Numerical Linear Algebra (BF 6.3, 6.5)

So, we often have a problem $Ax=b$ that we would like to solve, requiring a way to calculate A^{-1}, the inverse

The brute force method: Gaussian Elimination

For a matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

The inverse A^{-1} is a matrix which give

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This gives us 9 eqns with 9 unknowns to solve

Gaussian elimination is the process through which substitutions are performed to get all the a_{ij}

An example to show how exhausting Gaussian elimination is

$$\begin{bmatrix} 9 & 3 & 4 \\ 4 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$$

Augmented form:

$$\left[\begin{array}{ccc|ccc} 9 & 3 & 4 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Procedure: Get 1 in (1,1) by division

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{4}{9} & \frac{1}{9} & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Get zeros in (2,1), (3,1) by sub R_1

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{4}{9} & \frac{1}{9} & 0 & 0 \\ 0 & \frac{5}{3} & \frac{20}{9} & -\frac{4}{9} & 1 & 0 \\ 0 & \frac{2}{3} & \frac{5}{9} & -\frac{1}{9} & 0 & 0 \end{array}\right]$$

Get 1 in (2,2) by division

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{4}{9} & \frac{1}{9} & 0 & 0 \\ 0 & 1 & 10/9 & -\frac{2}{9} & \frac{3}{5} & 0 \\ 0 & \frac{2}{3} & \frac{5}{9} & -\frac{1}{9} & 0 & 0 \end{array}\right]$$

Get 0 in (1,2) and (3,2) by sub of R2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{27} & \frac{5}{27} & -\frac{1}{5} & 0 \\ 0 & 1 & 10/9 & -\frac{2}{9} & \frac{3}{5} & 0 \\ 0 & 0 & -\frac{5}{27} & \frac{1}{27} & -\frac{2}{5} & 0 \end{array}\right]$$

Get 1 in (3,3) by division

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{27} & \frac{5}{27} & -\frac{1}{5} & 0 \\ 0 & 1 & 10/9 & -2/9 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{54}{25} & 0 \end{array}\right]$$

Get 0 in (1,3) and (2,3) by sub by R3

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & 0 \\ 0 & -1 & 4 \\ -\frac{1}{5} & \frac{5}{4} & -3 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 9 & 3 & 4 \\ 4 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & 0 \\ 0 & -1 & 4 \\ -\frac{1}{5} & \frac{6}{5} & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

This is so tedious because it requires so many operations, generally n^3 for an $n \times n$ matrix.

If we always solved $Ax=b$ this way, everything you did on a computer would be waaay slower.

But, theres hope!

LU decomposition

If matrix A looks like (upper triangular matrix)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then our life is so much easier, because we know that

$$b_3 / a_{33} = x_3 \rightarrow a_{22}x_2 + a_{23}x_3 = b_2$$

etc etc

We only have to do backward substitution - way fewer operations, much faster.

The trick is then to turn any A into the product of lower and upper triangular matrices $\rightarrow A = LU$ which is generally possible if no row subs are needed during elim

→ This type of conversion (called LU factorization) is known as "pre-conditioning" and is used to make solving $Ax=b$ much faster (n^2 vs. n^3 → for a 100×100 matrix its 100x faster!)

→ Pre-conditioning requires many operations too, but can be worth it if we are going to solve $Ax=b$ many times with different b's and the same A

→ Factorization is built into MATLAB and many other programming languages are built upon algorithms for factorization

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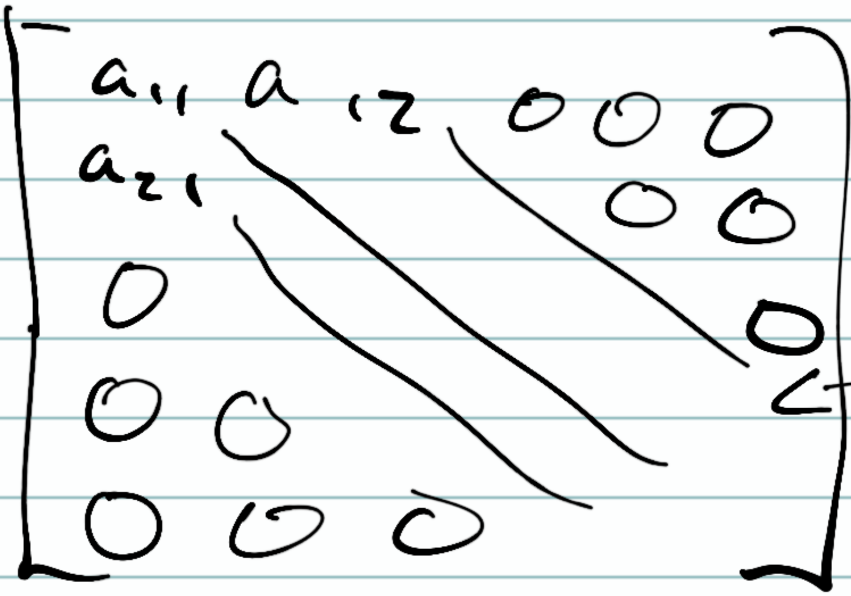
Another important type of matrix we will encounter are banded matrices where



These bands along the diagonal have values, but everything else is zero

Banded matrices are easier to solve because variables only interact with their neighbors:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$\vdots$$

The result is only $O(n)$ operations are needed to solve problems with tridiagonal matrices (3 bands)

-> These will be important when we talk about PDEs