Lecture 9

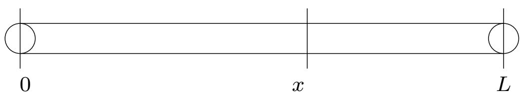
Applications of integration (cont'd)

One-dimensional continuous distributions of mass/charge: The "thin wire"

`This topic is not treated in the textbook by Stewart`

In this section, we consider the distribution of either mass or charge along a thin "one-dimensional" rod. By "one-dimensional", we mean that we may factor all variations in the rod along one direction. This may appear to be mathematical idealization – which, in fact, it is – but it does yield some important ideas that can be carried to higher dimensions. As well, it does serve as a good approximation in many cases.

In what follows, we'll consider the distribution of mass in a one-dimensional rod. Consider a straight rod of length L and constant cross-sectional area A. (Later, we can let the cross-sectional area A vary, i.e., $A(x)$ and then incorporate this variation into the linear density function.) Also let M denote the mass of this rod. It is useful to let $x \in [0, L]$ denote a position (actually a slice) on this rod, as shown below.



If the rod is **homogeneous**, that is, the same everywhere, then we can define its *linear density*, which is in units of mass per unit length, as follows,

$$\rho = \frac{M}{L}. \tag{1}$$

You are no doubt familiar with the idea of density as mass per unit volume. Here, the area has essentially been factored out so that the problem becomes one-dimensional.
This, of course, implies that the total mass M of the rod may be expressed as

$$M = \rho L. \tag{2}$$

Because of the homogeneity of the rod, any piece of the rod of length $l \leq L$ has mass m given by

$$m = \rho l = M\left(\frac{l}{L}\right). \tag{3}$$

And more to the "Spirit of Calculus" which will be used later, the mass Δm of a piece of the rod of length Δx will be given by

$$\Delta m = \rho \Delta x. \tag{4}$$

Now suppose that the rod is **inhomogeneous**, i.e., it is **not** the same everywhere, perhaps due to a difference in material. We expect that the linear density ρ is not necessarily constant throughout the rod. In fact, we expect that the linear density ρ in Eq. (4) will depend on x, the coordinate of a point on the rod. In other words,

$$\Delta m \approx \rho(x)\Delta x, \qquad x \in [0, L]. \tag{5}$$

This is indeed the case, but we shall derive it in terms of rates of change of mass.

The mass function $m(x)$

Let us define the function $m(x)$ to be the mass of the rod contained in the interval $[0, x]$, $x \in [0, L]$. As such, we definitely know two values of $m(x)$, namely,

$$m(0) = 0 \quad \text{(no mass)}, \qquad m(L) = M \quad \text{(total mass of rod)}. \tag{6}$$

Some important properties of the mass function

1. The function $m(x)$ must be an **increasing** function of x, i.e., if $x_1 < x_2$, we must have that $m(x_1) < m(x_2)$. This follows from the definition of $m(x)$. $m(x_1)$ is the amount of mass over the interval $[0, x_1]$. $m(x_2)$ is the amount of mass over the interval $[0, x_2]$. We must *add* mass to the first interval, precisely the mass over the interval $[x_1, x_2]$ to obtain the second interval.

2. The function $m(x)$ must be a **continuous** function of x. If $\lim_{x \to x_0^-} m(x) \neq m(x_0)$ at some $x_0 \in [0, L]$, this means that there is a sudden, nonzero jump of $m(x)$ at x_0. This implies that a nonzero amount of mass would be situated at a single point x_0. This would imply an infinite density of mass at x_0, which is not physically possible. In fact, we shall now investigate the idea of mass density further.

Let us now assume that there exists a function $\rho(x)$ defined on $[0, L]$ so that

$$m(x) = \int_0^x \rho(s)\, ds \quad 0 \leq x \leq L\,. \tag{7}$$

This is a way of defining the function $\rho(x)$, known as the **mass density function** of the rod. In what follows, we shall assume that $\rho(x)$ is not necessarily continuous on $[0, L]$ but at least piecewise continuous – we'll see why later.

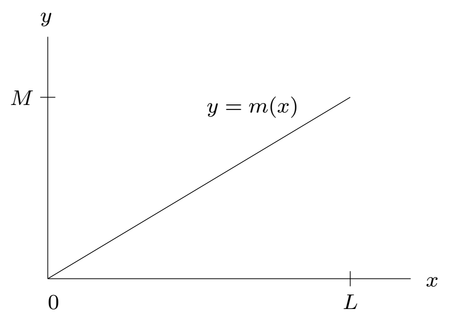
Example 1: In the special case of a homogeneous rod, we expect the density function $\rho(x)$ to be a constant, i.e., $\rho(x) = \rho_0$. In this case,

$$m(x) = \int_0^x \rho_0 \, ds = \rho_0 x \,, \tag{8}$$

i.e., $m(x)$ is a linear function of x. If the total mass of the rod is $m(L) = M$, we then have that

$$M = \rho_0 L \quad \Longrightarrow \quad \rho_0 = \frac{M}{L} \,, \tag{9}$$

which is consistent with our earlier discussion. The graph of $m(x)$ vs. x is sketched below.



Graph of $m(x)$ for a homogeneous rod. The slope of the graph is $\rho_0 = \frac{M}{L}$.

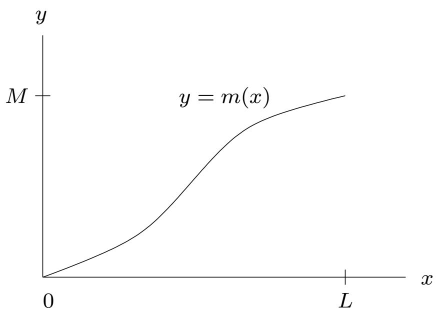
In the general case of an inhomogeneous rod, however, the function $m(x)$ will not necessarily be linear, due to variations in the materials. A generic situation is sketched in the figure below.
We now investigate this more general situation further. From FTC I applied to Eq. (7), it follows that

$$m'(x) = \rho(x) \,, \quad \text{i.e.,} \quad \frac{dm}{dx} = \rho(x) \,. \tag{10}$$

Once again in the "Spirit of Calculus," the above relation is telling us the following,

$$\rho(x) = \lim_{\Delta x \to 0} \frac{m(x + \Delta x) - m(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} \,, \tag{11}$$

provided that the limit exists. (Later, we'll see a situation where the limit does not exist.) Here Δm is the mass of the segment of the rod contained in the interval $[x, x + \Delta x]$. As such, the ratio $\Delta m/\Delta x$



Graph of $m(x)$ for an inhomogeneous rod.

is the mass of this segment divided by its length Δx. This means that $\rho(x)$ is the limiting ratio of mass per length at x.

From the above limit, it follows that for small Δx, the amount of mass Δm in a segment of the rod of length Δx centered at a point x is well approximated by

$$\Delta m \approx \rho(x)\Delta x \,. \tag{12}$$

Going one step further, the infinitesimal element of mass dm at x is related to the infinitesimal element of length dx by

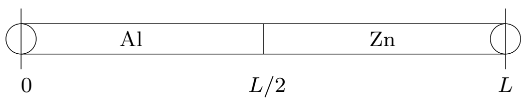
$$dm = \rho(x)\, dx \,. \tag{13}$$

Returning to Example 1, the special case of a homogeneous rod, from Eq. (8), it follows that

$$\rho(x) = \frac{dm}{dx} = \frac{M}{L} = \rho_0 \,, \tag{14}$$

the constant density that we obtained earlier. As expected, it is independent of x.

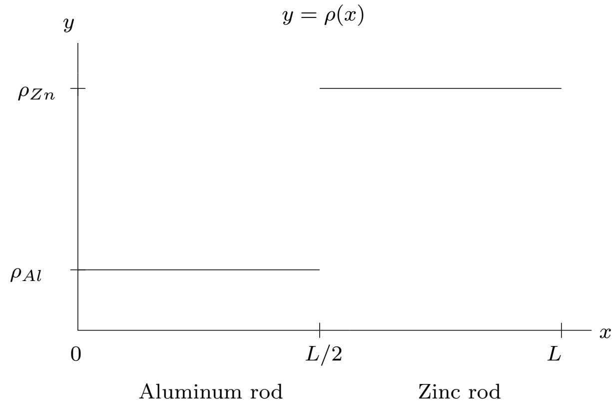
Example 2: Suppose that our rod of length L is composed of two smaller, homogeneous rods of length $L/2$ that are welded together, as sketched below.



The two smaller rods are composed of materials with substantially different densities, for example, aluminum (Al) and zinc (Zn). The mass density of zinc (7.14 g/cm^3) is about three times the mass density of aluminum (2.70 g/cm^3). We assume the weld at $x = L/2$ to be a "perfect weld", i.e., there is no intermingling of material across the weld so that the linear density function of the welded rod will be given by

$$\rho(x) = \begin{cases} \rho_{Al}, & 0 \le x < L/2, \\ \rho_{Zn}, & L/2 \le x \le L, \end{cases} \tag{15}$$

where ρ_{Al} and ρ_{Zn} denote the linear mass densities of, respectively, aluminum and zinc for this problem. A sketch of the graph of $\rho(x)$ for this welded rod is shown in the next figure. The most important feature of this graph is that $\rho(x)$ is discontinuous at $x = L/2$.



