Stochastic Models

-> Random Walk
-> Levy Flight
-> Simple SDE

Random Walk

The only variable is position and time

-> In a simple lattice random walk on each time step:
   -> Position changes to a random neighbor (including the possibility of not moving)

-> In general any random process changing a variable from one time to the next time where the probabibility of changing to some state does not depended on the current state is called a Markov Process

A sequence of Markov process steps is called a Markov chain

→ If you take a random walk with very small steps, you approach a Wiener process which is simply the most common continuous random process

→ Useful for modeling randomness in real-world systems which are continuous

→ Steps are drawn from a Normal Distribution

$$x(t+\Delta t) = x(t) + x'(t)$$

where $x' \sim N(0, \sigma)$

Noise in dynamical systems

→ The Wiener process is useful to represent the influence of randomness or uncertainty in real-world systems

→ Can be added to a differential equation making it a stochastic differential equation (SDE)

Ex] $\frac{dx}{dt} = f(x,t) + \eta(t)$

where $\eta(t) = \sigma \frac{dw}{dt}$

with w being a Wiener process

→ This can be solved using a Forward-Euler discretization (or others) where the noise term is scaled by the time step (to maintain the st. dev. regardless of time step - a hallmark of a Wiener process)

for $t = \ldots$

$\eta = \sigma (\Delta t)^{\frac{1}{2}} \times \mathrm{randn}(1)$

$x(t+\Delta t) = x(t) + f(x,t)\Delta t + \eta$

end

This is called the Euler-Maruyama method.

Example: A particle released in the atmosphere - where will it end up?

→ If we knew air velocity at every mm of space this problem would be easy → use the advection equation

→ But we only have course measurements of velocity (say every ~1 km from satellites)
  - How to represent uncertainty of unknown velocities
    - Noise!

① Write a FE scheme for particle advection in 2D with

$t_f = 100$

$$u(x,y) = \sin(\pi x)\cos(\pi y)$$
$$v(x,y) = \cos(\pi x)\sin(\pi y)$$

Try initial conditions: $(0,0), (1,0), (\frac{1}{2}, \frac{1}{4})$

② Add noise using approach outlined with standard deviation:

$0, 10^{-9}, 10^{-3}, 10^{-1}, 1$

Run the model ten times for each

③ What have we learned about the importance of incorporating noise?