

# More Gaussian Elimination and Matrix Inversion

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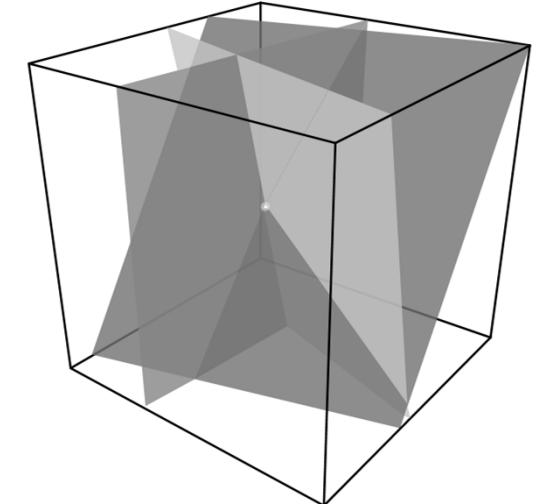
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# Opening Remarks

$$Ax = b$$



$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} \\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

- Gaussian Elimination
- Back substitution
- Check your answer

# When Gaussian Elimination Breaks Down

- When Gaussian Elimination Works
- The Problem
- Permutations
- Gaussian Elimination with Row Swapping (LU factorization with Partial Pivoting)
- When Gaussian Elimination Fails Altogether

# When Gaussian Elimination Breaks Down

- When Gaussian Elimination Works

$$Ux = b$$

**Algorithm:**  $[b] := \text{UTRSV\_UNB\_VAR1}(U, b)$

Partition  $U \rightarrow \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}$ ,  $b \rightarrow \begin{pmatrix} b_T \\ b_B \end{pmatrix}$

where  $U_{BR}$  is  $0 \times 0$ ,  $b_B$  has 0 rows

while  $m(U_{BR}) < m(U)$  do

Repartition

$$\begin{pmatrix} U_{TL} & U_{TR} \\ 0 & U_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} U_{00} & u_{01} & U_{02} \\ 0 & v_{11} & u_{12}^T \\ 0 & 0 & U_{22} \end{pmatrix}, \begin{pmatrix} b_T \\ b_B \end{pmatrix} \rightarrow \begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$$

$$\beta_1 := \beta_1 - u_{12}^T b_2$$

$$\beta_1 := \beta_1 / v_{11}$$

Continue with

$$\begin{pmatrix} U_{TL} & U_{TR} \\ 0 & U_{BR} \end{pmatrix} \leftarrow \begin{pmatrix} U_{00} & u_{01} & U_{02} \\ 0 & v_{11} & u_{12}^T \\ 0 & 0 & U_{22} \end{pmatrix}, \begin{pmatrix} b_T \\ b_B \end{pmatrix} \leftarrow \begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$$

endwhile

$$Lz = b$$

**Algorithm:**  $[b] := \text{LTSRV\_UNB\_VAR1}(L, b)$

Partition  $L \rightarrow \begin{pmatrix} L_{TL} & 0 \\ L_{BL} & L_{BR} \end{pmatrix}$ ,  $b \rightarrow \begin{pmatrix} b_T \\ b_B \end{pmatrix}$

where  $L_{TL}$  is  $0 \times 0$ ,  $b_T$  has 0 rows

while  $m(L_{TL}) < m(L)$  do

Repartition

$$\begin{pmatrix} L_{TL} & 0 \\ L_{BL} & L_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} L_{00} & 0 & 0 \\ l_{10}^T & \lambda_{11} & 0 \\ L_{20} & l_{21} & L_{22} \end{pmatrix}, \begin{pmatrix} b_T \\ b_B \end{pmatrix} \rightarrow \begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$$

where  $\lambda_{11}$  is  $1 \times 1$ ,  $\beta_1$  has 1 row

$$b_2 := b_2 - \beta_1 l_{21}$$

Continue with

$$\begin{pmatrix} L_{TL} & 0 \\ L_{BL} & L_{BR} \end{pmatrix} \leftarrow \begin{pmatrix} L_{00} & 0 & 0 \\ l_{10}^T & \lambda_{11} & 0 \\ L_{20} & l_{21} & L_{22} \end{pmatrix}, \begin{pmatrix} b_T \\ b_B \end{pmatrix} \leftarrow \begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$$

endwhile

# When Gaussian Elimination Breaks Down

## • When Gaussian Elimination Works

**Homework 7.2.1.1** Let  $L \in \mathbb{R}^{1 \times 1}$  be a unit lower triangular matrix.  $Lx = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.2** Give the solution of  $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

**Homework 7.2.1.3** Give the solution of  $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ .

(Hint: look carefully at the last problem, and you will be able to save yourself some work.)

