Linear Algebra Examples

Earth's gravity field (similar to SL equation from week 1)

The local acceleration due to the gravitational pull of an object is

$$d = \frac{Gp}{r^2}$$

$G \to$ Gravitational constant
$p \to$ mass of the object
$r \to$ distance to object
$d \to$ local grav. accel.

The sea level problem was a version of the forward problem:
- $\to$ we know $G, p, r \to$ calculate d

We can also consider the inverse problem:
- $\to$ We measure d, know $G, r \to$ want to know p

If the objects are discrete, we can write this as a matrix problem

Measurements, d_i: $\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$

Knowns, G, r_{ij} from location of each measurement to each object

Unknowns: object masses, p_i: $\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$

Problem can be written:

$$\underbrace{\begin{bmatrix} \frac{G}{r_{11}^2} & \frac{G}{r_{12}^2} & \cdots \\ \frac{G}{r_{21}^2} & \ddots & \\ \vdots & & \ddots \end{bmatrix}}_{F} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

This has the familiar form $Fp = d$ where we want to solve for $p = F^{-1}d$

Nuclear Decay Chain

→ Earlier we talked about the nuclear decay equation: $dN/dt = -\lambda N$ for a single radioactive element decay into another.

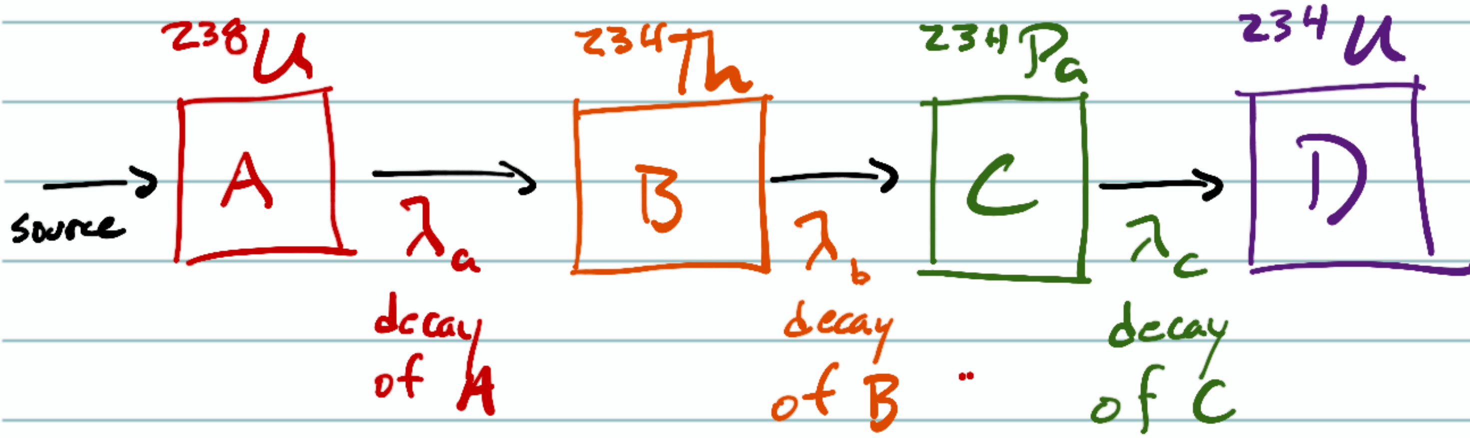
→ However, many radioactive elements undergo chains of decay through many other radioactive elements. These are especially important for <u>dating</u> materials in Earth science

Ex: Uranium Decay series

$$^{238}U \rightarrow {}^{234}Th \rightarrow {}^{234}Pa \rightarrow {}^{234}U \rightarrow \ldots \rightarrow {}^{206}Pb$$

many more things

In general, a decay series can be modeled as a series of boxes representing the concentrations of each radionuclide



$$\frac{dA}{dt} = -\lambda_a A + S \qquad \frac{dB}{dt} = -\lambda_b B + \lambda_a A$$

$$\frac{dC}{dt} = \lambda_b B + \lambda_c C \qquad \frac{dD}{dt} = -\lambda_d D$$

If $S = 0$, this is a closed system

To write this as a forward Euler system:

$$\frac{A(t+\Delta t) - A(t)}{\Delta t} = -\lambda_a A(t) + \cancel{S}^{\,0}$$

$$\frac{B(t+\Delta t) - B(t)}{\Delta t} = \lambda_a A(t) - \lambda_b B(t)$$

etc...

