From Matrix-Vector Multiplication to Matrix-Matrix Multiplication

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Opening Remarks

- Predicting the Weather
  - The following table tells us how the weather for any day (e.g., today) predicts the weather for the next day (e.g., tomorrow):

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

Opening Remarks

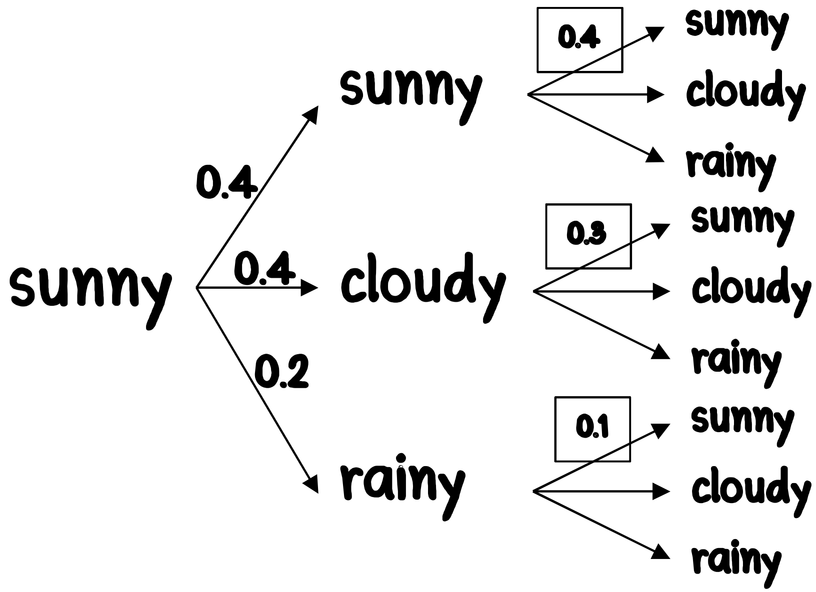
		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- Predicting the Weather
  - If today is cloudy, what is the probability that tomorrow is
    - sunny?
    - cloudy?
    - rainy?

Opening Remarks

		Tomorrow		
		sunny	cloudy	rainy
The day after tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- Predicting the Weather
  - If today is sunny,
    what is the probability that the day after tomorrow is
    - sunny?
    - cloudy?
    - rainy?



The probability

sunny->sunny->sunny = 0.4×0.4

sunny->cloudy->sunny = 0.4×0.3

sunny->rainy->sunny = 0.2×0.1

The probability that the day after tomorrow is : $0.4\times0.4 + 0.4\times0.3 + 0.2\times0.1$

: 0.3

Opening Remarks

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

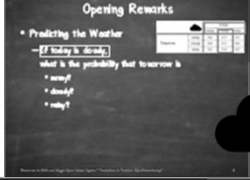
- Predicting the Weather
  - When things get messy, it helps to introduce some notation.
    - Let $\chi_s^{(k)}$ denote the probability that it will be sunny k days from now (on day k).
    - Let $\chi_c^{(k)}$ denote the probability that it will be cloudy k days from now.
    - Let $\chi_r^{(k)}$ denote the probability that it will be rainy k days from now.

Opening Remarks

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- Predicting the Weather
  - What is the probability that tomorrow is
    - sunny?
    - cloudy?
    - rainy?
  - The discussion so far motivate the equations (k : today $k+1$: tomorrow)
    - $\chi_s^{(k+1)} = 0.4 \times \chi_s^{(k)} + 0.3 \times \chi_c^{(k)} + 0.1 \times \chi_r^{(k)}$
    - $\chi_c^{(k+1)} = 0.4 \times \chi_s^{(k)} + 0.3 \times \chi_c^{(k)} + 0.6 \times \chi_r^{(k)}$
    - $\chi_r^{(k+1)} = 0.2 \times \chi_s^{(k)} + 0.4 \times \chi_c^{(k)} + 0.3 \times \chi_r^{(k)}$

Opening Remarks



		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- Predicting the Weather
  - If today is cloudy, what is the probability that tomorrow is sunny? cloudy? rainy?
    - today is cloudy : $\chi_s^{(0)} = 0$, $\chi_c^{(0)} = 1$ and $\chi_r^{(0)} = 0$
    - $$\begin{aligned}\chi_s^{(1)} &= 0.4 \times \chi_s^{(0)} + 0.3 \times \chi_c^{(0)} + 0.1 \times \chi_r^{(0)} \\ &= 0.4 \times 0 + 0.3 \times 1 + 0.1 \times 0 = 0.3\end{aligned}$$
    - $$\begin{aligned}\chi_c^{(1)} &= 0.4 \times \chi_s^{(0)} + 0.3 \times \chi_c^{(0)} + 0.6 \times \chi_r^{(0)} \\ &= 0.4 \times 0 + 0.3 \times 1 + 0.6 \times 0 = 0.3\end{aligned}$$
    - $$\begin{aligned}\chi_r^{(1)} &= 0.2 \times \chi_s^{(0)} + 0.4 \times \chi_c^{(0)} + 0.3 \times \chi_r^{(0)} \\ &= 0.2 \times 0 + 0.4 \times 1 + 0.3 \times 0 = 0.4\end{aligned}$$

Opening Remarks

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- Predicting the Weather
  - The probabilities that denote what the weather may be on day k and the table that summarizes the probabilities are often represented as a (state) vector, $x^{(k)}$, and (transition) matrix, P, respectively:
    - $x^{(k)} = \begin{pmatrix} \chi_s^{(k)} \\ \chi_c^{(k)} \\ \chi_r^{(k)} \end{pmatrix}$ and $P = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix}$
  - The transition from day k to day $k+1$ is then written as the matrix-vector product (multiplication)
    - $x^{(k+1)} = \begin{pmatrix} \chi_s^{(k+1)} \\ \chi_c^{(k+1)} \\ \chi_r^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(k)} \\ \chi_c^{(k)} \\ \chi_r^{(k)} \end{pmatrix}$

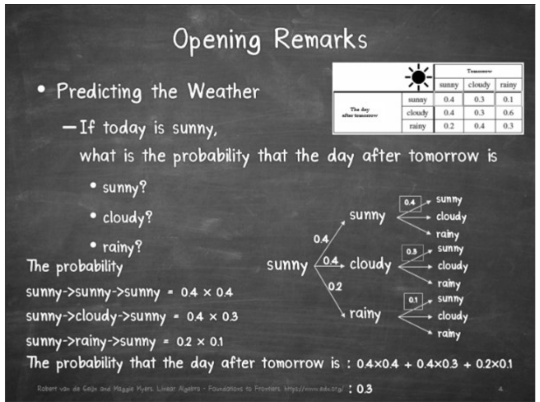
Opening Remarks

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- **Predicting the Weather**
  - **If today is cloudy, what is the probability that tomorrow is sunny? cloudy? rainy?**
    - **today is cloudy :** $x^{(0)} = \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
    - $x^{(1)} = \begin{pmatrix} \chi_s^{(1)} \\ \chi_c^{(1)} \\ \chi_r^{(1)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix}$

$$= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \times 0 + 0.3 \times 1 + 0.1 \times 0 \\ 0.4 \times 0 + 0.3 \times 1 + 0.6 \times 0 \\ 0.2 \times 0 + 0.4 \times 1 + 0.3 \times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}$$



Opening Remarks

		Tomorrow		
		sunny	cloudy	rainy
The day after tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