**Homework 7.2.1.4** Let  $L \in \mathbb{R}^{2 \times 2}$  be a unit lower triangular matrix.  $Lx = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.5** Let  $L \in \mathbb{R}^{3 \times 3}$  be a unit lower triangular matrix.  $Lx = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.6** Let  $L \in \mathbb{R}^{n \times n}$  be a unit lower triangular matrix.  $Lx = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

# When Gaussian Elimination Breaks Down

## • When Gaussian Elimination Works

**Homework 7.2.1.8** Let  $L \in \mathbb{R}^{n \times n}$  be a unit lower triangular matrix.  $Lx = 0$ , where 0 is the zero vector of size  $n$ , has the unique solution  $x = 0$ .

Always/Sometimes/Never

**Homework 7.2.1.9** Let  $U \in \mathbb{R}^{1 \times 1}$  be an upper triangular matrix with no zeroes on its diagonal.  $Ux = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.10** Give the solution of  $\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

**Homework 7.2.1.11** Give the solution of  $\begin{pmatrix} -2 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ .

**Homework 7.2.1.12** Let  $U \in \mathbb{R}^{2 \times 2}$  be an upper triangular matrix with no zeroes on its diagonal.  $Ux = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.13** Let  $U \in \mathbb{R}^{3 \times 3}$  be an upper triangular matrix with no zeroes on its diagonal.  $Ux = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.14** Let  $U \in \mathbb{R}^{n \times n}$  be an upper triangular matrix with no zeroes on its diagonal.  $Ux = b$ , where  $x$  is the unknown and  $b$  is given, has a unique solution.

Always/Sometimes/Never

# When Gaussian Elimination Breaks Down

- **The Problem**

A simple example where Gaussian elimination and LU factorization break down involves the matrix  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . In the first step, the multiplier equals  $1/0$ , which will cause a “division by zero” error.

Now,  $Ax = b$  is given by the set of linear equations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$$

so that  $Ax = b$  is equivalent to

$$\begin{pmatrix} \chi_1 \\ \chi_0 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

**Algorithm:**  $[b] := \text{UTRSV\_UNB\_VAR1}(U, b)$

**Partition**  $U \rightarrow \left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right)$ ,  $b \rightarrow \left( \begin{array}{c} b_T \\ b_B \end{array} \right)$   
where  $U_{BR}$  is  $0 \times 0$ ,  $b_B$  has 0 rows

**while**  $m(U_{BR}) < m(U)$  **do**

**Repartition**

$$\left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & v_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right), \left( \begin{array}{c} b_T \\ b_B \end{array} \right) \rightarrow \left( \begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

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$$\beta_1 := \beta_1 - u_{12}^T b_2$$

$$\beta_1 := \beta_1 / v_{11}$$


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**Continue with**

$$\left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & v_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right), \left( \begin{array}{c} b_T \\ b_B \end{array} \right) \leftarrow \left( \begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

**endwhile**

# When Gaussian Elimination Breaks Down

## • The Problem

**Homework 7.2.2.1** Solve the following linear system, via the steps in Gaussian elimination that you have learned so far.

$$2\chi_0 + 4\chi_1 + (-2)\chi_2 = -10$$

$$4\chi_0 + 8\chi_1 + 6\chi_2 = 20$$

$$6\chi_0 + (-4)\chi_1 + 2\chi_2 = 18$$

Mark all that are correct:

- (a) The process breaks down.
- (b) There is no solution.

(c) 
$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

**Homework 7.2.2.2** Perform Gaussian elimination with

$$0\chi_0 + 4\chi_1 + (-2)\chi_2 = -10$$

$$4\chi_0 + 8\chi_1 + 6\chi_2 = 20$$

$$6\chi_0 + (-4)\chi_1 + 2\chi_2 = 18$$

# When Gaussian Elimination Breaks Down

- Permutations

Homework 7.2.3.1 Compute

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_P \underbrace{\begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix}}_A =$$

# When Gaussian Elimination Breaks Down

## • Permutations

**Definition 7.1** *A vector with integer components*

$$p = \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_{n-1} \end{pmatrix}$$

*is said to be a permutation vector if*

- $k_j \in \{0, \dots, n-1\}$ , for  $0 \leq j < n$ ; and
- $k_i = k_j$  implies  $i = j$ .

