1 Clustering: Mixture of Multinomials

1.1 MLE for multinomial

Given:

$$P(x|\mu) = \frac{n!}{\prod_i x_i!} \prod_i \mu_i^{x_i}, \quad i = 1, \ldots d$$
$$s.t. \quad \sum_i \mu_i - 1 = 0$$

We derive the following log-likelihood:

$$\begin{aligned} L &= \log P(x|\mu) \\ &= \log(n!) - \sum_i log(x_i!) + \sum x_i * log(\mu_i) \end{aligned}$$

We introduce the langrangian to handle the constraint:

$$L(\mu, \lambda) = \log(n!) - \sum_i log(x_i!) + \sum x_i * log(\mu_i) + \lambda(1 - \sum \mu_i)$$

Next we compute the gradient of $L(\mu, \lambda)$ and set it equal to zero.

$$\begin{cases} \frac{\partial L(\mu,\lambda)}{\partial \mu} = 0 & \implies \sum(\frac{x_i}{\mu_i}) - \lambda = 0 \implies \mu = \frac{x}{\lambda} \\ \frac{\partial L(\mu,\lambda)}{\partial \lambda} = 0 & \implies 1 - \sum \mu_i = 0 \implies \mu = 1 \end{cases}$$

Substituting the result from the second equation into the first we obtain:

$$\frac{x}{\lambda} = 1 \implies \lambda = x = \sum_i x_i = n$$

Consequently,

$$\hat{\mu} = \frac{x}{n}$$

1.2 EM Mixture of Multinomials

$$
\begin{aligned}
P(d) &= \frac{n_d!}{\prod_w T_{dw}!}\sum_{k=1}^{K}\pi_k\prod_w \mu_{wk}^{T_{dw}} \\
\ell(\pi,\mu) &= \sum \ln p(d) \\
&= \sum \Big[\frac{n_d!}{\prod_w T_{dw}!} + \ln\big(\sum_{k=1}^{K}\pi_k\prod_w \mu_{wk}^{T_{dw}}\big)\Big]
\end{aligned}
$$

Introducing the Lagrangian:

$$\mathcal{L}(\pi,\mu,\lambda) = \ell(\pi,\mu) - \lambda_1\big(\sum_k^K \pi_k - 1\big) - \lambda_2\big(\sum_w \mu_{wk} - 1\big)$$

Hence,

$$\frac{\partial \mathcal{L}}{\partial \mu_{wk}} = \sum_d \frac{\pi_k\prod_w \mu_{wk}^{T_{dw}}}{\sum_{k=1}^{K}\pi_k\prod_w \mu_{wk}^{T_{dw}}} * \frac{T_{dw}}{\mu_{wk}} - \lambda_2$$

Substituting by γ_{dk}, hereby referred as responsibility:

$$\gamma_{dk} = \frac{\pi_k\prod_w \mu_{wk}^{T_{dw}}}{\sum_{k=1}^{K}\pi_k\prod_w \mu_{wk}^{T_{dw}}}$$

Then, setting the gradients equal zero:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mu_{wk}} &= \sum_d \gamma_{dk} * \frac{T_{dw}}{\mu_{wk}} - \lambda_2 = 0 \\
\sum_d \gamma_{dk} * T_{dw} &= \lambda_2\mu_{wk} \\
\sum_w\sum_d \gamma_{dk} * T_{dw} &= \lambda_2\mu_{wk} \\
\lambda_2 &= \sum_w\sum_d \gamma_{dw} * T_{dk} \\
\therefore\ \mu_{wk} &= \frac{\sum_d \gamma_{dk} * T_{dw}}{\sum_w\sum_d \gamma_{dw} * T_{dk}}
\end{aligned}
$$

And also:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \pi_k} = \sum_d \frac{\pi_k\prod_w \mu_{wk}^{T_{dw}}}{\sum_{k=1}^{K}\pi_k\prod_w \mu_{wk}^{T_{dw}}} * \frac{1}{\pi_k} - \lambda_1 &= 0 \\
\sum_d \gamma_{dk} &= \lambda_1\pi_k \\
\sum_k\sum_d \gamma_{dk} &= \lambda_1\sum_k \pi_k \\
\lambda_1 &= \sum_k\sum_d \gamma_{dk} \\
\therefore\ \pi_k = \frac{\sum_d \gamma_{dk}}{\sum_k\sum_d \gamma_{dk}} &= \frac{\sum_d \gamma_{dk}}{D}
\end{aligned}
$$

