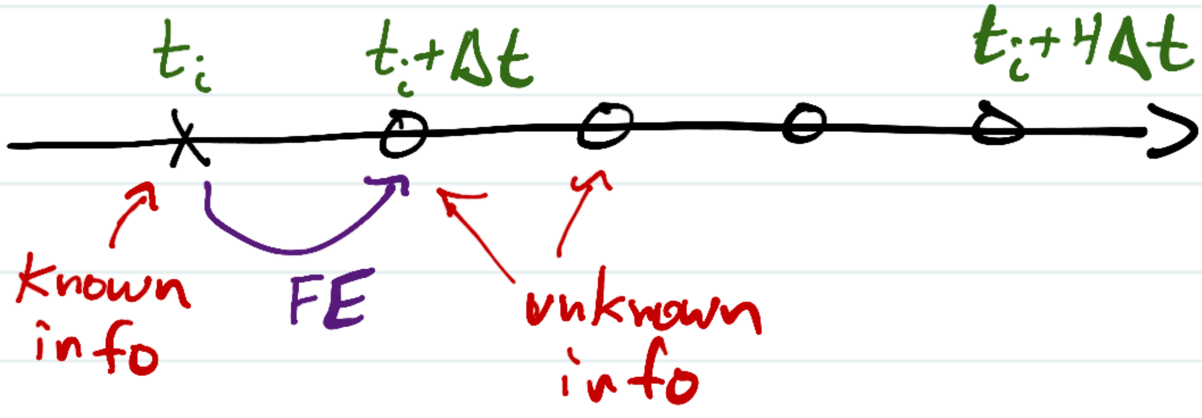
Higher-order ODE methods

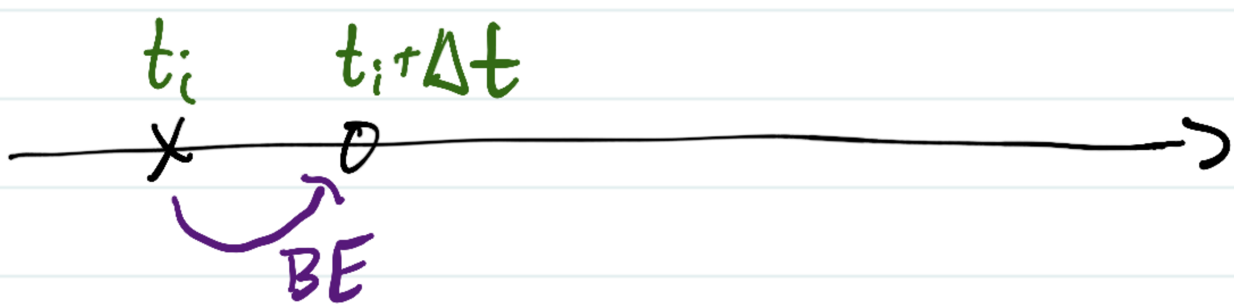
→ In thinking about these different methods for solving ODEs, it is useful to consider the stencil of a scheme:

Graphical representation of the times that info comes from

FE



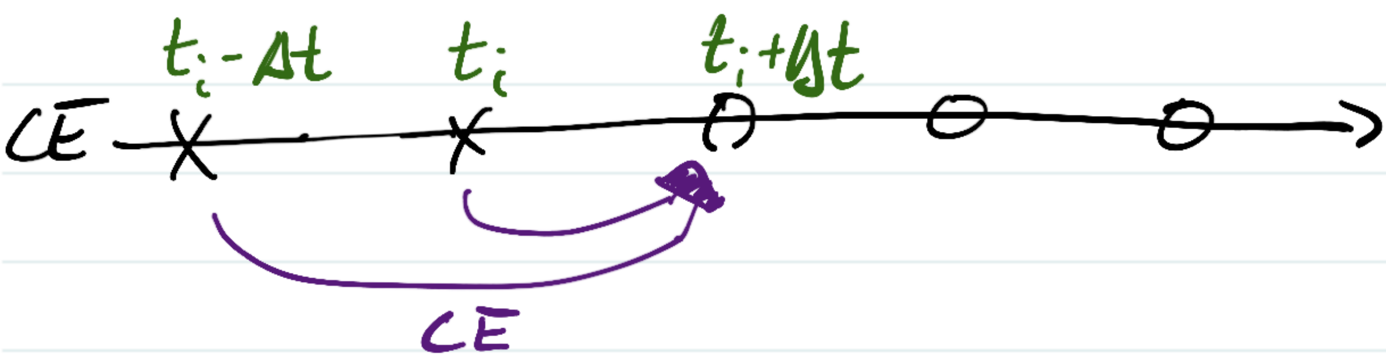
BE



Stencil is the same, we are just using the information differently

We can consider a scheme that uses more info

Centered Euler (aka the Leapfrog)



$$\frac{dx}{dt} \approx \underbrace{\frac{x(t+\Delta t) - x(t-\Delta t)}{2\Delta t}}_{\text{Wider stencil}} = f(x(t_i), t_i)$$

$$\boxed{\begin{aligned} x(t+\Delta t) &= x(t-\Delta t) + f(x_i, t_i)(2\Delta t) \\ x(0) &= x_0 \end{aligned}}$$

