Combinatorics HW 3.1 Albert Millan Trujols 2019280366

1) Let $|A_1|$ = # squares of integers $\Rightarrow$ $|A_1|^2 \leq 10000$ $\therefore$ $|A_1| = \lfloor 100 \rfloor = 100$

$|A_2|$ = # cubes of integers $\Rightarrow$ $|A_2|^3 = 10000$ $\therefore$ $|A_2| = \lfloor 21.54 \rfloor = 21$

For a number to be both the square and cube of integers, the number must be the sixth power of an integer:

$$N = A^6 = (A \cdot A)^3 = (A \cdot A \cdot A)^2$$

$\therefore$ $|A_1 \cap A_2|$ = # both squares & cubes of integers $\Rightarrow$ $|A_1 \cap A_2|^6 = 10000$ $\therefore$ $|A_1 \cap A_2| = \lfloor 4.64 \rfloor = 4$

$$|\overline{A_1} \cap \overline{A_2}| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$
$$= 10000 - 100 - 21 + 4 = \boxed{9883}$$

2) The question can be converted to '# of permutations of 1, 2, 3, ..., 9 where no odd number is in its natural position'. In this case, the answer would be the amount substracted to the universal set. Consider:

$$\underset{P_1}{\underline{1}} \ \underline{2} \ \underset{P_2}{\underline{3}} \ \underline{4} \ \underset{P_3}{\underline{5}} \ \underline{6} \ \underset{P_4}{\underline{7}} \ \underline{8} \ \underset{P_5}{\underline{9}}$$, then...

$\therefore |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$

$|A_1|$ = # perm where $P_1 = 1$
$|A_2|$ = # perm where $P_2 = 3$
$|A_3|$ = # perm where $P_3 = 5$
$|A_4|$ = # perm where $P_4 = 7$
$|A_5|$ = # perm where $P_5 = 9$

$$= |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \sum |A_i \cap A_j \cap A_k \cap A_l| - |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$$
$$= 9! - 5 \cdot (8!) + 10 \cdot (7!) - 10 \cdot (6!) + 5 \cdot (5!) - (4!)$$
$$= 362880 - 157824$$
$$= 205056$$

$\boxed{\therefore \text{Answer} = 157824}$

Q3) $x_1+x_2+x_3+x_4=20$

$1 \le x_1 \le 6,\ 0 \le x_2 \le 7$

$4 \le x_3 \le 8,\ 2 \le x_4 \le 6$

$y_1+y_2+y_3+y_4 = 20-1-4-2$

$y_1+y_2+y_3+y_4 = 13$

$0 \le y_1 \le 5,\ 0 \le y_2 \le 7$

$0 \le y_3 \le 4,\ 0 \le y_4 \le 4$

$|S| = \binom{13+4-1}{13} = \binom{16}{13} = \binom{16}{3}$

$z_1+6+y_2+y_3+y_4=13$

$z_1+y_2+y_3+y_4=7$

$\therefore |A_1| = \binom{7+4-1}{7} = \binom{10}{7} = \binom{10}{3}$

Similarly

$|A_2| = \binom{5+4-1}{5} = \binom{8}{5} = \binom{8}{3}$

$|A_3| = \binom{11}{8} = \binom{11}{3}$

$|A_4| = \binom{11}{8} = \binom{11}{3}$

$z_1+6+z_2+8+y_3+y_4=13$

$z_1+z_2+y_3+y_4=-1$

$\therefore |A_1 \cap A_2| = \binom{-1+4-1}{-1} = \binom{2}{-1} = 0$

Similarly

$|A_1 \cap A_3| = \binom{2+4-1}{2} = \binom{5}{2} = \binom{5}{3}$

$|A_1 \cap A_4| = \binom{5}{3}$

$|A_2 \cap A_3| = \binom{0+4-1}{0} = \binom{3}{0} = \binom{3}{3} = 1$

$|A_2 \cap A_4| = \binom{3}{0} = 1$

$|A_3 \cap A_4| = \binom{3+4-1}{3} = \binom{6}{3}$

$z_1+6+z_2+8+z_3+5+y_4=13$

$z_1+z_2+z_3+y_4=-6$

$\therefore |A_1 \cap A_2 \cap A_3| = \binom{-6+4-1}{-6} = 0$

Remaining cases add up to a number > 13 hence values equal zero. Hence:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = \binom{16}{3} - \left[\binom{10}{3}+\binom{8}{3}+\binom{11}{3}+\binom{11}{3}\right] + \left[0+\binom{5}{3}+\binom{5}{3}+1+1+\binom{6}{3}\right]$$

$$= 560-(120+56+165\cdot 2)+(10+10+1+1+20)$$

$$= \boxed{96}$$

Q4) $|A_1| = (P_1=2) = 3!$ $\quad |A_3| = (P_3=\{3,4\}) = 2\cdot 3!$

$|A_2| = (P_2=\{2,3\}) = 2\cdot 3!$ $\quad |A_4| = (P_4=4) = 3!$

This poses constraints that need to be considered. Ex. if $P_1=2$, then $P_3=3$ only. Need to adjust for those cases:

$|A_1 \cap A_2| = 1\cdot 2!$ $\quad |A_2 \cap A_3| = 3\cdot 2!$

$|A_1 \cap A_3| = 2\cdot 2!$ $\quad |A_2 \cap A_4| = 2\cdot 2!$

$|A_1 \cap A_4| = 2!$ $\quad |A_3 \cap A_4| = 1\cdot 2!$

$|A_1 \cap A_2 \cap A_3| = 1$ $\quad |A_1 \cap A_3 \cap A_4| = 1$

$|A_1 \cap A_2 \cap A_4| = 1$ $\quad |A_2 \cap A_3 \cap A_4| = 1$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

$$\therefore |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 24 - [2\cdot(3!) + (4\cdot 3!)] + [2+4+2+6+4+2] - [1+1+1+1]$$

$$= \boxed{4}$$