On the relation between the number of lines and their intersections in non-trivial arrangements

Mathematics Analysis and Approaches Higher Level Internal Assessment

Exam Session of 2025

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1 Introduction

Last year, in January, I participated in the first round of Netherland Mathematics Olympiad. While the result is satisfactory, there was a problem that no one but me got wrong. So I would really like to prove that I'm better than that, by making the problem harder, and then solve it. The problem is:

> Xander draws five points and a number of infinitely long lines on an infinite sheet of paper. He does this in such a way that on each line there are at least two of those points and that the lines intersect only at points that Xander has drawn.
>
> What is the maximum number of lines Xander could have drawn?[1]

The answer for this question is 6, as shown in Figure 1. The organization also gave a proof.[2]. But to prove my ability, I need to make it harder. The question asks for the maximal number of lines given 5 points. But I will try to answer the question for any number of points.

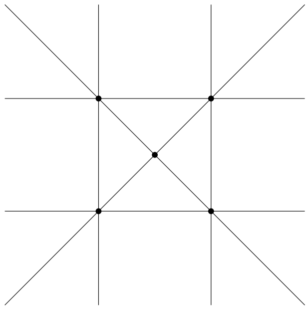

Figure 1: answer to the original problem

In this essay, I will use n to denote the number of points, m_n to denote the maximum number of lines one can draw with n points, while following the requirements from the question. I will use "point(s)" and "dot(s)" interchangeably, and "graph" will refer to a way to draw the lines and dots.

[1]https://wiskundeolympiade.nl/phocadownload/opgaven/1e-ronde/2024/Problems.pdf

[2]Which can be found at https://wiskundeolympiade.nl/phocadownload/opgaven/1e-ronde/2024/Solutions.pdf Later in this essay, I will provide a different proof.

2 Solving the problem

One method I found efficient in solving those type of problems is trying to come up with some hypothesis to prove. If the hypothesis is true, you are just doing what needs to be done. If it is false, it is much easier to find an exception to disprove the hypothesis after trying to prove it and struggling on the weak point of the logic. But to do that, one first needs to have an idea what the answer will be like. So I tried to solve the problem for a few given n. Soon, I managed to find a solution that seems to work for all $n \geq 3$, as shown in Figure 2.

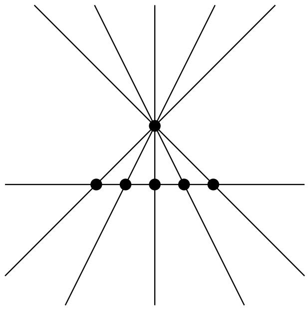

Figure 2: general solution with $n = 6$

By arranging $n - 1$ dots in a line and the last one above it, we can draw $n - 1$ lines between the last dot and the dots on the line. Together with the line by the $n - 1$ dots, we can get n lines in total. Hence, we can establish for $n \geq 3$, that we can draw at least n lines.

And now, I hypothesize that this is also the upper bound. As shown above, this does not apply to $n = 5$, where it is possible to draw 6 lines. But I can try to prove it is true for $n > 5$.

After some efforts, I managed to produce a proof for that. Since my process is a bit messy, I have rearranged the order in which lemmas are presented. For brevity, some less important and obvious conclusions will be explained in the footnote.

3 Proof

First, we shall reiterate what has been shown in section 2.

Lemma 1. *For $n \geq 3$, $m_n \geq n$.*

The proof is omitted, as it should be trivial to turn the previous example into an informal proof.

Next, we will go on to prove an interesting, but seemly unrelated statement.

Theorem 2. *Given $m > 0$ lines lying on a plane, if every line intersects at more than one point, then there exists an intersection of only two lines.*

Proof. Given $m > 0$ lines on a plane. We will choose one line as the base line(l_0) and construct a coordinate system with y axis parallel to l_0. Suppose there is no intersection of less than two lines.

Consider the intersection that is closest to l_0 but does not lie on it (if there are multiple, choose the one with the largest y value). We will name this point A. We can, without loss of generality, assume that point A is to the right of l_0. According to our assumption, at least three lines pass through point A.

Such a point must exist. To show so, suppose no such point exists, that is, every intersection lies on l_0. Since every line must intersect at no less than two places, there must be a line which intersect with l_0. However, since two lines can only intersect once, and every intersection in on l_0, the new line would intersect only once, creating a contradiction. Hence, the assumption must be false and such a point exists.

