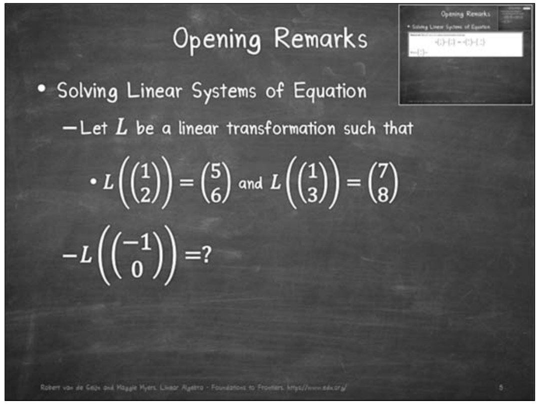
Gaussian Elimination

Jirasak Sittigorn

Department of Computer Engineering
Faculty of Engineering
King Mongkut's Institute of Technology Ladkrabang



Opening Remarks

- **Solving Linear Systems of Equation**

Homework 2.4.1.2 Let L be a linear transformation such that

$$L(\begin{pmatrix}1\\0\end{pmatrix})=\begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad L(\begin{pmatrix}0\\1\end{pmatrix})=\begin{pmatrix}2\\-1\end{pmatrix}.$$

Then $L(\begin{pmatrix}2\\3\end{pmatrix})=$

Opening Remarks

- **Solving Linear Systems of Equation**

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} =? \begin{pmatrix} 1 \\ 0 \end{pmatrix} +? \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \chi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \chi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Opening Remarks

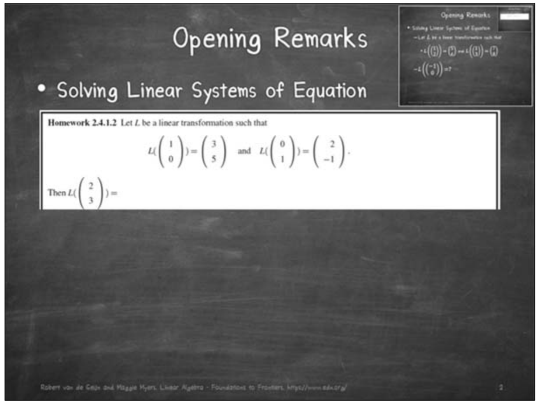
- **Solving Linear Systems of Equation**

For the next three exercises, let L be a linear transformation such that

$$L\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}5\\4\end{pmatrix}.$$

Homework 2.4.1.5 $L\left(\begin{pmatrix}2\\3\end{pmatrix}\right)=$

$$\begin{pmatrix}2\\3\end{pmatrix}=\chi_0\begin{pmatrix}1\\0\end{pmatrix}+\chi_1\begin{pmatrix}1\\1\end{pmatrix}$$



Opening Remarks

- Solving Linear Systems of Equation
  - Let L be a linear transformation such that
    - $L\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $L\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$
  - $L\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = ?$

Opening Remarks

- Solving Linear Systems of Equation

$$\begin{array}{rcrcr} \chi_0 & + & \chi_1 & = & -1 \\ 2\chi_0 & + & 3\chi_1 & = & 0 \end{array}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 3 & 0 \end{array}\right)$$

Outline

- Solving system of linear equation
- Representing system of linear equation as appended matrices
- Reducing matrices to row echelon form
- LU factorization

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System
- Appended Matrices
- Gauss Transforms
- Computing Separately with the Matrix and Right-Hand Side (Forward Substitution)
- Towards an Algorithm

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System

$$\left(\begin{array}{cccc|c} \chi_{0,0} & \chi_{0,1} & \cdots & \chi_{0,m} & \alpha_0 \\ 0 & \chi_{1,1} & \cdots & \chi_{1,1} & \alpha_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \chi_{m,m} & \alpha_m \end{array}\right)$$

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System
  - Gaussian elimination (transform linear system of equations to an upper triangular system)
    - Solving the above linear system relies on the fact that its solution does not change if
      - Equations are reordered (not used until next week);
      - An equation in the system is modified by subtracting a multiple of another equation in the system from it; and/or
      - Both sides of an equation in the system are scaled by a nonzero number.

