# The Wave Equation

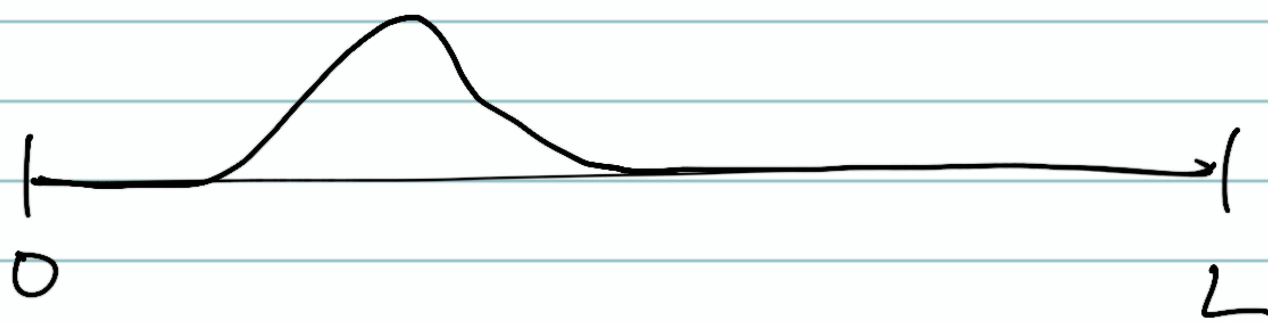
→ Reminder, the wave equation is an example of a hyperbolic PDE:

wave speed

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad u = u(x,t)$$

With some boundary conditions:

$$u(0,t) = a \qquad u(L,t) = b$$




And initial condition on both $u$ and $\frac{\partial u}{\partial t}$

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

→ So, we know that when $\alpha$ is constant in time and space, the wave equation has a solution of a traveling wave

i.e. $u(x,t) = f(x-at) + g(x+at)$

→ However this is not so simple if $\alpha$ is not constant for all $x, t$.

→ In Earth sciences, we frequently consider waves that travel through heterogeneous media:

Atmospheric gravity waves
Seismic waves
Ocean waves
Porosity waves etc etc

→ Such problems cannot be solved analytically for arbitrary $\alpha(x,t)$

A numerical method: discretize $u_{xx}, u_{tt}$ terms.

$x_i = a + i\Delta x$

$t_k = k\Delta t$

Then use centered difference formulas for both terms.

$$\frac{\partial^2 u}{\partial t^2} = \frac{u(x_i, t_{k+1}) - 2u(x_i, t_k) + u(x_i, t_{k-1})}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k)}{\Delta x^2}$$

We substitute $\lambda = \alpha \frac{\Delta t}{\Delta x}$ (similar to Courant number from adv eqn)

Leaving (after rearranging):

$$u_i^{k+1} = 2(1-\lambda^2) u_i^k + \lambda^2 (u_{i+1}^k + u_{i-1}^k) - u_i^{k-1}$$

In matrix form

$$\begin{bmatrix} \\ u_i^{k+1} \\ \\ \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & & \\ 0 & \lambda^2 & \ddots & \ddots & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \\ u_i^k \\ \\ \end{bmatrix} - \begin{bmatrix} \\ u_i^{k-1} \\ \\ \end{bmatrix}$$

(If $\alpha$ varies in space, we will have different $\lambda_i$ for each point)

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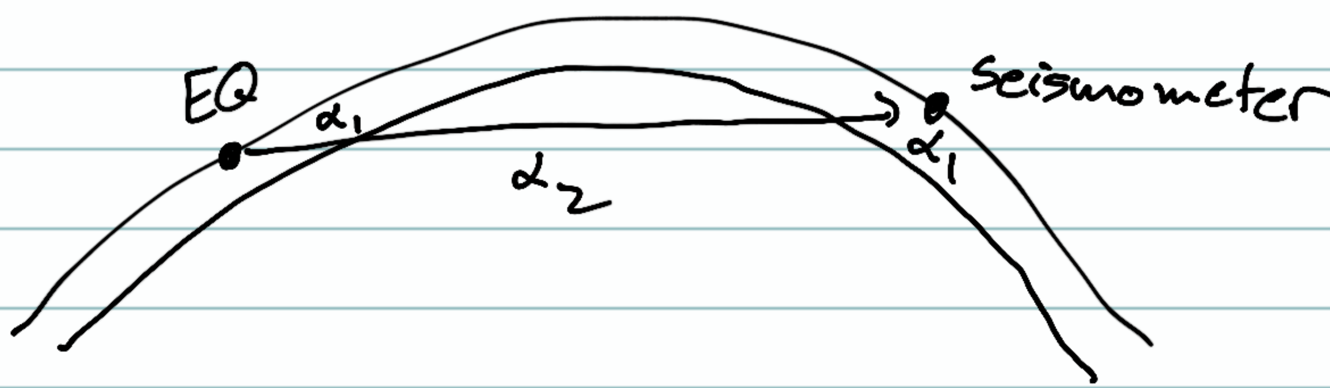
Live coding demo?

Wave propagation for arbitrary $\alpha(x,t)$?

(Refraction, etc.)




$$\alpha = \alpha_1 \quad \text{for} \quad 0 < x < x_1$$
$$\alpha = \alpha_2 \quad \text{for} \quad x_1 < x < x_2$$
$$\alpha = \alpha_1 \quad \text{for} \quad x_2 < x < x_3$$

$\alpha_1 = 1$ $\quad \alpha_2 = 2$ $\quad x_1 = 100$ $\quad x_2 = 200$ $\quad x_3 = 300$

IC
$$u(x, t=0) = \exp\left(-\frac{(x-10)^2}{10}\right)$$

$$u_t(x, t=0) = 0$$

BC
$$u(x=0, t) = 0$$
$$u(x=x_3, t) = 0$$