For single object:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} &= \boldsymbol{v}_i \\ \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} &= \sum_{j\neq i} \frac{Gm_j(\boldsymbol{p}_j-\boldsymbol{p}_i)}{|\boldsymbol{p}_j-\boldsymbol{p}_i|^3} \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{p}_{xi}}{\mathrm{d}t} &= \boldsymbol{v}_{xi} \\ \frac{\mathrm{d}\boldsymbol{v}_{xi}}{\mathrm{d}t} &= \sum_{j\neq i} \frac{Gm_j(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})^2+(\boldsymbol{p}_{yj}-\boldsymbol{p}_{yi})^2+(\boldsymbol{p}_{zj}-\boldsymbol{p}_{zi})^2}^3} \end{cases}$$

$$\frac{\mathrm{d}^2 \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}}{\mathrm{d}t^2} = \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p}' \\ \boldsymbol{v}' \end{bmatrix}}{\mathrm{d}t} = \sum_j \frac{\partial \begin{bmatrix} \boldsymbol{p}' \\ \boldsymbol{v}' \end{bmatrix}}{\partial \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j} \cdot \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j}{\mathrm{d}t}$$

$$\frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p}' \\ \boldsymbol{v}' \end{bmatrix}_i}{\mathrm{d}t} = \sum_j \frac{\partial \begin{bmatrix} \boldsymbol{p}' \\ \boldsymbol{v}' \end{bmatrix}_i}{\partial \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j} \cdot \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j}{\mathrm{d}t}$$

$$\begin{aligned} \frac{\mathrm{d}^2 \boldsymbol{p}_{xi}}{\mathrm{d}t^2} &= \sum_j \frac{\partial \boldsymbol{p}'_{xi}}{\partial \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j} \cdot \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j}{\mathrm{d}t} \\ &= \sum_j \frac{\partial \boldsymbol{v}_{xi}}{\partial \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j} \cdot \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_j}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\boldsymbol{v}_{xi}}{\mathrm{d}t} \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 \boldsymbol{v}_{xi}}{\mathrm{d}t^2} &= \sum_k \frac{\partial \boldsymbol{v}'_{xi}}{\partial \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_k} \cdot \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_k}{\mathrm{d}t} \\
&= \sum_k \frac{\partial \sum_{j \neq i} \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_k} \cdot \frac{\mathrm{d} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}_k}{\mathrm{d}t} \\
&= \sum_{k \neq i} \frac{\partial \sum_{j \neq i} \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{xk}} \cdot \frac{\mathrm{d}\boldsymbol{p}_{xk}}{\mathrm{d}t} \\
&\quad + \sum_{k \neq i} \frac{\partial \sum_{j \neq i} \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{y|z\,k}} \cdot \frac{\mathrm{d}\boldsymbol{p}_{y|z\,k}}{\mathrm{d}t} \\
&\quad + \frac{\partial \sum_{j \neq i} \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{xi}} \cdot \frac{\mathrm{d}\boldsymbol{p}_{xi}}{\mathrm{d}t} \\
&= \sum_{j \neq i} \frac{\partial \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{xj}} \cdot \frac{\mathrm{d}\boldsymbol{p}_{xj}}{\mathrm{d}t} \\
&\quad + \sum_{j \neq i} \frac{\partial \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{y|z\,j}} \cdot \frac{\mathrm{d}\boldsymbol{p}_{y|z\,j}}{\mathrm{d}t} \\
&\quad + \sum_{j \neq i} \frac{\partial \frac{G m_j (\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj} - \boldsymbol{p}_{xi})^2 + (\boldsymbol{p}_{yj} - \boldsymbol{p}_{yi})^2 + (\boldsymbol{p}_{zj} - \boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{xi}} \cdot \frac{\mathrm{d}\boldsymbol{p}_{xi}}{\mathrm{d}t}
\end{aligned}
$$

Let $C_x = \boldsymbol{p}_{xj} - \boldsymbol{p}_{xi}$, $C_y = \boldsymbol{p}_{yj} - \boldsymbol{p}_{yi}$ and $C_z = \boldsymbol{p}_{zj} - \boldsymbol{p}_{zi}$.

