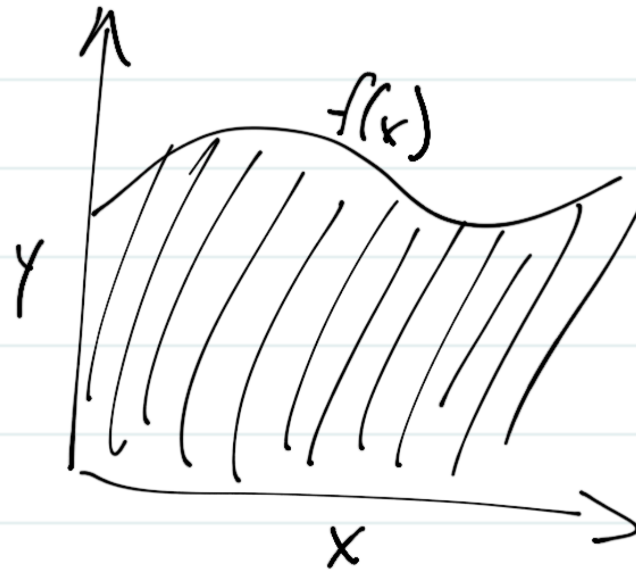
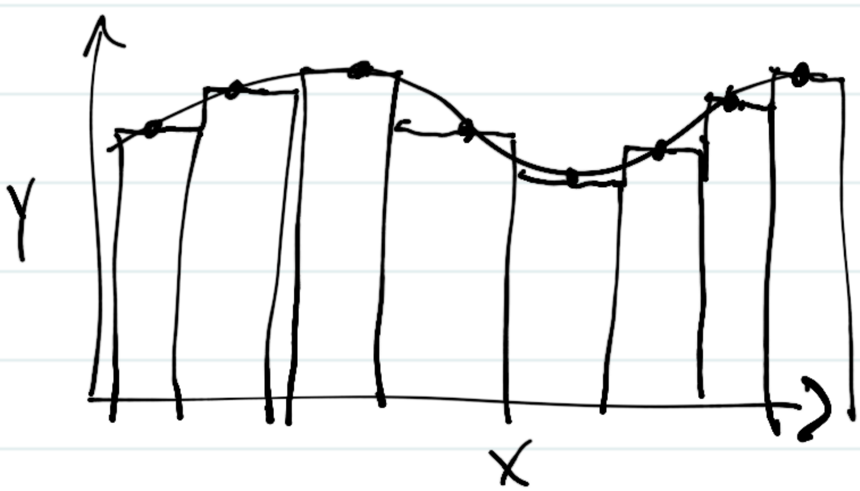
Integration Review

What is an integral?

Area under a curve



How to find this area if we dont know f(x), just have some points?



Draw rectangles, make an estimate of integral by summing area of rectangles

$$\sum_{i=0}^{n} f(x_i)\,\Delta x_i$$

→ The definition of a definite integral just comes from taking the continuous limit of this sum:

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

$$x_i \in [a, b]$$

$$\Delta x = \frac{b-a}{n}$$

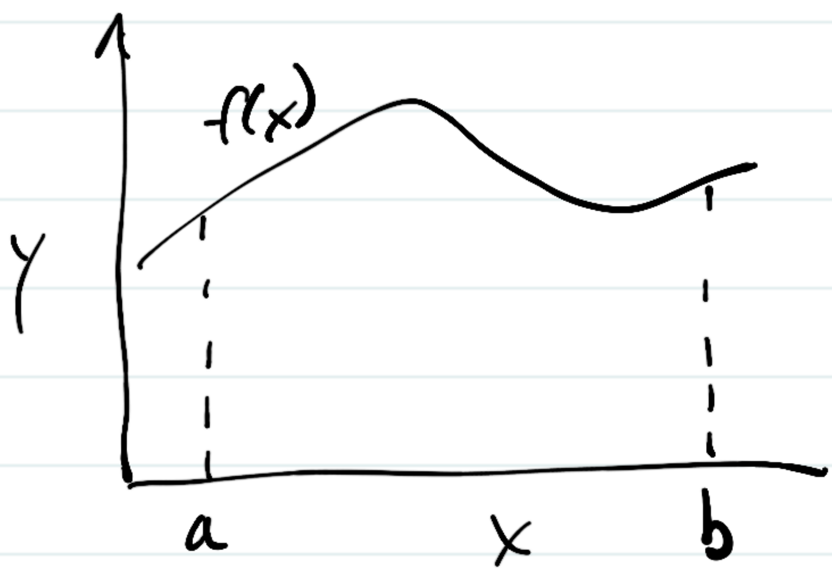
→ In calc you spent a lot of time learning ways to solve integrals exactly for different functions f(x)

→ But this may present problems:

- f(x) may not be analytically integrable
- we may not have a function for f(x)

Numerical integration provides a general approach for approximately finding any integral

So, lets take it back to our original method for thinking about the area under a curve



(1) Mid point rule

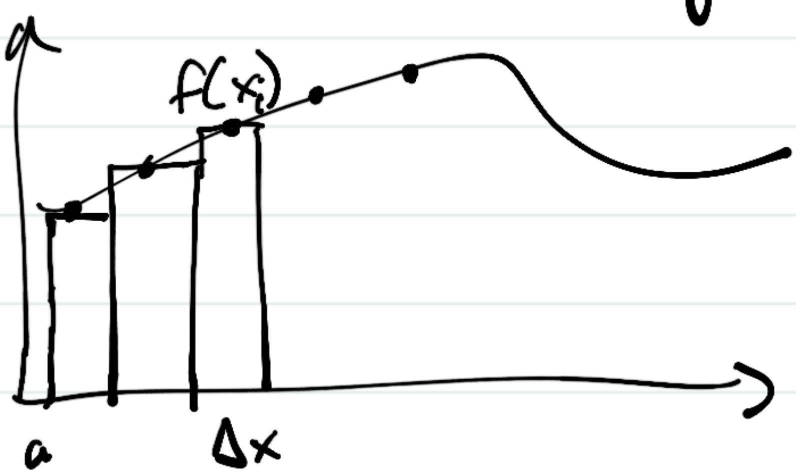
→ The midpoint rule is the same idea as the rectangles we drew when defining a derivative

