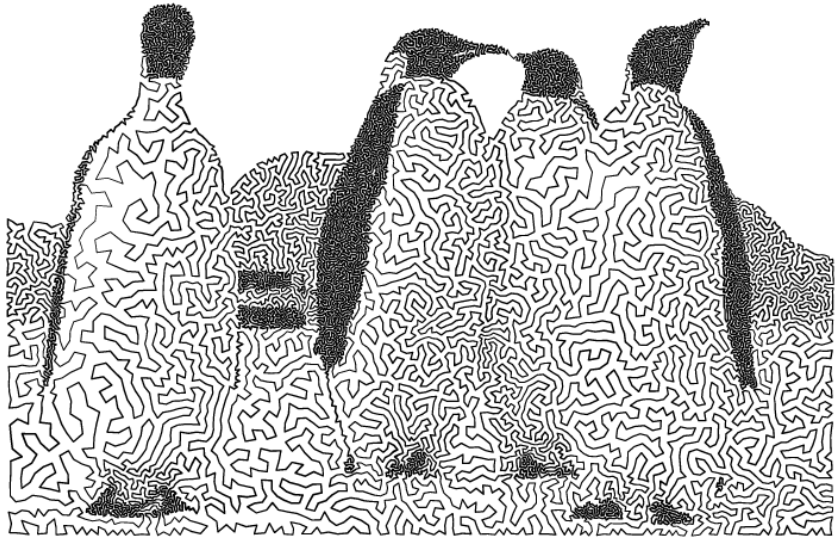
- Suppose after reduction, there is a TSP tour with cost ≤ 0. Let it be $\langle v_1, v_2, \ldots, v_n, v_1 \rangle$
- $\Rightarrow$ The tour contains only edges in E
- $\Rightarrow$ Thus, $\langle v_1, v_2, \ldots, v_n, v_1 \rangle$ is a Hamiltonian cycle ($x \in \mathrm{HAM{-}CYCLE}$).



u, y, v, w, z, u is a Hamiltonian cycle

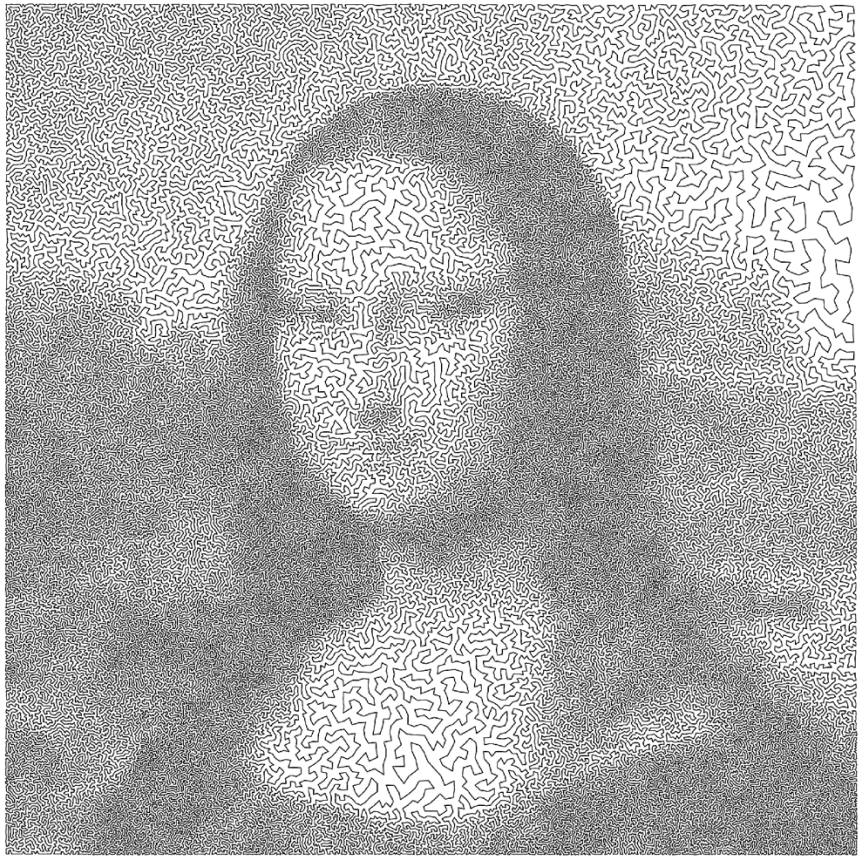
u, y, v, w, z, u is a traveling-salesman tour with cost 0

TSP arts and challenges



http://www.cgl.uwaterloo.ca/~csk/projects/tsp/

Mona Lisa TSP: $1,000 Prize for a 100,000-city challenge problem



http://www.math.uwaterloo.ca/tsp/

Coping with NP-Hard problems

- Since polynomial-time solutions are unlikely (unless P = NP), we must sacrifice either optimality, efficiency, or generality

- Approximation algorithms: guarantee to be a fixed percentage away from the optimum
- Local search: simulated annealing (hill climbing), genetic algorithms, etc.
- Heuristics: no formal guarantee of performance
- Randomized algorithms: make use of a randomizer (random number generator) for operation

Coping with NP-Hard problems

- Since polynomial-time solutions are unlikely (unless P = NP), we must sacrifice either optimality, efficiency, or generality

- Pseudo-polynomial time algorithms: e.g., dynamic programming for the 0-1 Knapsack problem
- Exponential algorithms/Intelligent exhaustive search: feasible only when the problem size is small
- Restriction: work on some special cases of the original problem. e.g., the maximum independent set problem in circle graphs