Mass density function $\rho(x)$ for the welded rod example.

We now consider the mass function $m(x)$ for this welded rod,

$$m(x) = \int_0^s \rho(s)\, ds\,. \tag{16}$$

First of all, as we stated earlier, the function $m(x)$ must be a continuous function on $[0, L]$. And recalling that $m'(x) = \rho(x)$, we may conclude that

$$m'(x) = \begin{cases} \rho_{Al}, & 0 \le x < L/2, \\ \text{undefined}, & x = L/2, \\ \rho_{Zn}, & L/2 < x \le L, \end{cases} \tag{17}$$

The fact that $m'(x)$ is undefined at $x = L/2$ follows from the fact that $\rho(x)$ is discontinuous at $x = L/2$.

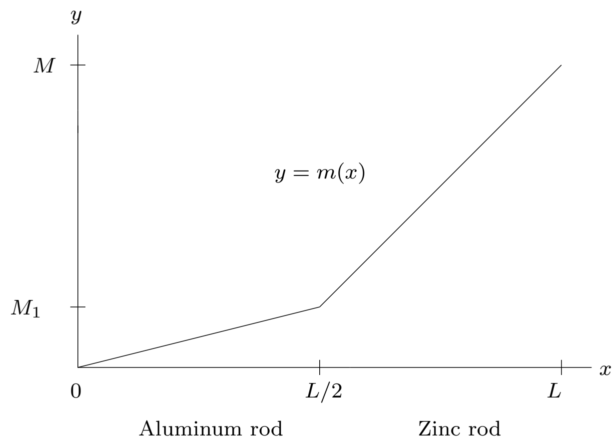
We can easily perform the integration for $m(x)$ involving the discontinuous function $\rho(x)$,

$$m(x) = \begin{cases} \rho_{Al} x\,, & 0 \le x < L/2 \\ M_1 + \rho_{Zn}(x - L/2) & L/2 \le x \le L\,. \end{cases} \tag{18}$$

Here,

$$M_1 = \rho_{Al} \frac{L}{2} \tag{19}$$

is the mass of the half-rod containing the aluminum. The graph of $m(x)$ vs. x will have following form,



Mass function $m(x)$ for the welded rod example.

Perhaps the most interesting aspect of this graph is that it is a continuous function (recall that it has to be), despite the fact that it was obtained by integrating the function $\rho(x)$ which had a discontinuity at $x = L/2$. This illustrates a characteristic property of integration: It is a kind of "smoothing" or "regularizing" operation.

A more general mass function $\mu([a, b])$

Let us now return to the definition of the mass function $m(x)$ in terms of the density function $\rho(x)$ in Eq. (7). Let $[a, b]$ be any subset of $[0, L]$, i.e., $[a, b] \subseteq [0, L]$. We'll now let $\mu([a, b])$ denote the mass of the portion of the rod that lies in the interval $[a, b]$. We use the letter "μ" instead of "m" so that there is no confusion with the original mass function $m(x)$ defined earlier. The relationship between the two is quite straightforward,

$$\mu([0, x]) = m(x) \, . \tag{20}$$

The following relation should also be quite easy to see:

$$\mu([a, b]) = m(b) - m(a) \, . \tag{21}$$

In other words, mass of the portion of the rod lying in the interval $[a, b]$ is the mass of the portion of the rod lying in the interval $[0, b]$ – $m(b)$ – minus the mass of the portion of the rod lying in the interval $[0, a]$ – $m(a)$. But from the definition of $m(x)$, it follows that

$$\begin{aligned} \mu([a,b]) &= \int_0^b \rho(s)\,ds - \int_0^a \rho(s)\,ds \\ &= \int_a^b \rho(s)\,ds\,. \end{aligned} \tag{22}$$

In other words, the mass of the rod on $[a, b]$ is simply the integral of the mass density function $\rho(x)$ over $[a, b]$. This result should not be surprising at all. The reason it has been presented in this way is to release us from the original definition of $m(x)$ using the reference point $x = 0$.

The "continuum approximation" behind the model of the rod: The idea of letting $\Delta x \to 0$ in a physical problem is, of course, a mathematical abstraction: Nature does not behave in such a continuous way in the limit. From a classical, i.e., everyday, point of view, matter may be treated as a continuous substance down to some small length scales such as 10^{-6} m. But when we get to 10^{-10} m (an Angstrom), the atomic nature of matter shows itself. The so-called "continuum approximation" consists of "smearing" out all matter, thus filling in all the "holes" between nuclei and electrons. This turns out to be an excellent approxmation for the treatment of classical phenomena. The same is true for electric charge, which we consider in the next section. Each fundamental unit of electric charge is very small. Nevertheless, the number of such charges in a substance is very high (on the order of Avogadro's number). For that reason, we may treat charge as a smeared-out distribution, i.e., like a fluid.

Note that in the continuum approach, **there are no point masses or charges**. Recall that

$$\Delta m \approx \rho(x)\,\Delta x\,, \tag{23}$$

and that $\Delta m \to 0$ as $\Delta x \to 0$. In the case of a point mass, Δm would be nonzero in the limit $\Delta x \to 0$, which implies that $\rho(x)$ would not be finite. In other words, the mass density at x would be infinite.

That being said, we shall occasionally be using point masses and charges in discussions and examples.

Charge distribution over a one-dimensional rod

Everything mentioned above for the problem of mass distribution over a thin rod also applies to **charge distributions**. We may define $q(x)$ as the amount of charge in the segment of the rod represented by the interval $[0, x]$ and assume that this charge distribution is characterized by a **linear charge density function** $\rho(x)$ such that

$$q(x) = \int_0^x \rho(x)\, dx\,. \tag{24}$$

Then, by FTC I, we have

$$q'(x) = \rho(x)\,. \tag{25}$$

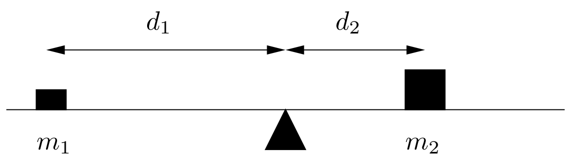
And as in the case for the mass density function, $q(x)$ represents the limiting amount of charge Δx per length Δx as $\Delta x \to 0$.

Center of mass of a thin wire

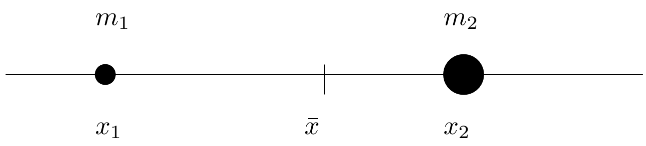
We now address the problem of finding the **center of mass** of a wire on $[0, L]$, with linear density function $\rho(x)$. Recall that the center of mass is the point at which the wire can be balanced. In order to solve this problem, it is helpful to return to the case of a finite number of masses.

The simplest problem is the two-mass case: the "teeter-totter," sketched below. Given two masses m_1 and m_2 located on opposite sides of the pivot point, and at distances of d_1 and d_2, respectively, from the pivot, balance is achieved when

$$m_1 d_1 = m_2 d_2. \tag{26}$$



We now reformulate this problem as follows: Suppose that masses m_1 and m_2 are located at coordinate positions x_1 and x_2, with $x_1 < x_2$. Where is the center of mass $\bar{x}$?



Once again, $\bar{x}$ is the location of the pivot point for perfect balance. Because $\bar{x}$ lies between x_1 and x_2, Eq. (26) translates to

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}). \tag{27}$$

We'll rewrite this equation as follows,

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0. \tag{28}$$

We can solve for $\bar{x}$:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \tag{29}$$

a formula with which you are no doubt familiar as the center of mass of a two-body system.

We can now generalize this result to the case of n masses on the line: For $k = 1, 2, \cdots, n$, a mass of m_k is situated at position x_k. Eq. (28) generalizes to

$$\sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0. \tag{30}$$

The LHS of this equation is known as the **first moment of the masses about** $\bar{x}$.

From this equation, we may easily solve for $\bar{x}$. First rewrite it as follows,

$$\sum_{k=1}^{n} m_k x_k - \bar{x} \sum_{k=1}^{n} m_k = 0. \tag{31}$$

The second sum on the LHS is the total mass M of the system. Therefore, the center of mass is given by

$$\bar{x} = \frac{1}{M} \sum_{k=1}^{n} m_k x_k. \tag{32}$$

Once again, you are most probably familiar with this equation.