- **Predicting the Weather**
  - **If today is sunny, what is the probability that the day after tomorrow is sunny? cloudy? rainy?**
    - today is sunny : $x^{(0)} = \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
    - $x^{(2)} = \begin{pmatrix} \chi_s^{(2)} \\ \chi_c^{(2)} \\ \chi_r^{(2)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(1)} \\ \chi_c^{(1)} \\ \chi_r^{(1)} \end{pmatrix}$

$$= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \left(\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \left(\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.4 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 \times 0.4 + 0.3 \times 0.4 + 0.1 \times 0.2 \\ 0.4 \times 0.4 + 0.3 \times 0.4 + 0.6 \times 0.2 \\ 0.2 \times 0.2 + 0.4 \times 0.4 + 0.3 \times 0.2 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix}$$

```
>> P = [
0.4 0.3 0.1
0.4 0.3 0.6
0.2 0.4 0.3
];
>>  x0 = [
1
0
0
];
>> x2 = P*(P*x0)

x2 =

    0.3000
    0.4000
    0.3000
```

```
>> P = [
0.4 0.3 0.1
0.4 0.3 0.6
0.2 0.4 0.3
];
>> x0 = [
1
0
0
];
>> x1 = P*x0

x1 =

    0.4000
    0.4000
    0.2000

>> x2 = P*x1

x2 =

    0.3000
    0.4000
    0.3000
```

Opening Remarks (MATLAB)

```
>> P = [
0.4 0.3 0.1
0.4 0.3 0.6
0.2 0.4 0.3
];
>> x0 = [
1
0
0
];

>> x1 = P * x0
x1 =
    0.4000
    0.4000
    0.2000
```

```
>> x2 = P * x1
x2 =
    0.3000
    0.4000
    0.3000

>> x3 = P * x2
x3 =
    0.2700
    0.4200
    0.3100

>> x4 = P * x3
x4 =
    0.2650
    0.4200
    0.3150
```

```
>> x5 = P * x4
x5 =
    0.2635
    0.4210
    0.3155

>> x6 = P * x5
x6 =
    0.2633
    0.4210
    0.3158

>> x7 = P * x6
x7 =
    0.2632
    0.4211
    0.3158

>>
```

Opening Remarks (MATLAB)

```
>> P = [
0.4 0.3 0.1
0.4 0.3 0.6
0.2 0.4 0.3
];
>> x = [
1
0
0
];
>> for i=1:7
i
x = P * x
end

i =
     1
x =
    0.4000
    0.4000
    0.2000
i =
     2
x =
    0.3000
    0.4000
    0.3000

i =
     3
x =
    0.2700
    0.4200
    0.3100

i =
     4
x =
    0.2650
    0.4200
    0.3150
i =
     5
x =
    0.2635
    0.4210
    0.3155

i =
     6
x =
    0.2633
    0.4210
    0.3158

i =
     7
x =
    0.2632
    0.4211
    0.3158

>>
```

Opening Remarks (MATLAB)

```
for i=1:365
    i
    x = P * x
end
```

```
for i=1:365
    i;
    x = P * x;
end
i, x
```

Opening Remarks

Homework 4.1.1.4 Given

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

fill in the following table, which predicts the weather the day after tomorrow given the weather today:

		Today		
		sunny	cloudy	rainy
Day after Tomorrow	sunny			
	cloudy			
	rainy			

Preparation

- Partitioned Matrix-Vector
- Transposing a Partitioned Matrix
- Matrix-Vector Multiplication, Again

Preparation

- **Partitioned Matrix-Vector Multiplication**
  - Motivation
    - Consider

$$A = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{rr|r|rr} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ \hline 2 & -1 & 3 & 1 & 2 \\ \hline 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{array}\right),$$

$$x = \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{c} 1 \\ 2 \\ \hline 3 \\ \hline 4 \\ 5 \end{array}\right), \quad \text{and} \quad y = \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right),$$

  - where $y_0, y_2 \in \mathbb{R}^2$. Then $y := Ax$ means that

$$y = \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right) = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{ccccc} A_{00}x_0 & + & a_{01}\chi_1 & + & A_{02}x_2 \\ \hline a_{10}^T x_0 & + & \alpha_{11}\chi_1 & + & a_{12}^T x_2 \\ \hline A_{20}x_0 & + & a_{21}\chi_1 & + & A_{22}x_2 \end{array}\right)$$

Preparation

$$A = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{rr|r|rr} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ \hline 2 & -1 & 3 & 1 & 2 \\ \hline 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{array}\right),$$

$$x = \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{c} 1 \\ 2 \\ \hline 3 \\ \hline 4 \\ 5 \end{array}\right), \quad \text{and} \quad y = \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right),$$

$$y = \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right) = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{ccccc} A_{00}x_0 & + & a_{01}\chi_1 & + & A_{02}x_2 \\ \hline a_{10}^T x_0 & + & \alpha_{11}\chi_1 & + & a_{12}^T x_2 \\ \hline A_{20}x_0 & + & a_{21}\chi_1 & + & A_{22}x_2 \end{array}\right)$$

$$= \left(\begin{array}{c} \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} 3 + \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ \hline \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \end{pmatrix} 3 + \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} 3 + \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \end{array}\right) =$$

$$\left(\begin{array}{c} \begin{pmatrix} (-1)\times(1)+(2)\times(2) \\ (1)\times(1)+(0)\times(2) \end{pmatrix} + \begin{pmatrix} (4)\times(3) \\ (-1)\times(3) \end{pmatrix} + \begin{pmatrix} (1)\times(4)+(0)\times(5) \\ (-2)\times(4)+(1)\times(5) \end{pmatrix} \\ \hline (2)\times(1)+(-1)\times(2)+ \quad (3)\times(3)+ \quad (1)\times(4)+(2)\times(5) \\ \hline \begin{pmatrix} (1)\times(1)+(2)\times(2) \\ (-1)\times(1)+(-2)\times(2) \end{pmatrix} + \begin{pmatrix} (3)\times 3 \\ (0)\times 3 \end{pmatrix} + \begin{pmatrix} (4)\times(4)+(3)\times(5) \\ (1)\times(4)+(2)\times(5) \end{pmatrix} \end{array}\right) =$$

$$\left(\begin{array}{c} (-1)\times(1)+(2)\times(2)+(4)\times(3)+(1)\times(4)+(0)\times(5) \\ (1)\times(1)+(0)\times(2)+(-1)\times(3)+(-2)\times(4)+(1)\times(5) \\ \hline (2)\times(1)+(-1)\times(2)+(3)\times(3)+(1)\times(4)+(2)\times(5) \\ \hline (1)\times(1)+(2)\times(2)+(3)\times(3)+(4)\times(4)+(3)\times(5) \\ (-1)\times(1)+(-2)\times(2)+(0)\times(3)+(1)\times(4)+(2)\times(5) \end{array}\right) = \left(\begin{array}{r} 19 \\ -5 \\ \hline 23 \\ \hline 45 \\ 9 \end{array}\right)$$

Preparation

Homework 4.2.1.1 Consider

$$A = \begin{pmatrix} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix},$$

and partition these into submatrices (regions) as follows:

$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right),$$

where $A_{00} \in \mathbb{R}^{3x3}$, $x_0 \in \mathbb{R}^3$, α_{11} is a scalar, and χ_1 is a scalar. Show with lines how A and x are partitioned:

$$\left(\begin{array}{ccc|c|c} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ \hline 1 & 2 & 3 & 4 & 3 \\ \hline -1 & -2 & 0 & 1 & 2 \end{array}\right) \qquad \left(\begin{array}{c} 1 \\ 2 \\ 3 \\ \hline 4 \\ \hline 5 \end{array}\right).$$

Preparation

- **Partitioned Matrix-Vector Multiplication**

Theory

Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$. Partition

$$A = \left(\begin{array}{c|c|c|c} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ \hline A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,N-1} \end{array}\right), \quad x = \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array}\right), \quad \text{and} \quad y = \left(\begin{array}{c} y_0 \\ \hline y_1 \\ \hline \vdots \\ \hline y_{M-1} \end{array}\right)$$

where

- $m = m_0 + m_1 + \cdots + m_{M-1}$,
- $m_i \geq 0$ for $i = 0, \ldots, M-1$,
- $n = n_0 + n_1 + \cdots + n_{N-1}$,
- $n_j \geq 0$ for $j = 0, \ldots, N-1$, and
- $A_{i,j} \in \mathbb{R}^{m_i \times n_j}$, $x_j \in \mathbb{R}^{n_j}$, and $y_i \in \mathbb{R}^{m_i}$.