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System

$$
\begin{array}{rcrcrcr}
2x & + & 4y & - & 2z & = & -10 \\
4x & - & 2y & + & 6z & = & 20 \\
6x & - & 4y & + & 2z & = & 18
\end{array}
$$

$$
\begin{array}{rcrcrcr}
2\chi_0 & + & 4\chi_1 & - & 2\chi_2 & = & -10 \\
4\chi_0 & - & 2\chi_1 & + & 6\chi_2 & = & 20 \\
6\chi_0 & - & 4\chi_1 & + & 2\chi_2 & = & 18
\end{array}
$$

Gaussian Elimination

- Linear Equations

$$\begin{array}{ccccccc} 2\chi_0 & + & 4\chi_1 & - & 2\chi_2 & = & -10 \\ 4\chi_0 & - & 2\chi_1 & + & 6\chi_2 & = & 20 \\ 6\chi_0 & - & 4\chi_1 & + & 2\chi_2 & = & 18 \end{array}$$

- Gaussian elimination
  - Following Step 1
    - $row1 - \lambda_{1,0} row0$
    - $row2 - \lambda_{2,0} row0$
  - Following Step 2
    - $row2 - \lambda_{2,1} row0$
- Back substitution
- Check your answer

Gaussian Elimination

- Practice with Gaussian Elimination
  - http://ulaff.s3.amazonaws.com/GaussianEliminationPractice/index.html

Homework 6.2.1.2 Compute the solution of the linear system of equations given by

$$\begin{array}{rcrcrcr} -2\chi_0 & + & \chi_1 & + & 2\chi_2 & = & 0 \\ 4\chi_0 & - & \chi_1 & - & 5\chi_2 & = & 4 \\ 2\chi_0 & - & 3\chi_1 & - & \chi_2 & = & -6 \end{array}$$

- $\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$

Gaussian Elimination

- **Appended Matrices**

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & -10 \\ 4 & -2 & 6 & 20 \\ 6 & -4 & 2 & 18 \end{array}\right)$$

Represent

$$\begin{array}{rcrcrcr} 2\chi_0 & + & 4\chi_1 & - & 2\chi_2 & = & -10 \\ 4\chi_0 & - & 2\chi_1 & + & 6\chi_2 & = & 20 \\ 6\chi_0 & - & 4\chi_1 & + & 2\chi_2 & = & 18 \end{array}$$

Gaussian Elimination

- Appended Matrices

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & -10 \\ 4 & -2 & 6 & 20 \\ 6 & -4 & 2 & 18 \end{array}\right)$$

- Gaussian elimination
  - Following Step 1
    - $row1 - \lambda_{1,0} row0$ & $row2 - \lambda_{2,0} row0$
  - Following Step 2
    - $row2 - \lambda_{2,1} row0$
- Back substitution
- Check your answer

Gaussian Elimination

- **Appended Matrice**

Homework 6.2.2.2 Compute the solution of the linear system of equations expressed as an appended matrix given by

$$\left(\begin{array}{rrr|r} -1 & 2 & -3 & 2 \\ -2 & 2 & -8 & 10 \\ 2 & -6 & 6 & -2 \end{array}\right)$$

- $\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$

Gaussian Elimination

- Gauss Transforms

$$0\quad\begin{pmatrix} 1 & 0 & 0 \\ \cancel{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & -2 \\ 4 & -2 & 6 \\ 6 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & -2 \\ 4 & -2 & 6 \\ 6 & -4 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{-\frac{4}{2}} & 1 & 0 \\ \boxed{-\frac{6}{2}} & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & -2 \\ 4 & -2 & 6 \\ 6 & -4 & 2 \end{pmatrix} = \begin{pmatrix} \boxed{2} & 4 & -2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-\frac{16}{10}} & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & -2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 & -2 \\ 0 & -10 & 10 \\ 0 & 0 & -8 \end{pmatrix}.$$

Gaussian Elimination

- ## Gauss Transforms

Theorem 6.1 *Let $\widehat{L}_j$ be a matrix that equals the identity, except that for $i > j$the (i,j) elements (the ones below the diagonal in the jth column) have been replaced with $-\lambda_{i,,j}$:*

$$\widehat{L}_j = \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $\widehat{L}_j A$ equals the matrix A except that for $i > j$ the ith row is modified by subtracting $\lambda_{i,j}$ times the jth row from it. Such a matrix $\widehat{L}_j$ is called a Gauss transform.