Before going on, let's rewrite Eq. (32) as follows,

$$\bar{x} = \sum_{k=1}^{n} \left(\frac{m_k}{M}\right) x_k, \tag{33}$$

or

$$\bar{x} = \sum_{k=1}^{n} p_k x_k, \tag{34}$$

where

$$p_k = \frac{m_k}{M} \quad \Rightarrow \quad \sum_{k=1}^{n} p_k = 1. \tag{35}$$

From Eq. (34), the center of mass $\bar{x}$ may be viewed as a **weighted** average of the x_k values. The greater the mass m_k, the greater the weighting factor p_k. In fact, because the weighting factors p_k are nonnegative and sum to 1, the weighted sum in Eq. (34) has a special name: it is called a **convex combination** of the x_k.

The next step is to carry this idea over to continuous distributions of mass. Our goal: to find the continuous version of Eq. (32) for the center of mass corresponding to a density function $\rho(x)$. The way to do this is to use – guess what? – the "Spirit of Calculus." Once again, we divide up the mass into a finite number n of tiny pieces, consider each piece as a point mass m_k, then compute the center of mass of this ensemble, and then let $n \to \infty$.

So, as before, for an $n > 0$, let $\Delta x = \dfrac{L}{n}$ (our interval $[a, b]$ is now $[0, L]$) and define the partition points

$$x_k = k\Delta x = \frac{k}{n}L, \quad 0 \le k \le n. \tag{36}$$

These partition points define a set of n subintervals $I_k = [x_{k-1}, x_k]$. We now let Δm_k denote the mass of wire over each subinterval I_k.

Once again, we choose sample points $x_k^* \in I_k$ from each subinterval. From our definition of the density function, the mass Δm_k of the wire on I_k is approximated as follows,

$$\Delta m_k \cong \rho(x_k^*)\Delta x. \tag{37}$$

The total mass of the wire, M, is then approximated as follows,

$$M = \sum_{k=1}^{n} \Delta m_k \cong \sum_{k=1}^{n} \rho(x_k^*)\Delta x \tag{38}$$

But the sum on the RHS is the Riemann sum for the function $\rho(x)$ over the interval $[0, L]$. As such, in the limit $n \to \infty$, we have

$$M = \int_0^L \rho(x)\, dx, \tag{39}$$

which is consistent with our earlier definition of the density function.

But we haven't finished! We still have to compute the continuous version of the sum in Eq. (32). Returning to the n masses Δm_k produced by our partition above, we shall consider them as point masses situated at the sample points x_k^*. Of course, this is an approximation, but as $n \to \infty$, this approximation gets better and better. The **moment** of these masses is then approximated by

$$\sum_{k=1}^{n} \Delta m_k x_k^* = \sum_{k=1}^{n} x_k^* \rho(x_k^*) \Delta x. \tag{40}$$

Note that the sum on the RHS has the form of a Riemann sum over the interval $[0, L]$, but it is the Riemann sum corresponding to the function $f(x) = x\rho(x)$. In the limit, this Riemann sum converges to the definite integral

$$\int_0^L x\rho(x)\, dx. \tag{41}$$

As a result, the center of mass $\bar{x}$ of the continuous distribution of mass corresponding to the density function $\rho(x)$, $0 \leq x \leq L$ is given by

$$\bar{x} = \frac{1}{M} \int_0^L x\rho(x)\, dx, \qquad \text{where} \quad M = \int_0^L \rho(x)\, dx. \tag{42}$$

We now compute the centers of mass of some simple continuous mass distributions. In all cases, the wire is located on the interval $[0, 1]$.

1. **Example 1:** The mass distribution $\rho(x) = 1$. Since the mass density function is constant, the wire may be considered **homogeneous**, i.e., identical composition throughout the wire. In this case, we would expect the center of mass to be located at its center point, i.e., $\bar{x} = 1/2$. The total mass of the wire is

   $$M = \int_0^1 \rho(x)\, dx = \int_0^1 dx = 1. \tag{43}$$

   The first moment of the wire with respect to the origin is given by

   $$M_x = \int_0^1 x\rho(x)\, dx = \int_0^1 x\, dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}. \tag{44}$$

   The center of mass of this wire is therefore

   $$\bar{x} = \frac{M_x}{M} = \frac{1/2}{1} = 1, \tag{45}$$

   as expected.

2. **Example 2:** We now consider a perturbation of the above mass distribution, the density function

$$\rho(x) = 1 + \frac{1}{2}x. \tag{46}$$

The density function $\rho(x)$ increases as x increases from 0 to 1. As such, the wire is heavier on the right side than on the left, and we expect the center of mass to lie to the right of the geometric center $x = 1/2$. The total mass of the wire is

$$M = \int_0^1 \rho(x)\,dx = \int_0^1 \left(1 + \frac{1}{2}x\right)\,dx = \left[x + \frac{1}{4}x^2\right]_0^1 = 1 + \frac{1}{4} = \frac{5}{4}. \tag{47}$$

The first moment of the wire with respect to the origin is

$$M_x = \int_0^1 x\rho(x)\,dx = \int_0^1 \left(x + \frac{1}{2}x^2\right)\,dx = \left[\frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}. \tag{48}$$

The center of mass of the wire is therefore

$$\bar{x} = \frac{M_x}{M} = \frac{2/3}{5/4} = \frac{8}{15}. \tag{49}$$

As expected, the center of mass lies to the right of the geometrical center point $x = 1/2$ (although not that far away from it).

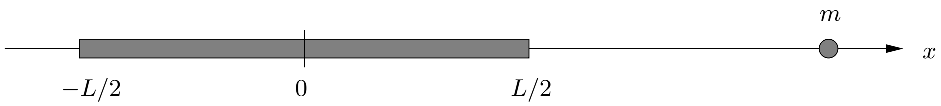
From an understanding of integration and the "Spirit of Calculus," we are now in a position to consider a wide variety of applications in physics that involve continuous mass distributions. The following is an example of such an application.

The total gravitational force exerted by a one-dimensional rod

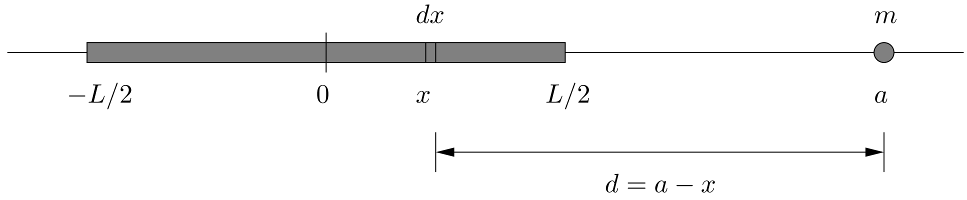
Consider a wire over the interval $\left[-\frac{L}{2}, \frac{L}{2}\right]$ with mass density function $\rho(x)$. Now let a point mass m be situated outside the wire, at position coordinate $a > \frac{L}{2}$, as sketched below. (By symmetry, the case $a < -L/2$ will yield the same magnitude.) We wish to find the total gravitational force exerted by the wire on the point mass.

We'll be using the following basic fact from Physics: The magnitude of the force of gravitational attraction between two point masses m_1 and m_2 situated a distance $d > 0$ apart is

$$F = \frac{Gm_1m_2}{d^2}. \tag{50}$$



In what follows, we'll avoid the partitioning/Riemann sum approach and simply present an abbreviated derivation that you will probably encounter (or may already have encountered) in your Physics courses. We simply go to the "infinitesimal limit" and consider an infinitesimal interval of width dx situated at $x \in [-L/2, L/2]$, as sketched below.



The infinitesimal mass dm of the element of wire situated in this interval is

$$dm = \rho(x)\, dx. \tag{51}$$

This comes from the definition of the mass density function,

$$\frac{dm}{dx} = \rho(x) \quad \Rightarrow \quad dm = \frac{dm}{dx} dx = \rho(x)\, dx. \tag{52}$$

The magnitude dF of the force between this infinitesimal mass element at x and the point mass at a is given by

$$dF = \frac{Gm\, dm}{(a-x)^2} = \frac{Gm\rho(x)\, dx}{(a-x)^2} = Gm \frac{\rho(x)}{(a-x)^2} dx. \tag{53}$$

The magnitude of the total force exerted by the rod on the point mass is obtained by integrating over all mass elements on $[-L/2, L/2]$:

$$F = Gm \int_{-L/2}^{L/2} \frac{\rho(x)}{(a-x)^2} dx\,. \tag{54}$$

Once we specify the mass density function $\rho(x)$, we may, at least in principle, compute the magnitude of the total force F. In what follows, we consider the particular case $\rho(x) = \rho_0$, constant, the case of a homogeneous wire. In this case, the total mass of the wire is

$$M = \int_{-L/2}^{L/2} \rho(x)\, dx = \int_{-L/2}^{L/2} \rho_0\, dx = \rho_0 L. \tag{55}$$

And since the wire is homogeneous, the center of mass is located at $\bar{x} = 0$.

