Ordinary differential equations: review

What is an ODE?

Remember:

$$\left.\frac{dx}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Draw graph rep

In essence:

rate of change = process + process + process

ODEs are a central part of modeling in the Earth sciences, because change is everywhere in the Earth system and we are often interested in determining <u>why</u> something changed and predicting how it will change in the future.

quant of interest

In terms of equations: $\frac{dx}{dt} = f(x,t)$

time

describes a generic ODE

process causing change

→ You took a whole class on how to solve these equations exactly, but what if you can't?

→ Generally, for most interesting Earth science problems $f(x,t)$ is complicated and may involve many nonlinear terms

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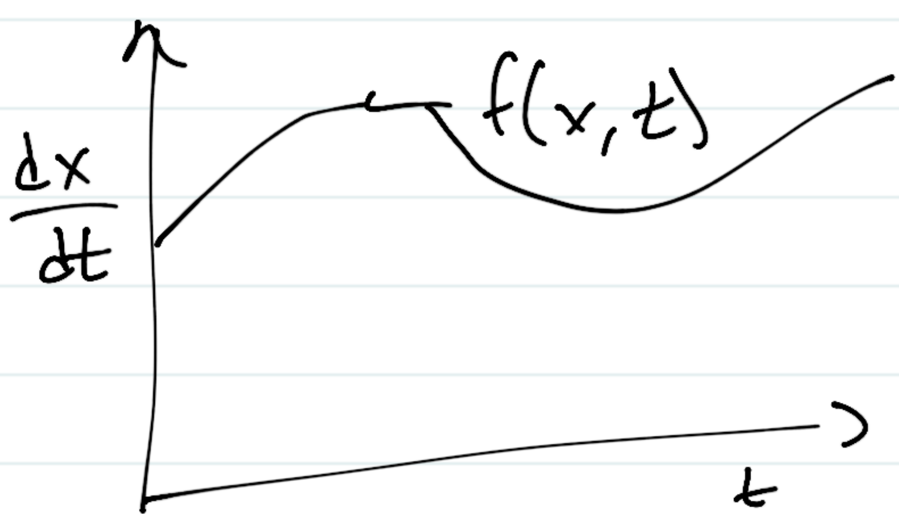
So, how to numerically solve

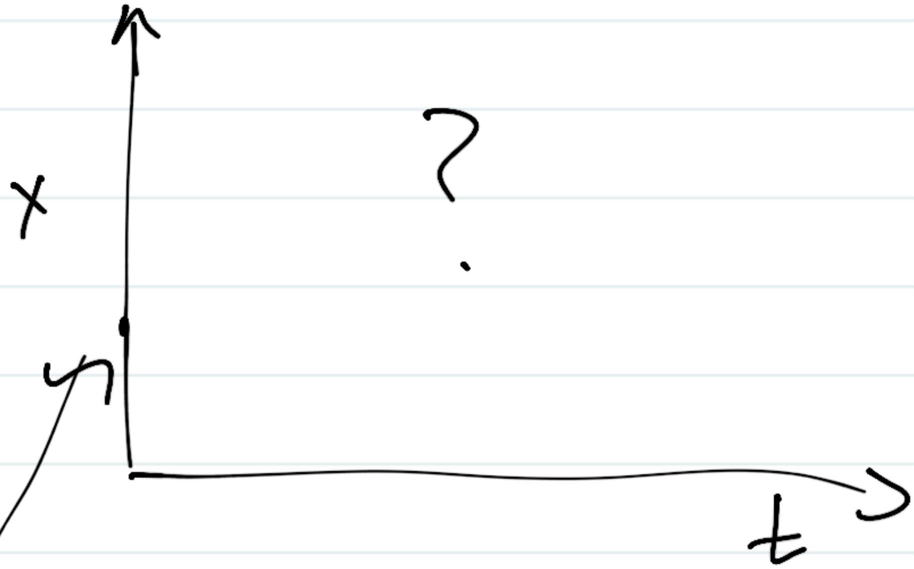
$$\frac{dx}{dt} = f(x,t) \quad ?$$
$$x(t=0) = x_0$$

