Advection Equation

- If we have a flow of the background fluid, mobile things can be transported

J: amount of some stuff (heat, chemicals, biology, etc.)

chain rule!

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial x}\boxed{\frac{\partial x}{\partial t}}$$

→ velocity of frame of reference (i.e. fluid flow)

$$\boxed{\frac{\partial J}{\partial t} + u\frac{\partial J}{\partial x} = S(x,t)}$$

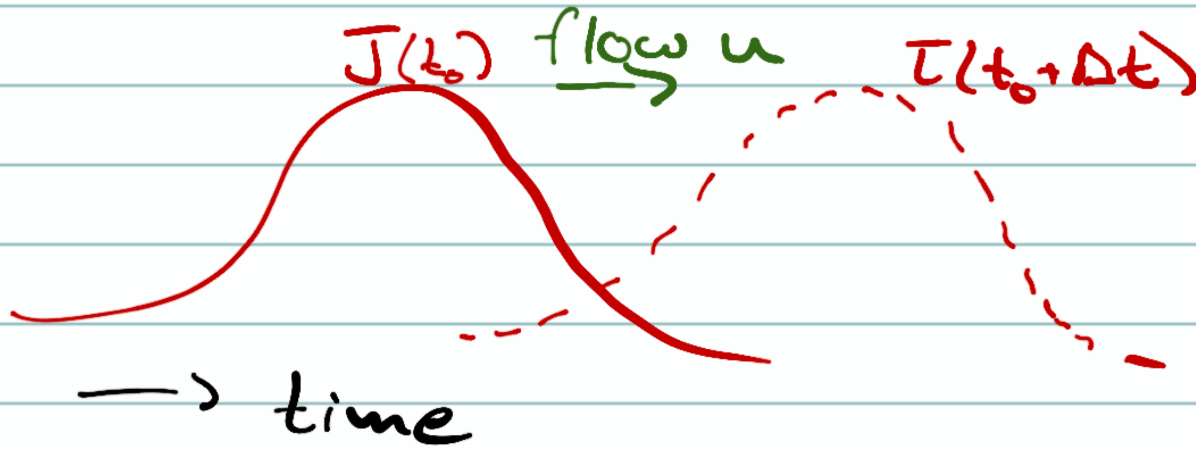
Advection Equation!

u: fluid velocity

$S(x,t)$: source of stuff

This form assumes u is constant, but it can also be written in more general flux form

$$\boxed{\frac{\partial J}{\partial t} + \frac{\partial}{\partial x}(uJ) = S(x,t)}$$



Advection equation describes how things are transported by <u>advection</u> of background flow

<u>Discretizing the advection equation</u>

Consider $J_i^k = J(x_i, t_k)$

$k \leftarrow$ time index

$i \leftarrow$ space index

$i \in [1, \ldots, n]$

$k \in [1, \ldots, m]$

The strategy: discretize each derivative seperately depending on type of variable and desired accuracy of method

Simplest: discretize time and space using forward Euler

Time

$$\frac{\partial J}{\partial t} = \frac{J_i^{k+1} - J_i^k}{\Delta t}$$

Space

$$\frac{\partial J}{\partial x} = \frac{J_i^k - J_{i-1}^k}{\Delta x}$$

Putting these together, the discretized advection eqn:

$$\frac{J_i^{k+1} - J_i^k}{\Delta t} = -u\,\frac{J_i^k - J_{i-1}^k}{\Delta x} + S(x_i, t_k)$$

In this scheme, this is the only unknown → we can rewrite to solve for this

$$\boxed{\bar{J}_i^{k+1} = \bar{J}_i^{k} - u\frac{\Delta t}{\Delta x}\left[\bar{J}_i^{k} - \bar{J}_{i-1}^{k}\right]}$$