$$
\begin{aligned}
\frac{\partial \frac{Gm_j(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})^2+(\boldsymbol{p}_{yj}-\boldsymbol{p}_{yi})^2+(\boldsymbol{p}_{zj}-\boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{xj}} &= \frac{\partial \frac{Gm_jC_x}{\sqrt{C_x^2+C_y^2+C_z^2}^3}}{\partial \boldsymbol{p}_{xj}} \\
&= \frac{\partial \frac{Gm_jC_x}{\sqrt{C_x^2+C_y^2+C_z^2}^3}}{\partial C_x} \cdot \frac{\mathrm{d}C_x}{\mathrm{d}\boldsymbol{p}_{x_j}} \\
&= \frac{Gm_j}{\sqrt{C_x^2+C_y^2+C_z^2}^3} - Gm_jC_x \cdot \left(\frac{3}{2} \frac{2C_x}{\sqrt{C_x^2+C_y^2+C_z^2}^5} \right) \cdot 1 \\
&= Gm_j \left(\frac{(C_x^2+C_y^2+C_z^2) - 3C_x^2}{\sqrt{C_x^2+C_y^2+C_z^2}^5} \right) \\
&= Gm_j \left(\frac{C_y^2+C_z^2-2C_x^2}{\sqrt{C_x^2+C_y^2+C_z^2}^5} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \frac{Gm_j(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})^2+(\boldsymbol{p}_{yj}-\boldsymbol{p}_{yi})^2+(\boldsymbol{p}_{zj}-\boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{yj}} &= \frac{\partial \frac{Gm_jC_x}{\sqrt{C_x^2+C_y^2+C_z^2}^3}}{\partial \boldsymbol{p}_{C_y}} \cdot \frac{\mathrm{d}C_y}{\mathrm{d}\boldsymbol{p}_{yj}} \\
&= -\frac{3}{2} \frac{Gm_jC_x \cdot 2C_y}{\sqrt{C_x^2+C_y^2+C_z^2}^5} \cdot 1 \\
&= -\frac{3Gm_jC_xC_y}{\sqrt{C_x^2+C_y^2+C_z^2}^5}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \frac{Gm_j(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})}{\sqrt{(\boldsymbol{p}_{xj}-\boldsymbol{p}_{xi})^2+(\boldsymbol{p}_{yj}-\boldsymbol{p}_{yi})^2+(\boldsymbol{p}_{zj}-\boldsymbol{p}_{zi})^2}^3}}{\partial \boldsymbol{p}_{xi}} &= \frac{\partial \frac{Gm_jC_x}{\sqrt{C_x^2+C_y^2+C_z^2}^3}}{\partial C_x} \cdot \frac{\mathrm{d}C_x}{\mathrm{d}\boldsymbol{p}_{xi}} \\
&= Gm_j \left(\frac{C_y^2+C_z^2-2C_x^2}{\sqrt{C_x^2+C_y^2+C_z^2}^5} \right) \cdot -1 \\
&= -Gm_j \left(\frac{C_y^2+C_z^2-2C_x^2}{\sqrt{C_x^2+C_y^2+C_z^2}^5} \right)
\end{aligned}
$$

So the original equation becomes

$$
\begin{aligned}
\frac{\mathrm{d}^2 \boldsymbol{v}_{xi}}{\mathrm{d}t^2} = &\sum_{j\neq i} Gm_j \left(\frac{C_y^2 + C_z^2 - 2C_x^2}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \right) \cdot \frac{\mathrm{d}\boldsymbol{p}_{xj}}{\mathrm{d}t} \\
&+ \sum_{j\neq i} - \frac{3Gm_j C_x C_y}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \cdot \frac{\mathrm{d}\boldsymbol{p}_{y|z\,j}}{\mathrm{d}t} \\
&+ \sum_{j\neq i} -Gm_j \left(\frac{C_y^2 + C_z^2 - 2C_x^2}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \right) \cdot \frac{\mathrm{d}\boldsymbol{p}_{xi}}{\mathrm{d}t} \\
= &\sum_{j\neq i} \left(Gm_j \left(\frac{C_y^2 + C_z^2 - 2C_x^2}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \right) \cdot \frac{\mathrm{d}\boldsymbol{p}_{xj}}{\mathrm{d}t} \right. \\
&\quad - \frac{3Gm_j C_x C_y}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \cdot \frac{\mathrm{d}\boldsymbol{p}_{y|z\,j}}{\mathrm{d}t} \\
&\quad \left. -Gm_j \left(\frac{C_y^2 + C_z^2 - 2C_x^2}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \right) \cdot \frac{\mathrm{d}\boldsymbol{p}_{xi}}{\mathrm{d}t} \right) \\
= &\sum_{j\neq i} \frac{Gm_j}{\sqrt{C_x^2 + C_y^2 + C_z^2}^5} \left(\left(C_y^2 + C_z^2 - 2C_x^2\right) \frac{\mathrm{d}\boldsymbol{p}_{xj}}{\mathrm{d}t} - 3C_x C_y \frac{\mathrm{d}\boldsymbol{p}_{y|z\,j}}{\mathrm{d}t} - \left(C_y^2 + C_z^2 - 2C_x^2\right) \frac{\mathrm{d}\boldsymbol{p}_{xi}}{\mathrm{d}t} \right)
\end{aligned}
$$