Preparation

If $y = Ax$ then

$$\left(\begin{array}{c|c|c|c} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ \hline A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,N-1} \end{array}\right)\left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array}\right)$$

$$= \left(\begin{array}{c} A_{0,0}x_0 + A_{0,1}x_1 + \cdots + A_{0,N-1}x_{N-1} \\ \hline A_{1,0}x_0 + A_{1,1}x_1 + \cdots + A_{1,N-1}x_{N-1} \\ \hline \vdots \\ \hline A_{M-1,0}x_0 + A_{M-1,1}x_1 + \cdots + A_{M-1,N-1}x_{N-1} \end{array}\right).$$

In other words,

$$y_i = \sum_{j=0}^{N-1} A_{i,j}x_j.$$

Preparation

- **Transposing a Partitioned Matrix**
  - **Motivation**
    - **Consider**

$$
\begin{aligned}
\left(\begin{array}{rrr|r} 1 & -1 & 3 & 2 \\ 2 & -2 & 1 & 0 \\ \hline 0 & -4 & 3 & 2 \end{array}\right)^T
&= \left(\begin{array}{c|c} \begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & -4 & 3 \end{pmatrix} & \begin{pmatrix} 2 \end{pmatrix} \end{array}\right)^T \\
&= \left(\begin{array}{c|c} \begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 1 \end{pmatrix}^T & \begin{pmatrix} 0 & -4 & 3 \end{pmatrix}^T \\ \hline \begin{pmatrix} 2 \\ 0 \end{pmatrix}^T & \begin{pmatrix} 2 \end{pmatrix}^T \end{array}\right) \\
&= \left(\begin{array}{c|c} \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \\ \hline \begin{pmatrix} 2 & 0 \end{pmatrix} & \begin{pmatrix} 2 \end{pmatrix} \end{array}\right) = \left(\begin{array}{rr|r} 1 & 2 & 0 \\ -1 & -2 & -4 \\ 3 & 1 & 3 \\ \hline 2 & 0 & 2 \end{array}\right).
\end{aligned}
$$

Preparation

- **Transposing a Partitioned Matrix**

Theory

Let $A \in \mathbb{R}^{m \times n}$ be partitioned as follows:

$$A = \left(\begin{array}{c|c|c|c} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ \hline A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,N-1} \end{array}\right),$$

where $A_{i,j} \in \mathbb{R}^{m_i \times n_j}$. Then

$$A^T = \left(\begin{array}{c|c|c|c} A^T_{0,0} & A^T_{1,0} & \cdots & A^T_{M-1,0} \\ \hline A^T_{0,1} & A^T_{1,1} & \cdots & A^T_{M-1,1} \\ \hline \vdots & \vdots & & \vdots \\ \hline A^T_{0,N-1} & A^T_{1,N-1} & \cdots & A^T_{M-1,N-1} \end{array}\right).$$

Preparation

- **Transposing a Partitioned Matrix (Special cases)**

Each submatrix is a scalar. If

$$A = \left(\begin{array}{c|c|c|c} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,N-1} \\ \hline \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,N-1} \\ \hline \vdots & \vdots & & \vdots \\ \hline \alpha_{M-1,0} & \alpha_{M-1,1} & \cdots & \alpha_{M-1,N-1} \end{array}\right)$$

then

$$A^T = \left(\begin{array}{c|c|c|c} \alpha_{0,0}^T & \alpha_{1,0}^T & \cdots & \alpha_{M-1,0}^T \\ \hline \alpha_{0,1}^T & \alpha_{1,1}^T & \cdots & \alpha_{M-1,1}^T \\ \hline \vdots & \vdots & & \vdots \\ \hline \alpha_{0,N-1}^T & \alpha_{1,N-1}^T & \cdots & \alpha_{M-1N-1}^T \end{array}\right) = \left(\begin{array}{cccc} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{M-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{M-1,1} \\ \vdots & \vdots & & \vdots \\ \alpha_{0,N-1} & \alpha_{1,N-1} & \cdots & \alpha_{M-1,N-1} \end{array}\right).$$

This is because the transpose of a scalar is just that scalar.

2×2 **blocked partitioning.** If

$$A = \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right),$$

then

$$A^T = \left(\begin{array}{c|c} A_{TL}^T & A_{BL}^T \\ \hline A_{TR}^T & A_{BR}^T \end{array}\right).$$

3×3 **blocked partitioning.** If

$$A = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

then

$$A^T = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)^T = \left(\begin{array}{c|c|c} A_{00}^T & (a_{10}^T)^T & A_{20}^T \\ \hline a_{01}^T & \alpha_{11}^T & a_{21}^T \\ \hline A_{02}^T & (a_{12}^T)^T & A_{22}^T \end{array}\right) = \left(\begin{array}{c|c|c} A_{00}^T & a_{10} & A_{20}^T \\ \hline a_{01}^T & \alpha_{11} & a_{21}^T \\ \hline A_{02}^T & a_{12} & A_{22}^T \end{array}\right).$$

The matrix is partitioned by rows. If

$$A = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \widetilde{a}_1^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array}\right),$$

where each $\widetilde{a}_i^T$ is a row of A, then

$$A^T = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \widetilde{a}_1^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array}\right)^T = \left(\begin{array}{c|c|c|c} (\widetilde{a}_0^T)^T & (\widetilde{a}_1^T)^T & \cdots & (\widetilde{a}_{m-1}^T)^T \end{array}\right) = \left(\begin{array}{c|c|c|c} \widetilde{a}_0 & \widetilde{a}_1 & \cdots & \widetilde{a}_{m-1} \end{array}\right).$$

This shows that rows of A, $\widetilde{a}_i^T$, become columns of A^T: $\widetilde{a}_i$.

The matrix is partitioned by columns. If

$$A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array}\right),$$

where each a_j is a column of A, then

$$A^T = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array}\right)^T = \left(\begin{array}{c} a_0^T \\ \hline a_1^T \\ \hline \vdots \\ \hline a_{n-1}^T \end{array}\right).$$

This shows that columns of A, a_j, become rows of A^T: a_j^T.

