Examples for higher-order ODE solvers

(a) Planetary orbits in a gravitional field

→ A planet of mass m at a position (x,y)

→ Sun, mass $M \gg m$, not moving at position $(x,y) = (0,0)$

Newton's law of gravitation

$$m \frac{d^2 r}{dt^2} = F = -G \frac{Mm}{r^2}$$

↓ force on planet

where $\vec{r} = (x,y)$

$$\frac{d^2 \vec{r}}{dt^2} = \frac{-GM}{r^2} \longrightarrow$$

$$\boxed{\begin{aligned} \frac{d}{dt}\vec{r} &= \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} &= \frac{-GM}{(x^2+y^2)^{\frac{3}{2}}} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}}$$

Result: 4 coupled ODEs

(1) $$\frac{dx}{dt} = v_x$$

(3) $$\frac{dv_x}{dt} = -\frac{GM}{(x^2+y^2)^{\frac{3}{2}}}\,x$$

(2) $$\frac{dy}{dt} = v_y$$

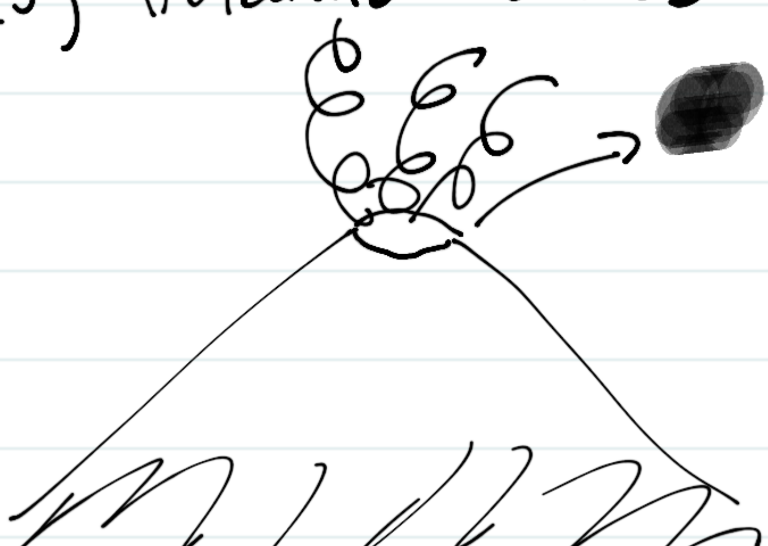
(4) $$\frac{dv_y}{dt} = -\frac{GM}{(x^2+y^2)^{\frac{3}{2}}}\,y$$

Need 4 initial condions

$x(t=0) = x_0$ $\qquad$ $v_x(t=0) = v_{x,0}$

$y(t=0) = y_0$ $\qquad$ $v_y(t=0) = v_{y,0}$

(b) Volcanic Bomb



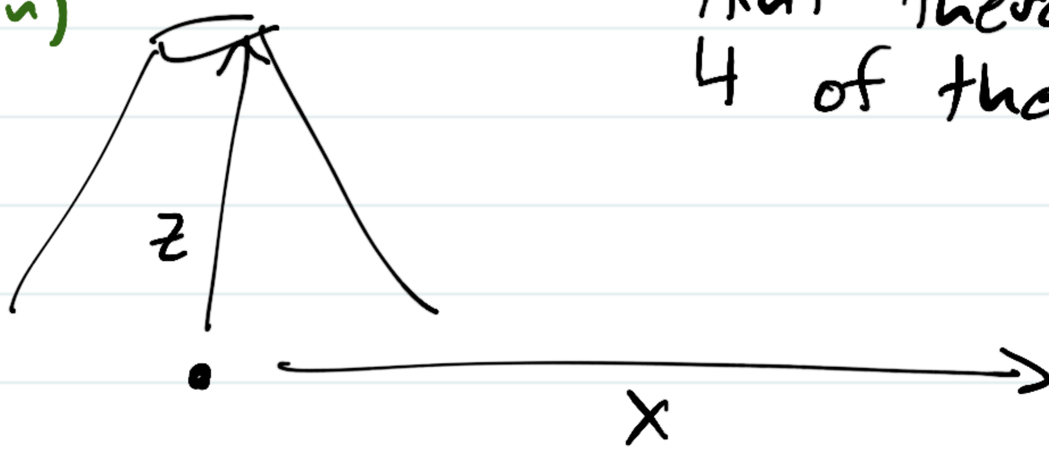
A volcanic bomb is a ballistic projectile that is thrown away from an erupting volcano

Activity

① ODEs needed to determine position in x and z?

(5 min)

Hint there are 4 of them



$$\frac{dv_x}{dt} = -\frac{\alpha}{m} v_x$$

(linear drag law)

$$\frac{dv_z}{dt} = -g - \frac{\alpha}{m} v_z \, \mathrm{sgn}(v_z)$$

$$\frac{dx}{dt} = v_x \qquad \frac{dz}{dt} = v_z$$

(2) Initial conditions:

How many do we need? 4

(2 min)

What are they

$$v_x(t=0) = v_{x,0}$$
$$v_z(t=0) = v_{z,0}$$

$$x(t=0) = 0$$
$$z(t=0) = H$$

(3) Now: use Predictor-Corrector method to write code - plot $x(t), z(t)$

(10 min) Use $v_{x,0} = 100$ m/s $\quad \frac{\alpha}{m} = 0.1\ s^{-1}$

$$v_{z,0} = 30 \text{ m/s}$$

$$H = 2000 \text{ m}$$

④ Answer question: at what x will $z=0$ (i.e. how far will rock be thrown?)

⑤ If you knew $x(z=0)$, could you determine v_x, v_y? What other info would you have to know?