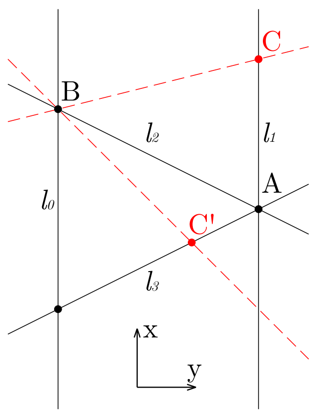

Figure 3: Contradicting points as a result of the assumption

If one of the lines that pass through point A is parallel to l_0, which we will refer to as l_1. Then since we chose A to be the point with the largest y value when distance to l_0 is the same, l_1 does not have any intersection above point A. At least two other lines pass through point A. Among those lines, we will denote the line with the lowest gradient (most negative) as line l_2, the line with the second lowest gradient as l_3, and the intersection of l_0 and l_2 as point B. There must be a third line that passes through point B. However, if the third line has a larger gradient, it would intersect with line l_1 at point C, above point A. And if the third line has a lowest gradient than line l_2, it would intersect with l_3 at point C', which is closer to line l_0 than point A, creating a contradiction.

Otherwise, all lines that pass through point A is not parallel to line l_0 and must intersect with it. We will call the line with the lowest gradient l_1, the line with the second lowest gradient l_2 and the line with the largest gradient l_3. l_2 intersect with l_0 at point B. There must be another line that passes through point B. However, such a line must intersect with l_1 or l_2 in the segment between l_0 and point A, creating an intersection closer to line l_0, as shown in Figure 4. This contradicts the assumption that point A is the closest point to l_0.

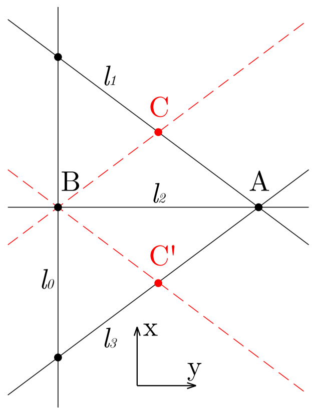

Figure 4: Contradicting points as a result of the assumption

In both cases, we end up with a paradox. Thus, the assumption must be false and an intersection of less than three lines must exist. □

Now, we propose that m does not increase faster than n, or formally,

Lemma 3. *For $n \geq 5$, $m_n \leq m_{n-1} + 1$.*

Proof. According to Theorem 2, given a graph with $n \geq 1$ and $m_n \geq 1$,[3] there exists a dot that is on no more than two lines. We will show that in every cases, it is possible to remove the dot and construct a new graph with more than $m_n - 1$ lines:

Case 1, the dot is on 0 lines. In this case, we can simply remove the dot. The new

[3]Note that since $n \geq 5$, $m_n \geq 6$ and thus the condition will always be met

graph will have $n-1$ dots and m_n lines. Therefore, $m_{n-1} \geq m_n$, or $m_n \leq m_{n-1}$.

Case 2, the dot is on 1 line. In this case, we can remove both the dot and the line. In this case, $m_{n-1} \geq m_n - 1$, or $m_n \leq m_{n-1} + 1$.

Case 3, the dot is on 2 lines, and one of the lines has more than 2 dots lying on it. In this case we can remove the dot and the other line. In this case, $m_{n-1} \geq m_n - 1$, or, $m_n \leq m_{n-1} + 1$.[4]

Case 4, the dot is on 2 lines, and both lines have only 2 dots on them. In this case, there can be only up to 5 lines, as shown in Figure 5. To prove this is so, consider line l_0. Any line not parallel to it must intersect with it at point B. The same reasoning applies to line l_1 and point C. Therefore, only a maximum of three lines could be added. Thus, in this case, $m_n \leq 5$.

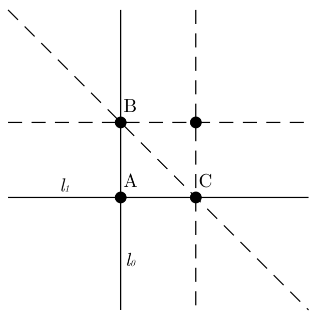

Figure 5: 3 potential lines when both line have only 2 dots

Thus, $m_n \leq \max(m_{n-1}+1, 5)$. For $m_{n-1} \geq 5$, $m_n \leq m_{n-1} + 1$. According to Lemma 1, for $n \geq 5 \geq 3$, $m_{n-1} + 1 \geq (n-1) + 1 \geq 5$, so $m_n \leq m_{n-1} + 1$. □

For $n = 3$, it is trivial to see that $m_n \leq 3$.[5] From the above proof, $m_4 \leq \max(m_3+1, 5) = 5$ and $m_5 \leq m_4 + 1 \leq 5 + 1 = 6$. To show those upper bound are also the lower bound, there are examples in Appendix A.