*In other words,  $p$  is a rearrangement of the numbers  $0, \dots, n-1$  (without repetition).*

We will often write  $(k_0, k_1, \dots, k_{n-1})^T$  to indicate the column vector, for space considerations.

# When Gaussian Elimination Breaks Down

- ## Permutations

**Definition 7.2** Let  $p = (k_0, \dots, k_{n-1})^T$  be a permutation vector. Then

$$P = P(p) = \begin{pmatrix} e_{k_0}^T \\ e_{k_1}^T \\ \vdots \\ e_{k_{n-1}}^T \end{pmatrix}$$

is said to be a permutation matrix.

In other words,  $P$  is the identity matrix with its rows rearranged as indicated by the permutation vector  $(k_0, k_1, \dots, k_{n-1})$ . We will frequently indicate this permutation matrix as  $P(p)$  to indicate that the permutation matrix corresponds to the permutation vector  $p$ .

# When Gaussian Elimination Breaks Down

## • Permutations

**Homework 7.2.3.2** For each of the following, give the permutation matrix  $P(p)$ :

- If  $p = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$  then  $P(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ,

- If  $p = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$  then  $P(p) =$

**Homework 7.2.3.3** Let  $p = (2, 0, 1)^T$ . Compute

- $P(p) \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} =$

- $P(p) \begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix} =$

**Homework 7.2.3.4** Let  $p = (2, 0, 1)^T$  and  $P = P(p)$ . Compute

$$\begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix} P^T =$$

# When Gaussian Elimination Breaks Down

## • Permutations

**Homework 7.2.3.5** Let  $p = (k_0, \dots, k_{n-1})^T$  be a permutation vector. Consider

$$x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}.$$

Applying permutation matrix  $P = P(p)$  to  $x$  yields

$$Px = \begin{pmatrix} \chi_{k_0} \\ \chi_{k_1} \\ \vdots \\ \chi_{k_{n-1}} \end{pmatrix}.$$

**Homework 7.2.3.6** Let  $p = (k_0, \dots, k_{n-1})^T$  be a permutation. Consider

$$A = \begin{pmatrix} \tilde{a}_0^T \\ \tilde{a}_1^T \\ \vdots \\ \tilde{a}_{n-1}^T \end{pmatrix}.$$

Applying  $P = P(p)$  to  $A$  yields

$$PA = \begin{pmatrix} \tilde{a}_{k_0}^T \\ \tilde{a}_{k_1}^T \\ \vdots \\ \tilde{a}_{k_{n-1}}^T \end{pmatrix}.$$

**Homework 7.2.3.7** Let  $p = (k_0, \dots, k_{n-1})^T$  be a permutation,  $P = P(p)$ , and  $A = \left( \begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$ .

$$AP^T = \left( \begin{array}{c|c|c|c} a_{k_0} & a_{k_1} & \cdots & a_{k_{n-1}} \end{array} \right).$$

Aways/Sometimes/Never

# When Gaussian Elimination Breaks Down

- Permutations

**Definition 7.3** Let us call the special permutation matrix of the form

$$\tilde{P}(\pi) = \begin{pmatrix} e_{\pi}^T \\ e_1^T \\ \vdots \\ e_{\pi-1}^T \\ e_0^T \\ e_{\pi+1}^T \\ \vdots \\ e_{n-1}^T \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

a pivot matrix.

# When Gaussian Elimination Breaks Down

- Permutations

**Homework 7.2.3.9** Compute

$$\tilde{P}(1) \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \quad \text{and} \quad \tilde{P}(1) \begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix} = .$$

**Homework 7.2.3.10** Compute

$$\begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix} \tilde{P}(1) = .$$

# When Gaussian Elimination Breaks Down

- Gaussian Elimination with Row Swapping (LU factorization with Partial Pivoting)

Homework 7.2.4.1 Compute

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 8 & 6 \\ 6 & -4 & 2 \end{pmatrix} =$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & -16 & 8 \\ 0 & 0 & 10 \end{pmatrix} =$$

- What do you notice?

