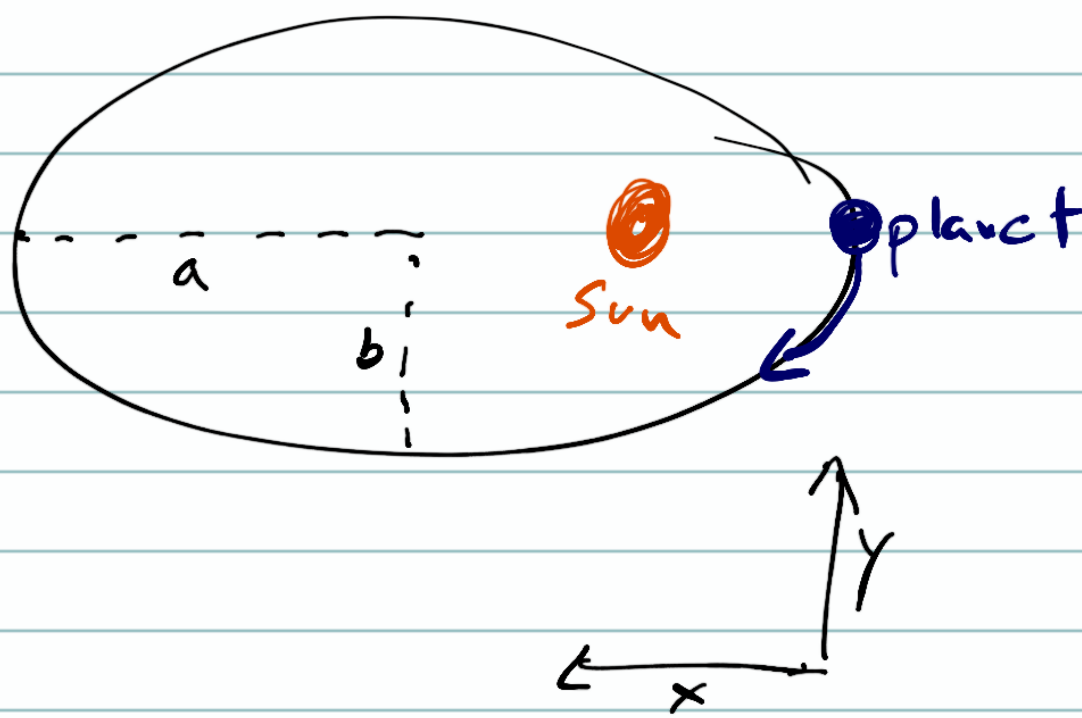
Root-finding examples

→ Kepler discovered that planets have elliptical orbits.



He found that the orbit could be described by these equations

$$\left.\begin{aligned} x(t) &= a\left[\cos(E) - e\right] \\ y(t) &= a\sqrt{1-e^2}\,\sin E \end{aligned}\right]$$

where $E = wt + e\sin(E)$

→ Great! Except that the equation for E is implicit and not (analytically soluble!

So, in practice a root-finding method is used to solve this equation:

$$0 = \omega t + e\sin(E) - E = f(E)$$

$$\frac{dF}{dE} = e\cos(E) - 1$$

Pseudocode for Newton-Raphson

Pick E guess (as long as $e<1$, should be safe that $F' \neq 0$)

$E_n = E_{guess}$

error = Inf

while error > 10^{-3}
  calc $f(E_n)$, $\frac{dF}{dE}\Big|_{E_n}$
  $E_{n+1} = E_n - f(E_n) / \frac{dF}{dE}\Big|_{E_n}$

$$error = |E_{n+1} - E_n|$$

end

Two other common uses for root-finding methods:

1. Backward Euler solves:

$$\frac{x_{i+1} - x_i}{\Delta t} = f(x_{i+1})$$

If we can't rearrange this equation to get

$x_{i+1} \propto g(x_i)$, then we need

a method to solve

$$x_{i+1} - f(x_{i+1})\Delta t - x_i = 0$$

2. Finding the steady-state of systems of coupled ODEs

An example: Lotka-Volterra model (aka Predator-Prey model)

Consider an ecosystem with two species

→ Species x always has enough food and grows exponentially due to reproduction:

$$\frac{dx}{dt} = \alpha x$$

→ Species y eats species x and grows according to how much it eats and its current size: $\frac{dy}{dt} = \delta \underbrace{xy}_{\text{eating}}$

→ Species x only dies by being eaten by y: βxy

→ Species y dies naturally due to its own population size (i.e. how much the ecosystem can support): γy

The model for population sizes of species x and y:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

<u>Steady states of population</u>

$$\frac{dx}{dt} = 0 = \alpha x - \beta xy$$

$$\frac{dy}{dt} = 0 = \delta xy - \gamma y$$

(We can solve)

$$0 = \alpha - \beta y \rightarrow \boxed{y = \alpha / \beta}$$

$$0 = \delta x - \gamma \rightarrow \boxed{x = \gamma / \delta}$$

Also, $\boxed{x = y = 0}$

But, if this model was slightly more complicated

$$\frac{dx}{\partial t} = \alpha x - \beta xy - \phi y$$

$$\frac{dy}{\partial t} = \delta xy - \gamma y + \lambda x$$

Suddenly not so easy to solve

$$f_1(x,y) = \alpha x - \beta xy - \phi y$$

$$f_2(x,y) = \delta xy - \gamma y + \lambda x$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha - \beta y & \delta y + \lambda \\ -\beta x - \phi & \delta x - \gamma \end{bmatrix}$$

Live coding example (Or Rothman's ocean from prev week)