Let $C = \begin{bmatrix} C_x & C_y & C_z \end{bmatrix}^T$ and $C_l = \sqrt{C_x^2 + C_y^2 + C_z^2}$

$$
\frac{\mathrm{d}^2 \boldsymbol{v}_i}{\mathrm{d}t^2} = \sum_{j\neq i} \frac{Gm_j}{C_l^5} \left(C_l^2 \frac{\mathrm{d}\boldsymbol{p}_j}{\mathrm{d}t} - 3(CC^T) \cdot \frac{\mathrm{d}\boldsymbol{p}_j}{\mathrm{d}t} - C_l^2 \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} + 3C \odot C \odot \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} \right)
$$

For single object:

$$\begin{cases} \frac{d\boldsymbol{x}_i}{dt} & = -\boldsymbol{v}_i \\ \frac{d\boldsymbol{v}_i}{dt} & = \sum_{j \neq i} \frac{Gm_j(\boldsymbol{x}_j - \boldsymbol{x}_i)}{|\boldsymbol{x}_j - \boldsymbol{x}_i|^3} \end{cases}$$

$$\frac{d\boldsymbol{s}}{dt} = f(\boldsymbol{s})$$

where

$$f(\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{v}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{x}_{n-1} \\ \boldsymbol{v}_{n-1} \end{bmatrix}) = \begin{bmatrix} -\boldsymbol{v}_0 \\ \sum_{i \neq 0} \frac{Gm_i(\boldsymbol{x}_i - \boldsymbol{x}_0)}{|\boldsymbol{x}_i - \boldsymbol{x}_0|^3} \\ -\boldsymbol{v}_1 \\ \sum_{i \neq 1} \frac{Gm_i(\boldsymbol{x}_i - \boldsymbol{x}_1)}{|\boldsymbol{x}_i - \boldsymbol{x}_1|^3} \\ \vdots \\ -\boldsymbol{v}_{n-1} \\ \sum_{i \neq n-1} \frac{Gm_i(\boldsymbol{x}_i - \boldsymbol{x}_{n-1})}{|\boldsymbol{x}_i - \boldsymbol{x}_{n-1}|^3} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{s}_1 \\ \sum_{i \neq 0} \frac{Gm_i(\boldsymbol{s}_{2i} - \boldsymbol{s}_0)}{|\boldsymbol{s}_{2i} - \boldsymbol{s}_0|^3} \\ -\boldsymbol{s}_3 \\ \sum_{i \neq 1} \frac{Gm_i(\boldsymbol{s}_{2i} - \boldsymbol{s}_2)}{|\boldsymbol{s}_{2i} - \boldsymbol{s}_2|^3} \\ \vdots \\ -\boldsymbol{s}_{2n-1} \\ \sum_{i \neq n-1} \frac{Gm_i(\boldsymbol{s}_{2i} - \boldsymbol{s}_{2n-2})}{|\boldsymbol{s}_{2i} - \boldsymbol{s}_{2n-2}|^3} \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{s} &= \int f(\boldsymbol{s}(t))dt \\ \Delta\boldsymbol{s} &= \int \sum_{i=0}^{\infty} \frac{(f \circ \boldsymbol{s})^{(i)}(t_0)}{i!}(t - t_0)^i dt \\ \Delta\boldsymbol{s} &= \sum_{i=0}^{\infty} \int \frac{(f \circ \boldsymbol{s})^{(i)}(t_0)}{i!}(t - t_0)^i dt \\ \Delta\boldsymbol{s} &= \sum_{i=0}^{\infty} \frac{(f \circ \boldsymbol{s})^{(i)}(t_0)}{(i+1)!}\Delta t^{(i+1)} \\ \Delta\boldsymbol{s} &\approx f(\boldsymbol{s}(t_0))\Delta t + \frac{\boldsymbol{s}'(t_0)f'(\boldsymbol{s}(t_0))}{2}\Delta t^2 \end{aligned}$$

$$\frac{df}{dt} = \sum_{i=0}^{2n-1} \frac{\partial f}{\partial \boldsymbol{s}_i}\frac{d\boldsymbol{s}_i}{dt}$$