Centered Euler Scheme

Two notes about this scheme:

1. When we get started, we don't have info from two time steps. So we need to use FE or BE for the first time step, then CE.
2. The number of points in the scheme's stencil generally determines the scheme's <u>order of accuracy</u>
   - ↳ this can be determined mathematically, we'll cover that next week

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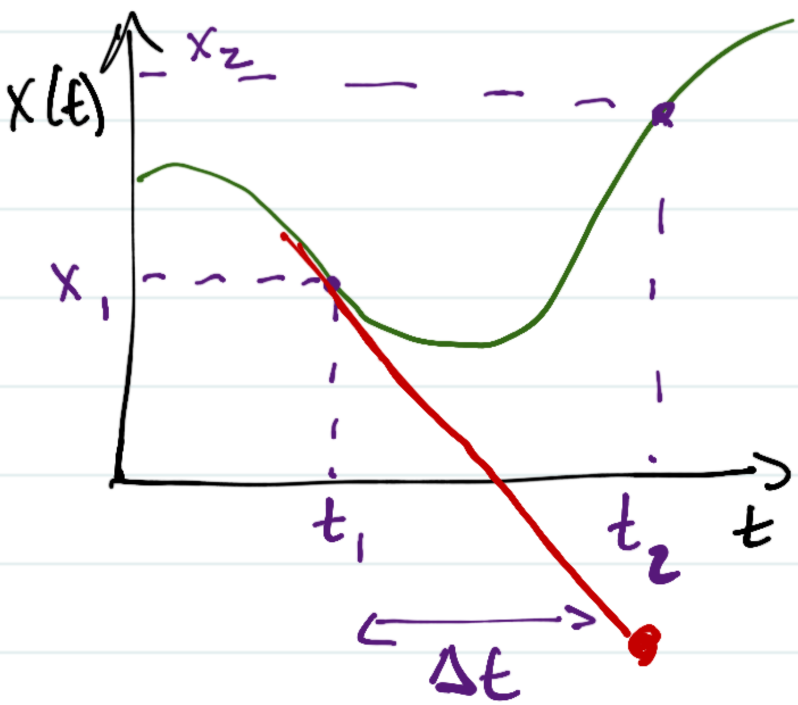
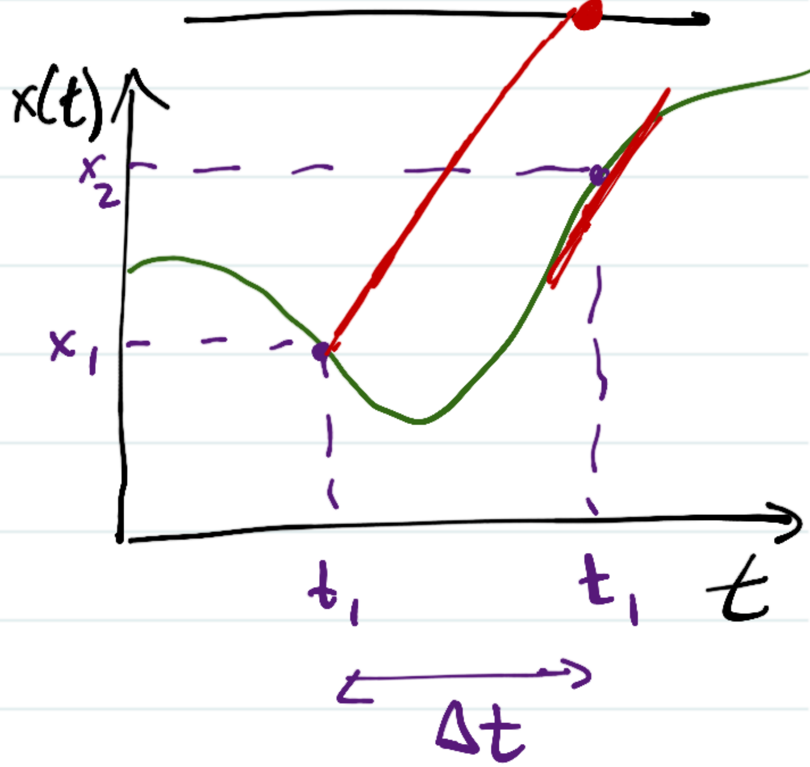
Different schemes work by approximating the slope of the solution using different approaches

Forward Euler



Approximate x_2 from slope at t_1

Backward Euler



Approximate x_2 from slope at t_2

→ Centered Euler approximates over a wider stencil, but still just slope at one point.