Preparation

- **Matrix-Vector Multiplication, Again**
  - Consider $y = Ax + y$
    - $A = \begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$,
    - $x = \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\begin{pmatrix} 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ \hline 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ \hline 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 & 3 & 2 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 1$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\begin{pmatrix} 1 \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ \hline 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ \hline 0 \\ \hline 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ \hline -1 \\ \hline 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & -2 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -1$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 1 \\ -1 \\ \hline 0 \\ \hline 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{ccccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ \hline 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ 0 \\ \hline 2 \\ \hline -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} 1 \\ -1 \\ \hline 5 \\ \hline 0 \\ 0 \end{array}\right)$

$$\begin{pmatrix} 0 & -4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 5$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ \hline 0 \\ \hline 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ \hline 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ \hline -1 \\ \hline 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ \hline -8 \\ \hline 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -8$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 1 \\ -1 \\ 5 \\ -8 \\ \hline 0 \end{array}\right)$	$\left(\begin{array}{ccccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ 0 \\ 2 \\ -1 \\ \hline 1 \end{array}\right)$	$\left(\begin{array}{c} 1 \\ -1 \\ 5 \\ -8 \\ \hline 2 \end{array}\right)$

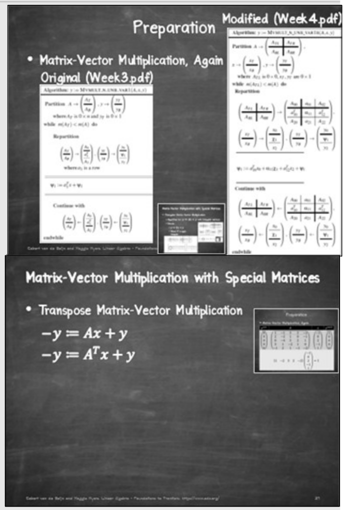
$$\begin{pmatrix} -1 & 2 & 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 2$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ \underline{2} \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ \underline{-1} & \underline{2} & \underline{1} & \underline{-1} & \underline{-2} \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ \underline{1} \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ \underline{2} \end{pmatrix}$

$$\psi_i := \alpha_i^T x + \psi_i$$



Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{c\|cccc} 1 & -1 & 3 & 2 & -2 \\ \hline 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ \hline 0 \\ 2 \\ -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$

$$(1) \times (-1) + \left(\overset{-1}{\cancel{-2}} \quad 3 \quad 2 \quad -2\right) \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 1$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 1 \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{c\|c\|ccc} 1 & -1 & 3 & 2 & -2 \\ \hline 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ \hline 0 \\ \hline 2 \\ -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} 1 \\ \hline -1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$

$$(2)(-1) + \overset{-2}{\cancel{(2)}} \times \overset{0}{\cancel{(-2)}} + \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = -1$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 1 \\ -1 \\ \hline 0 \\ \hline 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{cc\|c\|cc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ \hline 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ 0 \\ \hline 2 \\ \hline -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} 1 \\ -1 \\ \hline 5 \\ \hline 0 \\ 0 \end{array}\right)$

$$\begin{pmatrix} 0 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + (3) \times (2) + \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 5$$

Preparation

- **Matrix-Vector Multiplication, Again**

<table>
<tr><th>y^{cur}</th><th>A</th><th>x</th><th>y^{next}</th></tr>
<tr>
<td>$\left(\begin{array}{c} 1 \\ -1 \\ 5 \\ \hline 0 \\ \hline 0 \end{array}\right)$</td>
<td>$\left(\begin{array}{ccc|c|c} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ \hline 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{array}\right)$</td>
<td>$\left(\begin{array}{c} -1 \\ 0 \\ 2 \\ \hline -1 \\ \hline 1 \end{array}\right)$</td>
<td>$\left(\begin{array}{c} 1 \\ -1 \\ 5 \\ \hline -8 \\ \hline 0 \end{array}\right)$</td>
</tr>
</table>

$$\begin{pmatrix} 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + (1) \times (-1) + (0)(-1) = -8$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 1 \\ -1 \\ 5 \\ -8 \\ \hline 0 \end{array}\right)$	$\left(\begin{array}{cccc\|c} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ 0 \\ 2 \\ -1 \\ \hline 1 \end{array}\right)$	$\left(\begin{array}{c} 1 \\ -1 \\ 5 \\ -8 \\ \hline 2 \end{array}\right)$

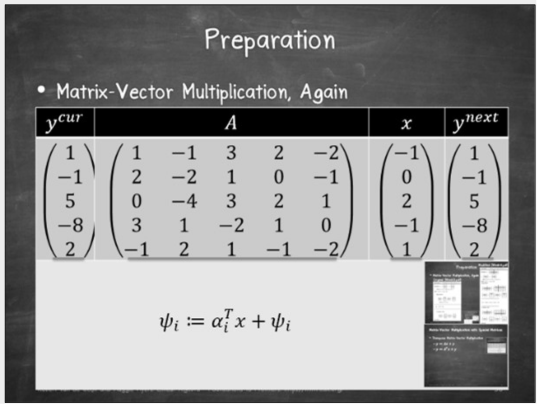
$$\begin{pmatrix} -1 & 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix} + (-2) \times (1) = 2$$

Preparation

- **Matrix-Vector Multiplication, Again**

y^{cur}	A	x	y^{next}
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ \underline{0} \end{pmatrix}$	$\left(\begin{array}{rrrrr\|} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ \underline{1} \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ \underline{2} \end{pmatrix}$

$$\psi_i := a_{10}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_i$$



Preparation

Modified (Week4.pdf)

Algorithm: $y := \text{MVMULT\_N\_UNB\_VAR1B}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$,

$x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_{TL} is 0×0, x_T, y_T are 0×1

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$\psi_1 := a_{10}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_1$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

- **Matrix-Vector Multiplication, Again**

Original (Week3.pdf)

Algorithm: $y := \text{MVMULT\_N\_UNB\_VAR1}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_T is $0 \times n$ and y_T is 0×1

while $m(A_T) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \rightarrow \left(\begin{array}{c} A_0 \\ \hline a_1^T \\ \hline A_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

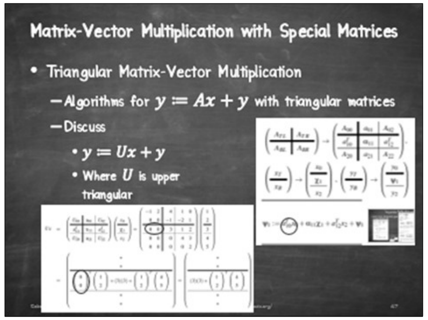
where a_1 is a row

$$\psi_1 := a_1^T x + \psi_1$$

Continue with

$$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \leftarrow \left(\begin{array}{c} A_0 \\ \hline a_1^T \\ \hline A_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile



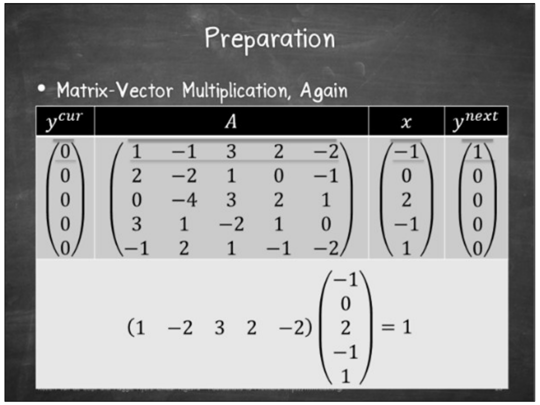
Robert van de Geijn and Maggie Myers. Linear Algebra - Foundations

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication
- Triangular Matrix-Vector Multiplication
- Symmetric Matrix-Vector Multiplication .