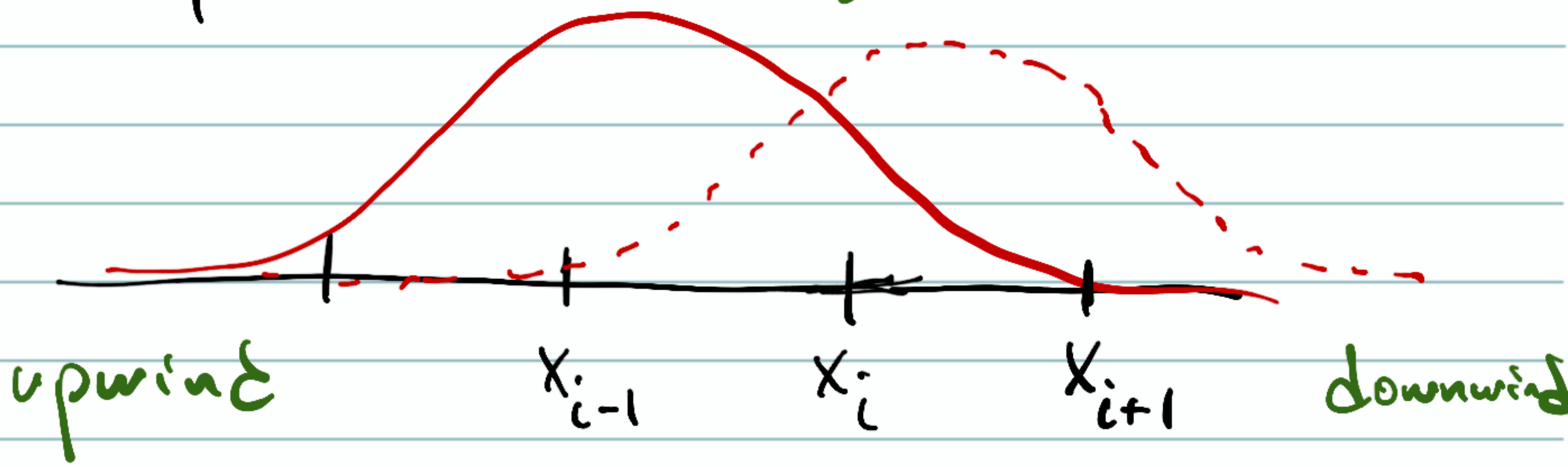
Marching Equation

$C = \frac{u\,\Delta t}{\Delta x}$ → A dimensionless parameter known as the Courant number (must be small to ensure stability)

This method is know as the <u>upwind scheme</u>

why?

u flow →



When $u > 0$ and $\bar{J}_i - \bar{J}_{i-1} < 0$ then $\frac{\partial \bar{J}}{\partial t} > 0$

Or in other words, information is carried from upwind to downwind (the dir of u)

→ To ensure that information is always upwinded → we must specify that if $u<0$, then

$$\frac{\partial J}{\partial x} \approx \frac{J_{i+1} - J_i}{\Delta x}$$

Pros and cons of upwind scheme

$$J_i^{k+1} - J_i^k = -u\frac{\Delta t}{\Delta x}\left(J_i^k - J_{i-1}^k\right)$$

$$-\frac{u\Delta t}{2\Delta x}\left(2J_i^k - 2J_{i-1}^k\right)$$

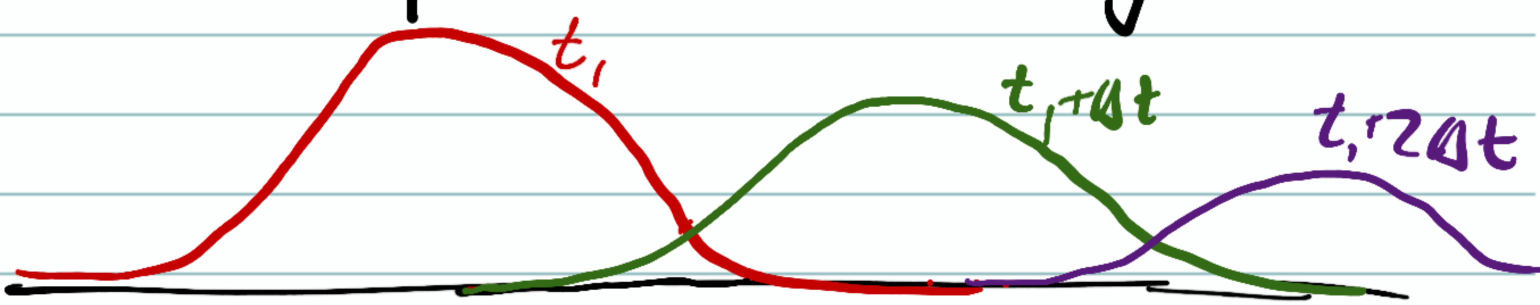
$$\underbrace{-\frac{u\Delta t}{2\Delta x}\left(+J_{i+1}^k - J_{i-1}^k\right)}_{\text{Term 1}} + \underbrace{\frac{u\Delta t}{2\Delta x}\left(J_{i+1}^k - 2J_i^k + J_{i-1}^k\right)}_{\text{Term 2}}$$

Term 1 Term

Centered difference approx for $\frac{\partial \bar{J}}{\partial x}$

Diffusion $\frac{\partial^2 \bar{J}}{\partial x^2} \Delta x^2$

The prominence of this diffusion term in the upwind scheme can be a problem - leading to:



Dissipation of concentrations of $\bar{J}$ $\longrightarrow$ time, x

Solution: be careful with picking $\Delta x, \Delta t$

$C = u \frac{\Delta t}{\Delta x}$ $\longrightarrow$ when $C > 1 \rightarrow$ method unstable
when $C < 1 \rightarrow$ method diffusive
when $C = 1 \rightarrow$ method stable + non-diffusive