For $j \in [n]$,

$$\frac{\partial f}{\partial \boldsymbol{s}_{2j}} = \begin{bmatrix} \frac{\partial -\boldsymbol{s}_1}{\partial \boldsymbol{s}_{2j}} \\ \sum_{i \neq 0} \frac{\partial \frac{Gm_i(\boldsymbol{s}_{2i}-\boldsymbol{s}_0)}{|\boldsymbol{s}_{2i}-\boldsymbol{s}_0|^3}}{\partial \boldsymbol{s}_{2j}} \\ \frac{\partial -\boldsymbol{s}_3}{\partial \boldsymbol{s}_{2j}} \\ \sum_{i \neq 1} \frac{\partial \frac{Gm_i(\boldsymbol{s}_{2i}-\boldsymbol{s}_2)}{|\boldsymbol{s}_{2i}-\boldsymbol{s}_2|^3}}{\partial \boldsymbol{s}_{2j}} \\ \vdots \\ \frac{\partial -\boldsymbol{s}_{2j+1}}{\partial \boldsymbol{s}_{2j}} \\ \sum_{i \neq j} \frac{\partial \frac{Gm_i(\boldsymbol{s}_{2i}-\boldsymbol{s}_{2j})}{|\boldsymbol{s}_{2i}-\boldsymbol{s}_{2j}|^3}}{\partial \boldsymbol{s}_{2j}} \\ \vdots \\ \frac{\partial -\boldsymbol{s}_{2n-1}}{\partial \boldsymbol{s}_{2j}} \\ \sum_{i \neq n-1} \frac{\partial \frac{Gm_i(\boldsymbol{s}_{2i}-\boldsymbol{s}_{2n-2})}{|\boldsymbol{s}_{2i}-\boldsymbol{s}_{2n-2}|^3}}{\partial \boldsymbol{s}_{2j}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial \frac{Gm_j(\boldsymbol{s}_{2j}-\boldsymbol{s}_0)}{|\boldsymbol{s}_{2j}-\boldsymbol{s}_0|^3}}{\partial \boldsymbol{s}_{2j}} \\ 0 \\ \frac{\partial \frac{Gm_j(\boldsymbol{s}_{2j}-\boldsymbol{s}_2)}{|\boldsymbol{s}_{2j}-\boldsymbol{s}_2|^3}}{\partial \boldsymbol{s}_{2j}} \\ \vdots \\ 0 \\ \sum_{i \neq j} \frac{\partial \frac{Gm_i(\boldsymbol{s}_{2i}-\boldsymbol{s}_{2j})}{|\boldsymbol{s}_{2i}-\boldsymbol{s}_{2j}|^3}}{\partial \boldsymbol{s}_{2j}} \\ \vdots \\ 0 \\ \frac{\partial \frac{Gm_j(\boldsymbol{s}_{2j}-\boldsymbol{s}_{2n-2})}{|\boldsymbol{s}_{2j}-\boldsymbol{s}_{2n-2}|^3}}{\partial \boldsymbol{s}_{2j}} \end{bmatrix}$$

$$\frac{df}{dt}(\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{v}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{x}_n \\ \boldsymbol{v}_n \end{bmatrix}) = \begin{bmatrix} -\boldsymbol{v}_0' \\ \sum_{i \neq 0} \frac{Gm_i(\boldsymbol{x}_i - \boldsymbol{x}_0)}{|\boldsymbol{x}_i - \boldsymbol{x}_0|^3} \\ -\boldsymbol{v}_1' \\ \sum_{i \neq 1} \frac{Gm_i(\boldsymbol{x}_i - \boldsymbol{x}_1)'}{|\boldsymbol{x}_i - \boldsymbol{x}_1|^3} \\ \vdots \\ -\boldsymbol{v}_n' \\ \sum_{i \neq n} \frac{Gm_i(\boldsymbol{x}_i - \boldsymbol{x}_n)'}{|\boldsymbol{x}_i - \boldsymbol{x}_n|^3} \end{bmatrix}$$