The total force exerted by this homogeneous wire is then given by the integral

$$F = Gm\rho_0 \int_{-L/2}^{L/2} \frac{1}{(a-x)^2}\, dx. \tag{56}$$

The integral is easy to compute:

$$\begin{aligned}
\int_{-L/2}^{L/2} \frac{1}{(a-x)^2} dx &= \left[\frac{1}{a-x}\right]_{-L/2}^{L/2} \\
&= \frac{1}{a-L/2} - \frac{1}{a+L/2} \\
&= \frac{L}{a^2 - L^2/4}.
\end{aligned} \tag{57}$$

Therefore,

$$F = \frac{Gm\rho_0 L}{a^2 - L^2/4}. \tag{58}$$

Recalling that the total mass of the wire is $M = \rho_0 L$, we have the final result,

$$F = \frac{GMm}{a^2 - L^2/4}. \tag{59}$$

This is a very interesting result, and worthy of some comment and analysis. First of all, the most obvious observation is that the force is **not** given by

$$F = \frac{GMm}{a^2}, \tag{60}$$

the case if the rod were replaced by a point mass M at its center of mass $x = 0$. Many of you may be aware of the result that the gravitational force exerted by a **three-dimensional spherical and homogeneous** mass M is the same as the force due to a point mass M located at the center of the sphere. But this is **not** the case in one-dimension. Nor is it the case in two-dimensions. And even in three dimensions, the mass must be spherical and homogeneous (or at least have a spherically symmetric mass density function ρ) for the ability to replace it by a point mass at its center.

Note that for a very large, the term $L^2/4$ in the denominator of Eq. (59) is negligible, in which case the magnitude of the force is well approximated by Eq. (60). Perhaps it is helpful to characterize how large a would have to be for the approximation to be valid. We can do this by rewriting Eq. (59) as follows,

$$F = \frac{GMm}{a^2} \frac{1}{1 - \left(\frac{L}{2a}\right)^2}. \tag{61}$$

For any given $L > 0$, we see that the ratio $\dfrac{L}{2a}$ must be sufficiently small.

One-dimensional charge distributions

Because the classical electrostatic force between two charges is also inversely proportional to the square of their separation, the gravitational example examined above has an electrostatic counterpart. The rod now supports a one-dimensional distribution of charge with linear density $\rho(x)$, $x \in [-L/2, L/2]$. And the mass m at $x = a$ is now a test charge q. There is one important difference, however – the electrostatic force can be either (i) repulsive or (ii) attractive, depending on whether the charge q has the (i) same or (ii) opposite sign to that of the rod. For simplicity, we'll assume that the charges have the same sign so that the force is repulsive.

We start with the electrostatic analogy to Eq. (50), namely, "Coulomb's Law", in which the electrostatic force between two charges q_1 and q_2 a distance d apart is given by

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 d^2}. \tag{62}$$

Here, ϵ_0 denotes the permittivity of the vacuum.

We proceed as before, considering the electrostatic force between an infinitesimal element of charge $\rho(x)\,dx$ situated at $x \in [-L/2, L/2]$ and the test charge q at $x = a$. Integration over the entire rod yields the net force

$$F = \frac{q}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\rho(x)}{(a-x)^2}\,dx. \tag{63}$$

This result may be compared with its gravitational counterpart in Eq. (54).

In the special case that the $\rho(x) = \rho_0$, a constant, we have the result,

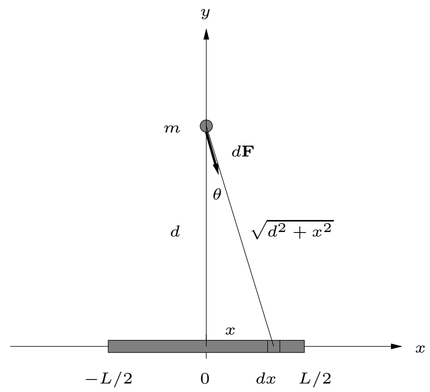
$$F = \frac{qQ}{4\pi\epsilon_0} \cdot \frac{1}{a^2 - L^2/4}, \tag{64}$$

where $Q = \rho_0 L$ is the total charge on the rod. This result may be compared to its gravitational counterpart in Eq. (59).

Appendix: Material covered in Monday, January 23, 2016 tutorial

Derivation of integral in Assignment No. 3, Question 7

Let $d\mathbf{F}$ denote the infinitesimal force exerted on the point mass m (at $(0,d)$) by an (infinitesimal) element $dm = \rho_0\, dx$ of the rod at position $x \in [-L/2, L/2]$, as sketched in the figure below.



The magnitude of this force is

$$\|d\mathbf{F}\| = \frac{Gm\,dm}{d^2+x^2} = \frac{Gm\rho_0}{d^2+x^2}\,dx\,. \tag{65}$$

Because of the symmetry of this problem, the net horizontal force will be zero – for each element $x \in [0, L/2]$, exerting a force with a positive x component, there is an element $x \in [-L/2, 0]$ exerting force with an equally negative x component. (We'll verify this statement later.)

As such, we'll consider only the y-component of the force $d\mathbf{F}$, i.e., its projection on the y-axis. Introducing the angle θ as shown in the figure, the magnitude of this component will be

$$\begin{aligned}
\|d\mathbf{F}\|\cos\theta &= \|d\mathbf{F}\|\,\frac{d}{\sqrt{d^2+x^2}} \\
&= \frac{Gm\rho_0}{d^2+x^2}\,\frac{d}{d^2+x^2} \\
&= Gm\rho_0 d\frac{1}{[d^2+x^2]^{3/2}}\,dx\,.
\end{aligned} \tag{66}$$

Integrating over all $x \in [-L/2, L/2]$, we arrive at the desired result,

$$F = Gm\rho_0 d\int_{-L/2}^{L/2} \frac{1}{[d^2+x^2]^{3/2}}\,dx\,. \tag{67}$$

Let's now go back and confirm that the net contribution of the force in the x direction is zero. The magnitude of the component in the x direction is

$$\begin{aligned}
\|d\mathbf{F}\| \sin\theta &= \|d\mathbf{F}\| \frac{x}{\sqrt{d^2+x^2}} \\
&= \frac{Gm\rho_0}{d^2+x^2} \frac{x}{d^2+x^2} \\
&= Gm\rho_0 d \frac{x}{[d^2+x^2]^{3/2}}\, dx\,.
\end{aligned} \tag{68}$$

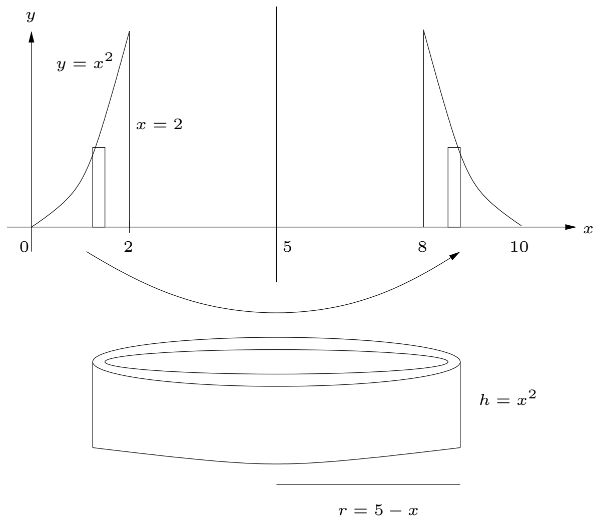
Integrating over all $x \in [-L/2, L/2]$, we obtain the integral

$$Gm\rho_0 d \int_{-L/2}^{L/2} \frac{x}{[d^2+x^2]^{3/2}}\, dx\,. \tag{69}$$

Since the integrand is an odd function of x, the integral vanishes.

Solid of revolution obtained by revolving about a line that is not the x or y axis

We consider the region D in the first quadrant enclosed by the curve $y = x^2$, the line $x = 2$ and the x-axis. We'll now rotation this region, not about the x or the y axis, but about the vertical line $x = 5$, to produce a solid of revolution S. The goal is to compute the volume V of this solid. The components are sketched below.



The approach is basically the same as for the problems considered earlier. We'll consider a vertical strip of width Δx in region D, as shown in the diagram, and revolve it about the line $x = 5$ to produce a cylindrical shell as shown in the figure. The volume of this shell will be

$$\begin{aligned}
\Delta V &= 2\pi r h \\
&= 2\pi(5-x)x^2\,\Delta x \\
&= 2\pi(5x^2 - x^3)\,\Delta x\,.
\end{aligned} \tag{70}$$

Or we can consider the infinitesimal volume element associated with this shell,

$$dV = 2\pi(5x^2 - x^3)\,dx\,. \tag{71}$$

The total volume will then be given by

$$\begin{aligned} V &= \int_0^2 dV \\ &= \int_0^2 (5x^2 - x^3)\,dx \\ &= 2\pi\left[\frac{5}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \\ &\vdots \\ &= \frac{56}{3}\pi\,. \end{aligned} \tag{72}$$

Lecture 10

Applications of integration (cont'd)

Centers of mass/centroids of planar regions

`Relevant section of Stewart: 8.3`

In this section, we consider the problem of finding the **centroid**, or "geometric center," of a bounded region D in the plane. If this region is considered to be a thin and homogeneous plate (i.e., constant density ρ - mass/area), then the centroid $(\bar{x}, \bar{y})$ is also the **center of mass** of the region. It is the point of balance of the region. Since the density is assumed to be constant we can, without loss of generality, set it to $\rho_0 = 1$.
It is instructive to recall a couple of basic results from the previous lecture.