Gaussian Elimination

Proof: Let

$$\widehat{L}_j = \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T \\ \check{a}_{j+2}^T \\ \vdots \\ \check{a}_{m-1}^T \end{pmatrix},$$

where I_k equals a $k \times k$ identity matrix, $A_{s:t,:}$ equals the matrix that consists of rows s through t from matrix A, and $\check{a}_k^T$ equals the kth row of A. Then

$$\begin{aligned} \widehat{L}_j A &= \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T \\ \check{a}_{j+2}^T \\ \vdots \\ \check{a}_{m-1}^T \end{pmatrix} \\ &= \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ -\lambda_{j+1,j}\check{a}_j^T + \check{a}_{j+1}^T \\ -\lambda_{j+2,j}\check{a}_j^T + \check{a}_{j+2}^T \\ \vdots \\ -\lambda_{m-1,j}\check{a}_j^T + \check{a}_{m-1}^T \end{pmatrix} = \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T - \lambda_{j+1,j}\check{a}_j^T \\ \check{a}_{j+2}^T - \lambda_{j+2,j}\check{a}_j^T \\ \vdots \\ \check{a}_{m-1}^T - \lambda_{m-1,j}\check{a}_j^T \end{pmatrix}. \end{aligned}$$

Gaussian Elimination

- **Gauss Transforms**

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & -10 \\ 4 & -2 & 6 & 20 \\ 6 & -4 & 2 & 18 \end{array}\right)$$

- **Gaussian elimination**

$$- \begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 2 & 4 & -2 & -10 \\ 4 & -2 & 6 & 20 \\ 6 & -4 & 2 & 18 \end{array}\right)$$

$$- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-16/-10) & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 0 & -10 & 10 & 40 \\ 0 & -16 & 8 & 48 \end{array}\right)$$

- **Back substitution**
- **Check your answer**

Gaussian Elimination

- **Computing Separately with the Matrix and Right-Hand Side (Forward Substitution)**

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 & -10 \\ 0 & -10 & 10 & 40 \\ 0 & -16 & 8 & 48 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 6 \\ 6 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ \cancel{-4}/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -10 \\ 20 \\ 18 \end{pmatrix} = \begin{pmatrix} -10 \\ 40 \\ 48 \end{pmatrix}$$

-6

Gaussian Elimination

$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & \boxed{a_{21}} & \boxed{A_{22}} \end{array}\right)$$

- **Towards an Algorithm**

Before

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & -2 & 6 \\ 6 & -4 & 2 \end{array}\right)$$

After

$$\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ 3 & -16 & 8 \end{array}\right)$$

Before

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -1.6 & 1 \end{array}\right) \left(\begin{array}{c|c|c} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ \hline 3 & -16 & 8 \end{array}\right)$$

After

$$\left(\begin{array}{ccc} 2 & 4 & -2 \\ 2 & -10 & 10 \\ 3 & 1.6 & -8 \end{array}\right)$$

Gaussian Elimination

$$
\left(\begin{array}{ccc|c|ccc}
1 & & & & & & \\
& \ddots & & 0 & & 0 & \\
& & 1 & & & & \\
\hline
& 0 & & 1 & & 0 & \\
\hline
& & & \times & 1 & & \\
& 0 & & \vdots & & \ddots & \\
& & & \times & & & 1
\end{array}\right)
\left(\begin{array}{ccc|c|ccc}
\times & \cdots & \times & \times & \times & \cdots & \times \\
& \ddots & \times & \vdots & \times & \ddots & \times \\
& & \times & \times & \times & \cdots & \times \\
\hline
& & & \times & \times & \cdots & \times \\
\hline
& & & \times & \times & \cdots & \times \\
& & & \vdots & \times & \ddots & \times \\
& & & \times & \times & \cdots & \times
\end{array}\right)
$$

$$
= \left(\begin{array}{ccc|c|ccc}
\times & \cdots & \times & \times & \times & \cdots & \times \\
& \ddots & \times & \vdots & \times & \ddots & \times \\
& & \times & \times & \times & \cdots & \times \\
\hline
& & & \times & \times & \cdots & \times \\
\hline
& & & 0 & \times & \cdots & \times \\
& & & \vdots & \times & \ddots & \times \\
& & & 0 & \times & \cdots & \times
\end{array}\right)
$$

$$
\underbrace{\left(\begin{array}{c|c|c}
I & 0 & 0 \\
\hline
0 & 1 & 0 \\
\hline
0 & -l_{21} & I
\end{array}\right)
\left(\begin{array}{c|c|c}
U_{00} & u_{01} & U_{22} \\
\hline
0 & \alpha_{11} & a_{12}^T \\
\hline
0 & a_{21} & A_{22}
\end{array}\right)}
= \left(\begin{array}{c|c|c}
U_{00} & u_{01} & U_{22} \\
\hline
0 & \upsilon_{11} & u_{12}^T \\
\hline
0 & 0 & A_{22}^{\mathrm{new}}
\end{array}\right)
$$