For $n = 6$, it can be proven that $m_n = 6$. However, since the proof is long and tedious, it can be found in appendix B.

Since for $n = 6$, $m_n = 6 = n$, by induction, for any $n \geq 6$, $n \leq m_n \leq n$. Thus, $m_n = n$.

[4] The reason this proof can not be used for situation in Figure 5 is that it would leave the line with only one dot on it.

[5] Since each line must pass through at least two points, there can be a maximum of $\frac{3\times(3-1)}{2} = 3$ lines. And an example of $n = 3$, $m = 3$ could easily be drawn.

Hence, the answer to our question is:

$$m_n = \begin{cases} 0, & \text{if } n = 1 \\ 1, & \text{if } n = 2 \\ 3, & \text{if } n = 3 \\ 5, & \text{if } n = 4 \\ 6, & \text{if } n = 5 \\ n, & \text{otherwise} \end{cases} \tag{3.1}$$

4 conclusion

There you have it. We have not only found the answer to the problem, but also answer for any value of n if they change it. Although it seems that the only value that is of interest is $n = 4$ or $n = 5$. The proof for that was not easy to come up with, though. Many times I thought I have gotten a good proof, but then found that I didn't consider how multiple lines could intersect at the same point. This caused me to realize that I need to put an upper bound on the number of lines that could intersect, leading to the proof of Theorem 2. Regarding Theorem 2, while I feel that the proof is a bit messy, I'm really proud of the theorem itself, which is very elegant and interesting. Many proof here rely heavily on images and exhaustive search through possibility. For example, I had to move the proof for $m_6 = 6$ to an appendix because the proof would be too long and disrupt the chain of deduction. A better proof, while not necessary, would be preferred. Overall, I believe this proof is rather rigorous, and the journey to get here was quite enjoyable.