1. For a distribution of **discrete masses** m_i on the x-axis, $1 \leq i \leq n$, situated at coordinates x_i, the (first) moment of the mass ensemble with respect to the point $x = 0$ is

$$M_y = \sum_{k=1}^{n} m_k x_k \,. \tag{73}$$

The reason that the subscript is y instead of x, is that the x_k represent the distances of the masses from the y-axis.

The center of mass $\bar{x}$ of this set of masses is determined by the so-called "zero moment" condition, i.e., the net (first) moment of the mass ensemble with respect to the point $x = \bar{x}$ is zero, i.e.,

$$\sum_{k=1}^{n} m_k (x_k - \bar{x}) = 0 \,. \tag{74}$$

From this condition, we can easily determine $\bar{x}$,

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} = \frac{M_y}{M} \,, \tag{75}$$

where

$$M = \sum_{k=1}^{n} m_k \tag{76}$$

is the total mass of the system.

2. For a **continuous distribution** of mass, as determined by a density function $\rho(x)$ for a thin rod on the interval $[a, b]$, the center of mass $\bar{x}$ is determined by the (continuous) zero-moment condition,

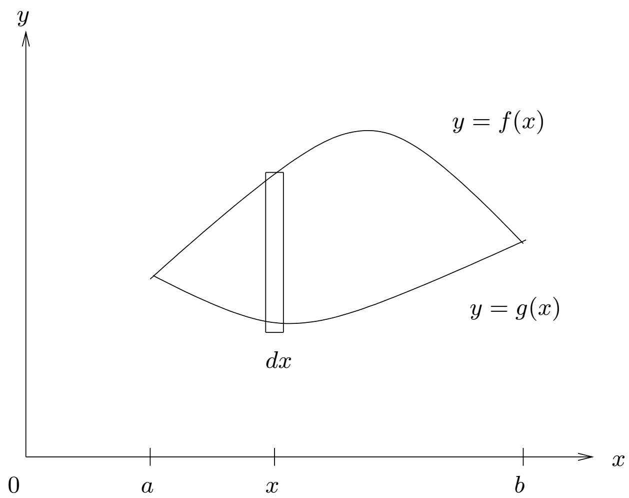
$$\int_a^b (x - \bar{x})\rho(x)\, dx = 0\,, \tag{77}$$

which allows us to solve for $\bar{x}$:

$$\bar{x} = \frac{\int_a^b x\rho(x)\, dx}{\int_a^b \rho(x)\, dx} = \frac{M_y}{M}\,. \tag{78}$$

Here, M_y and M are the continuous versions of the moment and mass, respectively, given earlier for the discrete case.

We now wish to carry these ideas over to the problem of finding the centroid/center of mass of a planar region D. As sketched below, suppose that D lies within the interval $a \le x \le b$ and that the boundary of region D is composed of an upper curve $y = f(x)$ and a lower curve $y = g(x)$ which meet at a and b. (They really don't have to, though).



We first try to determine $\bar{x}$, the x-coordinate of the centroid. First of all, let's compute the total moment M_y of the region with respect to the y-axis. It is convenient to do this using vertical strips of width dx centered at $x \in [a, b]$, as sketched in the figure.

The (infinitesimal) moment of this strip with respect to the y-axis is

$$dM_y = x dA\,, \tag{79}$$

where dA is the differential area of the strip. We can write this because all points on the strip are the same distance from the y-axis, i.e., x. And recalling 1A Calculus,

$$dA = [f(x) - g(x)]\, dx\,. \tag{80}$$

Therefore,

$$dM_y = x[f(x) - g(x)]\,dx\,. \tag{81}$$

To obtain the total moment of the region, we integrate over all vertical strips, i.e., over all $x \in [a, b]$:

$$\begin{aligned} M_y &= \int_a^b dM_y \\ &= \int_a^b x dA \\ &= \int_a^b x[f(x) - g(x)]\,dx\,. \end{aligned} \tag{82}$$

But just as in the discrete and continuous one-dimensional cases, we want to find the "balancing point" $\bar{x}$ so that the total moment with respect to $\bar{x}$ is zero, i.e.,

$$\begin{aligned} \int_a^b (x - \bar{x}) dA &= \int_a^b (x - \bar{x})[f(x) - g(x)]\,dx \\ &= 0\,. \end{aligned} \tag{83}$$

Once again, we can solve for $\bar{x}$:

$$\begin{aligned} \bar{x} &= \frac{\int_a^b x[f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx} \\ &= \frac{M_y}{A}\,, \end{aligned} \tag{84}$$

where

$$A = \int_a^b dA = \int_a^b [f(x) - g(x)]\,dx \tag{85}$$

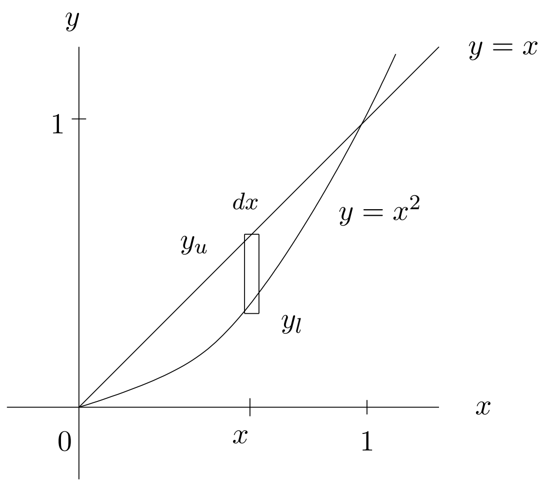
is the total area of region D.

That's fine - we have, at least in principle, computed $\bar{x}$. How do we compute $\bar{y}$? One way is to employ, if possible, horizontal strips as sketched below and integrate over y. But instead of doing this in detail and writing any more formulas, perhaps it's better to consider a concrete example.

Let us consider the region D which lies between the curves $y = x$ and $y = x^2$ in the first quadrant. (This region was used in the section on volumes of revolution.)

Method No. 1: We'll use strips that are perfectly suited for the computation of each coordinate of the centroid.

Computation of $\bar{x}$: We'll use vertical strips as in the discussion above. Consider a vertical strip of width dx at x as illustrated below.



The (infinitesimal) area of this strip is

$$\begin{aligned} dA &= (y_u - y_l)\,dx \\ &= (x - x^2)\,dx\,. \end{aligned} \tag{86}$$

Therefore, the total area of the region is,

$$\begin{aligned} A &= \int_0^1 dA \\ &= \int_0^1 (x - x^2)\,dx \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6}\,. \end{aligned} \tag{87}$$

We now compute the total first moment M_y of the region with respect to the y-axis. The infinitesimal moment dM_y of the vertical strip at x is

$$dM_y = x\,dA = x(x - x^2)\,dx\,. \tag{88}$$

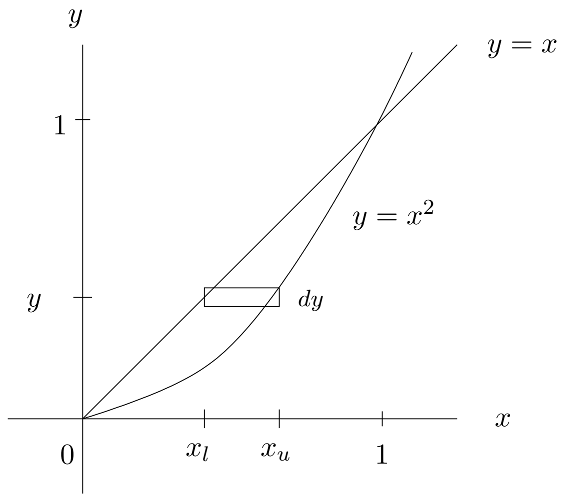
Remember that this is because every point on this strip lies a distance x from the y-axis. The total moment is then

$$
\begin{aligned}
M_y &= \int_0^1 x\,dA \\
&= \int_0^1 x(x - x^2)\,dx \\
&= \int_0^1 (x^2 - x^3)\,dx \\
&= \frac{1}{3} - \frac{1}{4} \\
&= \frac{1}{12}\,.
\end{aligned}
\tag{89}
$$

We now compute $\bar{x}$ as

$$
\bar{x} = \frac{M_y}{A} = \frac{1/12}{1/6} = \frac{1}{2}\,. \tag{90}
$$

Computation of $\bar{y}$: We'll now use horizontal strips. Consider a horizontal strip of width dy at y as sketched below.



The (infinitesimal) area of this strip is

$$
\begin{aligned}
dA &= (x_u - x_l)\,dy \\
&= (y^{1/2} - y)\,dy\,,
\end{aligned}
\tag{91}
$$

where we have used the formulas $y = x^2$ and $y = x$ to express the upper and lower x values of the strip in terms of y. Let's just check this result by using it to compute the total area A (which we know to be $\frac{1}{6}$.

$$\begin{aligned} A &= \int_0^1 dA \\ &= \int_0^1 (y^{1/2} - y)\, dy \\ &= \frac{2}{3} - \frac{1}{2} \\ &= \frac{1}{6}, \end{aligned} \tag{92}$$

which is the correct answer.