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication
  - $y := Ax + y$
  - $y := A^T x + y$



Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $\boldsymbol{A^T x + y}$

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{c\|cccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ \hline 0 \\ 2 \\ -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} -5 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \\ -1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -5$$

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $A^T x + y$

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} -5 \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{c\|c\|ccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ \hline 0 \\ \hline 2 \\ -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} -5 \\ \hline -6 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$

$$\begin{pmatrix} -1 \\ -2 \\ -4 \\ 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -6$$

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $\boldsymbol{A^T x + y}$

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} -5 \\ -6 \\ \hline 0 \\ \hline 0 \\ 0 \end{array}\right)$	$\left(\begin{array}{cc\|c\|cc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ 0 \\ \hline 2 \\ \hline -1 \\ 1 \end{array}\right)$	$\left(\begin{array}{c} -5 \\ -6 \\ \hline 6 \\ \hline 0 \\ 0 \end{array}\right)$

$$\begin{pmatrix} 3 \\ 1 \\ 3 \\ -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 6$$

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $\boldsymbol{A^T x + y}$

y^{cur}	A	x	y^{next}
$\begin{pmatrix} -5 \\ -6 \\ 6 \\ \hline 0 \\ \hline 0 \end{pmatrix}$	$\left(\begin{array}{ccc\|c\|c} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ \hline -1 \\ \hline 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 6 \\ \hline 0 \\ \hline 0 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $\boldsymbol{A^T x + y}$

y^{cur}	A	x	y^{next}
$\left(\begin{array}{c} -5 \\ -6 \\ 6 \\ 0 \\ \hline 0 \end{array}\right)$	$\left(\begin{array}{cccc\|c} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right)$	$\left(\begin{array}{c} -1 \\ 0 \\ 2 \\ -1 \\ \hline 1 \end{array}\right)$	$\left(\begin{array}{c} -5 \\ -6 \\ 6 \\ 0 \\ \hline 2 \end{array}\right)$

$$\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ -2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 2$$

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $\boldsymbol{A^T x + y}$

y^{cur}	A	x	y^{next}
$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 2 \end{pmatrix}$

Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication : $A^T x + y$

  Original (Week3.pdf)

Algorithm: $y := \mathrm{MVMULT\_N\_UNB\_VAR1}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_T is $0 \times n$ and y_T is 0×1

while $m(A_T) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \rightarrow \left(\begin{array}{c} A_0 \\ \hline a_1^T \\ A_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ y_2 \end{array}\right)$$

where a_1 is a row

$$\psi_1 := a_1^T x + \psi_1$$

Continue with

$$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \leftarrow \left(\begin{array}{c} A_0 \\ a_1^T \\ \hline A_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Algorithm: $y := \mathrm{MVMULT\_T\_UNB\_VAR1}(A, x, y)$

Partition $A \rightarrow \left(A_L \,\middle|\, A_R\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_L is $m \times 0$ and y_T is 0×1

while $m(y_T) < m(y)$ **do**

Repartition

$$\left(A_L \,\middle|\, A_R\right) \rightarrow \left(A_0 \,\middle|\, a_1 \,\middle|\, A_2\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ y_2 \end{array}\right)$$

$$\psi_1 := a_1^T x + \psi_1$$

Continue with

$$\left(A_L \,\middle|\, A_R\right) \leftarrow \left(A_0 \,\middle|\, a_1 \,\middle|\, A_2\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

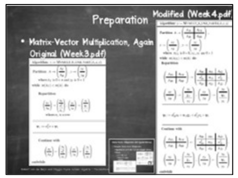
Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication
  - Algorithms for $y := Ax + y$ with triangular matrices
  - Discuss
    - $y := Ux + y$
    - Where U is upper triangular

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \quad \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$\psi_1 := a_{10}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_1$$



$$Ux = \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & U_{22} \end{array}\right) \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{cc|c|cc} -1 & 2 & 4 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 \\ \hline 0 & 0 & 3 & 1 & 2 \\ \hline 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) \left(\begin{array}{c} 1 \\ 2 \\ \hline 3 \\ \hline 4 \\ 5 \end{array}\right)$$

$$= \left(\begin{array}{c} \star \\ \star \\ \hline \left(\begin{array}{c} 0 \\ 0 \end{array}\right)^T \left(\begin{array}{c} 1 \\ 2 \end{array}\right) + (3)(3) + \left(\begin{array}{c} 1 \\ 2 \end{array}\right)^T \left(\begin{array}{c} 4 \\ 5 \end{array}\right) \\ \hline \star \\ \star \end{array}\right) = \left(\begin{array}{c} \star \\ \star \\ \hline (3)(3) + \left(\begin{array}{c} 1 \\ 2 \end{array}\right)^T \left(\begin{array}{c} 4 \\ 5 \end{array}\right) \\ \hline \star \\ \star \end{array}\right)$$

Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

Algorithm: $y := \text{MVMULT\_N\_UNB\_VAR1B}(A,x,y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$,
$x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$
where A_{TL} is 0×0, x_T, y_T are 0×1

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$\psi_1 := a_{10}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_1$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Algorithm: $y := \text{TRMVP\_UN\_UNB\_VAR1}(U,x,y)$

Partition $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$,
$x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$
where U_{TL} is 0×0, x_T, y_T are 0×1

while $m(U_{TL}) < m(U)$ **do**

Repartition

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & U_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$\psi_1 := u_{10}^T x_0 + \upsilon_{11}\chi_1 + u_{12}^T x_2 + \psi_1$$

Continue with

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

Algorithm: $y := \text{TRMVP\_UN\_UNB\_VAR1}(U, x, y)$

Partition $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$, $x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$
  where U_{TL} is 0×0, x_T, y_T are 0×1
while $m(U_{TL}) < m(U)$ **do**
  Repartition

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & U_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$\psi_1 := u_{10}^T x_0 + \upsilon_{11}\chi_1 + u_{12}^T x_2 + \psi_1$

  Continue with

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Algorithm: $y := \text{TRMVP\_UN\_UNB\_VAR2}(U, x, y)$

Partition $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$, $x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$
  where U_{TL} is 0×0, x_T, y_T are 0×1
while $m(U_{TL}) < m(U)$ **do**
  Repartition

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & U_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$y_0 := \chi_1 u_{01} + y_0$
$\psi_1 := \chi_1 \upsilon_{11} + \psi_1$
$y_2 := \chi_1 u_{21} + y_2$

  Continue with

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication
  - $y := Ux + y =$

$$\begin{array}{ccc} k & 1 & n\text{-}k\text{-}1 \end{array}$$

$$y := Ux + y = \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & U_{22} \end{array}\right)\left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) + \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

    - $u_{10}^T x_0 + v_{11}\chi_1 + u_{12}^T x_2 + \psi_1$
    - 0 + 2 + 2(n-k-1)
    - $\sum_{k=0}^{n-1}\left(2 + 2(n-k-1)\right)$
    - $= \sum_{k=0}^{n-1}\left(2(n-k)\right) = 2\sum_{k=0}^{n-1} n - k$
    - $= 2(n + (n-1) + \cdots + (n-(n-1)))$
    - $= 2(n + (n-1) + \cdots + (1)) = 2\sum_{j=1}^{n} i = 2\left(\frac{n(n+1)}{2}\right)$

Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

$$
-\ y := Ux + y = \begin{array}{ccc} k & 1 & n\text{-}k\text{-}1 \\ \end{array}
$$

$$
\left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ \hline U_{20} & u_{21} & U_{22} \end{array}\right)\left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) + \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)
$$

  - $y_0 := \chi_1 u_{01} + y_0$
  - $\psi_1 := \chi_1 v_{11} + \psi_1$
  - $y_2 := \chi_1 u_{21} + y_2$
  - 2k+2+0
  - $\sum_{k=0}^{n-1}(2k+2) = 2\sum_{k=0}^{n-1}(k+1) = 2\sum_{k=1}^{n}(k)$

Matrix-Vector Multiplication with Special Matrices

- Symmetric Matrix-Vector Multiplication

$$- \; y := Ax + y = \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) + \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$- = \left(\begin{array}{cc|c|cc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ \hline 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array}\right) \left(\begin{array}{c} -1 \\ 0 \\ \hline 2 \\ \hline -1 \\ 1 \end{array}\right) + \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ 0 \end{array}\right)$$

Matrix-Vector Multiplication with Special Matrices

- Matrix-Vector Multiplication
  - $y := Ax + y$
  - $\psi_i := a_{10}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_i$
- Symmetric Matrix-Vector Multiplication
  - $\psi_i := a_{01}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_i$