Remember



We know the times over which to solve this problem, but we don't know $x(t)$ to

But what is $x(t)$? evaluate $f(x,t)\ldots$



If we don't know x_i yet, we need to start at the only place where we do know x, which is $t=0$ $\quad \underline{x(t=0)=x_0}$

initial condition

We must discretize the solution in time:

$$\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x(t_i+\Delta t)-x(t_i)}{\Delta t} = f(x(t_i), t_i)$$

$$\boxed{\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x(t_i+\Delta t)-x(t_i)}{\Delta t} = f(x(t_i),t_i) \qquad x(t=0)=x_0}$$

Forward-Euler Method

If we rewrite this method

$$\boxed{x(t_i+\Delta t) \doteq x(t_i)+f(x(t_i),t_i)\Delta t}$$

This gives us a way to solve for $x(t)$ by taking time steps from the known initial condition through the duration of time we want a solution for.

<u>Note</u>: the RHS of this equation only uses info from the past time step t_i

We can turn this into an algorithm for solving a general ODE

$x(1) = x_{init}$ ← initial condition

n = number of time steps to use

$$\Delta t = \frac{t_f - t_i}{n}$$

for $i = 1, \ldots, n$

$t = t_i + i\Delta t$

can be coded here or defined elsewhere for a more general algorithm

$x(i+1) = x(i) + f(x(i), t)\Delta t$

end

The smaller Δt → time step the more accurate this solution will be.

Since as $\Delta t \to 0$, this goes to the exact definition of a derivative

We can also make up a slightly different method:

Again discretize $\frac{dx}{dt} \approx \frac{x(t_i+\Delta t)-x(t_i)}{\Delta t}$

But now use information from the current time step for $f(x,t)$

$$\frac{x(t_i+\Delta t)-x(t_i)}{\Delta t} = f(x(t_i+\Delta t), t+\Delta t)$$

Rewrite this

$$\boxed{\underbrace{x(t_i+\Delta t) - f(x(t_i+\Delta t), t+\Delta t)\Delta t} = \overset{\text{known}}{x(t_i)}}$$

Backward Euler method

We have to solve this part for $x(t_i+\Delta t)$.
Can be hard depending on how complex $f(x,t)$ is

Example: Nuclear Decay

$N \rightarrow$ concentration of a radioactive element

Rate of change in concentration related to current concentration:

$$\frac{dN}{dt} = -\lambda N \qquad N(t=0) = N_0$$

→ λ is the decay constant which is related to half life by $t_{HL} = \frac{\ln(2)}{\lambda}$

→ This can be solved analytically

$$\int_0^N \frac{dN}{N} = -\lambda \int_0^t dt$$

$$\ln(N) = -\lambda t \rightarrow \boxed{N(t) = N_0 e^{-\lambda t}}$$

We'll use this exact solution as a benchmark to compare against our two methods.

Forward-Euler Method

for $i = 1, \ldots, n$

$$N(i+1) = N(i) + \left(-\lambda N(i)\right)\Delta t.$$

end

Backward-Euler Method

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = -\lambda N(t+\Delta t)$$

$$N(t+\Delta t) + \lambda N(t+\Delta t)\Delta t = N(t)$$
$$(1+\lambda\Delta t)N(t+\Delta t) = N(t)$$

$$N(t+\Delta t) = \frac{N(t)}{1+\lambda\Delta t}$$ ← $f(x,t)$ is simple enough that this is easy to solve

for $i=1, \ldots, n$

$$N(i+1) = N(i) / (1+\lambda\Delta t)$$

end

→ Compare each method to exact solution for different time step size

Slightly more complicated

Radiocarbon content of one-box biosphere
(S+K p. 40)

$$\frac{dM}{dt} = \underbrace{-kM}_{\text{decay}} + \underset{\substack{\uparrow \\ \text{production rate}}}{P}$$

Possibilities:

$P = \text{constant}$

$P = P_0 + b\sin(\omega t)$ periodic due to sunspots (e.g.)