We now compute the total moment M_x using these strips:

$$\begin{aligned} M_x &= \int_0^1 y\, dA \\ &= \int_0^1 y(y^{1/2} - y)\, dy \\ &= \int_0^1 (y^{3/2} - y^2)\, dy \\ &= \frac{2}{5} - \frac{1}{3} \\ &= \frac{1}{15}. \end{aligned} \tag{93}$$

We now compute $\bar{y}$ to be

$$\bar{y} = \frac{M_x}{A} = \frac{1/15}{1/6} = \frac{2}{5}. \tag{94}$$

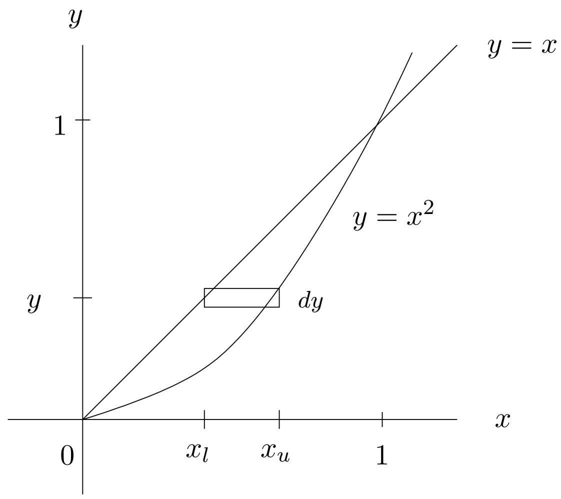
In summary, the centroid of the region D is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right). \tag{95}$$

It is easy to see that this point lies inside the region D: The $\bar{y}$ value $\frac{2}{5}$ lies above the y-value of $\frac{1}{4}$ for the curve $y = x^2$ and below the y-value of $\frac{1}{2}$ for the curve $y = x$.

Method No. 2: We'll now use strips that will seem to be completely counterintuitive for the computation of $\bar{x}$ and $\bar{y}$.

Computation of $\bar{x}$: We'll now use horizontal strips of width dy centered at a given y, as sketched below once again.



This seems quite strange since all points on this strip have different distances from the y-axis. How can we possibly compute the net moment dM_y of this strip with respect to the y-axis. A saving grace is provided by the assumption that the density ρ of the plate is constant. This means that the entire mass/area dA of the strip, spread out over an interval of x values, can be replaced by a point mass dA situated at the centroid of the strip, namely, the **midpoint** of the strip.

The midpoint of this strip is located at

$$\frac{1}{2}(x_{\mathrm{l}} + x_{\mathrm{u}}) = \frac{1}{2}(y + y^{1/2}) \,, \tag{96}$$

where we have once again used the relations $y = x$ and $y = x^2$ to express x in terms of y. The (infinitesimal) moment dM_y of this strip is then given by

$$\begin{aligned} dM_x &= (\text{midpoint of strip } (y))\, dA \\ &= \frac{1}{2}(y + y^{1/2})\, dA \\ &= \frac{1}{2}(y + y^{1/2})(y^{1/2} - y)\, dy \\ &= \frac{1}{2}(y^2 - y)\, dy \,. \end{aligned} \tag{97}$$

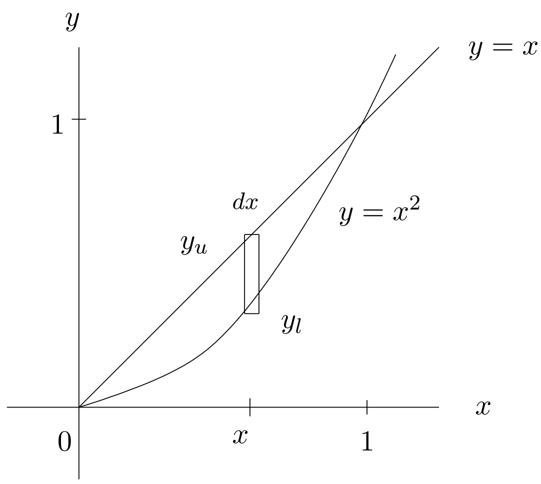
A product of the form $(A + B)(A - B)$ will always occur in this "counterintuitive" formulation, since the length of the interval is given by the difference, and the midpoint is given by (one-half times) the sum.

From the above result, we may easily compute the total moment M_x:

$$\begin{aligned}
M_x &= \int_0^1 \frac{1}{2}(y + y^{1/2})(y^{1/2} - y)\,dy \\
&= \frac{1}{2}\int_0^1 (y^2 - y)\,dy \\
&= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right) \\
&= \frac{1}{12}\,,
\end{aligned} \tag{98}$$

which is in agreement with the result obtained by Method No. 1.

Computation of $\bar{y}$: We'll now use vertical strips of width dx centered at a given x, as sketched below once again.



The basic strategy of using the midpoint of the strip was discussed earlier, so we'll simply proceed with the computations.

The midpoint of this vertical strip is located at

$$\frac{1}{2}(y_l + y_u) = \frac{1}{2}(x + x^2)\,. \tag{99}$$

The (infinitesimal) moment dM_x of this strip is then given by

$$\begin{aligned}
dM_y &= (\text{midpoint of strip } (x))\,dA \\
&= \frac{1}{2}(x + x^2)\,dA \\
&= \frac{1}{2}(x + x^2)(x - x^2)\,dx \\
&= \frac{1}{2}(x^2 - x^4)\,dx\,.
\end{aligned} \tag{100}$$

The total moment M_y is then given by

$$\begin{aligned} M_y &= \int_0^1 \frac{1}{2}(x^2 - x^4)\,dx \\ &= \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx \\ &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) \\ &= \frac{1}{15}\,, \end{aligned} \tag{101}$$

which is in agreement with the result obtained by Method No. 1.

The above example illustrates that either method – "intuitive" or "counterintuitive" – can be used, at least in principle, to obtain the area or the moments. That being said, in some problems one method may be much easier to use than the other.

Computation of arclength

`Relevant section of Stewart:  8.1`

The very brief and quick discussion near the end of the lecture period followed the textbook discussion quite closely. Very briefly, our goal is to compute, if possible, the length L of the curve $y = f(x)$ from $x = a$ to $x = b$.

Once again, the "Spirit of Calculus," we construct a partition of the interval $[a, b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$ by defining the partition points,

$$x_k = a + k\Delta x \qquad k = 0, 1, 2, \cdots, n\,. \tag{102}$$

Well let C denote the curve $y = f(x)$, i.e., the graph of f, from $x = a$ to $x = b$. Now define the following points on C,

$$\text{Point } P_k : \qquad (x_k, f(x_k))\,, \quad k = 0, 1, 2, \cdots, n\,. \tag{103}$$

We'll also let C_k denote the piece of curve C which connects point P_{k-1} to P_k, i.e., the portion of the graph of f which lies over the interval $I_k = [x_{k-1}, x_k]$.

The next step is to approximate the length L_k of curve C_k by the length of the straight line segment $\bar{P_{k-1}}P_k$, i.e.,

$$\begin{aligned} L_k &\approx \|P_{k-1}P_k\| \\ &= \sqrt{(\Delta x)^2 + (\Delta y_k)^2} \\ &= \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2} \\ &= \Delta x \sqrt{1 + \left[\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right]^2}\,. \end{aligned} \tag{104}$$

If we now assume that $f(x)$ is continuously differentiable on $[a, b]$, i.e., the function $f'(x)$ is continuous, then we may apply the Mean Value Theorem to the final term in the square root, i.e.,

$$\frac{f(x_k) - f(xk-1)}{x_k - x_{k-1}} = f'(x_k^*) \qquad \text{for some} \;\; x_k^* \in [x_k, x_{k-1}]\,. \tag{105}$$

The expression for L_k becomes

$$L_k \approx \Delta x \sqrt{1 + [f'(x_k^*)]^2}\,. \tag{106}$$

We now sum over all pieces C_k in order to obtain an estimate of the total length L of the curve C,

$$\begin{aligned} L &= \sum_{k=1}^{n} L_k \\ &\approx \sum_{k=1}^{n} \sqrt{1+[f'(x_k^*)]^2}\,\Delta x\,. \end{aligned} \tag{107}$$

The right side may be viewed as a Riemann sum of the function,

$$g(x) = \sqrt{1+[f'(x)]^2}\,, \tag{108}$$

where the x_k^* are sample points taken from the subintervals $I_k = [x_k, x_{k-1}]$. Since $f'(x)$ is assumed to be continuous, the function $g(x)$ is continuous. As such, we may conclude that in the limit $n \to \infty$, with $\Delta x \to 0$, the above Riemann sum converges to the desired arclength,

$$L = \int_a^b g(x) = \int_a^b \sqrt{1+[f'(x)]^2}\,dx\,.$$

We have arrived at our goal: If $f'(x)$ is continuous (or at least piecewise continuous), then the length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by the integral,

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx\,. \tag{109}$$

Lecture 11

Applications of integration (cont'd)

Computation of arclength (cont'd)

We continue our discussion of arclength with an example.