$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

Matrix-Vector Multiplication with Special Matrices

Algorithm: $y := \textsc{Symv\_u\_unb\_var1}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_{TL} is 0×0, x_T, y_T are 0×1

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$\psi_1 := \overbrace{a_{10}^T}^{a_{01}^T} x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 + \psi_1$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Algorithm: $y := \textsc{Symv\_u\_unb\_var2}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_{TL} is 0×0, x_T, y_T are 0×1

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

$$y_0 := \chi_1 a_{01} + y_0$$
$$\psi_1 := \chi_1 \alpha_{11} + \psi_1$$
$$y_2 := \chi_1 \underbrace{a_{21}}_{a_{12}} + y_2$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Matrix-Matrix Multiplication (Product)

- Motivation
- From Composing Linear Transformations to Matrix-Matrix Multiplication
- Computing the Matrix-Matrix Product
- Special Shapes

Matrix-Matrix Multiplication (Product)

Given

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

fill in the following table, which predicts the weather the day after tomorrow given the weather today:

		Today		
		sunny	cloudy	rainy
Day after Tomorrow	sunny			
	cloudy			
	rainy			

Matrix-Matrix Multiplication (Product)

The entries in the table turn out to be the entries in the transition matrix Q that was described just above the exercise:

$$\begin{aligned}
\begin{pmatrix} \chi_s^{(2)} \\ \chi_c^{(2)} \\ \chi_r^{(2)} \end{pmatrix} &= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(1)} \\ \chi_c^{(1)} \\ \chi_r^{(1)} \end{pmatrix} \\
&= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \left(\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right) = Q \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix},
\end{aligned}$$

Now, those of you who remembered from, for example, some other course that

$$\begin{aligned}
&\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \left(\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right) \\
&\qquad = \left(\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \right) \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix}
\end{aligned}$$

would recognize that

$$Q = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix}.$$

Matrix-Matrix Multiplication (Product)

- $L_P(x) = P \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix}$

- $L_Q(x) = Q \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} = P \left(P \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right) = L_P(L_P(x))$

Matrix-Matrix Multiplication (Product)

- Question
  - Is $L_Q(x) = L_P\left(L_P(x)\right)$ a linear transformation?
  - Is $L_C(x) = L_B\left(L_A(x)\right)$ a linear transformation?
  - If it is, how are matrices A, B and C related?

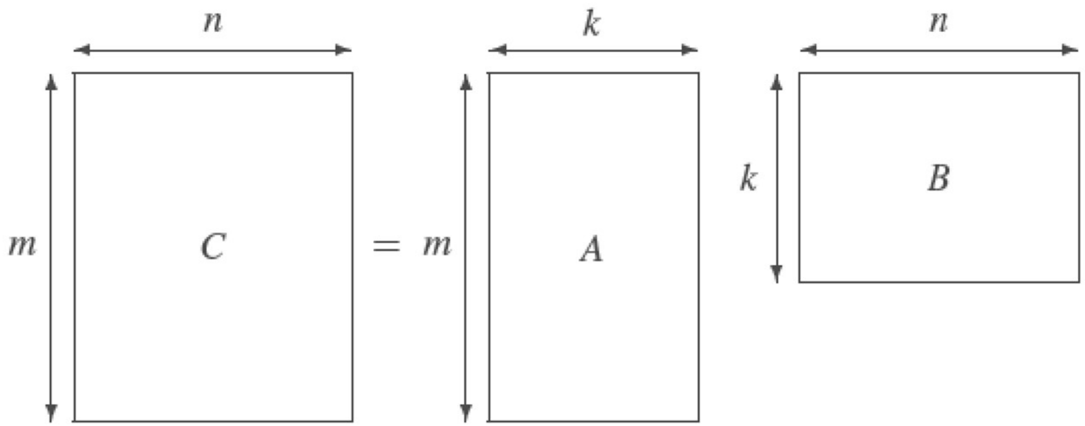
Matrix-Matrix Multiplication (Product)

- From Composing Linear Transformations to Matrix-Matrix Multiplication
  - Homework
    - Let $L_A: \mathbb{R}^k \to \mathbb{R}^m$ and $L_B: \mathbb{R}^n \to \mathbb{R}^k$ both be linear transformations and, for all $x \in \mathbb{R}^n$, define the function $L_C: \mathbb{R}^n \to \mathbb{R}^m$ by $L_C(x) = L_B\big(L_A(x)\big)$. $L_C(x)$ is a linear transformations.
      - Always
      - Sometimes
      - Never

Matrix-Matrix Multiplication (Product)

- From Composing Linear Transformations to Matrix-Matrix Multiplication

If A is $m_A \times n_A$ matrix, B is $m_B \times n_B$ matrix, and C is $m_C \times n_C$ matrix, then for $C = AB$ to hold it must be the case that $m_C = m_A$, $n_C = n_B$, and $n_A = m_B$. Usually, the integers m and n are used for the sizes of C: $C \in \mathbb{R}^{m \times n}$ and k is used for the "other size": $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$:



Matrix-Matrix Multiplication (Product)

- Summary
  - Let $L_A: \mathbb{R}^k \to \mathbb{R}^m$ and $L_B: \mathbb{R}^n \to \mathbb{R}^k$ are linear transformations and define $L_C(x) = L_A\big(L_B(x)\big)$. Then
    - $L_C: \mathbb{R}^n \to \mathbb{R}^m$ is linear transformations
    - There are $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$ that represent L_A and L_B respectively.
    - There is a matric, $C \in \mathbb{R}^{m\times n}$ that represent $L_C(x) = L_A\big(L_B(x)\big) = A\big(B(x)\big)$.
    - The operation that computes C from A and B is called <u>Matrix-Matrix Multiplication</u>.
    - Notaton: $\mathrm{C} = AB$ and $\mathrm{C}x = (AB)x = A\big(B(x)\big)$

Matrix-Matrix Multiplication (Product)

- Computing the Matrix-Matrix Product
  - Consider the following. Let
    - $C \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$; and $\mathrm{C} = AB$; and
    - $L_C: \mathbb{R}^n \rightarrow \mathbb{R}^m$ equal the linear transformation such that $L_C(x) = Cx$; and
    - $L_A: \mathbb{R}^k \rightarrow \mathbb{R}^m$ equal the linear transformation such that $L_A(x) = Bx$; and
    - $L_B: \mathbb{R}^n \rightarrow \mathbb{R}^k$ equal the linear transformation such that $L_B(x) = Bx$; and
    - e_j denote the j th unit basis vector; and
    - c_j denote the j th column of C; and
    - b_j denote the j th column of B.

$$c_j = Ce_j = L_C(e_j) = L_A\left(L_B(e_j)\right) = L_A\left(B(e_j)\right) = L_A(b_j) = Ab_j$$

Matrix-Matrix Multiplication (Product)

From this we learn that

> If $C = AB$ then the jth column of C, c_j, equals Ab_j, where b_j is the jth column of B.

Since by now you should be very comfortable with partitioning matrices by columns, we can summarize this as

$$\left(\begin{array}{c|c|c} c_0 & c_1 \; \cdots & c_{n-1} \end{array}\right) = C = AB = A\left(\begin{array}{c|c} b_0 \quad b_1 & \cdots \quad b_{n-1} \end{array}\right) = \left(\begin{array}{c|c} Ab_0 \quad Ab_1 & \cdots \quad Ab_{n-1} \end{array}\right).$$

Now, let's expose the elements of C, A, and B.

$$C = \begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \\ \gamma_{1,0} & \boxed{\gamma_{1,1}} & \cdots & \gamma_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{m-1,0} & \gamma_{m-1,1} & \cdots & \gamma_{m-1,n-1} \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,k-1} \end{pmatrix},$$

$$\text{and} \quad B = \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \beta_{1,1} & \cdots & \beta_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{k-1,0} & \beta_{k-1,1} & \cdots & \beta_{k-1,n-1} \end{pmatrix}.$$

> We are going to show that
>
> $$\gamma_{i,j} = \sum_{p=0}^{k-1} \alpha_{i,p}\beta_{p,j},$$
>
> which you may have learned in a high school algebra course.

