Boundary Value Problems

→ Before considering fully complicated PDEs, lets distinguish between two types of PDE problems:

→ Until now, we have considered Initial Value Problems (IVPs) with ODEs where we know the solution at some initial time and then use numerical methods to find the solution at other time

→ Boundary value problems (BVPs) are those in which the solution is known at many "boundaries" of the system (physical or in time) and we search for a solution everywhere else

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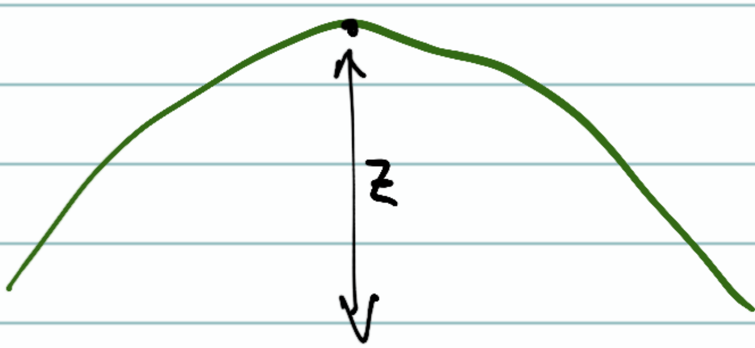
Example | Heat conduction in the Earth



We can describe heat conduction through a solid medium (in 1D) using:

Heat/Diffusion Equation : $\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial z^2} - h(z)$

diffusivity

heat source

At steady-state: $\boxed{\frac{\partial^2 T}{\partial z^2} = h(z)}$

Consider <u>two</u> types of boundary conditions:

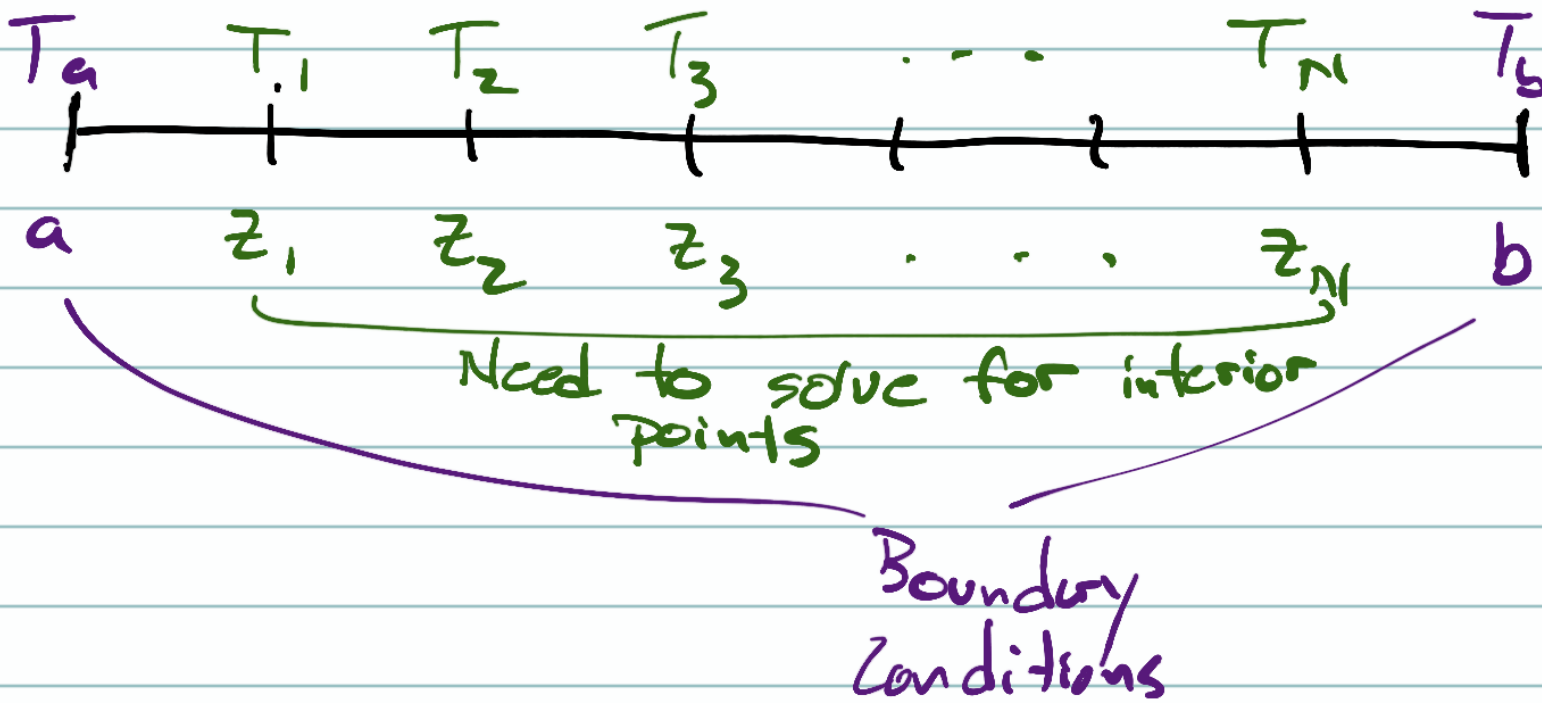
① Dirichlet boundary conditions
→ set values of T @ bndry points
$T(z=a) = T_a$ and $T(z=b) = T_b$