Example: Compute the length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

We recall the integral formula for the length L of the curve $y = f(x)$ from $x = a$ to $x = b$:

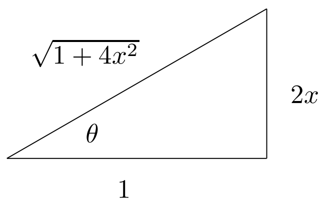
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,. \tag{110}$$

Here, $f(x) = x^2$, $a = 0$, $b = 1$ and $f'(x) = 2x$ so that

$$L = \int_0^1 \sqrt{1 + 4x^2}\, dx\,. \tag{111}$$

This integral can be evaluated by means of the following trigonometric substitution,

$$x = \frac{1}{2}\tan\theta \quad \Longrightarrow \quad dx = \frac{1}{2}\sec^2\theta\, d\theta\,, \tag{112}$$



so that the integrand is transformed to

$$\sqrt{1 + 4x^2} = \sqrt{1 + \tan^2\theta} = \sec\theta\,. \tag{113}$$

The indefinite integral is then transformed to

$$\begin{aligned} \int \sqrt{1 + 4x^2}\, dx &= \int (\sec\theta)\left(\frac{1}{2}\sec^2\theta\right) d\theta \\ &= \frac{1}{2}\int \sec^3\theta\, d\theta\,. \end{aligned} \tag{114}$$

We've encountered this integral in previous lectures:

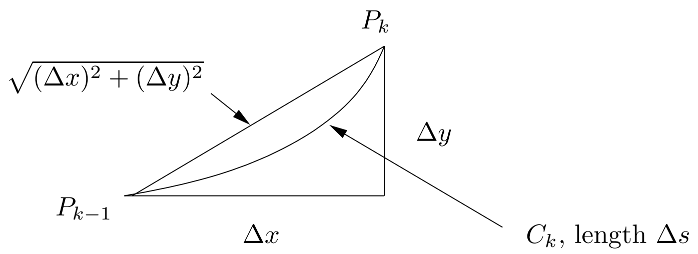
$$\int \sec^3\theta\, d\theta = \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]\,. \tag{115}$$

We use this result and resubstitute for x, using $\tan\theta = 2x$ and $\sec\theta = \sqrt{1+4x^2}$ to obtain,

$$\begin{aligned}
\int_0^1 \sqrt{1+4x^2}\,dx &= \frac{1}{4}\Big[2x\sqrt{1+4x^2} + \ln\left|\sqrt{1+4x^2}+2x\right|\Big]_0^1 \\
&= \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(\sqrt{5}+2) \\
&\approx 1.47894\,. \qquad (116)
\end{aligned}$$

This result "seems" to make sense since the length of the parabola from $(0,0)$ to $(1,1)$ is expected to be a little larger than the length, $\sqrt{2} \approx 1.414$, of the straight line between the two points.

Let us now return to the basic idea behind the derivation of the arclength formula: the approximation of the lenght of each subcurve C_k over the subinterval $[x_{k-1}, x_k]$ connecting points P_{k-1} and P_k with the length of the straight line connecting them:



If we let Δs denote the arclength of subcurve C_k, then this amounts to the aproximation,

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2\,. \qquad (117)$$

In the infinitesimal limit, these changes become infinitesimals and the relation becomes and equality,

$$ds^2 = dx^2 + dy^2 \qquad (118)$$

or

$$ds = \sqrt{dx^2 + dy^2}\,. \qquad (119)$$

The symbol ds represents the **infinitesimal element of arc** or simply **infinitesimal arclength**. The total length L of the curve C may be written as

$$L = \int_C ds\,. \qquad (120)$$

where $\int_C$ denotes integration over the curve C.

If we consider y, and therefore dy and ds to be a function of x, then the above relation may be written as

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{121}$$

If $y = f(x)$, then this relation becomes,

$$ds = \sqrt{1 + [f'(x)]^2}\, dx\,. \tag{122}$$

If curve C lies between $x = a$ and $x = b$, then the arclength L is given by the integral,

$$L = \int_C ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,, \tag{123}$$

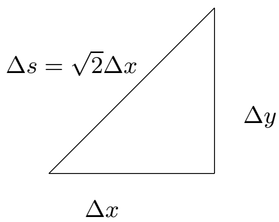
the formula which we obtained earlier.

Example: The simple function $f(x) = x$. Then $f'(x) = 1$ and

$$ds = \sqrt{1 + [f'(x)]^2}\, dx = \sqrt{2}\, dx\,. \tag{124}$$

This result shows how the arclength is related to the change dx in x. Of course, this is a consequence of the fact that the graph of $f(x)$ is the straight line $y = x$. Indeed, in this special case, because $f'(x)$ is constant, the above equality holds for non-infinitesimal changes, i.e.,

$$\Delta s = \sqrt{2}\Delta x\,. \tag{125}$$

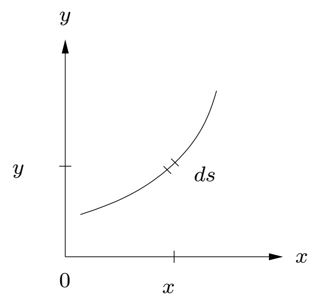


Example: The function $f(x) = x^2$. Then $f'(x) = 2x$ and we have (we have already seen this example),

$$ds = \sqrt{1 + 4x^2}\, dx\,. \tag{126}$$

The proportionality factor $\sqrt{1 + 4x^2}$ is, of course, not constant here. As x increases, this factor increases, which is reflected in the increasing steepness of the curve $y = x^2$ as x increases.

We may now employ the ideas used in our discussion of centroids for planar regions to compute centroids/centers of mass of curves. Consider a curve C as sketched below with infinitesimal element of arc ds situated at (x, y).



We'll assume that the curve represents a thin wire with linear density $\rho_0 = 1$ so that its mass $dm = ds$. Then the infinitesimal moments of this element dM_y and dM_x with respect to the y- and x-axes, respectively, are

$$dM_y = x\,ds \quad \text{and} \quad dM_x = y\,ds\,. \tag{127}$$

This implies that the total moments with respect to the y- and x-axes are

$$M_y = \int_C x\,ds \quad \text{and} \quad dM_x = y\,ds\,. \tag{128}$$

Now assume that $(\bar{x}, \bar{y})$ is the centroid/center of mass of curve C. Then the "balance conditions" for the centroid are given by

$$\int_C (x - \bar{x})\,ds = 0 \quad \text{and} \quad \int_C (y - \bar{y})\,ds = 0\,. \tag{129}$$

Let's take the first relation and rewrite it as

$$\int_C x\,ds - \bar{x}\int_C ds = 0 \quad \Longrightarrow \int_C x\,ds - \bar{x}L = 0\,, \tag{130}$$

from which we may solve for $\bar{x}$,

$$\bar{x} = \frac{1}{L}\int_C x\,ds\,, \quad L = \int_C ds\,. \tag{131}$$

In a similar fashion for $\bar{y}$, we obtain

$$\bar{y} = \frac{1}{L}\int_C y\,ds\,. \tag{132}$$

Example: We now return to the previous example, $y = x^2$ from $x = 0$ to $x = 1$. We computed the length L of this curve to be

$$\begin{aligned} L &= \int_C ds \\ &= \int_0^1 \sqrt{1+[f'(x)]^2}\, dx \\ &= \int_0^1 \sqrt{1+4x^2}\, dx \\ &\approx 1.47894\,. \end{aligned} \tag{133}$$

We now compute the moments of the curve and the corresponding centroid coordinates.

$$\begin{aligned} M_y &= \int_C x\, ds \\ &= \int_0^1 x\sqrt{1+[f'(x)]^2}\, dx \\ &= \int_0^1 x\sqrt{1+4x^2}\, dx \\ &= \frac{2}{3}\frac{1}{8}\left(1+4x^2\right)^{3/2}\Big|_0^1 \\ &= \frac{1}{12}\left[5^{3/2}-1\right] \\ &= 0.848\,. \end{aligned} \tag{134}$$

(Did you notice that one obtains the integral for M_y by inserting an x into the integrand of the integral for L presented earlier.) This implies that

$$\bar{x} = \frac{1}{L}M_y \approx \frac{0.848}{1.479} \approx 0.574\,. \tag{135}$$

Furthermore,

$$\begin{aligned} M_x &= \int_C y\, ds \\ &= \int_0^1 x^2\sqrt{1+[f'(x)]^2}\, dx \\ &\vdots \\ &= -\frac{1}{32}\ln 2 + \frac{9\sqrt{5}}{32} - \frac{1}{64}\ln\left[\frac{1}{2}+\frac{1}{4}\sqrt{5}\right] \\ &= 0.606\,, \end{aligned} \tag{136}$$

which implies that

$$\bar{y} = \frac{1}{L}M_x \approx \frac{0.606}{1.479} \approx 0.409\,. \tag{137}$$

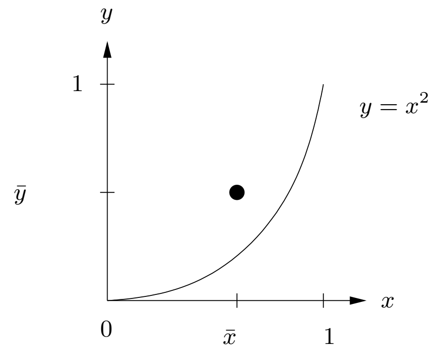
In summary, the centroid/center of mass of the curve is situated approximately at

$$(\bar{x}, \bar{y}) \approx (0.574, 0.409)\,. \tag{138}$$

Note that

$$\bar{y} \approx 0.409 > 0.25 \approx \bar{x}^2\,, \tag{139}$$

which implies that the centroid lies above the curve, as expected, since the parabolic curve is concave upward – see sketch below.