Matrix-Matrix Multiplication (Product)

We reasoned that $c_j = Ab_j$:

$$\begin{pmatrix} \gamma_{0,j} \\ \gamma_{1,j} \\ \vdots \\ \boxed{\gamma_{i,j}} \\ \vdots \\ \gamma_{m-1,j} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{i,0} & \alpha_{i,1} & \cdots & \alpha_{i,k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,k-1} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} \beta_{0,j} \\ \beta_{1,j} \\ \vdots \\ \beta_{k-1,j} \end{matrix}} \end{pmatrix}.$$

Here we highlight the ith element of c_j, $\gamma_{i,j}$, and the ith row of A. We recall that the ith element of Ax equals the dot product of the ith row of A with the vector x. Thus, $\gamma_{i,j}$ equals the dot product of the ith row of A with the vector b_j:

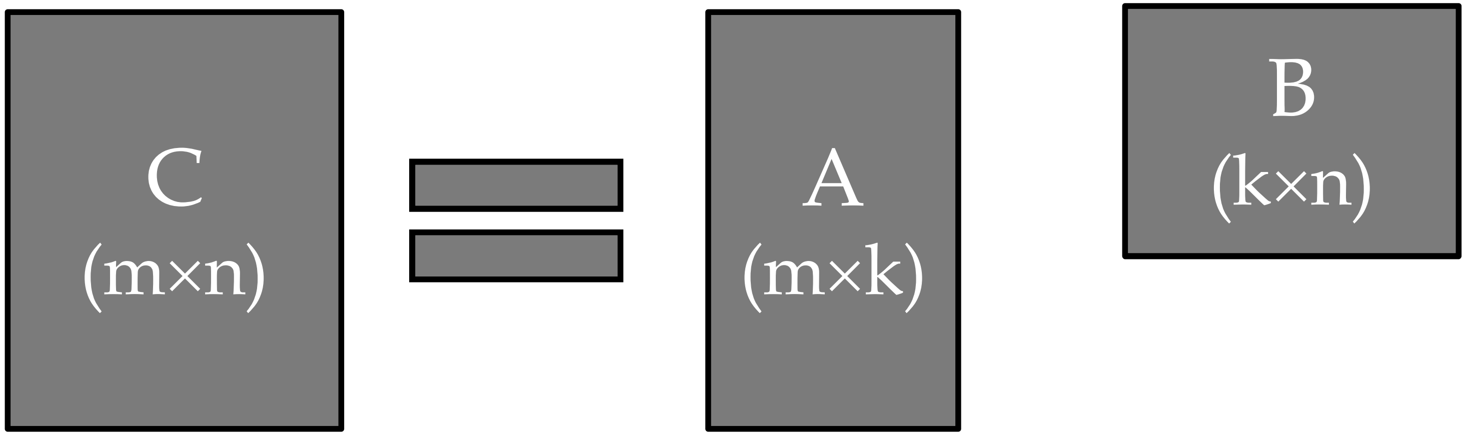
$$\gamma_{i,j} = \sum_{p=0}^{k-1} \alpha_{i,p}\beta_{p,j}.$$

Let $A \in \mathbb{R}^{m\times k}$, $B \in \mathbb{R}^{k\times n}$, and $C \in \mathbb{R}^{m\times n}$. Then the matrix-matrix multiplication (product) $C = AB$ is computed by

$$\gamma_{i,j} = \sum_{p=0}^{k-1} \alpha_{i,p}\beta_{p,j} = \alpha_{i,0}\beta_{0,j} + \alpha_{i,1}\beta_{1,j} + \cdots + \alpha_{i,k-1}\beta_{k-1,j}.$$

Matrix-Matrix Multiplication (Product)

- Special Shapes



Matrix-Matrix Multiplication (Product)

- Special Shapes : m=n=k=1 (scalar multiplication)

$m = n = k = 1$ **(scalar multiplication)**

$$1 \updownarrow \overset{\overset{1}{\leftrightarrow}}{\boxed{C}} \; = \; 1 \updownarrow \overset{\overset{1}{\leftrightarrow}}{\boxed{A}} \quad 1 \updownarrow \overset{\overset{1}{\leftrightarrow}}{\boxed{B}}$$

In this case, all three matrices are actually scalars:

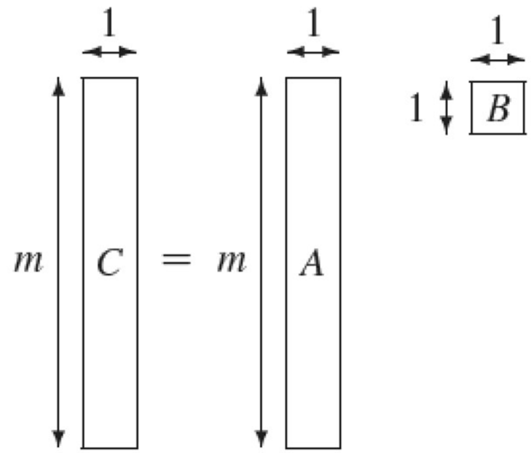
$$\left(\gamma_{0,0} \right) = \left(\alpha_{0,0} \right) \left(\beta_{0,0} \right) = \left(\alpha_{0,0}\beta_{0,0} \right)$$

so that matrix-matrix multiplication becomes scalar multiplication.

Matrix-Matrix Multiplication (Product)

- Special Shapes : n=1, k=1 (SCAL)

$n = 1, k = 1$ (SCAL)



Now the matrices look like

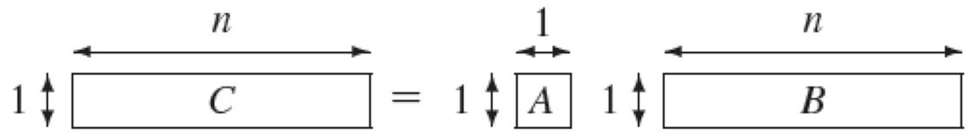
$$\begin{pmatrix} \gamma_{0,0} \\ \gamma_{1,0} \\ \vdots \\ \gamma_{m-1,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix} \begin{pmatrix} \beta_{0,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0}\beta_{0,0} \\ \alpha_{1,0}\beta_{0,0} \\ \vdots \\ \alpha_{m-1,0}\beta_{0,0} \end{pmatrix} = \begin{pmatrix} \beta_{0,0}\alpha_{0,0} \\ \beta_{0,0}\alpha_{1,0} \\ \vdots \\ \beta_{0,0}\alpha_{m-1,0} \end{pmatrix} = \beta_{0,0} \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix}.$$

In other words, C and A are vectors, B is a scalar, and the matrix-matrix multiplication becomes scaling of a vector.