Can rewrite these:

$$A(t+\Delta t) = (1-\lambda_a \Delta t)A(t)$$
$$B(t+\Delta t) = \lambda_a \Delta t A(t) + (1-\lambda_b \Delta t)B(t)$$
$$\vdots$$

This can be written as a matrix problem (where R_i are the diff radionuclides)

$$\begin{bmatrix} \vdots \\ R_i(t+\Delta t) \\ \vdots \\ \\ \end{bmatrix} = \begin{bmatrix} 1-\lambda_a \Delta t & 0 & \cdots & 0 \\ \lambda_a \Delta t & 1-\lambda_b \Delta t & \ddots & \vdots \\ 0 & \lambda_b \Delta t & \ddots & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & 1 \\ 0 & 0 & & \end{bmatrix} \times \begin{bmatrix} \\ R_i(t) \\ \\ \end{bmatrix}$$

Banded matrix!

This is the forward problem, where we know the initial concentration and want to find out some future concentration.

$$\vec{R}(t+\Delta t) = \underline{\underline{M}}(\lambda_i, \Delta t)\,\vec{R}(t)$$

But we can also consider the more likely scenario of the <u>inverse</u> problem where we know the current concentrations and want to know the concentrations at some point in the past.

We can do this by stepping through matrix inverses

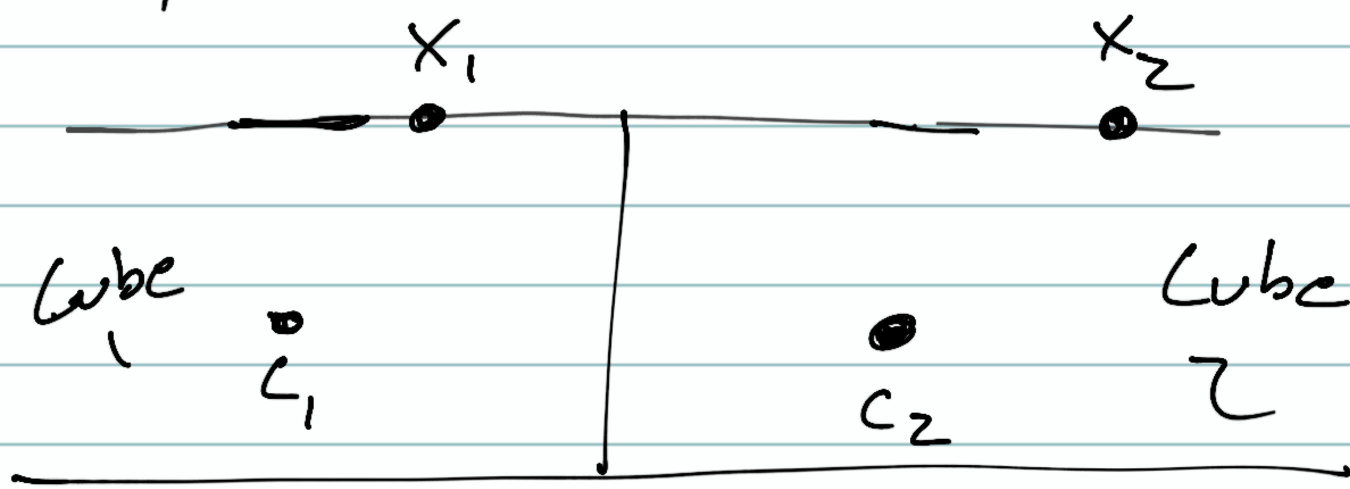
$$\boxed{\underbrace{(M^{-1})^n}_{\overbrace{M^{-1}M^{-1}\dots M^{-1}}^{n \text{ times}}} R(t_{now}) = R(t_{now} - n\Delta t)}$$

The inverse problem

If the R_i/R_j ratio of two of the nuclides at a specific time is known, we can also re-write this problem to solve for Δt (dating!)

In-class gravity inversion problem

① Consider two measurements of gravity at the Earth's surface



Write down the equations describing how the center-of-mass of each cube of earth is related to the measurements of gravitational accel

How many equation are there

② Now write this as a matrix problem

$$G\begin{bmatrix}(x_1-c_1)^{-2} & (x_1-c_2)^{-2}\\(x_2-c_1)^{-2} & (x_2-c_2)^{-2}\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}=\begin{bmatrix}d_1\\d_2\end{bmatrix}$$

③ Solve the inverse problem for the earth structure (p_1, p_2) @ $\begin{Bmatrix}(-1,-1)\\(1,-1)\end{Bmatrix}$

given $G=1$ and measurements of

$d_1 = 2.5$ @ $x_1 = (-1, 0)$

$d_2 = 1.2$ @ $x_2 = (1, 0)$