A final note: The moment integrals,

$$M_y = \int_C x\,ds \qquad M_x = \int_C y\,ds\,, \tag{140}$$

computed above are special cases of **line** or **path integrals** over the curve C. We can replace x and y in the integrands by general functions $g(x, y)$, i.e.,

$$\int_C g(x, y)\,ds\,, \tag{141}$$

with the understanding that the point (x, y) used to evaluate $g(x, y)$ must lie on curve C given by $y = f(x)$. If we wish to compute the above integral over the curve $y = f(x)$ from $x = a$ to $x = b$, then the integral to be computed is

$$\int_C g(x, y)\,ds = \int_a^b g(x, f(x))\sqrt{1 + [f'(x)]^2}\,dx\,. \tag{142}$$

This will be discussed in more detail in AMATH 231, “Vector Calculus and Fourier Series.”

Improper Integrals

`Relevant section of Stewart: 7.8`

Improper Integrals of Type I: Unbounded domains of integration

In many applications in science and engineering, we must integrate functions over infinite domains, e.g.,

$$I = \int_a^\infty f(x)\,dx\,, \quad J = \int_{-\infty}^a f(x)\,dx\,, \quad K = \int_{-\infty}^\infty f(x)\,dx\,. \tag{143}$$

These are called "improper integrals". In Stewart's text as well as many other books, these improper integrals are said to be **of Type 1**.

The natural question is, "How to we define or compute them?" (In other words, how do we evaluate antiderivatives at infinity?)

The answer is, "Using appropriate limits." For the integral I above, we define the **truncated integral** $I(b)$ as follows: For $b > a$,

$$I(b) = \int_a^b f(x)\,dx \quad \text{for} \quad b > a\,. \tag{144}$$

We then define the improper integral I as

$$I = \int_a^\infty f(x)\,dx = \lim_{b\to\infty} I(b)\,, \tag{145}$$

provided that the limit exists. If the limit exists, i.e., it is a finite number, then the improper integral I is said to be **convergent**. If it doesn't exist, then the improper integral I is said to be **divergent**.

Example 1: The improper integral,

$$\int_1^\infty \frac{1}{x^2}\,dx\,. \tag{146}$$

The associated truncated integral is

$$\begin{aligned} I(b) &= \int_1^b \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\Big|_1^b \qquad \text{(FTC II)} \\ &= 1 - \frac{1}{b}\,. \end{aligned} \tag{147}$$

Since

$$\lim_{b\to\infty} I(b) = \lim_{b\to\infty}\left[1-\frac{1}{b}\right] = 1\,, \tag{148}$$

the improper integral is convergent and its value is 1, i.e.,

$$\int_1^\infty \frac{1}{x^2}\,dx = 1\,. \tag{149}$$

We could, as is done in Stewart, perform the entire process in one sequence and avoid the writing of "$I(b)$" as follows,

$$\begin{aligned}
\int_1^\infty &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2} \\
&= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b \qquad \text{(FTC II)} \\
&= \lim_{b\to\infty}\left[1-\frac{1}{b}\right] \\
&= 1\,.
\end{aligned} \tag{150}$$

Example 2: The improper integral,

$$I = \int_1^\infty \frac{1}{x}\,dx$$

The associated truncated integral is

$$\begin{aligned}
I(b) &= \int_1^b \frac{1}{x}\,dx \\
&= \ln x\Big|_1^b \qquad \text{(FTC II)} \\
&= \ln b - \ln 1 \\
&= \ln b\,.
\end{aligned} \tag{151}$$

Since

$$\lim_{b\to\infty} I(b) = \lim_{b\to\infty} \ln b = +\infty \tag{152}$$

the improper integral is divergent. Once again, we could, as is done in Stewart, perform the entire process in one sequence and avoid the writing of "$I(b)$" as follows,

$$\begin{aligned}
\int_1^\infty &= \lim_{b\to\infty}\int_1^b \frac{1}{x} \\
&= \lim_{b\to\infty}\,[\ln x]_1^b \qquad \text{(FTC II)} \\
&= \lim_{b\to\infty}\,[\ln b - \ln 1] \\
&= \infty\,.
\end{aligned} \tag{153}$$

Important remark: Note that the function $f(x) = \dfrac{1}{x^2}$ decays more rapidly to 0 as $x \to \infty$ than the function $g(x) = \dfrac{1}{x}$. We'll see that the decay of the integrand $f(x)$ in the improper integral,

$$\int_a^\infty f(x)\,dx\,, \tag{154}$$

must be sufficiently rapid for the integral to converge.

Example 3: The improper integral,

$$I = \int_0^\infty e^{-x}\,dx\,. \tag{155}$$

Define its associated truncated integral,

$$\begin{aligned} I(b) &= \int_0^b e^{-x}\,dx \\ &= -e^{-x}\Big|_0^b \qquad \text{(FTC II)} \\ &= 1 - e^{-b}\,. \end{aligned} \tag{156}$$

Since

$$\lim_{b\to\infty} I(b) = \lim_{b\to\infty} [1 - e^{-b}] = 1\,, \tag{157}$$

the improper integral converges so that

$$\int_0^1 e^{-x}\,dx = 1\,. \tag{158}$$

Example 4: The improper integral,

$$I = \int_0^\infty \sin x\,dx\,. \tag{159}$$

Its truncated integral is

$$\begin{aligned} I(b) &= \int_0^b \sin x\,dx \\ &= -\cos x\Big|_0^b \\ &= 1 - \cos b\,. \end{aligned} \tag{160}$$

Since $I(b)$ oscillates between 0 and 2 as $b \to \infty$, $\lim\limits_{b\to\infty} I(b)$ does not exist. Therefore the improper integral diverges.

The improper integral,

$$J = \int_{-\infty}^a f(x)\,dx \tag{161}$$

may be examined by means of its improper integral,

$$I(b) = \int_b^a f(x)\,dx\,. \tag{162}$$

We then investigate whether or not $\lim_{b\to\infty} I(b)$ exists. See Example 2 of Stewart, Section 7.8, p. 529.

Finally, we must consider improper integrals over the entire real line, i.e.,

$$\int_{-\infty}^{\infty} f(x)\,dx\,. \tag{163}$$

For reasons that will become clear later, we should consider this integral in two parts by writing it as follows,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx \\ &= I_1 + I_2\,,\end{aligned} \tag{164}$$

where $a \in \mathbb{R}$ is a convenient point. If both of the improper integrals on the right converge, then the improper integral on the left converges.

Example 5: The improper integral,

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx\,. \tag{165}$$

We'll consider the following two improper integrals,

$$I_1 = \int_0^{\infty} \frac{1}{1+x^2}\,dx \tag{166}$$

and

$$I_2 = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx \tag{167}$$

We'll now use the quicker method:

$$\begin{aligned} I_1 &= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \mathrm{Tan}^{-1}(x)\Big|_0^b \\ &= \lim_{b\to\infty} \left[\mathrm{Tan}^{-1}(b) - \mathrm{Tan}^{-1}(0)\right] \\ &= \lim_{b\to\infty} \mathrm{Tan}^{-1}(b) \\ &= \frac{\pi}{2}\,. \end{aligned} \tag{168}$$

Therefore I_1 converges.

$$\begin{aligned}
I_2 &= \lim_{b\to-\infty} \int_b^0 \frac{1}{1+x^2}\,dx \\
&= \lim_{b\to-\infty} \mathrm{Tan}^{-1}(x)\Big|_b^0 \\
&= \lim_{b\to-\infty} \left[\mathrm{Tan}^{-1}(0) - \mathrm{Tan}^{-1}(b)\right] \\
&= \lim_{b\to-\infty} \left[-\mathrm{Tan}^{-1}(b)\right] \\
&= -\left(-\frac{\pi}{2}\right) \\
&= \frac{\pi}{2}\,. \qquad (169)
\end{aligned}$$

Therefore I_2 converges. As a result, the improper integral I converges,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi\,. \qquad (170)$$