$$
\underbrace{\left(\begin{array}{c|c}
U_{00} & u_{01} \quad U_{22} \\
\hline
0 & \left(\begin{array}{c|c}
1 & 0 \\
\hline
-l_{21} & I
\end{array}\right)
\left(\begin{array}{c|c}
\alpha_{11} & a_{12}^T \\
\hline
a_{21} & A_{22}
\end{array}\right)
\end{array}\right)}
$$

$$
\left(\begin{array}{c|c}
U_{00} & u_{01} \quad U_{22} \\
\hline
0 & \left(\begin{array}{c|c}
\alpha_{11} & a_{12}^T \\
\hline
-l_{21}\alpha_{11} + a_{21} & -l_{21}a_{12}^T + A_{22}
\end{array}\right)
\end{array}\right)
$$

Gaussian Elimination

Before

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & -2 & 6 \\ 6 & -4 & 2 \end{array}\right)$$

Before

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -1.6 & 1 \end{array}\right) \left(\begin{array}{c|c|c} 2 & 4 & -2 \\ \hline {\scriptstyle 2} & -10 & 10 \\ \hline {\scriptstyle 3} & -16 & 8 \end{array}\right)$$

Algorithm: $A := \text{GAUSSIAN\_ELIMINATION}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

$$a_{21} := a_{21}/\alpha_{11} \qquad (= l_{21})$$

$$A_{22} := A_{22} - a_{21}a_{12}^T \qquad (= A_{22} - l_{21}a_{12}^T)$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Gaussian Elimination

$$\underbrace{\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right)\left(\begin{array}{c} y_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)} = \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline b_2^{\text{new}} \end{array}\right)$$

$$\underbrace{\left(\begin{array}{c} y_0 \\ \hline \left(\begin{array}{c|c} 1 & 0 \\ \hline -l_{21} & I \end{array}\right)\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right) \end{array}\right)}$$

$$\left(\begin{array}{c} y_0 \\ \hline \left(\begin{array}{c} \beta_1 \\ \hline -l_{21}\beta_{11} + b_2 \end{array}\right) \end{array}\right)$$

Algorithm: $b := \text{FORWARD\_SUBSTITUTION}(A, b)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where A_{TL} is 0×0, b_T has 0 rows

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$b_2 := b_2 - \beta_1 a_{21} \qquad (= b_2 - \beta_1 l_{21})$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Solving $Ax = b$ via LU Factorization

- LU factorization (Gaussian elimination)
- Solving $Lz = b$ (Forward substitution)
- Solving $Ux = b$ (Back substitution)
- Putting it all together to solve $Ax = b$
- Cost

Solving Ax = b via LU Factorization

- LU factorization (Gaussian elimination)
  - Idea
    - A matrix $A \in \mathbb{R}^{n \times n}$ can be factored into the product of two matrices $\mathbf{L}, \mathbf{U} \in \mathbb{R}^{n \times n}$

$$A = LU$$

      - where L is unit lower triangular and U is upper triangular.

Solving Ax = b via LU Factorization

- **LU factorization (Gaussian elimination)**

Assume $A \in \mathbb{R}^{n\times n}$ is given and that L and U are to be computed such that $A = LU$, where $L \in \mathbb{R}^{n\times n}$ is unit lower triangular and $U \in \mathbb{R}^{n\times n}$ is upper triangular. We derive an algorithm for computing this operation by partitioning

$$A \rightarrow \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right), \quad L \rightarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right), \quad \text{and} \quad U \rightarrow \left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right).$$

Now, $A = LU$ implies (using what we learned about multiplying matrices that have been partitioned into submatrices)

$$\begin{aligned}
\overbrace{\left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right)}^{A} &= \overbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right)}^{L} \overbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)}^{U} \\
&= \overbrace{\left(\begin{array}{c|c} 1 \times \upsilon_{11} + 0 \times 0 & 1 \times u_{12}^T + 0 \times U_{22} \\ \hline l_{21}\upsilon_{11} + L_{22} \times 0 & l_{21}u_{12}^T + L_{22}U_{22} \end{array}\right)}^{LU}. \\
&= \overbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline l_{21}\upsilon_{11} & l_{21}u_{12}^T + L_{22}U_{22} \end{array}\right)}^{LU}.
\end{aligned}$$