Appendices

A Examples

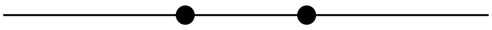

Figure 6: $n = 2$, $m_n = 1$

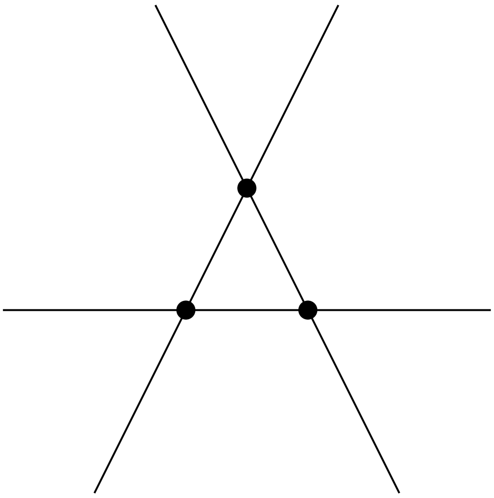

Figure 7: $n = 3$, $m_n = 3$

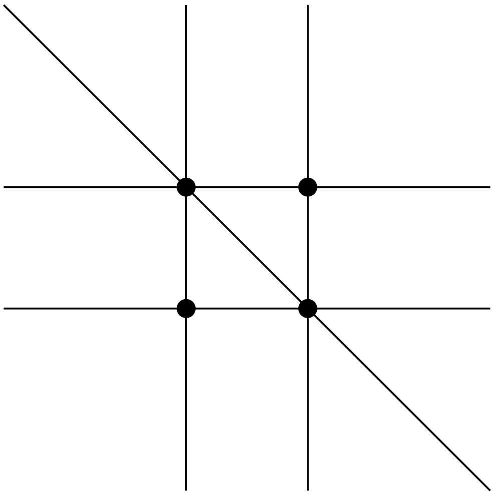

Figure 8: $n = 4$, $m_n = 5$

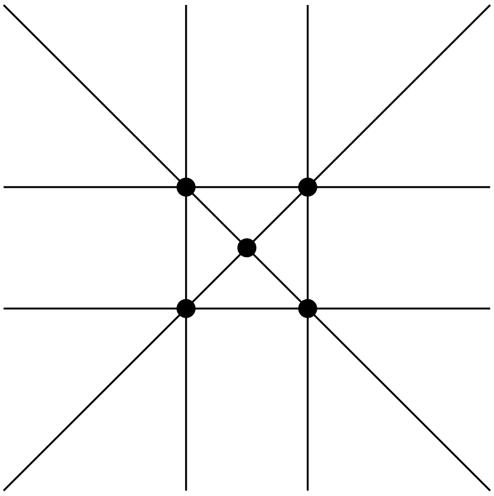

Figure 9: $n = 5$, $m_n = 6$

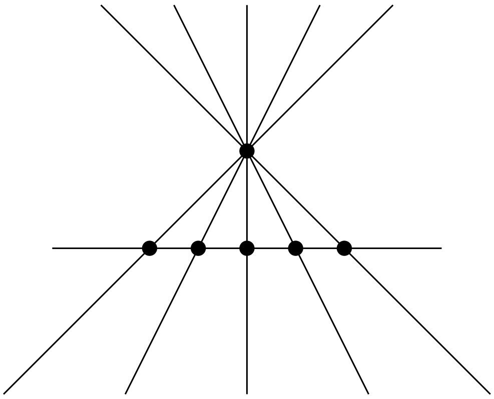

Figure 10: $n = 6$, $m_n = 6$

B Proof for n=6

We have shown $m_3 = 3$, $m_4 = 5$ and $m_5 = 6$. Here, I will first show that the solution for $n = 4$ and $n = 5$ are unique.

Consider when $n = 4$. In the proof for lemma 3, we considered four cases. In case 1, $m_n \leq m_{n-1}$ and in case 2 and 3, $m_n \leq m_{n-1} + 1$. Given $m_3 = 3$ and $m_4 = 5$, none of those cases could hold true. It follows that case 4 must hold true here, and the only graph that allows for $m_4 = 5$ is Figure 8.[6]

Now consider the process of turning from $n = 4$ to $n = 5$. Since $m_5 = 6 \geq m_4 = 5$, case 1 and case 4 would fail. Hence, we must be able to turn Figure 8 to Figure 9 through reversing the methods listed in case 2 or 3, that is, by adding a line and a dot.

Since we will only add one dot, and the new line must go through at least two dots, the new line must pass through one of the pre-existing points. Since the graph is highly symmetric, point D is equivalent to point A, as is point B to C, so we only need to consider one point from each pair, and we only need to consider line with gradient in the range of $[-1, 1]$.

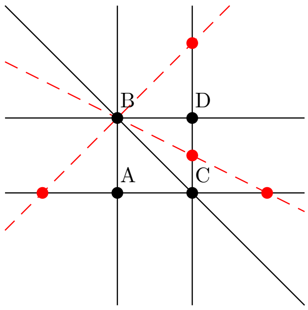

Figure 11: If the extra line passes through point B.

If the new line passes through point B, as shown in Figure 11, at least two new dots would be formed. Hence both case 2 and case 3 would fail.

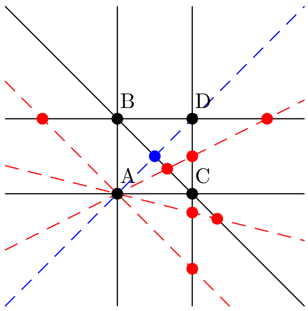

Figure 12: If the extra line passes through point A.

If the new line passes through point A, as shown in Figure 12, at least two new dots would be formed for all line except the blue line. Thus, the only way to construct a graph with $n = 5$ and $m = 6$ is Figure 9.

[6]Here, we will consider two graph that differ by a affine transformation as the same graph.

Then, we consider going from $n = 5$ to $n = 6$. Suppose that $m_6 \geq 7 = m_5 + 1$, then case 2 or 3 must hold in lemma 3. Like before, this means the extra line must pass through at least one pre-existing point. There are only 2 ways of achieving so.

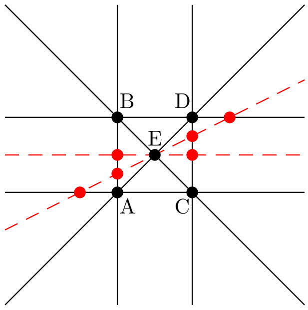

Figure 13: If the extra line passes through point E.

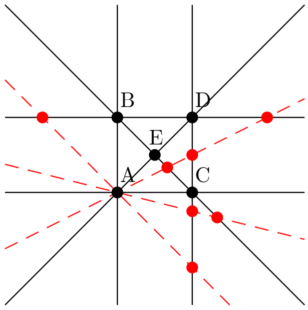

Figure 14: If the extra line passes through point A.

If the new line pass through the center point, denoted by point E as shown in Figure 13, then at least two dots would form, failing the requirement for case 2 and 3. As seen in Figure 14, the same thing happens when the line pass through point A (and since the graph is symmetric, point B, C and D as well). Thus, the assumption must be false. Therefore, $m_6 < 7$. Since $m_6 \geq 6$ (lemma 1), $m_6 = 6$.