② Neumann boundary conditions
→ set derivatives of T @ bndry points

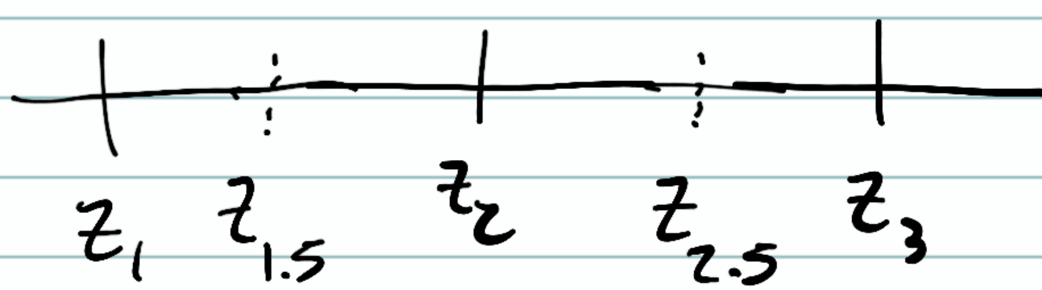
$\left.\frac{dT}{dz}\right|_{z=a} = F_a \qquad \left.\frac{dT}{dz}\right|_{z=b} = F_b$

So, how to solve $\frac{\partial^2 T}{\partial z^2} = h(z)$?

As always $\rightarrow$ DISCRETIZE!



The finite difference method - the second derivative is the difference of derivatives on either side:



$$\left.\frac{\partial T}{\partial z}\right|_{z_{2.5}} = \frac{T_3 - T_2}{\Delta z} \qquad \left.\frac{\partial T}{\partial z}\right|_{z_{1.5}} = \frac{T_2 - T_1}{\Delta z}$$

$$\left.\frac{\partial^2 T}{\partial z^2}\right|_{z_2} = \frac{\left.\frac{\partial T}{\partial z}\right|_{z_{2.5}} - \left.\frac{\partial T}{\partial z}\right|_{z_{1.5}}}{\Delta z}$$

$$= \frac{T_3 - T_2 - (T_2 - T_1)}{(\Delta z)^2}$$

$$\left.\frac{\partial^2 T}{\partial z^2}\right|_{z_2} = \frac{T_3 - 2T_2 + T_1}{\Delta z^2}$$

Given a function for the heat source we have a system of equations to solve:

$$(T_2 - 2T_1 + T_0)/\Delta z^2 = h(z_1)$$

$$(T_3 - 2T_2 + T_1)/\Delta z^2 = h(z_2)$$

$\vdots$

$$\left(T_{N+1} - 2T_N + T_{N-1}\right)/\Delta z^2 = h(z_n)$$

The first and last equations include the temperatures at the boundaries: T_0 and T_{N+1}

- These equations must be modified to account for the bndry conditions

→ If we have Dirichlet BCs, then we simply set $T_0 = T_a$, $T_{N+1} = T_b$ and we are left with N equations and N unknowns $(T_1, \dots, T_N)$