$i$	$L_i$	$\tilde{P}$	$A$	$p$
0		$\begin{array}{ c c c } \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 2 & 4 & -2 \\ \hline 4 & 8 & 6 \\ \hline 6 & -4 & 2 \\ \hline \end{array}$	$\begin{array}{c} 0 \\ \cdot \\ \cdot \end{array}$
		$\begin{array}{ c c c } \hline 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ \hline -3 & 0 & 1 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 2 & 4 & -2 \\ \hline 4 & 8 & 6 \\ \hline 6 & -4 & 2 \\ \hline \end{array}$	$\begin{array}{c} 0 \\ \cdot \\ \cdot \end{array}$
1		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline 0 & 1 \\ \hline   &   \\ \hline 1 & 0 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 2 & 4 & -2 \\ \hline 2 & 0 & 10 \\ \hline 3 & -16 & 8 \\ \hline \end{array}$	$\begin{array}{c} 0 \\ 1 \\ \cdot \end{array}$
		$\begin{array}{ c c c } \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -0 & 1 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 2 & 4 & -2 \\ \hline 3 & -16 & 8 \\ \hline 2 & 0 & 10 \\ \hline \end{array}$	$\begin{array}{c} 0 \\ 1 \\ \cdot \end{array}$
2			$\begin{array}{ c c c } \hline 2 & 4 & -2 \\ \hline 3 & -16 & 8 \\ \hline 2 & 0 & 10 \\ \hline \end{array}$	$\begin{array}{c} 0 \\ 1 \\ 0 \end{array}$

frontiers

$i$	$L_i$	$\tilde{P}$	$A$	$p$
0		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline   &   \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 0 & 4 & -2 \\ \hline 4 & 8 & 6 \\ \hline 6 & -4 & 2 \\ \hline \end{array}$	$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$
		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline   &   \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 0 & 4 & -2 \\ \hline 4 & 8 & 6 \\ \hline 6 & -4 & 2 \\ \hline \end{array}$	$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$
1		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline   &   \\ \hline \end{array}$	$\begin{array}{ c c c } \hline   &   &   \\ \hline   &   &   \\ \hline   &   &   \\ \hline \end{array}$	$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$
		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline   &   \\ \hline \end{array}$	$\begin{array}{ c c c } \hline   &   &   \\ \hline   &   &   \\ \hline   &   &   \\ \hline \end{array}$	$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$
2		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline   &   \\ \hline \end{array}$	$\begin{array}{ c c c } \hline   &   &   \\ \hline   &   &   \\ \hline   &   &   \\ \hline \end{array}$	$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$
		$\begin{array}{ c c } \hline   &   \\ \hline   &   \\ \hline   &   \\ \hline \end{array}$	$\begin{array}{ c c c } \hline   &   &   \\ \hline   &   &   \\ \hline   &   &   \\ \hline \end{array}$	$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$

# When Gaussian Elimination Breaks Down

Algorithm:  $[A, p] := \text{LU\_PIV}(A, p)$

$$\text{Partition } A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right), p \rightarrow \left( \begin{array}{c} p_T \\ p_B \end{array} \right)$$

where  $A_{TL}$  is  $0 \times 0$  and  $p_T$  has 0 components

while  $m(A_{TL}) < m(A)$  do

Repartition

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left( \begin{array}{c} p_T \\ p_B \end{array} \right) \rightarrow \left( \begin{array}{c} p_0 \\ \pi_1 \\ p_2 \end{array} \right)$$

$$\pi_1 = \text{PIVOT} \left( \left( \frac{\alpha_{11}}{a_{21}} \right) \right)$$

$$\left( \begin{array}{c|c|c} a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) := P(\pi_1) \left( \begin{array}{c|c|c} a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

$a_{21} := a_{21}/\alpha_{11}$  (  $a_{21}$  now contains  $l_{21}$  )

$$\left( \begin{array}{c} a_{12}^T \\ A_{22} \end{array} \right) = \left( \begin{array}{c} a_{12}^T \\ A_{22} - a_{21}a_{12}^T \end{array} \right)$$