Summarizing, we have:

- **E-step:** For each document d, update the responsibility:

$$\gamma_{dk} = \frac{\pi_k \prod_w \mu_{wk}^{T_{dw}}}{\sum_{k=1}^{K} \pi_k \prod_w \mu_{wk}^{T_{dw}}}$$

  This corresponds to the posterior probability of topic $c_d = k$ for generating the document.

- **M-step:** For each $c_d = k$, update parameter estimates:

$$\pi_k = \frac{\sum_d \gamma_{dk}}{D}$$
$$\mu_{wk} = \frac{\sum_d \gamma_{dk} * T_{dw}}{\sum_w \sum_d \gamma_{dw} * T_{dk}}$$

2 PCA

$$
\begin{aligned}
J &= \frac{1}{N}\sum_{n=1}^{N} ||x_n - \hat{x_n}||^2 \\
&= \frac{1}{N}\sum_{n=1}^{N} ||x_n - \sum_{i=1}^{d} z_{ni}\mu_i + \sum_{i=d+1}^{p} b_i\mu_i||^2
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial J}{\partial z_{ni}} &= \frac{1}{N}\sum_{n=1}^{N} \frac{\partial}{\partial z_{ni}} ||x_n - \sum_{i=1}^{d} z_{ni}\mu_i - \sum_{i=d+1}^{p} b_i\mu_i||^2 \\
0 &= \frac{2}{N}\sum_{n=1}^{N} (x_n - \sum_{i=1}^{d} z_{ni}\mu_i - \sum_{i=d+1}^{p} b_i\mu_i)(-\mu_i) \\
0 &= \sum_{n=1}^{N} (x_n^T\mu_i - z_{ni})
\end{aligned}
$$

As z_{ni} is unique for each data point n, we can simply look at each term in the summation. Therefore,

$$
z_{ni} = x_n^T\mu_i, \quad \forall i = 1, \ldots, d \tag{1}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial J}{\partial b_i} &= \frac{1}{N}\sum_{n=1}^{N} \frac{\partial}{\partial b_i} ||x_n - \sum_{i=1}^{d} z_{ni}\mu_i - \sum_{i=d+1}^{p} b_i\mu_i||^2 \\
0 &= \frac{2}{N}\sum_{n=1}^{N} (x_n - \sum_{i=1}^{d} z_{ni}\mu_i - \sum_{i=d+1}^{p} b_i\mu_i)(-\mu_i) \\
0 &= \sum_{n=1}^{N} (x_n^T\mu_i - b_i) \\
\sum_{n=1}^{N} b_i &= \sum_{n=1}^{N} x_n^T\mu_i \\
\therefore b_i &= \bar{x}_n^T\mu_i, \quad \forall i = d+1, \ldots, p
\end{aligned}
$$

3 Reinforcement Learning

The 5 remaining equations are the following:

$$
\begin{aligned}
0 + 0.9 * [0.95 * (-1) + 0.05 * (35) + 0 * (50)] = 0.72 \\
15 + 0.9 * [0.1 * (-1) + 0.2 * (35) + 0.7 * (50)] = 52.71 \\
20 + 0.9 * [0.2 * (-1) + 0.75 * (35) + 0.05 * (50)] = 45.695 \\
80 + 0.9 * [0.05 * (-1) + 0.15 * (35) + 0.8 * (50)] = 137.71 \\
100 + 0.9 * [0.1 * (-1) + 0.2 * (35) + 0.7 * (50)] = 120.68
\end{aligned}
$$

4 CVAE

The algorithm is derived as per [1], with an implementation provided in the python file 'CVAE.py'. The output generations of the model are in the file results, showing results for the first 100 iterations.

References

[1] https://zhusuan.readthedocs.io/en/latest/tutorials/vae.html