Frustrated Bound

A. Samoilenka[†] and Ya. Shnir[†⋆]

[†]Department of Theoretical Physics and Astrophysics
Belarusian State University, Minsk 220004, Belarus
[⋆]BLTP, JINR, Dubna, Russia

May 7, 2018

Abstract

1 Frustrated magnets model

We have energy density [1]:

$$\mathcal{E} = -\frac{I_1}{2}(\partial_i m_j)^2 + \frac{I_2}{2}(\Delta m_i)^2 + H(1-m_3) \tag{1}$$

Where $i, j = 1, 2, 3$ and $m_i m_i = 1$. Recall that hence

$$\partial_i m_j \partial_k m_j = -m_j \partial_i \partial_k m_j \tag{2}$$

Lets introduce matrix

$$M_{ij} = m_i \Delta m_j \tag{3}$$

Thus for example we get $\partial_i m_j \partial_i m_j = -m_j \partial_i^2 m_j = -TrM$. In the same manner we can rewrite energy as:

$$\mathcal{E} = \frac{I_1}{2}TrM + \frac{I_2}{2}Tr(MM^T) + HV(m_3) \tag{4}$$

Let's do now little trick which will help us get better bound later. If m_i minimizes functional (1), then it should be minimal also relative scaling $x \to \lambda x$:[1]

$$\left.\frac{d\mathcal{E}}{d\lambda}\right|_{\lambda=1} = \frac{I_1}{2}TrM - \frac{I_2}{2}Tr(MM^T) + 3HV(m_3) = 0 \tag{5}$$

Hence we can rewrite energy introducing arbitrary parameter α:

$$\mathcal{E} = \mathcal{E} + \alpha\left(\left.\frac{d\mathcal{E}}{d\lambda}\right|_{\lambda=1}\right) = \frac{I_1}{2}(1+\alpha)TrM + \frac{I_2}{2}(1-\alpha)Tr(MM^T) + (1+3\alpha)HV(m_3) \tag{6}$$

Actually this trick shoves the way to potentially improve any topological bound: if one thinks that formula for bound is not the best he can simply redefine parameters in the formula by this procedure and maximize it with respect to α.

[1]Here and farther we drop the integrals just for the sake of brevity

2 Usual Faddeev-Skyrme model

Now lets consider usual Faddeev-Skyrme model with Skyrme term quartic in derivatives and positive parameter for second in derivatives term. Following notation of [2]:[2]

$$\mathcal{E}_{FS} = \alpha_2(\partial_i m_j)^2 + \frac{\alpha_4}{2}\left[(\partial_i m_j \partial_i m_j)^2 - \partial_i m_k \partial_j m_k \partial_j m_n \partial_i m_n\right] + \alpha_0 V \tag{7}$$

Then using (2) and (3) we obtain the next form for $\mathcal{E}_{FS}$

$$\mathcal{E}_{FS} = -\alpha_2 TrM + \frac{\alpha_4}{2}\left[(TrM)^2 - (m_k\partial_i\partial_j m_k)^2\right] + \alpha_0 V \tag{8}$$

As $(m_k\partial_i\partial_j m_k)^2 \geq 0$ we get:

$$\mathcal{E}_{FS} \leq -\alpha_2 TrM + \frac{\alpha_4}{2}(TrM)^2 + \alpha_0 V \tag{9}$$

Cauchy-Schwartz inequality states that $(TrM)^2 \leq d\ TrM^2$, where $d = dim(M) = 3$ – dimension of the matrix. Thus we get:

$$\mathcal{E}_{FS} \leq -\alpha_2 TrM + \frac{3\alpha_4}{2} TrM^2 + \alpha_0 V \tag{10}$$

Next we can recall that any matrix could be represented as $M = M_s + M_a$, where $M_s^T = M_s$ is a symmetric part and $M_a^T = -M_a$ antisymmetric. And $TrM_a = Tr(M_s M_a) = 0$ because $Tr(M_s M_a) = (M_s)_{ij}(M_a)_{ji} = -(M_s)_{ji}(M_a)_{ji}$. Then we get

$$TrM^2 = TrM_s^2 + TrM_a^2$$

$$Tr(MM^T) = TrM_s^2 - TrM_a^2 \tag{11}$$

Where $TrM_a^2 = (M_a)_{ij}(M_a)_{ji} = -(M_a)_{ij}(M_a)_{ij} \leq 0$, hence $TrM^2 \leq Tr(MM^T)$ and we have new upper bound for $\mathcal{E}_{FS}$:

$$\mathcal{E}_{FS} \leq -\alpha_2 TrM + \frac{3\alpha_4}{2} Tr(MM^T) + \alpha_0 V \tag{12}$$