Solving Ax = b via LU Factorization

- **LU factorization (Gaussian elimination)**

For two matrices to be equal, their elements must be equal, and therefore, if they are partitioned conformally, their submatrices must be equal:

$$\begin{array}{c|c} \alpha_{11} = \upsilon_{11} & a_{12}^T = u_{12}^T \\ \hline a_{21} = l_{21}\upsilon_{11} & A_{22} = l_{21}u_{12}^T + L_{22}U_{22} \end{array}$$

or, rearranging,

$$\begin{array}{c|c} \upsilon_{11} = \alpha_{11} & u_{12}^T = a_{12}^T \\ \hline l_{21} = a_{21}/\upsilon_{11} & L_{22}U_{22} = A_{22} - l_{21}u_{12}^T \end{array}.$$

This suggests the following steps for **overwriting** a matrix A with its LU factorization:

- Partition

$$A \rightarrow \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array} \right).$$

- Update $a_{21} = a_{21}/\alpha_{11} (= l_{21})$. (Scale a_{21} by $1/\alpha_{11}$!)

Solving Ax = b via LU Factorization

- LU factorization (Gaussian elimination)

Algorithm: $A := \text{LU\_UNB\_VAR5}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

where α_{11} is 1×1

$a_{21} := a_{21}/\alpha_{11} \qquad (= l_{21})$

$A_{22} := A_{22} - a_{21}a_{12}^T \qquad (= A_{22} - l_{21}a_{12}^T)$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Step	$\left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$	a_{21}/α_{11}	$A_{22} - a_{21}a_{12}^T$
1-2	$\left(\begin{array}{c\|cc} -2 & -1 & 1 \\ \hline 2 & -2 & -3 \\ -4 & 4 & 7 \end{array}\right)$	$\begin{pmatrix} 2 \\ -4 \end{pmatrix} / (-2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -2 & -3 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 6 & 5 \end{pmatrix}$
3	$\left(\begin{array}{c\|c\|c} -2 & -1 & 1 \\ \hline -1 & -3 & -2 \\ \hline 2 & 6 & 5 \end{array}\right)$	$\begin{pmatrix} 6 \end{pmatrix} / (-3) = \begin{pmatrix} -2 \end{pmatrix}$	$\begin{pmatrix} 5 \end{pmatrix} - \begin{pmatrix} -2 \end{pmatrix} \begin{pmatrix} -2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$
	$\left(\begin{array}{cc\|c} -2 & -1 & 1 \\ -1 & -3 & -2 \\ \hline 2 & -2 & 1 \end{array}\right)$		

Step	Current system	Multiplier	Operation
1	$\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 2 & -2 & -3 & 3 \\ -4 & 4 & 7 & -3 \end{array}\right)$	$\frac{2}{-2} = -1$	$\begin{array}{r} 2 \quad -2 \quad -3 \quad 3 \\ -1 \times (-2 \quad -1 \quad 1 \quad 6) \\ \hline 0 \quad -3 \quad -2 \quad 9 \end{array}$
2	$\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ -4 & 4 & 7 & -3 \end{array}\right)$	$\frac{-4}{-2} = 2$	$\begin{array}{r} -4 \quad 4 \quad 7 \quad -3 \\ -(2) \times (-2 \quad -1 \quad 1 \quad 6) \\ \hline 0 \quad 6 \quad 5 \quad -15 \end{array}$
3	$\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 6 & 5 & -15 \end{array}\right)$	$\frac{6}{-3} = -2$	$\begin{array}{r} 0 \quad 6 \quad 5 \quad -15 \\ -(-2) \times (0 \quad -3 \quad -2 \quad 9) \\ \hline 0 \quad 0 \quad 1 \quad 3 \end{array}$
4	$\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right)$		

Solving Ax = b via LU Factorization

- **Solving Lz = b (Forward substitution)**

Given a unit lower triangular matrix $L \in \mathbb{R}^{n\times n}$ and vectors $z, b \in \mathbb{R}^n$, consider the equation $Lz = b$ where L and b are known and z is to be computed. Partition

$$L \rightarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right), \quad z \rightarrow \left(\begin{array}{c} \zeta_1 \\ \hline z_2 \end{array}\right), \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right).$$