→ We can do better! We can calculate the exact slope at 'many diff points'.

$f(x_1, x_2, x_3, \ldots)$

Higher order methods

Define $\Delta x_{21} = x_2 - x_1$

$$\frac{\Delta x_{21}}{\Delta t} = f(x_1, t_1) \leftarrow \text{FE approx}$$

Define $k_1 = f(x_1, t_1)\Delta t$

Define $k_2 = f(x_1 + \beta k_1, t_1 + \alpha \Delta t)$

This is the slope at some other location on solution: time $t_1 + \alpha\Delta t$ and approx value $x_1 + \beta k_1$

k_1 and k_2 are two possible approximations

We assign them some weights: $\boxed{x_2 = x_1 + a k_1 + b k_2}$

To have any potential numerical method be accurate we would want it to at least match the general Taylor Expansion of x:

$$x_{i+1} = x_i + \underbrace{\frac{dx}{dt}}_{f}\Delta t + \frac{1}{2}\underbrace{\frac{d^2x}{dt^2}}(\Delta t)^2 + \underbrace{O(\Delta t^3)}_{\text{higher order terms}}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}f(x,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}f$$

$$\boxed{x_{i+1} = x_i + f(x,t)\Delta t + \frac{1}{2}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}f(x,t)\right]\Delta t^2 + O(\Delta t^3)}$$

Now we want our approx from above to match this.

$$f(t_i+\alpha\Delta t, x_i+\beta k_1) = f(t_i, x_i) + \left.\frac{\partial f}{\partial t}\right|_{t_i, x_i}\alpha\Delta t$$

$$\underbrace{+\left.\frac{\partial f}{\partial x}\right|_{t_i, x_i}\beta k_1 + O(\Delta t^2)}$$

$$k_2 = \Delta t \times f(t_i+\alpha\Delta t, x_i+\beta k_1)$$

Remember: $x_2 = x_1 + ak_1 + bk_2$

$$x_2 = x_1 + a\Delta t f(t_1, x_1) + b\Delta t\left[f(t_1, x_1) + \left.\frac{\partial f}{\partial t}\right|_{t_1, x_1}\alpha\Delta t + \left.\frac{\partial f}{\partial x}\right|_{t_1, x_1}\beta\Delta t f(x_1, y_1)\right] + O(\Delta t^3)$$

$$x_2 = x_1 + \Delta t f(x_1, t_1)(a+b) + \Delta t^2 b\left(\alpha\frac{\partial f}{\partial t} + \beta f\frac{\partial f}{\partial x}\right) + O(\Delta t^3)$$

To match Taylor expansion:

$$\boxed{\begin{aligned} a+b &= 1 \\ \alpha b &= \tfrac{1}{2} \\ \beta b &= \tfrac{1}{2} \end{aligned}}$$

Any method we construct must follow these rules, but there are infinite solns.

Predictor-Corrector Method

$\alpha = \beta = 1 \qquad a = b = \frac{1}{2}$

$$k_1 = \Delta t f(t_i, x_i)$$
$$k_2 = \Delta t f(t_i + \Delta t, x_i + k_1)$$

$$x_{i+1} = x_i + \frac{1}{2}(k_1 + k_2)$$

or $$\boxed{x_{i+1} = x_i + \frac{\Delta t}{2}\underbrace{\left[f(t_i, x_i) + f(t_{i+1}, x_i + f(x_i, t_i)\Delta t)\right]}_{\text{effectively an avg of FE and BE}}}$$

Euler-Richardson Method

$a = 0 \quad b = 1 \quad \alpha = \beta = \frac{1}{2}$

$$k_1 = \Delta t f(t_i, x_i)$$

$$k_2 = \Delta t f(t_i + \tfrac{1}{2}\Delta t, x_i + \tfrac{1}{2}k_1)$$

$$x_{i+1} = x_i + k_2$$

$$\hookrightarrow \boxed{x_{i+1} = x_i + f(t_i + \tfrac{1}{2}\Delta t, x_i + \tfrac{1}{2}\Delta t f(x_i, t_i))\Delta t}$$

$\frac{1}{2}$ time step with FE

Take it to the max!

Runge-Kutta - 4th order (aka RK4)

$$k_1 = f(t_i, x_i)\Delta t$$

$$k_2 = f(t_i + \tfrac{1}{2}\Delta t, x_i + \tfrac{1}{2}k_1)\Delta t$$

$$k_3 = f(t_i + \tfrac{1}{2}\Delta t, x_i + \tfrac{1}{2}k_2)\Delta t$$

$$k_4 = f(t_i + \Delta t, x_i + k_3)\Delta t$$

← must be computed in sequence

$$x_{i+1} = x_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$