3 Bound for frustrated magnets

Now comparing (6) and (12) we see that in order for RHS to be equal we have to set:

$$\alpha_2 = -\frac{I_1}{2}(1+\alpha)$$

$$\alpha_4 = \frac{I_2}{3}(1-\alpha) \geq 0 \tag{13}$$

$$\alpha_0 = (1+3\alpha)H$$

Thus we obtained the bound for the Frustrated magnets model with respect to Faddeev-Skyrme model, but with $I_1 < 0$, and hence positive sign for first term:

$$\mathcal{E} \geq \mathcal{E}_{FS} \tag{14}$$

[2]we keep the notation for the field m_i instead of usual symbol $\vec{\phi}$

and corresponding redefinition of the parameters (13). Though we still have free parameter α, lets try to fix it. Obviously we should pick α so that $\mathcal{E}_{FS}$ would be maximal for the given set of parameters in the model (1).

Now lets use general formula for the topological bound by Harland [2]:

$$\mathcal{E} \geq 3^{3/8} 16\pi^2 \sqrt{\alpha_2 \alpha_4} Q^{3/4} \left(1 + \frac{1}{3} \frac{k}{1+\sqrt{1+k}}\right) \sqrt{\frac{2}{1+\sqrt{1+k}}} \tag{15}$$

where

$$k = \left(\frac{2^7 3^{11/4}}{7^3 \pi^{3/2}}\right)^2 \frac{\alpha_4 \alpha_0}{\alpha_2^2} \tag{16}$$

4 Generalization of scaling theorem

We can redefine so that

$$\mathcal{E} = -(\partial_i m_j)^2 + (\Delta m_i)^2 + HV \tag{17}$$

Let's consider more general changes for x, in form

$$x^i \to f^i(x^j) \tag{18}$$

so that now it has non equal diagonal elements, non-diagonal elements and depends on x^i, but still close to x, with small λ_{ij}:

$$\frac{\partial x^j}{\partial f^i} = \delta_{ij} + \lambda_{ij} \tag{19}$$

then $d^3 f = (1 - Tr\lambda) d^3 x$ and $\frac{\partial}{\partial f^i} = \partial_i + \lambda_{ij}\partial_j$. For solution first variation of energy is zero hence:

$$\int d^3x (-2\partial_i \vec{m} \partial_j \vec{m} + 4\partial_i \partial_j \vec{m} \Delta \vec{m} + 2\partial_j \vec{m} \Delta \vec{m} \partial_i - \delta_{ij}\mathcal{E})\lambda_{ij} = 0 \tag{20}$$

then

$$-2\partial_i \vec{m} \partial_j \vec{m} + 4\partial_i \partial_j \vec{m} \Delta \vec{m} - 2\partial_i(\partial_j \vec{m} \Delta \vec{m}) - \delta_{ij}\mathcal{E} = 0 \tag{21}$$

in particular we get

$$\partial_i \vec{m}^2 + \Delta \vec{m}^2 - 3HV = 2\partial_i(\partial_i \vec{m} \Delta \vec{m}) \tag{22}$$

5 Negative energy density

Let's consider $m_1 = f(r), m_2 = 0$, then

$$\mathcal{E} \sim -f'^2 + (f'' + 2\frac{f'}{r})^2 + \frac{H}{2} f^2 \tag{23}$$

for small f. In order for second to be zero we obtain $f' r^2 = c_1$, and hence $f = -\frac{c_1}{r} + c_2$. let's set $c_2 = 0$. then $\mathcal{E} \sim -\frac{c_1^2}{r^4} + \frac{H}{2}\frac{c_1^2}{r^2}$. Hence for small r energy density will be negative!

6 Conclusions

7 To do

-Can I prove that energy can be negative globally, not only locally?
-Can I make Wick rotation and analitically continue solution for $I_1 > 0$?

Acknowledgements

A.S. is very sorry to professors R. Poghossian and A. Belavin who's lectures during SymPhys XIV he entirely wasted on the derivation of this bound.

References

[1] P. Sutcliffe, Skyrmion knots in frustrated magnets, Phys. Rev. Lett. **118** (2017) no.24, 247203

[2] Harland, D. (2014). Topological energy bounds for the Skyrme and Faddeev models with massive pions. Physics Letters B, 728, 518-523.