(Recall: the horizontal line here partitions the result. It is *not* a division.) Now, $Lz = b$ implies that

$$\begin{aligned} \overbrace{\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right)}^{b} &= \overbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right)}^{L} \overbrace{\left(\begin{array}{c} \zeta_1 \\ \hline z_2 \end{array}\right)}^{z} \\ &= \overbrace{\left(\begin{array}{c} 1 \times \zeta_1 + 0 \times z_2 \\ \hline l_{21}\zeta_1 + L_{22}z_2 \end{array}\right)}^{Lz} = \overbrace{\left(\begin{array}{c} \zeta_1 \\ \hline l_{21}\zeta_1 + L_{22}z_2 \end{array}\right)}^{Lz} \end{aligned}$$

so that

$$\begin{array}{c} \beta_1 = \zeta_1 \\ \hline b_2 = l_{21}\zeta_1 + L_{22}z_2 \end{array} \quad \text{or, equivalently,} \quad \begin{array}{c} \zeta_1 = \beta_1 \\ \hline L_{22}z_2 = b_2 - \zeta_1 l_{21} \end{array}.$$

Solving Ax = b via LU Factorization

- Solving Lz = b (Forward substitution)

Algorithm: $[b] := \text{LTRSV\_UNB\_VAR1}(L, b)$

Partition $L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where L_{TL} is 0×0, b_T has 0 rows

while $m(L_{TL}) < m(L)$ **do**

Repartition

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

where λ_{11} is 1×1 , β_1 has 1 row

$b_2 := b_2 - \beta_1 l_{21}$

Continue with

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Step	$\left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right)$	$\left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$	$b_2 - l_{21}\beta_1$
1-2	$\left(\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{array}\right)$	$\left(\begin{array}{r} 6 \\ 3 \\ -3 \end{array}\right)$	$\left(\begin{array}{r} 3 \\ -3 \end{array}\right) - \left(\begin{array}{r} -1 \\ 2 \end{array}\right)(6) = \left(\begin{array}{r} 9 \\ -15 \end{array}\right)$
3	$\left(\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{array}\right)$	$\left(\begin{array}{r} 6 \\ 9 \\ -15 \end{array}\right)$	$(-15) - (-2)(9) = (3)$
	$\left(\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{array}\right)$	$\left(\begin{array}{r} 6 \\ 9 \\ 3 \end{array}\right)$	

Solving Ax = b via LU Factorization

- **Solving Ux = b (Back substitution)**

Given upper triangular matrix $U \in \mathbb{R}^{n\times n}$ and vectors $x, b \in \mathbb{R}^n$, consider the equation $Ux = b$ where U and b are known and x is to be computed. Partition

$$U \rightarrow \left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right), \quad x \rightarrow \left(\begin{array}{c} \chi_1 \\ \hline x_2 \end{array}\right) \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right).$$

Now, $Ux = b$ implies

$$\begin{aligned} \overbrace{\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right)}^{b} &= \overbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)}^{U} \overbrace{\left(\begin{array}{c} \chi_1 \\ \hline x_2 \end{array}\right)}^{x} \\ &= \overbrace{\left(\begin{array}{c} \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline 0 \times \chi_1 + U_{22}x_2 \end{array}\right)}^{Ux} = \overbrace{\left(\begin{array}{c} \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline U_{22}x_2 \end{array}\right)}^{Ux} \end{aligned}$$

so that

$$\begin{array}{c} \beta_1 = \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline b_2 = U_{22}x_2 \end{array} \quad \text{or, equivalently,} \quad \begin{array}{c} \chi_1 = (\beta_1 - u_{12}^T x_2)/\upsilon_{11} \\ \hline U_{22}x_2 = b_2 \end{array}.$$

Solving Ax = b via LU Factorization

- Solving Ux = b (Back substitution)

Algorithm: $[b] := \text{UTRSV\_UNB\_VAR1}(U, b)$

Partition $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

  where U_{BR} is 0×0, b_B has 0 rows

while $m(U_{BR}) < m(U)$ **do**

  Repartition

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$\beta_1 := \beta_1 - u_{12}^T b_2$

$\beta_1 := \beta_1 / \upsilon_{11}$

  Continue with

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Solving Ax = b via LU Factorization

- **Putting it all together to solve Ax = b**

Want to solve:	$Ax = b.$
We can now find triangular L and U so that	$A = LU.$
Substitute:	$(LU)x = b.$
Matrix multiplication is associative:	$L(Ux) = b.$
We don't know x but we can create a dummy vector $y = Ux$.	$L \underbrace{y}_{Ux} = b.$
Solve Solving a (lower) triangular system is easy!	$Ly = b$ for y
Solve Solving an (upper) triangular system is easy!	$Ux = y$ for x

Questions and Answers