Review of linear algebra

Vector and matrices - a useful way of organizing information about a model: variables, constants, initial values, ...

Matrix: $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ $m=3$ rows, $n=2$ columns

3×2 matrix

(Column) Vector: $\begin{bmatrix} g \\ h \\ i \end{bmatrix}$ A matrix with one dimension (typically columns) being $n=1$

The classic matrix problem: $Ax = b$

matrix of constants (A)

vector of constants (b)

vector of unknown variables (x)

This matrix problem is equivalent to a system of linear equations:

$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = l$$

Which can be written:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

Reminder about how matrix mult works

The question: how to solve this matrix problem for unknowns (x, y, z)?

Invert A:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

The definition of the matrix inverse is simply that: $A^{-1}A = I$

where I is the identity matrix: $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$

We will return later to the question of how to calculate the matrix inverse
(BF p.376)

Matrix transpose: switching elements (i,j) of matrix with elements (j,i)

For example

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

In MATLAB → ' is the transpose

Matrix determinant

In the matrix multiplication Ax, the $\det(A)$ tells us how much A makes vector x "bigger" in the geometric sense.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\vdots$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

I wont make you calc these, bc MATLAB will efficiently do this, but it good to remember what it means

$$= \ldots$$

Eigenvalues and eigenvectors

Back to our matrix problem:

$$Ax = b$$

The solutions, λ_i, to

$$\det(A - \lambda \underset{\substack{\uparrow \\ \text{identity matrix}}}{I}) = 0$$

are the $\boxed{\text{eigenvalues}}$

And each eigenvalue has a corresponding $\boxed{\text{eigenvector}}$, v_i, which is the solution to

$$(A - \lambda_i I) v_i = 0$$

Geometrically, $\boxed{\text{eigenvectors}}$ are the characteristic vectors that point in the direction that A stretches vectors, the $\boxed{\text{eigenvalue}}$ is the amount by which things are stretched

Why do we care about eigenvalues and eigenvectors?

because in models with coupled linear ODEs the solutions can be written in terms of exponential eigenfunctions

Example: One our our coupled box models:

$$\frac{dM_1}{dt} = k_{21} M_2 + k_{31} M_3 + (k_{12}+k_{13}) M_1$$

$$\frac{dM_2}{dt} = k_{12} M_1 + k_{32} M_3 - (k_{21}+k_{23}) M_2$$

$$\frac{dM_3}{dt} = k_{23} M_2 + k_{13} M_1 - (k_{32}+k_{31}) M_1$$

$$\frac{d}{dt}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

The eigenvalues and eigenvectors of K are λ_i and v_i

In general for any closed box model, the solution is

$$M_i(t) = \sum_i^N \left[E^{-1} M(t=0)\right]_i e^{\lambda_i t} v_j$$

where $E = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots \end{bmatrix}$

Or in other words, the solution is a bunch of exponential functions (eigenfuctions) with decay rates equal to the eigenvalves of K