→ If we have Neumann BCs, then we must set the derivative at the boundaries like so:

$$\frac{T_2 - 2T_1 + T_0}{\Delta z^2} \rightarrow \frac{T_2 - T_1}{\Delta z^2} - \frac{1}{\Delta z}\left[\cancel{\left.\frac{dT}{dz}\right|_{z_a}} \rightarrow F_a\right]$$

$$\frac{T_2 - T_1}{\Delta z^2} - \frac{F_a}{\Delta z} = h(z_1)$$

... and same on other bndry

$$\frac{F_b}{\Delta z} - \frac{T_N - T_{N-1}}{\Delta z^2} = h(z_N)$$

So now we have a linear system of equations that we would like to solve → matrix form!

(Dirichlet BCs)

$$\begin{bmatrix} -2/\Delta z^2 & \frac{1}{\Delta z^2} & 0 & \cdots \\ 1/\Delta z^2 & -2/\Delta z^2 & 1/\Delta z^2 & \ddots \\ 0 & 1/\Delta z^2 & -2/\Delta z^2 & \ddots \\ \vdots & 0 & 1/\Delta z^2 & \ddots \\ \vdots & \vdots & 0 & \ddots \end{bmatrix} \begin{bmatrix} T_1 \\ \vdots \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} h_1 - \frac{T_a}{\Delta z^2} \\ h_2 \\ \vdots \\ h_{N-1} \\ h_N - \frac{T_b}{\Delta z^2} \end{bmatrix}$$

$$M \cdot \vec{T} = \vec{H}$$

Now we simply solve the matrix problem

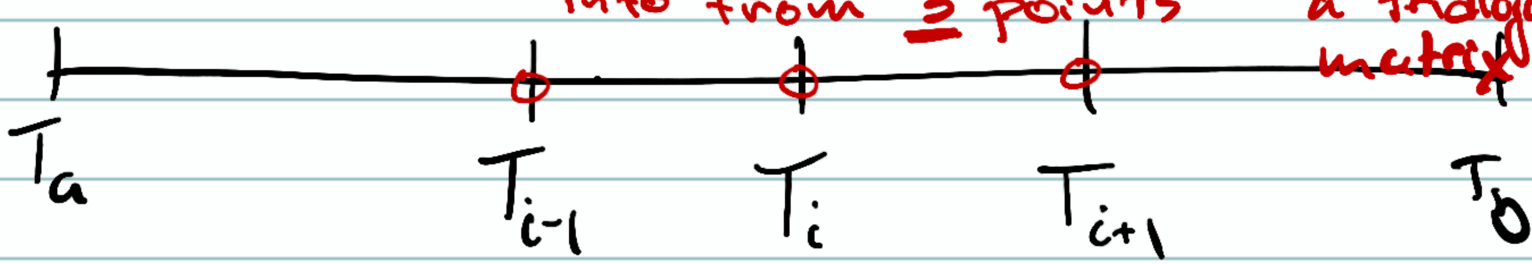
→ The nice thing is that M is a tridiagonal matrix, so it will generally be invertible and relatively easy to solve.

For Neumann BCs, only small changes

$$\begin{bmatrix} -1/\Delta z^2 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & -1/\Delta z^2 \end{bmatrix} \begin{bmatrix} T_1 \\ \vdots \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} h_1 - \frac{F_a}{\Delta z} \\ \\ \\ h_N - \frac{F_b}{\Delta z} \end{bmatrix}$$

Returning to the idea of the stencil:

Each equation uses info from $\underline{3}$ points → produces a tridiagonal matrix



Returning to the problem of the temperature of the solid Earth, we can solve for temperature through a vertical profile assuming:

→ We know $h(z)$ → the rate at which radiogenic heat is generated in the Earth

→ $D\frac{\partial T}{\partial z}\Big|_{z=0}$ = heat flux at the core

→ $D\frac{\partial T}{\partial z}\Big|_{z=z_s}$ = heat flux at the surface (i.e. geothermal heat flux measurement)

In general to solve a DE of the form

$$y^{(n)}=f(x,y,y',\ldots y^{(n-1)})$$

Requires n boundary/initial conditions