Matrix-Matrix Multiplication (Product)

- **Special Shapes : m=1, k=1 (SCAL)**

$m = 1, k = 1$ (SCAL)



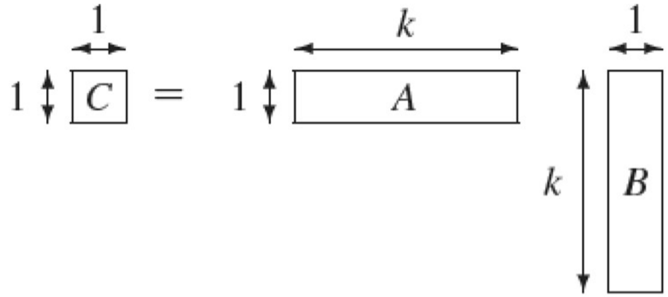
Now the matrices look like

$$
\begin{aligned}
\begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \end{pmatrix} &= \begin{pmatrix} \alpha_{0,0} \end{pmatrix} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \end{pmatrix} \\
&= \alpha_{0,0} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{0,0}\beta_{0,0} & \alpha_{0,0}\beta_{0,1} & \cdots & \alpha_{0,0}\beta_{0,n-1} \end{pmatrix}.
\end{aligned}
$$

In other words, C and B are just row vectors and A is a scalar. The vector C is computed by scaling the row vector B by the scalar A.

Matrix-Matrix Multiplication (Product)

- **Special Shapes : m=1, n=1 (DOT)**



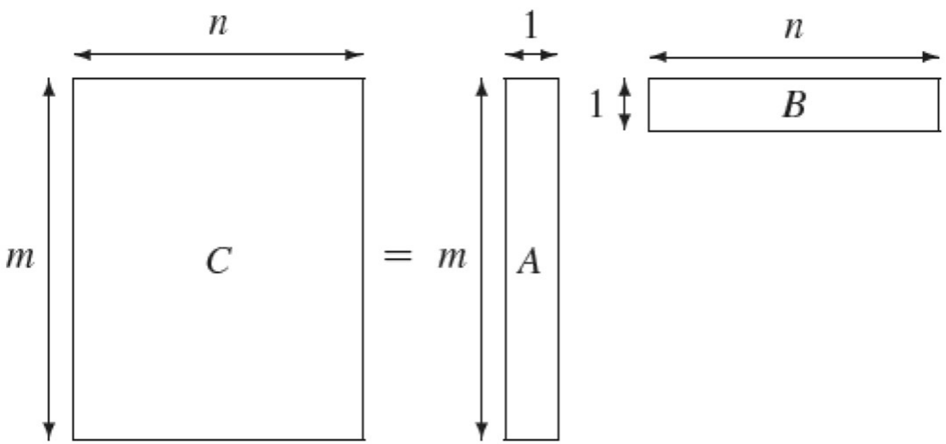
The matrices look like

$$\left(\ \gamma_{0,0}\ \right) = \left(\ \alpha_{0,0} \quad \alpha_{0,1} \quad \cdots \quad \alpha_{0,k-1}\ \right) \begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \vdots \\ \beta_{k-1,0} \end{pmatrix} = \sum_{p=0}^{k-1} \alpha_{0,p}\beta_{p,0}.$$

In other words, C is a scalar that is computed by taking the dot product of the one row that is A and the one column that is B.

Matrix-Matrix Multiplication (Product)

- Special Shapes : k=1 (outer product)

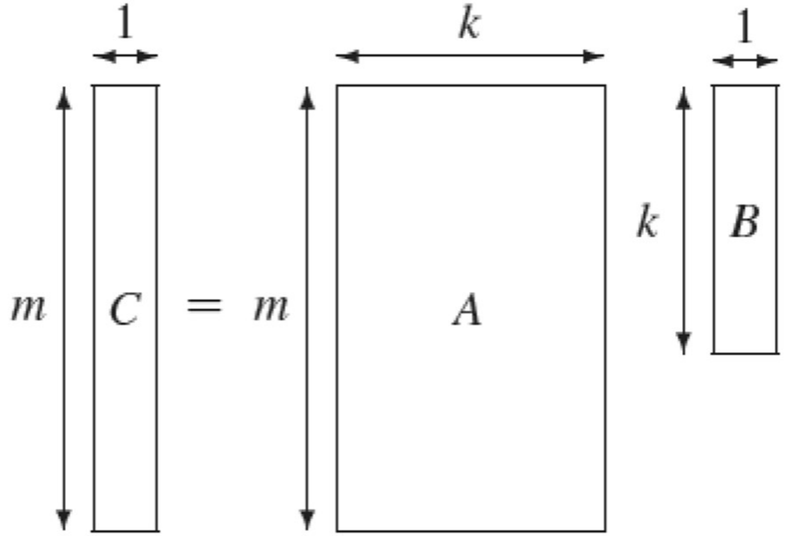


$$
\begin{pmatrix}
\gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \\
\gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{m-1,0} & \gamma_{m-1,1} & \cdots & \gamma_{m-1,n-1}
\end{pmatrix}
=
\begin{pmatrix}
\alpha_{0,0} \\
\alpha_{1,0} \\
\vdots \\
\alpha_{m-1,0}
\end{pmatrix}
\begin{pmatrix}
\beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\alpha_{0,0}\beta_{0,0} & \alpha_{0,0}\beta_{0,1} & \cdots & \alpha_{0,0}\beta_{0,n-1} \\
\alpha_{1,0}\beta_{0,0} & \alpha_{1,0}\beta_{0,1} & \cdots & \alpha_{1,0}\beta_{0,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m-1,0}\beta_{0,0} & \alpha_{m-1,0}\beta_{0,1} & \cdots & \alpha_{m-1,0}\beta_{0,n-1}
\end{pmatrix}
$$

Matrix-Matrix Multiplication (Product)

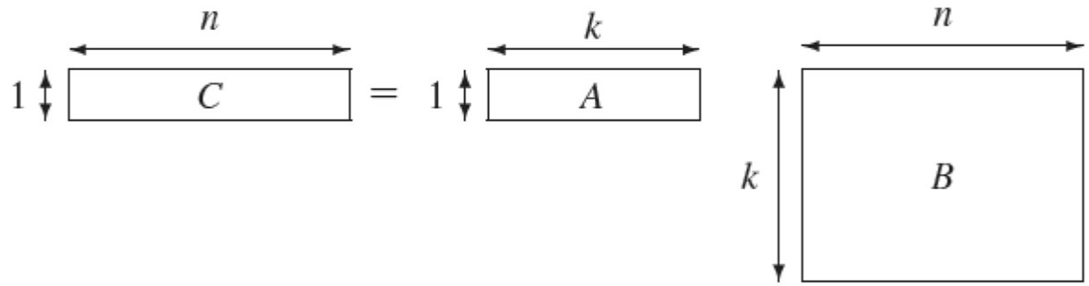
- Special Shapes : n=1 (matrix-vector product)



$$\begin{pmatrix} \gamma_{0,0} \\ \gamma_{1,0} \\ \vdots \\ \gamma_{m-1,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \vdots \\ \beta_{k-1,0} \end{pmatrix}$$

Matrix-Matrix Multiplication (Product)

- Special Shapes : m = 1 (row vector-matrix product)



$$\begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \beta_{1,1} & \cdots & \beta_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1,0} & \beta_{k-1,1} & \cdots & \beta_{k-1,n-1} \end{pmatrix}$$

so that $\gamma_{0,j} = \sum_{p=0}^{k-1} \alpha_{0,p}\beta_{p,j}$. To emphasize how it relates to have matrix-matrix multiplication is computed, consider the following:

$$\begin{aligned} &\begin{pmatrix} \gamma_{0,0} & \cdots & \boxed{\gamma_{0,j}} & \cdots & \gamma_{0,n-1} \end{pmatrix} \\ &= \left(\boxed{\begin{matrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \end{matrix}}\right) \begin{pmatrix} \beta_{0,0} & \cdots & \beta_{0,j} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \cdots & \beta_{1,j} & \cdots & \beta_{1,n-1} \\ \vdots & & \vdots & & \vdots \\ \beta_{k-1,0} & \cdots & \beta_{k-1,j} & \cdots & \beta_{k-1,n-1} \end{pmatrix}. \end{aligned}$$

Questions and Answers