Continue with

$$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left( \begin{array}{c} p_T \\ p_B \end{array} \right) \leftarrow \left( \begin{array}{c} p_0 \\ \pi_1 \\ p_2 \end{array} \right)$$

endwhile

Algorithm:  $b := \text{APPLY\_PIV}(p, b)$

$$\text{Partition } p \rightarrow \left( \begin{array}{c} p_T \\ p_B \end{array} \right), b \rightarrow \left( \begin{array}{c} b_T \\ b_B \end{array} \right)$$

where  $p_T$  and  $b_T$  have 0 components

while  $m(b_T) < m(b)$  do

Repartition

$$\left( \begin{array}{c} p_T \\ p_B \end{array} \right) \rightarrow \left( \begin{array}{c} p_0 \\ \pi_1 \\ p_2 \end{array} \right), \left( \begin{array}{c} b_T \\ b_B \end{array} \right) \rightarrow \left( \begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

$$\left( \begin{array}{c} \beta_1 \\ b_2 \end{array} \right) := P(\pi_1) \left( \begin{array}{c} \beta_1 \\ b_2 \end{array} \right)$$

Continue with

$$\left( \begin{array}{c} p_T \\ p_B \end{array} \right) \leftarrow \left( \begin{array}{c} p_0 \\ \pi_1 \\ p_2 \end{array} \right), \left( \begin{array}{c} b_T \\ b_B \end{array} \right) \leftarrow \left( \begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

endwhile

# When Gaussian Elimination Breaks Down

- When Gaussian Elimination Fails Altogether
  - Gaussian elimination (LU factorization) with  $Ax = b$  where  $A$  is a square matrix, one of three things can happen:
    - The process completes with no zeroes on the diagonal of the resulting matrix  $U$ . Then  $A = LU$  and  $Ax = b$  has a unique solution, which can be found by solving  $Lz = b$  followed by  $Ux = z$ .
    - The process requires row exchanges, completing with no zeroes on the diagonal of the resulting matrix  $U$ . Then  $PA = LU$  and  $Ax = b$  has a unique solution, which can be found by solving  $Lz = Pb$  followed by  $Ux = z$ .
    - The process requires row exchanges, but at some point no row can be found that puts a nonzero on the diagonal, at which point the process fails (unless the zero appears as the last element on the diagonal, in which case it completes, but leaves a zero on the diagonal).

# The Inverse Matrix

- Inverse Functions in 1D
- Back to Linear Transformations
- Simple Examples
- More Advanced (but Still Simple) Examples
- Properties

# The Inverse Matrix

- Inverse Functions in 1D
  - $f: \mathbb{R} \rightarrow \mathbb{R}$  maps a real to a real; and
  - it is a bijection (both one-to-one and onto)
- then
  - $f(x) = y$  has a unique solution for all  $y \in \mathbb{R}$ .
  - The function that maps  $y$  to  $x$  so that  $g(y) = x$  is called the inverse of  $f$ .
  - It is denoted by  $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ .
  - Importantly,  $f(f^{-1}(x)) = x$  and  $f^{-1}(f(x)) = x$ .

# The Inverse Matrix

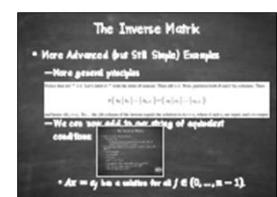
- Back to Linear Transformations
  - Theorem 7.5
    - Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a vector function. Then  $f$  is one-to-one and onto (a bijection) implies that  $m = n$ . The proof of this hinges on the dimensionality of  $\mathbb{R}^m$  and  $\mathbb{R}^n$ . We won't give it here.
  - Corollary 7.6
    - Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a vector function that is a bijection. Then there exists a function  $f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , which we will call its inverse, such that  $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ .

# The Inverse Matrix

- Back to Linear Transformations
  - Theorem 7.7
    - Let  $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation, and let  $A$  be the matrix that represents  $L$ . If there exists a matrix  $LB$  such that  $AB = BA = I$ , then  $L$  has an inverse,  $L^{-1}$ , and  $B$  equals the matrix that represents that linear transformation.

# The Inverse Matrix

- Back to Linear Transformations
  - Definition 7.8
    - A matrix  $A$  is said to be invertible if the inverse,  $A^{-1}$ , exists. An equivalent term for invertible is nonsingular.
- The following statements are equivalent statements about  $A \in \mathbb{R}^{n \times n}$ :
  - $A$  is nonsingular.
  - $A$  is invertible.
  - $A^{-1}$  exists.
  - $AA^{-1} = A^{-1}A = I$ .
  - $A$  represents a linear transformation that is a bijection.
  - $Ax = b$  has a unique solution for all  $b \in \mathbb{R}^n$ .
  - $Ax = 0$  implies that  $x = 0$ .