Chop up interval $[a, b]$ into N smaller intervals:

$$\Delta x = \frac{b-a}{N}$$

→ This step is called "discretization" - it is key to almost all numerical methods because computers are very good at solving arithmetic problems quickly, repeatedly. They aren't so good at solving abstract analytical problems.

→ The area of each rectangle is then



$$\text{area} = f(x_i)\,\Delta x$$

What is $f(x_i)$?

$$x_i = a + \frac{2i+1}{2}\Delta x \text{ for } i = 0, \ldots, N-1$$

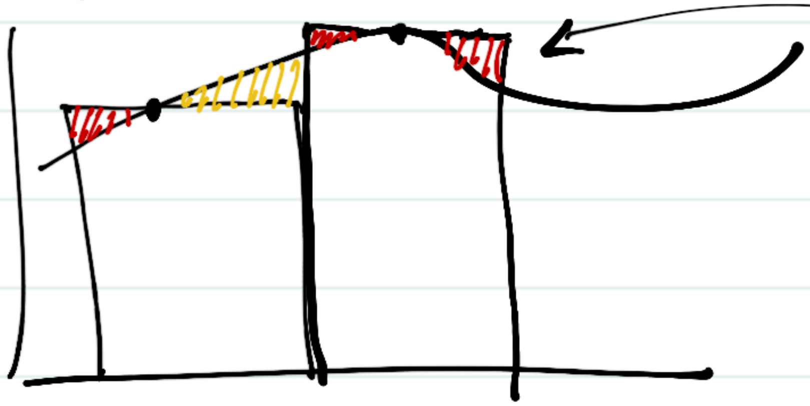
Then, we sum over all rectangles:

$$\boxed{I = \sum_{i=0}^{N-1} f(x_i)\,\Delta x \quad \text{where } \Delta x = \frac{b-a}{N} \text{ and } x_i = a + \frac{2i+1}{2}\Delta x}$$

As $N \to \infty$, we approach the definition of definite integral, meaning the approximation justs gets better and better

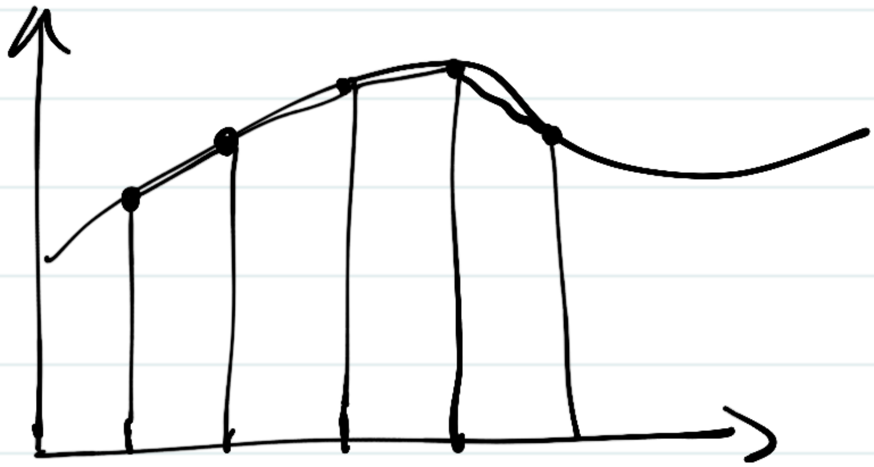
(2) Trapezoid Rule

The problem with the midpoint rule:

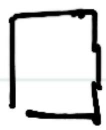


errors accumulate

Solution: use more information about function values (if you have it)



trapezoids instead of rectangles

Area of trapezoid:  =  + 

$$= (x_2 - x_1) f(x_1) + \frac{(x_2 - x_1)\left[f(x_2) - f(x_1)\right]}{2}$$

$$= \frac{1}{2} \underbrace{(x_2 - x_1)}_{\Delta x} \left[f(x_1) + f(x_2)\right]$$

Now, sum over many trapezoids

$$\boxed{I = \sum_{i=0}^{N-1} \frac{1}{2}\Delta x \left[f(x_i) + f(x_{i+1}) \right] \quad \text{where } \Delta x = \frac{b-a}{N} \qquad x_i = a + i\Delta x}$$

(3) Simpson's Rule

→ In midpoint rule, we use <u>one</u> point on $f(x)$ in each sub-interval, and in trapezoid rule we used two points.

Next logical step → <u>Three</u> points to approximate a parabola on each sub-interval:

area under parabola $= \Delta x \left(ax^2 + bx + c \right)$

What are a, b, c? Extra-credit for how to show this

$$\boxed{I = \sum_{i=0}^{N/2} \frac{\Delta x}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right] \quad \text{for } \Delta x = \frac{b-a}{N} \quad x_i = a + i\Delta x}$$

→ In general these methods are known as **Newton-Cotes formulas**.

→ In general, the more points used, the more accurate the method will be for the same number of sub-intervals (N)