# The Inverse Matrix

- Simple Examples
  - General principles
    - $AB = BA = I$
  - Inverse of the Identity matrix

Homework 7.3.3.1 If  $I$  is the identity matrix, then  $I^{-1} = I$ .

True/False

- Inverse of a diagonal matrix

Homework 7.3.3.2 Find

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} =$$

# The Inverse Matrix

- Simple Examples
  - Inverse of a Gauss transform

Homework 7.3.3.4 Find

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}^{-1} =$$

Important: read the answer!

- Inverse of a permutation

Homework 7.3.3.7 Find

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

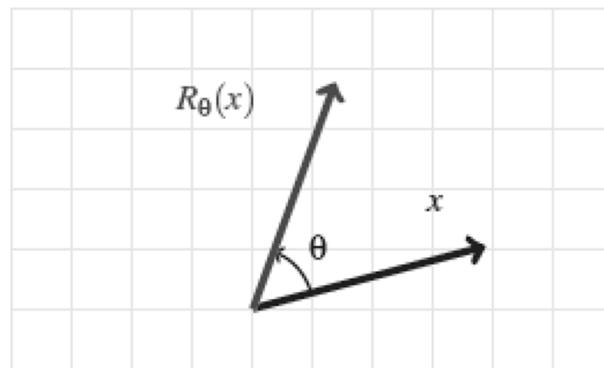
Homework 7.3.3.8 Find

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} =$$

# The Inverse Matrix

- Simple Examples
  - Inverting a 2D rotation

Homework 7.3.3.11 Recall from Week 2 how  $R_\theta(x)$  rotates a vector  $x$  through angle  $\theta$ :



$R_\theta$  is represented by the matrix

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

What transformation will “undo” this rotation through angle  $\theta$ ? (Mark all correct answers)

(a)  $R_{-\theta}(x)$

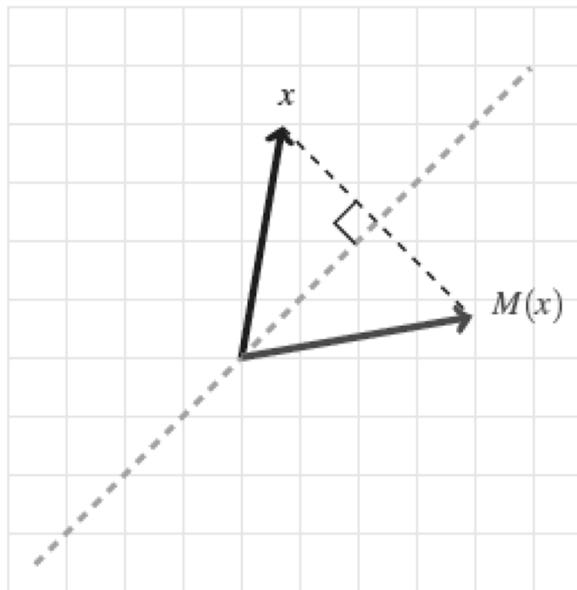
(b)  $Ax$ , where  $A = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

(c)  $Ax$ , where  $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

# The Inverse Matrix

- Simple Examples
  - Inverting a 2D reflection

Homework 7.3.3.12 Consider a reflection with respect to the 45 degree line:



If  $A$  represents the linear transformation  $M$ , then

(a)  $A^{-1} = -A$

(b)  $A^{-1} = A$

(c)  $A^{-1} = I$

(d) All of the above.

# The Inverse Matrix

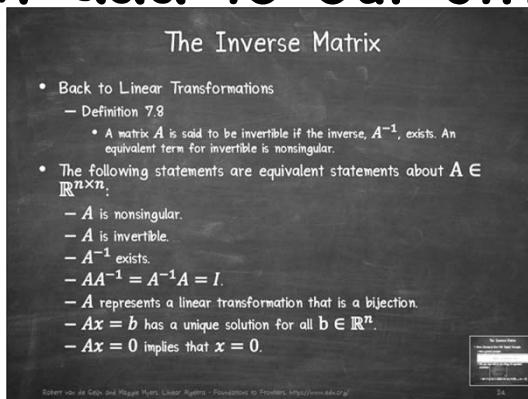
- More Advanced (but Still Simple) Examples
  - More general principles

Notice that  $AA^{-1} = I$ . Let's label  $A^{-1}$  with the letter  $B$  instead. Then  $AB = I$ . Now, partition both  $B$  and  $I$  by columns. Then

$$A \left( \begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right) = \left( \begin{array}{c|c|c|c} e_0 & e_1 & \cdots & e_{n-1} \end{array} \right)$$

and hence  $Ab_j = e_j$ . So.... the  $j$ th column of the inverse equals the solution to  $Ax = e_j$  where  $A$  and  $e_j$  are input, and  $x$  is output.

- We can now add to our string of equivalent conditions:



- $Ax = e_j$  has a solution for all  $j \in \{0, \dots, n - 1\}$ .

# The Inverse Matrix

- More Advanced (but Still Simple) Examples
  - Inverse of a triangular matrix

Homework 7.3.4.1 Compute  $\begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}^{-1} =$

Homework 7.3.4.2 Find

$$\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}^{-1} =$$

# The Inverse Matrix

- More Advanced (but Still Simple) Examples
  - Inverting a  $2 \times 2$  matrix

Homework 7.3.4.7 Find

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} =$$

Homework 7.3.4.8 If  $\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1} \neq 0$  then

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix}^{-1} = \frac{1}{\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1}} \begin{pmatrix} \alpha_{1,1} & -\alpha_{0,1} \\ -\alpha_{1,0} & \alpha_{0,0} \end{pmatrix}$$

(Just check by multiplying... Deriving the formula is time consuming.)

True/False

Homework 7.3.4.9 The  $2 \times 2$  matrix  $A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix}$  has an inverse if and only if  $\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1} \neq 0$ .

True/False

# The Inverse Matrix

The expression  $\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1} \neq 0$  is known as the *determinant* of

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix}.$$

This  $2 \times 2$  matrix has an inverse if and only if its determinant is nonzero. We will see how the determinant is useful again later in the course, when we discuss how to compute eigenvalues of small matrices. The determinant of a  $n \times n$  matrix can be defined and is similarly a condition for checking whether the matrix is invertible. For this reason, we add it to our list of equivalent conditions:

The following statements are equivalent statements about  $A \in \mathbb{R}^{n \times n}$ :

- $A$  is nonsingular.
- $A$  is invertible.
- $A^{-1}$  exists.
- $AA^{-1} = A^{-1}A = I$ .
- $A$  represents a linear transformation that is a bijection.
- $Ax = b$  has a unique solution for all  $b \in \mathbb{R}^n$ .
- $Ax = 0$  implies that  $x = 0$ .
- $Ax = e_j$  has a solution for all  $j \in \{0, \dots, n-1\}$ .
- The determinant of  $A$  is nonzero:  $\det(A) \neq 0$ .

# The Inverse Matrix

- Properties

- Inverse of product

**Homework 7.3.5.1** Let  $\alpha \neq 0$  and  $B$  have an inverse. Then

$$(AB)^{-1} = \frac{1}{\alpha}B^{-1}.$$

True/False

**Homework 7.3.5.2** Which of the following is true regardless of matrices  $A$  and  $B$  (as long as they have an inverse and are of the same size)?

(a)  $(AB)^{-1} = A^{-1}B^{-1}$

(b)  $(AB)^{-1} = B^{-1}A^{-1}$

(c)  $(AB)^{-1} = B^{-1}A$

(d)  $(AB)^{-1} = B^{-1}$

**Homework 7.3.5.3** Let square matrices  $A, B, C \in \mathbb{R}^{n \times n}$  have inverses  $A^{-1}$ ,  $B^{-1}$ , and  $C^{-1}$ , respectively. Then  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ .

Always/Sometimes/Never

# The Inverse Matrix

- Properties
  - Inverse of transpose

Homework 7.3.5.4 Let square matrix  $A$  have inverse  $A^{-1}$ . Then  $(A^T)^{-1} = (A^{-1})^T$ .

Always/Sometimes/Never

- Inverse of inverse

Homework 7.3.5.5

$$(A^{-1})^{-1} = A$$

Always/Sometimes/Never

# Questions and Answers

